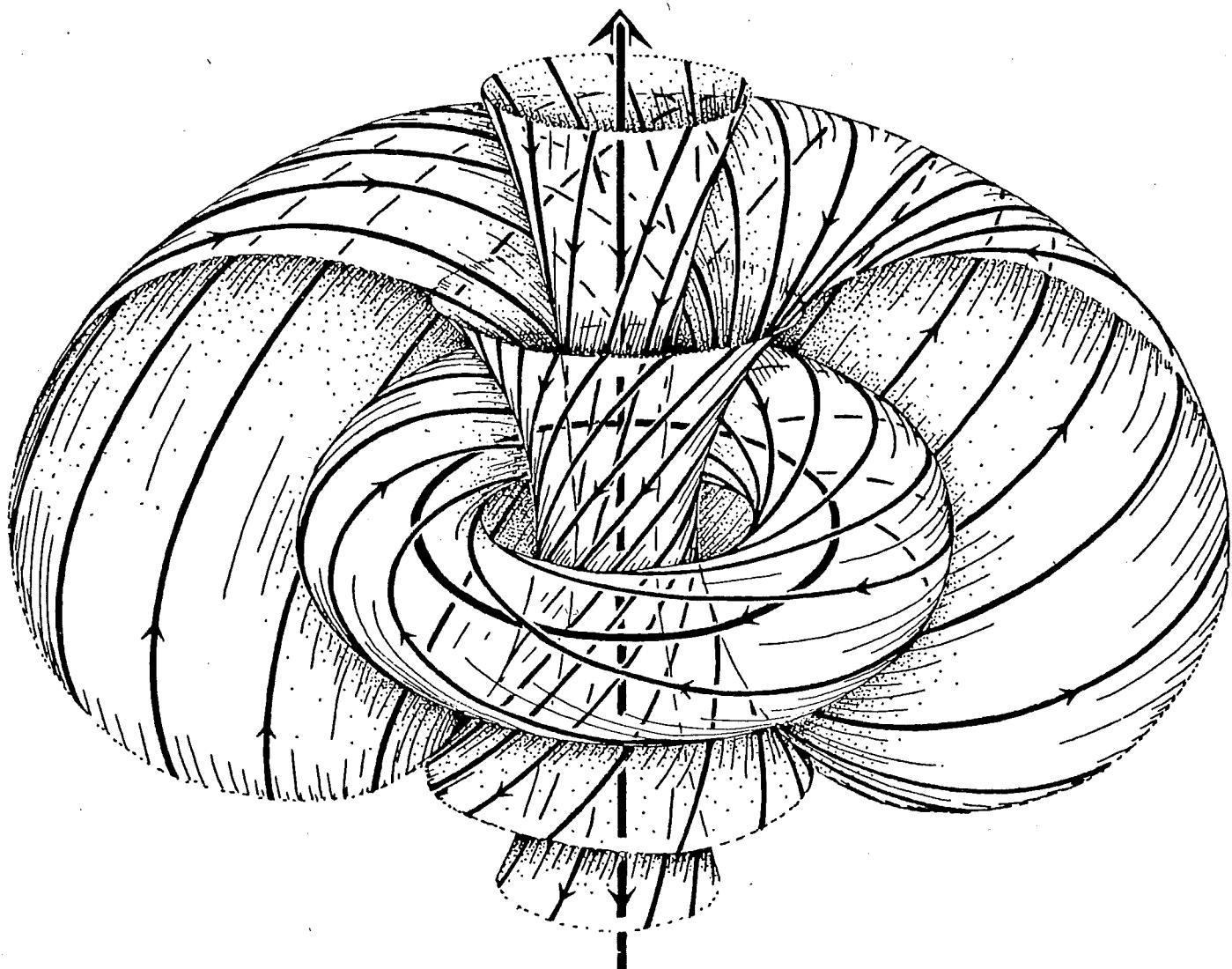


Twistor Newsletter

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Objective State-Vector Reduction?

It has been my opinion for a considerable number of years that the resolution of the problem of state-vector reduction in quantum-mechanics should be one of the major tasks of a (successful) quantum gravity theory. This seems to be very much a minority view. Most physicists seem reluctant to monkey with the standard rules of quantum theory (though they are less inhibited about monkeying with the rules of general relativity!). It seems that many physicists do not even accept that there is a problem of state-vector reduction! Here I shall just indicate some of my more recent thoughts on the subject, since there is a vague possibility of a link with twistor theory. Most of what I shall say is spelled out at rather greater length in my contribution to the O.Q.G.D.C. volume: 'Gravity and state-vector reduction' in Quantum Concepts in Space and Time, eds. C. J. Isham & R. Penrose (O.U.P., 1985).

I have written at length elsewhere (cf. R.P.: 'singularities and time-asymmetry' in General Relativity - an Einstein Centenary Survey, eds. S.W. Hawking & W. Israel (C.U.P., 1979); R.P.: 'Time-asymmetry and quantum gravity' in Quantum Gravity 2, eds. C.J. Isham, R. Penrose & D.W. Sciama (O.U.P. 1981)) explaining my views that the 2nd Law of Thermodynamics and cosmological uniformity must arise because of a constraint placed on initial space-time singularities (vanishing of Weyl curvature) which should be one implication of a (correct) quantum gravity theory; and that the 'other

side of the coin' is that quantum gravity should be playing an essential role in an objective state-vector reduction. An important ingredient of both these roles for quantum gravity would be a concept of gravitational entropy. The idea, in essence, is that a classical space-time geometry M would have (perhaps in relation to a spacelike hypersurface $H \subset M$) an entropy which measures (the expectation value of) the number of quantum states which contribute to the given classical geometry.

In relation to state-vector reduction, the idea is that a spreading wave function can (always will?) reduce as soon as by doing so, the gravitational entropy goes up by at least as much as the entropy comes down in the wave-function collapse. This is still rather vague, but a rough calculation can be done to see whether these ideas are 'on the right planet'. We anticipate some integral expression involving the Weyl curvature which should have, as two special cases, the Bekenstein-Hawking formula

$$\frac{1}{4} A \quad \dots \dots \dots \quad (1)$$

for the entropy of a stationary black hole of surface area A (where Boltzmann's constant, the velocity of light, $\frac{1}{2\pi} \times$ Planck's constant and the gravitational constant are all set equal to unity) and an expression for the 'graviton number' of a weak (dispersed) gravitational wave. For the latter, we envisage an integral over H of a 'density' of the general form

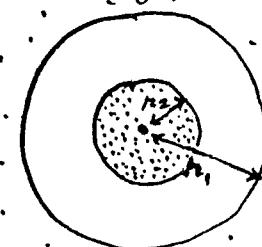
$$i \int \Psi^+ \iint \overline{\Psi^+} - i \iint \Psi^+ \int \overline{\Psi^+} \quad \dots \quad (2)$$

where Ψ^+ is the positive frequency part of the anti-self-dual linearized Weyl curvature.

Let us imagine the following situation: a nucleus emits a particle in a spherically symmetrical wave function, and this particle is then detected in a cloud chamber or bubble chamber. The detection involves a collapse of the wave function and the idea is that this collapse takes place objectively as soon as the clumping involved in the formation of the droplet or bubble entails an increase of gravitational entropy of at least the amount of the entropy reduction of the wave function collapse. Consider the contraction of a sphere of radius r_1 to one of radius r_2 (where, for simplicity, I provisionally assume that a vacuum is left between them). The effective gravitational entropy 'density' would appear to be something like $\frac{m}{r_2^2} \cdot \frac{m}{r_1}$ (since $\Psi \sim \frac{m}{r^3}$) where m is the mass inside r_2 — though it is quite unclear to me as yet how one could rigorously apply these ideas (or what to do with 'positive frequency' when sources are present) — which, when integrated between r_1 and r_2 , gives

$$\text{const.} \times m^2 \log\left(\frac{r_1}{r_2}\right). \quad \text{--- (3)}$$

This can't be quite right (e.g. for r_2 near the Schwarzschild radius we require $4\pi m^2$, not $m^2 \log(2m)$) but will serve as a rough guide. If we require a gravitational entropy of ~ 100 to reduce the spherical wave to a closely



linear track, we would need a single droplet size of something like a millimeter across — or a string of droplets of somewhat smaller size.

In the case of bubbles the result is similar since they can be considered as droplets of negative mass superimposed on a background of uniform density. The m^2 in the formula counts positively ('positive clumping') even when m is negative.

Though this is rather large, by comparison with one's prior expectations as to when systems become 'classical' it is not contradictory and not totally outlandish (i.e. 'on the right planet') — and, in view of all the uncertainties involved, I would call this 'reasonable encouragement' for the idea. (It is the logarithm in ③ that saves things. Gravitational self-energy, rather than entropy, would lead to $(\frac{1}{r_2} - \frac{1}{r_1})^{m^2}$ instead which would be much much too small!)

What has this got to do with twistor theory? Well, perhaps by analogy with the quasi-local mass expression, twistor ideas might turn out to be useful for defining a decent gravitational entropy formula.

Gravitational energy is already non-local but 2-surface twistor ideas seem to provide a means of getting at the concept in a quasi-local way.* Gravitational entropy should be 'one stage' less local (in the sense of one more \int in ②). Non-locality seems important for 'reduction' generally. Can twistors help here? ~Rigatree

*A comment on the status of quasi-local mass is in order. As of now, there is no evidence against the 'new improved' expression of TN1.8. But the proof of 'Property K' for Schwarzschild is still incomplete.

Quasi-local mass

In $\text{PTN } 18$, KPT¹ gave the results of some calculations that he and Ron Kelly had done on the quasi-local mass inside a small sphere in vacuum. I have checked their results by another method that can also be applied to small surfaces of other shapes.

Let Σ be a space-like hypersurface in a vacuum space-time, with unit normal t^A . Suppose that we have a family of 2-surfaces in Σ given by $f = \text{const.}$ and that we solve the 2-surface twistor equation on each surface. Then, provided that the solutions vary smoothly from surface to surface, we shall have

$$\nabla_{AA'} \omega^B = -i\varepsilon_A{}^B \pi_{A'} + (\nabla_{AA'} f) \Omega^B \text{ mod } t_{AA'} \quad (1)$$

for some Ω^B .

For the small sphere calculation, we can take $2f$ to be the square of the geodesic distance from a point $0 \in \Sigma$. Then

$$\nabla_{AA'} f = 0 \text{ and } \nabla_a \nabla_b f = -g_{ab} + t_a t_b \quad (2)$$

at 0. Of course, there is no obvious justification for the assumption that Ω^B is nonsingular at 0, although some thought shows that it is not implausible (and, in fact, may not be necessary).

Successive differentiation of (1) leads to a sequence of restrictions on the derivatives of Ω^B , in the same way that differentiating the 3-surface twistor equation leads to curvature obstructions. By taking two derivatives, one finds that the values of Ω^A and $\nabla_{AA'} \Omega^A$ at 0 may be chosen arbitrarily (this freedom arises because one can make independent linear transformations in the 2-surface twistor space on each $f = \text{const.}$), but that if

$$\alpha_{ABC} = -\sqrt{2} (\nabla_{(A}^A \Omega_{B)} C)_{A'}|_0, \quad (3)$$

then

$$\sqrt{2} t_{A'}^A \alpha_{ABC} = \frac{1}{3} H_{BCA'B'} \pi^{B'} - \frac{1}{3} F_{BCA'D} \omega^D \Big|_0 \quad (4)$$

where, with $V_{ABA'B'} = 2 t_{A'}^C t_{B'}^D \Psi_{ABCD}$,

$$H_{ABA'B'} = i (\bar{V}_{ABA'B'} - V_{ABA'B'}) \quad (5)$$

$$F_{ABA'C} = \nabla_{B'}(A V_B)_{CA'B'} + 2 t_{(A}{}^{E'} t_{B)}{}^{F'} \nabla_{BE'} V_{CA'F'}^B$$

The choices made for $\Sigma^A(0)$ and $\nabla_{AA'} \Sigma^A(0)$ do not affect the value of $A_{\alpha\beta} Z^\alpha Z^\beta$. One finds that

$$\begin{aligned} A_{\alpha\beta} Z^\alpha Z^\beta &= \frac{i}{2\pi} \int_{f \leq \varepsilon^2} \Psi_{ABCD} \omega^A (\nabla_C^C \omega^B) t^{C'D} d\tilde{\sigma} \\ &= -\frac{\sqrt{2} i V \varepsilon^2}{20\pi} \Psi_{ABCD} \omega^A \alpha^{BCD} \Big|_0 \end{aligned} \quad (6)$$

for the surface $f = \varepsilon^2$, which yields the (corrected version) of KPT's expression for $P_{AA'}$. Here $V = \text{Vol}\{f \leq \varepsilon^2\}$.

This can be repeated for a family of small ellipsoids, by replacing f by f' where

$$2 t_C^{A'} t_D^{B'} \nabla_{AA'} \bar{V}_{BB'} (f' - f) = P_{ABCD} = P_{(ABCD)} \quad (7)$$

In this case, (4) becomes

$$\begin{aligned} \sqrt{2} t_{A'}^A (\alpha_{EF(A} \beta_{B)C}^{EF} + \alpha_{ABC}) \\ = \frac{1}{3} H_{BCA'B'} \pi^{B'} - \frac{1}{3} F_{BCA'D} \omega^D, \end{aligned} \quad (8)$$

which determines α_{ABC} ; and (6) becomes

$$A_{\alpha\beta} Z^\alpha Z^\beta = -\frac{\sqrt{2} i V' \varepsilon^2}{20\pi} \Psi_{ABCD} \omega^A \alpha^{BCD}; \quad V' = \text{Vol}\{f' \leq \varepsilon^2\} \quad (9)$$

What about the 'improved' construction? It is straightforward to calculate KPT's function \det in this framework. Its gradient is given by taking the trace of the right-hand side of

$$\nabla_{AA'} \pi_B = -i V_{ABA'B'} \omega^B + i (\nabla_{BB'} f) \lambda_A{}^B{}_{A'} + i (\nabla_{BA'} f) \lambda_A{}^B{}_{B'} \text{ mod } t_{AA'} \quad (10)$$

considered as a linear function of $\pi_{A'}$, with $\omega^A = 0$.

The result of inserting det into the integrand in the definition of $A_{\alpha\beta} z^\alpha z^\beta$ as a 2-surface integral is to make $P_{AA'}$ zero in the case of a small sphere (to order ε^5), as KPT found. But when $\beta^{ABCD} \neq 0$, $P_{AA'}$ cannot be killed off in this way. In the special case that β^{ABCD} is of the form

$$\beta^{ABCD} = \beta^{0A 0B 0C 0D} + \bar{\beta}^{l_A l_B l_C l_D} \quad (11)$$

(where $0^A = \sqrt{2} t_A{}^A t_A{}^{A'}$), the 'improved' construction gives

$$P_A{}^{A'} = \frac{i \sqrt{\varepsilon^2}}{15\pi(3-\beta\bar{\beta})} \sum_{ABCD} t^{BA'} \beta^c{}_{EFG} H^{DEF} \quad (12)$$

where

$$H_{ABCD} = 2 t_c{}^{A'} t_D{}^{B'} H_{ABA'B'} \quad (13)$$

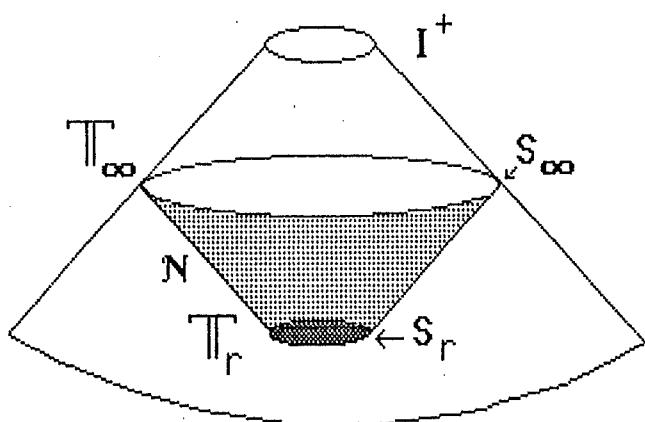
Apart from the fact that this leaves us with a generally nonzero $P_A{}^{A'}$ at order ε^5 , the result is not very transparent.

[1] 'More on Quasi-Local Mass'. K.P. Tod, in Twistor Newsletter No. 18, July 1984.

Nick Woodhouse.

2-Surface Twistors for Large Spheres

In TN 18 Paul Tod discussed the behaviour of 2-surface twistors on small spheres, including the effects of the contorsion factor, also described in TN 18 by Roger Penrose. In this note some results on the structure of 2-surface twistors on large spheres will be described. By a large sphere I mean the following : Let S_∞ be a cross-section of future null infinity I^+ , and let \mathcal{N} be the associated outgoing null surface. Choose an affine parameter r for \mathcal{N} asymptotic to an area coordinate for the 2-surface cross-sections (of constant r) S_r of \mathcal{N} . The picture to bear in mind is the following :



If one is given solutions to the 2-surface twistor equations on S_∞ then one can seek a power series expansion in r for the solutions on S_r . The contorsion factor, the twistor norm and the angular momentum twistor can then all be given in powers of r . The equations are difficult to solve in general, but a detailed analysis of all stationary Einstein-Maxwell space-times has been made. In this case, the ease with which one can solve the equations depends on the shear of S_∞ . So far, I have found solutions to first and second order in r^{-1} for S_∞ a general cross-section of infinity, but third order solutions have only been found for the case when S_∞ is a good cut.

The degree of contorsion depends on the magnetic multipole structure of the space-time. If J is the angular momentum then a scalar function J is given by $J = \underline{J} \cdot \underline{e}$, where \underline{e} is a triplet of basis functions given by $e_1 = \sin\theta \cos\phi$, $e_2 = \sin\theta \sin\phi$, $e_3 = \cos\theta$.

The contorsion factor C is then given by the following expression : $C = 1 + 6iJr^{-2} - 2i(B_M + 2qm)r^{-3} + O(r^{-4})$, where B_M is a function describing the total magnetic gravitational quadrupole, q is the charge and m (similarly to J) describes the magnetic dipole moment of the Maxwell field. Like all formulae in this note, it is correct to second order for arbitrary S_∞ and correct to third order provided S_∞ is a good cut.

The norm structure presents considerable problems. Firstly, suppose one computes the standard candidate :

$$H = H_{\alpha\alpha'} Z^{\alpha} \bar{Z}^{\alpha'} = \omega^A \pi_A + \bar{\omega}^{A'} \pi_{A'}$$

Then one can expand this in a sequence of matrices as :

$$H = H_{\alpha\alpha'}^{\infty} Z^{\alpha} \bar{Z}^{\alpha'} + r^{-2} H_{\alpha\alpha'}^2 Z^{\alpha} \bar{Z}^{\alpha'} + r^{-3} H_{\alpha\alpha'}^3 Z^{\alpha} \bar{Z}^{\alpha'}$$

in terms of suitable coordinates for the 2-surface twistor space on S_{∞} . The first term (the norm at infinity) is constant since the shear of S_{∞} is necessarily electric. However, the higher order terms are not constant on the sphere. In TN18 K.P.Tod suggested applying a complex conformal transformation to remove the non-constant terms. Here, if one chooses a rescaling given by :

$$\hat{\varepsilon}_{AB} = \Omega \varepsilon_{AB} ; \quad \hat{\varepsilon}_{A'B'} = \bar{\Omega} \varepsilon_{A'B'} ,$$

$$\text{where } \Omega = 1 - i[3Jr^{-2} - (B_M + 2qm)r^{-3}] ,$$

then the contorsion is removed and several of the terms in the norm are eliminated. However, to obtain a constant norm one must go further. In fact, in general one must apply

- (i) a complex supertranslation,
- (ii) a complex coordinate transformation.

The implications of (ii) are not fully understood, but at least one obtains a constant norm, which can be described as follows. Let S' be the unique shear-free cross-section of I^+ nearest to S_{∞} (with "nearest" defined in the conformal scale for I^+ induced by the Killing vector). One can regard S' as a cross-section of the I^+ of flat space-time, and is given as the intersection of the future light-cone of some point O with I^+ . One can extend the 2-surface twistors on S_{∞} to O using the twistor equation for flat space-time. This gives natural coordinates :

$$\overset{\infty}{Z}^{\alpha} = (\omega^A(0), \pi_A)$$

The norm is then given by :

$$g_{\alpha\alpha'} Z^{\alpha} \bar{Z}^{\alpha'} = G_{\alpha\alpha'} \overset{\infty}{Z}^{\alpha} \bar{Z}^{\alpha'} ,$$

$$G_{\alpha\beta'} = \begin{bmatrix} -3J_{AB}r^{-3} & S_A^B \\ S_B^A & 0 \end{bmatrix} .$$

Now consider the angular momentum twistor. To first order, the only contributions are from the electromagnetic stress tensor, and with 0 as above one finds :

$$A_{\alpha\beta} = \begin{bmatrix} 0 & t_A^{B'}(M-q^2/2r) \\ t_B^{A'}(M-q^2/2r) & 2i[\mu^{AB'}(0)-\frac{3q}{2r}\underline{\mu}^{AB'}(0)] \end{bmatrix}$$

where $Mt^{AA'}$ is the total energy-momentum, $\mu^{AB'}(0)$ is the total angular momentum and $\underline{\mu}^{AB'}(0)$ is the total electromagnetic dipole moment. Now let $D(0)$ and $d(0)$ denote the total mass and electric dipole moment respectively. Then the orbital and dipole angular momenta at radius r are:

$$\underline{J}(r) = \underline{J} - (2q/3r) \underline{m} ; \quad \underline{D}(r) = \underline{D}(0) - (2q/3r) \underline{d}(0).$$

The angular momentum twistor has 10 real components and the Penrose mass is just : $m_p = M - (1/2r)q^2$.

At second order gravitational non-linearities become important. The detailed structure of the angular momentum twistor depends on whether the original or "new improved" definition is applied. Suppose the integrand for the original construction is multiplied (c.f. K.P.T. and R.P. in TN18) by a factor

$$v = 1 - n + nC$$

so that $n=0,1$ corresponds to the original and modified construction respectively. One finds that the angular momentum twistor still has a zero in the top left corner, but the "momentum" component is now

$$P^a = t^a(M - (1/2r)q^2) + i(2n+1)MJ^ar^{-2} +$$

However, the imaginary term here can be removed by a transformation of the twistor coordinates for S_∞ . In any case, since $J_a t^a = 0$, the mass is still $m_p = M - (1/2r)q^2 + O(r^{-3})$, and this holds for all S_∞ .

The angular momentum components can also be evaluated. The results simplify considerably if one takes S_∞ to be shear-free. The second order contributions to the orbital and dipole terms are just :

$$\Delta \underline{J} = q(q\underline{J} - M\underline{m})r^{-2},$$

$$\Delta \underline{D} = q(q\underline{D} - M\underline{d})r^{-2} + [(1/3)\underline{m} \wedge \underline{d} + 3(n+1)\underline{J} \wedge \underline{D}]r^{-2}.$$

In the absence of an infinity twistor the identification of these terms as "angular momenta" is tentative, but the first terms in each come directly from the electromagnetic stress tensor, and are absent in flat space-time.

The combinations $(q\underline{J} - M\underline{m})$ and $(q\underline{D} - M\underline{d})$ are the 6 "Newman-Penrose" constants for the Maxwell field of a stationary space-time (2). Now we see that these quantities have a physical interpretation as part of the field's angular momentum.

At third order (S_∞ shear-free) the angular momentum twistor has 20 real components, but they are not algebraically independent. The details of a suitable untwisting procedure and the structure of the infinity twistor have yet to be worked out. At third order the NP constants also contribute to the quasi-local momentum terms. The mass can be computed, using the norm given above, and one finds

$$m_p = M - (1/2r)q^2 - (1/3)[\underline{m}^2 + \underline{d}^2]r^{-3} - 8\underline{J}^2r^{-3}.$$

The electromagnetic terms are those found in flat space-time, but the modified construction has a further contribution from the rotation of the space-time. A physical interpretation of the term $8\underline{J}^2r^{-3}$ has yet to be found.

An encouraging feature of these results is that when it can be computed, the mass formula displays the same dependence on S_∞ as one would expect in flat space-time. This compares very favourably with the results from some other definitions, where the mass is sensitive to the choice of S_∞ . For example, Ludvigsen and Vickers (1983)(3) have given a definition of a quasi-local momentum for cross-sections of outgoing null hypersurfaces. From this one can compute a mass (it makes no difference whether one computes the energy component or the momentum norm to these orders), and one finds,

$$m_{LV} = M - (1/2r)q^2 - [(1/3)(\underline{m}^2 + \underline{d}^2) + (1/4)(\underline{J}^2 + \underline{D}^2)]r^{-3},$$

where the third order terms are given for S_∞ shear-free. For example, in the Schwarzschild space-time the mass to third order is

$$m_{LV} = M - (1/4)\underline{D}^2 r^{-3}$$

which is sensitive to the choice of S_∞ . In the absence of a monopole energy density (as there is for the electromagnetic field) the presence of such a cross-section dependent term is unphysical.

For the Hawking (1968) definition (4) the sensitivity is worse. In this case one recovers the first order result $m_H = M - (1/2r)q^2$ only if S_∞ is a good cut.

Clearly the twistor analysis requires some more work. In particular, the structure of the infinity twistor requires elucidation. The extension of the results to the general case (radiating) is in progress. There are some similarities with the results of Paul Tod and Ron Kelly in this analysis, in that it is the magnetic parts of the curvature which are causing

all the problems. However, the details of the complex conformal transformations required are slightly different, in that unprimed and primed spinors have scalings which are conjugates here, but this is not the case in the computations described by Paul Tod.

Many thanks to Roger Penrose, Paul Tod, Ben Jeffryes and Ron Kelly for useful discussions.

William Shaw

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- 3) Ludvigsen, M. and Vickers, J.A.G. 1983. J. Phys. A 16, 1155-1168.
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SUPERSYMMETRY, TWISTORS, AND THE YANG-MILLS EQUATIONS

Michael Eastwood

Abstract This article investigates a supersymmetric proof due to Witten of the twistor description of general Yang-Mills fields due to Green, Isenberg, and Yasskin. In particular, some rigour is added and the rather complicated calculations are given in detail.

2 surface twistors and Killing vectors

So far little has been published on how the presence of Killing vectors affects two-surface twistor space, the only example being Paul Tod's work on spherically symmetric two-surfaces in his 'Some examples' paper (Tod 1983). If the Killing vectors are neither tangential nor orthogonal to the two-surface we still know nothing of their influence, however for a tangential (rotational) Killing vector or an orthogonal and hyper-surface orthogonal Killing vector simplifications result. Regrettably it appears that no similar simplifications hold if there is a two-surface orthogonal Killing vector which is not hypersurface orthogonal.

Before discussing Killing vectors some words on dual two-surface twistor space ($\mathbb{P}^*(\mathcal{S})$) are in order (see \mathbb{P}^N 16).

Two surface twistor space \mathbb{P} is the four dimensional (we hope) space of solutions to the equations

$$\mathcal{D}\omega^0 = \sigma'\omega^1 \quad \mathcal{D}\omega^1 = \sigma\omega^0 \quad 1,2.$$

Where ω^0 is of type $\{-1,0\}$, ω^1 of type $\{1,0\}$.

The corresponding \mathbb{P} 's are then defined by

$$-\bar{i}\mathbb{P}_{0^*} = \mathcal{D}'\omega^1 - p'\omega^0 \quad -\bar{i}\mathbb{P}_{1^*} = \mathcal{D}\omega^0 - p\omega^1 \quad 3,4$$

We may find expressions for $\mathcal{D}\mathbb{P}_{0^*}$ and $\mathcal{D}'\mathbb{P}_{1^*}$ by applying the commutator $[\mathcal{D}, \mathcal{D}']$ to ω^0 and ω^1 , but all we know of $\mathcal{D}'\mathbb{P}_{0^*}$ and $\mathcal{D}\mathbb{P}_{1^*}$ is that expressions of the form.

$$-i(\mathcal{D}'\mathbb{P}_{1^*} + \bar{s}\mathbb{P}_{0^*} - B'\mathbb{P}_{1^*}) = C'\omega^1 + D'\omega^0 \quad 5,6$$

$$-i(\mathcal{D}'\mathbb{P}_{0^*} + \bar{s}'\mathbb{P}_{1^*} - B\mathbb{P}_{0^*}) = C\omega^0 + D\omega^1$$

must exist for some $\bar{s}, \bar{s}', B, B', C, C', D, D'$, because \mathbb{P} is four dimensional. In a conformally flat space we would have $\bar{s} = \bar{\sigma}$, $\bar{s}' = \bar{\sigma}'$, $B = B' = 0$. Note that $B = \mathcal{D}' \ln \text{Det} \mathbb{P}$, $B = \mathcal{D} \ln \text{Det} \mathbb{P}$. \mathbb{P}^* is defined simply as the space algebraically dual to \mathbb{P} , ie $(\tilde{\omega}^A, \mathbb{P}_A)$ s.t.

$\sum = \omega^A \mathbb{P}_A + \tilde{\omega}^A \mathbb{P}_A$ is constant. Clearly in conformally flat space $\tilde{\omega}^A = \bar{\omega}^A$, $\mathbb{P}_A = \bar{\mathbb{P}}_A$, and more generally we find.

$$\mathcal{D}\tilde{\omega}^0 = \bar{s}'\tilde{\omega}^1 \quad \mathcal{D}'\tilde{\omega}^1 = \bar{s}\tilde{\omega}^0 \quad 7,8$$

and also $\mathcal{D}\mathcal{D}' \ln \text{Det} \mathbb{P} = \bar{\psi}_2 - \bar{\psi}_2 + \bar{\omega}^1 - \bar{s}s'$

Note that the r.h.s of 9 is in general complex.

Killing vectors.

Consider a Killing vector of the form

$$k_a = \bar{\zeta}' l_a + \bar{\zeta} n_a - \eta' m_a - \bar{\eta} \bar{m}_a$$

10.

The tangential components of the Killing vector equation $\nabla_a k_b = 0$ are

$$\bar{\zeta}'\eta = \sigma \bar{\zeta}' + \bar{\sigma}' \bar{\zeta}$$

$$\bar{\zeta}'\eta' = \bar{\sigma} \bar{\zeta}' + \sigma' \bar{\zeta}$$

$$\bar{\zeta}'\eta' + \bar{\zeta}\eta' = 2p \bar{\zeta}' + 2p' \bar{\zeta}$$

11,12,13

If the K.V. is tangential to \mathcal{S} then the r.h.sides of 11-13 vanish automatically, if orthogonal the l.h.sides vanish.

Tangential Killing vectors

If there is a Killing vector tangential to \mathcal{S} then we may make a choice of phase (except at the poles), such that $m_a - \bar{m}_a$ is aligned along the Killing vector direction, and thus regarding ϕ as the K.V. coordinate and θ as another suitably chosen coordinate we have

$$m^a \nabla_a = \frac{\partial}{\partial \theta} + \frac{i}{f(\theta)} \frac{\partial}{\partial \phi}$$

14.

Putting this into 11-13 we see that

$$\beta + \bar{\beta}' = \bar{\beta} + \beta'$$

15

It is convenient also to choose a boost such that

$$\bar{\beta} - \beta' = i \frac{d\gamma}{d\theta} = \bar{\beta}' - \beta \quad (\gamma \in \mathbb{R}).$$

16

$$\text{where } -2i \bar{\zeta} \bar{\zeta}' \gamma = \bar{\psi}_2 - \bar{\psi}_2' + \bar{\omega} - \bar{\omega}'$$

17

We write solutions to 1,2 as

$$\omega^\circ = e^{\pm \frac{i}{2}\phi} W_\pm^\circ(\theta) \quad \omega' = e^{\pm \frac{i}{2}\phi} W_\pm^1(\theta)$$

18

and thus obtain equations for $\frac{dW_\pm^\circ}{d\theta}$ in terms of W_\pm° and W_\pm^1 , and then write Π_0, Π_1 in terms of W_\pm°, W_\pm^1 . These expressions may be substituted into the two sides of 5,6 and coefficients compared. One finds

$$\sigma = \bar{\zeta}, \quad \sigma' = \bar{\zeta}', \quad B = -2(\bar{\beta}' - \beta), \quad B' = -2(\bar{\beta} - \beta')$$

19,20

Firstly we see that $\ln \text{Det} T \in i\mathbb{R}$, and thus $\text{Det} T$ has constant modulus. Comparing 1,2 with 7,8 we also find a direct correspondence between ω^A and $\tilde{\omega}^A$. We find solutions for $\tilde{\omega}^0, \tilde{\omega}^1$ are.

$$\tilde{\omega}^0 = e^{\mp \frac{1}{2}\psi} e^{-i\gamma} \omega_+^0, \quad \tilde{\omega}^1 = e^{\mp \frac{1}{2}\psi} e^{-i\gamma} \omega_+^1 \quad 21,22.$$

Since $\tilde{\omega}^0 = e^{\mp \frac{1}{2}\psi} \bar{\omega}_-^0$, etc this gives an isomorphism from the complex conjugate to the dual, and thus a norm on axis-symmetric two-surface.

If J is not only axisymmetric but spatial reflection symmetric (ie invariant under $\phi \rightarrow -\phi$) then the shear must be aligned along the chosen basis, and $\psi_2 \in \mathbb{R}$, and hence $\sigma = \bar{\sigma} = \bar{s}$, $\sigma' = \bar{\sigma}' = \bar{s}'$, $\text{Det} T$ is constant and J is uncontorted.

If we consider $A_{\alpha\beta}$ for axisymmetric J then clearly only four (complex) components survive. If reflection symmetry is added then these components are real and $A_{\alpha\beta}$ consists only of a real 4-momentum.

Hypersurface Orthogonal K.Vs orthogonal to J

Let us assume there is a hypersurface orthogonal K.v. orthogonal to J , ie in addition to $\nabla_a k_b = 0$ we have.

$$k^a * (\nabla_a k_b) = 0. \quad 23$$

and thus $k^a k^b * R_{cadb} = 0$ 24.

Let us assume k^a is timelike (the spacelike case involves a few sign changes), and fix the boost so that

$$k_a = \frac{1}{\sqrt{2}} |k| (l_a + n_a)$$

and thus from 11-13 & 24 we find.

$$\psi_2 - \bar{\psi}_2 = \psi_1 + \bar{\psi}_3 = \phi_{01} + \phi_{12} = 0 \quad 25$$

$$\sigma + \bar{\sigma}' = \rho - \bar{\rho}' = 0. \quad 26$$

and using 23 we find $\beta - \bar{\beta}' = 0$. 27.

Now compare the equations for T and \bar{T} .

$$\cancel{\partial} \omega^0 = -\bar{\sigma} \omega^1 \quad \cancel{\partial} \bar{\omega}^0 = -\sigma \bar{\omega}^1 \quad 28-31$$

$$\cancel{\partial} \omega^1 = \sigma \omega^0 \quad \cancel{\partial} \bar{\omega}^1 = \bar{\sigma} \bar{\omega}^0$$

since $\beta = \bar{\beta}'$ $\cancel{\partial}'$ acting on $\omega^0 = \cancel{\partial}'$ acting on $\bar{\omega}^1$, and thus if

$(\omega^0, \omega^1, \bar{\Pi}_0, \bar{\Pi}_1) \in T$, $(-\bar{\omega}^1, \bar{\omega}^0, \bar{\Pi}_1, -\bar{\Pi}_0) \in \bar{T}$. Since complex conjugates are both elements of T the same must be true for T^{**} , and thus

$S + \bar{S}' = 0$, and $\Im \operatorname{Det} \tilde{\Pi} = m S\bar{S} - \sigma\bar{\sigma} \in \mathbb{R}$ 32
 hence $\operatorname{Det} \tilde{\Pi}$ is real. In addition for either the modified or original version of $A_{\alpha\beta}$ (see Roger's article in $\tilde{\Pi}^N 18$). we see that if $\underline{W}_3^\alpha = \bar{Z}_1^\alpha$, $\underline{W}_4^\alpha = \bar{Z}_2^\alpha$

then $A_{\alpha\beta} \underline{Z}_1^\alpha \underline{Z}_2^\beta = A_{\alpha\beta} \underline{W}_3^\alpha \underline{W}_4^\beta$, and thus $A_{\alpha\beta}$ has four complex and two real components.

If the hypersurface orthogonality condition is relaxed then we still have $\sigma\sigma' = \bar{\sigma}\bar{\sigma}'$ (using 11,12), but we do not have the algebraic restriction on $\nabla_a k_b$ which enabled us to find that $\sigma + \bar{\sigma}' = 0 \Leftrightarrow \beta - \bar{\beta}' = 0$. More generally we may choose a boost so that $\sigma + \bar{\sigma}' = 0$, or we may choose another boost so that $\beta - \bar{\beta}' = i\gamma\delta$, $\gamma \in \mathbb{R}$; in order that we may find a similar relation between $\tilde{\Pi}$ and $\tilde{\Pi}'$ the two boosts must coincide. Writing $\nabla_a k_b = F_{ab}$ we find this condition is equivalent to

$$\gamma_a = \frac{k^b * F_{ab}}{k^c k_e} \text{ being a gradient, or } \nabla_{[a} \gamma_{b]} = 0$$

implying

$$0 = \|k\|^2 k^e k^f * R_{e[a b]f} + 2k^e F_{e[a} * F_{b]f} k^f \quad 33.$$

The term involving $* R_{eabf}$ vanishes if $k^a R_{ab} \propto k_b$, hence if this holds (eg if $R_{ab} = 0$) then a very stringent condition on $\nabla^a \|k\|$ remains, and most K.v.s orthogonal to J do not satisfy this condition.

I hope to write up all this work at greater length soon. Thanks to K.P.T for discussions.

Tod K.P 83 Proc R Soc Lond A 388, p 457

Ben Jeffye

An example of a two-surface twistor space
with complex determinant. Ben Jeffres

For terminology regarding the dual space see the other article by me
in this TN.

If we consider a two-surface near \mathcal{S} then we find that, to leading
order

$$\sigma = \frac{\sigma^0}{r^2}, \quad \sigma' = -\frac{\dot{\sigma}^0}{r}. \quad \psi_2 = \frac{\psi_2^0}{r^3}$$

$$\psi_2^0 - \bar{\psi}_2^0 = \bar{\sigma}^0 \dot{\sigma}^0 - \sigma^0 \dot{\bar{\sigma}}^0 + \gamma_0'^2 \sigma^0 - \gamma_0^2 \bar{\sigma}^0$$

where r is an affine parameter on the null ray approaching \mathcal{S} , and dot (\cdot)
denotes time derivative, and γ_0 & γ_0' are those for a metric sphere
ie $[\gamma_0, \gamma_0'] = g-p$.

On examining the equations obeyed by the dual shears \bar{s}, \bar{s}' we find
that

$$\bar{s} = \frac{\bar{s}^0}{r^2}, \quad \bar{s}' = \frac{\bar{s}'^0}{r}, \quad \sigma^0 = \gamma_0^2 \alpha, \quad \bar{s}^0 = \gamma_0'^2 \bar{\alpha}$$

$$\dot{\bar{s}}^0 = \gamma_0'^2 \dot{\bar{\alpha}}, \quad \bar{s}'^0 = -\gamma_0^2 \dot{\alpha}.$$

Substituting into the equation for $\text{Det} \bar{T}$ we find.

$$r^3 \gamma_0 \gamma_0' \ln \text{Det} \bar{T} = \gamma_0^2 \gamma_0'^2 (\alpha - \bar{\alpha}) - (\gamma_0^2 \alpha) (\gamma_0'^2 \bar{\alpha}) + (\gamma_0'^2 \alpha) (\gamma_0^2 \bar{\alpha}).$$

In general this is complex, but there are a number of circumstances where
it is imaginary (ie $\text{Det} \bar{T}$ has constant modulus)

i) non radiating $\alpha = 0$

$$\text{ii) axi-symmetry} \quad \frac{\gamma_0^2 \alpha}{\gamma_0'^2 \bar{\alpha}} = \frac{\gamma_0^2 \dot{\bar{\alpha}}}{\gamma_0'^2 \dot{\alpha}}$$

$$\text{iii)} \quad \frac{\alpha}{\bar{\alpha}} = \frac{\dot{\alpha}}{\dot{\bar{\alpha}}} = e^{2ik} \quad , k \in \mathbb{R}, \text{constant}$$

(Case iii) is interesting, since a norm naturally arises, which coincides with the usual norm if $\alpha = \bar{z}$ and $\beta_2 = \sqrt{2}^\circ$.

If $(\bar{\omega}^0, \bar{\omega}')$ is the principal part of a complex conjugate twistor then $(e^{ik}\bar{\omega}^0, e^{-ik}\bar{\omega}')$ is the principal part of a dual twistor, and

$$\sum - e^{-ik} (\omega^0 \bar{T}_0 + T_1 \bar{\omega}'') + e^{ik} (\bar{\omega}^0 T_0 + \omega' \bar{T}_1) \\ + ie^{ik} B \omega' \bar{\omega}'' + ie^{-ik} B' \omega'' \bar{\omega}'$$

is constant, where $B = \gamma \ln \text{Det} \bar{T}^1$, $B' = \gamma' \ln \text{Det} \bar{T}^1$.

Thanks to W.T. Shaw for the spin-coefficients

Ben Jeffys

Thickenings and Supersymmetric Extensions of Complex Manifolds

by Michael Eastwood and Claude LeBrun

Abstract: There is a close analogy between the theory of supermanifolds and the theory of thickened complex manifolds: abstract versions of codimension one infinitesimal neighbourhoods. Both theories are defined by augmenting the structure sheaf of an underlying complex manifold. Also, both structures naturally arise in twistor theory.

Starting with an arbitrary complex manifold, we develop cohomological machinery for investigating these augmentations. First we give a geometrical argument for the case of thickenings. Afterwards we derive the same results by using the Batchelor-Green approach to supermanifolds. Presumably the two cases can be unified.

The Twistor Transform and Conformally Invariant Operators

I shall try to outline a new proof of the twistor transform which makes direct use of the double-fibration $\mathbb{P} \xleftarrow{\cong} A \xrightarrow{\beta} \mathbb{P}^*$ of ambitwistor space. I shall also describe how the technique gives rise to (or at least hints at) the existence of) conformally invariant differential operators on vector bundles on spacetime.

(a). Basic Geometry

Consider the diagram

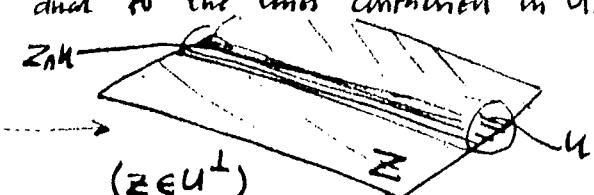
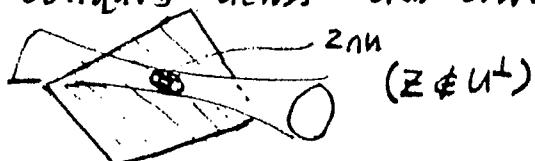
$$\begin{array}{ccc} \hat{U} & \xrightarrow{\alpha} & U \\ \bar{\alpha} \swarrow \quad \searrow \beta & \longleftarrow & \alpha \swarrow \quad \searrow \beta \\ \mathbb{P} & & \mathbb{P} \quad \mathbb{P}^* \end{array}$$

$\hat{U} = \beta^{-1}(U)$
 $\bar{\alpha}: \hat{U} \rightarrow \mathbb{P}$ subjective

where U is a nice open subset of \mathbb{P}^* (e.g. \mathbb{P}^{*-} , a neighbourhood of a line etc.) which corresponds to a nice convex open subset of \hat{U} . It is not hard to see that $\bar{\alpha} = \alpha|_{\hat{U}}$ is a map onto \mathbb{P} , because the fibre $\bar{\alpha}^{-1}(Z)$ is biholomorphic to $Z \cap U$ (regarding Z as a plane in \mathbb{P}^*) and this is always non-empty since U contains projective lines. Moreover, by considering the way in which the projective plane Z intersects U , it is seen that the fibre of $\bar{\alpha}$ varies as follows:

$$\bar{\alpha}^{-1}(Z) \text{ is homeomorphic to } \begin{cases} S^2 & \text{if } Z \in U^\perp \\ \{\text{point}\} & \text{otherwise.} \end{cases}$$

U^\perp is the subset of \mathbb{P} swept out by lines dual to the lines contained in U . For example, if U is a neighbourhood of a line in \mathbb{P}^* a plane can cut 'through' U : or obliquely across U as below:



Thus the generalised Penrose Transform as usually formulated does not apply because of the exciting topology of the fibres of $\bar{\alpha}$. This affects the 'push back mechanism' (whereby cohomology elements on \mathbb{P} are lifted to A) quite drastically, but a careful analysis of the situation enables one to derive the twistor transform.

(b) The map $\tilde{\alpha}$

The way in which cohomology on \mathbb{P}^n can be related by $\tilde{\alpha}$ to cohomology on $\hat{U} \subset A$ can be studied by putting back the Dolbeault resolution (E a holomorphic v.b. on \mathbb{P}^n)

$$0 \rightarrow \mathcal{O}(E) \rightarrow \mathcal{E}^{0,0}(E) \xrightarrow{\bar{\delta}} \mathcal{E}^{0,1}(E) \xrightarrow{\bar{\delta}} \mathcal{E}^{0,2}(E) \rightarrow \dots$$

on \mathbb{P}^n to the resolution

$$0 \rightarrow \tilde{\alpha}^{-1}\mathcal{O}(E) \rightarrow \tilde{\alpha}^{-1}\mathcal{E}^{0,0}(E)$$

on \hat{U} . It has been proved by Bartholomé [1] that if the fibres of $\tilde{\alpha}$ are topologically trivial, then this resolution will be acyclic so that there is an isomorphism on cohomology. In our situation, however, it can be proved that there's a long exact sequence [2]

$$\begin{aligned} 0 &\rightarrow H^2(\mathbb{P}^n, \mathcal{O}(E)) \rightarrow H^2(\hat{U}, \tilde{\alpha}^{-1}\mathcal{O}(E)) \rightarrow H^1(U^\perp, \mathcal{O}(E)) \rightarrow \\ &\quad \curvearrowright H^3(\mathbb{P}^n, \mathcal{O}(E)) \rightarrow H^3(\hat{U}, \tilde{\alpha}^{-1}\mathcal{O}(E)) \rightarrow H^1(U^\perp, \mathcal{O}(E)) \rightarrow \dots \quad (*) \\ &\quad \curvearrowright 0 \end{aligned}$$

This should be compared with the case of an ordinary S^2 -bundle $E \xrightarrow{\pi} M$ (M some smooth manifold) for which there is (usually) the Gysin sequence [3]

$$\dots \rightarrow H^k(M) \xrightarrow{\pi^*} H^k(E) \xrightarrow{\pi_*} H^{k-2}(M) \xrightarrow{\wedge \omega} H^{k+1}(M) \rightarrow \dots$$

relating the de Rham cohomology of E and M . Here, π^* is the natural pull-back, π_* is integration along the fibre and ω is the Euler class of E . Similar interpretations can be given to the maps occurring in $(*)$ but we shall not need to do this.

As a particular case of $(*)$ put $\mathcal{O}(E) = \mathcal{O}(k)$ and get

$$H^3(\hat{U}, \tilde{\alpha}^{-1}\mathcal{O}(k)) \cong H^1(U^\perp, \mathcal{O}(k)) \quad (k \geq -3)$$

$$0 \rightarrow H^3(\mathbb{P}^n, \mathcal{O}(k)) \rightarrow H^3(\hat{U}, \tilde{\alpha}^{-1}\mathcal{O}(k)) \rightarrow H^1(U^\perp, \mathcal{O}(k)) \rightarrow 0 \quad (\text{exact}) \quad (k < -3)$$

It is thus the purpose of the next section to relate the group $H^3(\tilde{\alpha}^{-1}\mathcal{O}(k))$ to cohomology on \mathbb{P}^n .

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(c) The Completion of the Twistor Transform

On \hat{U} we have the holomorphic relative de Rham sequence:

$$0 \rightarrow \tilde{\alpha}^{-1}(O(k)) \rightarrow \Omega_{\tilde{\alpha}}^0(k) \xrightarrow{d} \Omega_{\tilde{\alpha}}^1(k) \xrightarrow{d} \Omega_{\tilde{\alpha}}^2(k) \rightarrow 0$$

Abstract nonsense, in the shape of the Leray and hyperhomology spectral sequences tells us that

$$(i) \text{ If } k \leq -4, \quad H^3(\hat{U}, \tilde{\alpha}^{-1}(O(k))) = \ker\{H^1(U, \beta_*^2 \Omega_{\tilde{\alpha}}^0(k)) \rightarrow H^1(U, \beta_*^2 \Omega_{\tilde{\alpha}}^1(k))\}$$

$$H^3(\hat{U}, \tilde{\alpha}^{-1}(O(-3))) = H^1(U, \beta_*^2 \Omega_{\tilde{\alpha}}^0(-3))$$

$$(ii) \quad H^3(\hat{U}, \tilde{\alpha}^{-1}(O(-2))) = H^1(U, \beta_*^1 \Omega_{\tilde{\alpha}}^1(-2))$$

$$(iii) \quad H^3(\hat{U}, \tilde{\alpha}^{-1}(O(-1))) = H^1(U, \beta_*^0 \Omega_{\tilde{\alpha}}^2(-1))$$

$$\text{If } k \geq 0, \quad H^3(\hat{U}, \tilde{\alpha}^{-1}(O(k))) \cong \text{coker}\{H^1(U, \beta_*^0 \Omega_{\tilde{\alpha}}^1(k)) \rightarrow H^1(U, \beta_*^0 \Omega_{\tilde{\alpha}}^2(k))\}$$

Here $\beta_*^p \mathcal{G}$ for a sheaf \mathcal{G} on \hat{U} is the p^{th} direct image sheaf, i.e. the sheaf associated to the presheaf $V \mapsto H^p(f^{-1}(V), \mathcal{G})$ (V open in U). These computations are most conveniently done by making use of MGE's homogeneous bundle formalism [4]. An essential tool in the identification of the right hand sides of (i), (ii) and (iii) is the generalised de Rham sequence [5, 2] on P^*

$$0 \rightarrow V \rightarrow \Omega_V^0 \rightarrow \Omega_V^1 \rightarrow \Omega_V^2 \rightarrow \Omega_V^3 \rightarrow 0.$$

V is the constant sheaf of global sections of any homogeneous bundle on P^* and Ω_V^k is obtained by decomposing $\Omega^k \otimes V$ into irreducibles [4, 5]. Then the holomorphic de Rham sequence is obtained by putting $V = \mathbb{C}$, for example.

It can now be shown that

$$H^3(\hat{U}, \tilde{\alpha}^{-1}(O(k))) = H^1(U, \Omega^3(-k)) \quad \text{for } k = -3, -2, -1$$

and

$$(i) \text{ If } k = -n-2, \quad n \geq 2, \quad \beta_*^2 \Omega_{\tilde{\alpha}}^0(-n-2) = \Omega_{V_n}^{p+1} \quad \text{for appropriate } V_n;$$

$$(ii) \text{ If } k = n-2, \quad n \geq 2, \quad \beta_*^0 \Omega_{\tilde{\alpha}}^p(n-2) = \Omega_{W_n}^p \quad \text{for appropriate } W_n.$$

Thus

$$\left. \begin{aligned} H^3(\hat{U}, \tilde{\alpha}^{-1}(O(-n-2))) &= H^1(U, \Omega_{V_n}^0/V_n) \\ H^3(\hat{U}, \tilde{\alpha}^{-1}(O(n-2))) &= H^1(U, \Omega_{W_n}^3) \end{aligned} \right\} \quad n \geq 2 \quad (***)$$

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The remarkable thing is that W_n is such that $\Omega_{W_n}^3 = \Omega^3(-n+2) \cong \mathcal{O}(-n-2)$ while $V_n = \Gamma(\mathbb{P}^*, \Omega^2(n+2))$ and $\Omega_{V_n}^0 = \Omega^3(n+2) \cong \mathcal{O}(n-2)$, where \cong means "canonically isomorphic after choice of twistor ϵ ".

Next we use the fact that U is isomorphic to S^2 to deduce that

$$0 \rightarrow H^1(U, \Omega_{V_n}^0) \rightarrow H^1(U, \Omega_{V_n}^0 / V_n) \xrightarrow{\quad \parallel \quad} H^2(U, V_n) \rightarrow 0$$

is exact. So combining this with $(**)$ and $(***)$ we obtain

$$H^1(U^\perp, \mathcal{O}(n-2)) \cong H^1(U, \mathcal{O}(-n-2)) \quad n \geq -1 \quad (***)$$

and exact sequences

$$0 \rightarrow H^1(U, \mathcal{O}(n-2)) \rightarrow H^3(\hat{U}, \tilde{\alpha}^{-1}\mathcal{O}(-n-2)) \rightarrow \Gamma(\mathbb{P}^*, \mathcal{O}(n-2)) \rightarrow 0$$

$$0 \rightarrow H^3(\mathbb{P}, \mathcal{O}(-n-2)) \rightarrow H^3(\hat{U}, \tilde{\alpha}^{-1}\mathcal{O}(-n-2)) \rightarrow H^1(U^\perp, \mathcal{O}(-n-2)) \rightarrow 0.$$

By Serre duality, $H^3(\mathbb{P}, \mathcal{O}(-n-2)) \cong H^0(\mathbb{P}^*, \mathcal{O}(n-2))$. If the composite map

$$H^3(\mathbb{P}, \mathcal{O}(-n-2)) \rightarrow H^3(\hat{U}, \tilde{\alpha}^{-1}\mathcal{O}(-n-2)) \rightarrow \Gamma(\mathbb{P}^*, \mathcal{O}(n-2))$$

induces such an isomorphism, then a short exact sequence in diagram-chasing proves that the composite $H^1(U, \mathcal{O}(n-2)) \rightarrow H^1(U^\perp, \mathcal{O}(-n-2))$ is also an isomorphism. The lazy can also use $(****)$ and say 'By symmetry...'. Either way, we've arrived at the twistor transform.

(d) Conformally Invariant Operators

One of the interesting features of this approach is that one can see the isomorphism "fields \cong potentials/gauge" in the appearance of the columns in (b)(iii) above. Thus it would appear that the Minkowski space interpretation of the generalized de Rham sequence for W_n would be as follows:

$$\begin{array}{ccccccc} H^1(U, \Omega_{W_n}^1) & \rightarrow & H^1(U, \Omega_{W_n}^2) & \rightarrow & H^1(U, \Omega_{W_n}^3) & = & H^1(U, \mathcal{O}(-n-2)) \rightarrow 0 \\ \parallel & & \parallel & & \parallel & & \\ \{ \text{gauge} \} & & \{ \text{potentials} \} & & \{ \text{fields} \} & & \end{array}$$

However, careful analysis reveals that the space-time interpretation is that $H^1(U, \Omega_{W_n}^2) \cong \{ \text{potentials } \phi \text{ satisfying } D\phi = 0 \}$ and the "differentiation operator D " can often (always?) be modified to a conformally invariant operator. One of the authors (MGE) and RP think this has something to do with local twistor transport.

One example in which the details have been worked out in full is the following. On a space-time (\mathcal{M}, g) consider the operator

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$$D^a = \nabla_b (\nabla^b V^a + S^{ab}) : \Omega_{\text{gr}}^1 \rightarrow \Omega_{\text{gr}}^4$$

where

$$S^{ab} = -2R^{ab} + \frac{2}{3}Rg^{ab}.$$

Then application of the usual conformal variance formulae show that if $\hat{g}_{ab} = \Omega^2 g_{ab}$

$$\hat{D}^a \hat{\Phi}_a = \Omega^{-4} (D^a \Phi_a + 2r^b \nabla^a F_{ab})$$

where $F_{ab} = 2\nabla_{[a}\Phi_{b]}$. In particular, if Φ_a is a potential for a free Maxwell field, the equation

$$D^a \Phi_a = 0$$

is conformally invariant. See [6] for further details.

Michael Singer and Miles Eastwood
(in absentia)

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On Michael Murray's Twistor Correspondence

In a recent preprint [3] MKIII proves the following. Suppose $p(z_\alpha)$ is a complex homogeneous polynomial of degree k in $n+1$ variables and let P denote the corresponding homogeneous differential operator on \mathbb{C}^{n+1} . Suppose also that p defines without multiplicity a smooth subvariety $X \subset \mathbb{P}_n$. Let Z denote the total space of the hyperplane section bundle over X . To avoid degeneracies suppose $n \geq 2$. Then, if $k \leq n$, there is a natural isomorphism $H^{n-1}(Z, \mathcal{O}(-n-1+k)) \xrightarrow{\sim} \{\text{holomorphic } u \text{ s.t. } Pu=0\}$. This result provides a very pleasing extension of his motivating example which is that of a quadric in \mathbb{P}_2 . This example is the minitwistor description of harmonic functions.

MKIII's method of proof is to use the standard arguments (e.g. [2]) for: $X \times \mathbb{C}^{n+1}$
 In this diagram ν is projection on the second factor and μ is evaluation $\mathbb{C}^{n+1} \xrightarrow{\nu} Z$
 where \mathbb{C}^{n+1} is regarded as $T^*(\mathbb{P}_n, \mathcal{O}(1))$. In fact the restriction $k \leq n$ $\mathbb{C}^{n+1} \xrightarrow{\mu} Z$
 can be removed still using this method but, as shown below, there is an alternative approach which also allows X to be singular and unreduced. This alternative also allows a generalization in which p is no longer homogeneous.

So let $p(z_\alpha)$ be a polynomial of degree k for $z_\alpha \in V$ a complex vector space of dimension $n+1$. Write P for the corresponding differential operator on V^* . Write $\tilde{p}(z_\alpha, t)$, $(z_\alpha, t) \in V \otimes \mathbb{C}$, for p homogenized i.e. let $p = p_k + p_{k-1} + \dots + p_0$ (homogeneous bits) and $\tilde{p} = p_k + p_{k-1}t + \dots + p_0 t^k$. Let \tilde{P} denote the corresponding homogeneous differential operator on $\mathbb{P}(V^* \otimes \mathbb{C}) = \mathbb{P}^*$. Embed $V^* \hookrightarrow \mathbb{P}^*$ by $z^\alpha \mapsto [z^\alpha, 1]$ as usual and write I for the hyperplane "at infinity". I may also be regarded as a point in $\mathbb{P} \cong \mathbb{P}(V \otimes \mathbb{C})$. Let $Z = \{\tilde{p}=0\} - I$, noting that this agrees with the notation above for p homogeneous.

Now recall the twistor transform $\mathbb{I}: H^n(\mathbb{P}-I, \mathcal{O}(-n-1+l)) \xrightarrow{\sim} H^0(\mathbb{P}^*-I, \mathcal{O}(-l-1))$ for $l \geq 0$ (for $l < 0$ r.h.s. must be factored by $H^0(\mathbb{P}^*, \mathcal{O}(l-l-1))$) and for the minitwistor case this shows up as the correct additive constant freedom in case $l=-1$ (Abelian monopole). This is well-known in terms of twistor diagrams (see [4]) but can also be proved by the usual arguments (details in [1]). The twistor transform behaves much like the Fourier transform, turning multiplication by polynomials into the corresponding differential operators (often called "twistor quantization"). In particular, the short exact sequence of sheaves $0 \rightarrow \mathcal{O}(-n-1) \xrightarrow{\tilde{P}} \mathcal{O}(-n-1+k) \rightarrow \mathcal{O}_Z(-n-1+k) \rightarrow 0$ on $\mathbb{P}-I$ gives rise to the short exact sequence $0 \rightarrow H^{n-1}(Z, \mathcal{O}(-n-1+k)) \rightarrow T^*(\mathbb{P}^*-I, \mathcal{O}(-1)) \xrightarrow{\tilde{P}} T^*(\mathbb{P}^*-I, \mathcal{O}(-k-1)) \rightarrow 0$.

It remains to identify the differential operator \tilde{P} under the trivialization of $\mathcal{O}(1)$ on $V^* = \mathbb{P}^*-I$ given by $f(z^\alpha) \mapsto sf(z^\alpha/s)$ where (z^α, s) are coords for $V^* \otimes \mathbb{C}$. It is easy to verify that the answer is $P_k - P_{k-1}(X+1) + P_{k-2}(X+2)(X+1) - \dots + (-1)^k P_0(X+k)\dots(X+1)$ where X is the Euler operator $z^\alpha \partial / \partial z^\alpha$. It is amusing to compute for other homogeneities: $H^{n-1}(Z, \mathcal{O}(-n-1+l))$ for $l \geq k$ say. They are related by multiplication by t (which becomes $X + l - k + 1$ on V^*). In particular it turns out that if p is homogeneous then $H^{n-1}(Z, \mathcal{O}(-n-1+l)) \cong \{\text{Sol}'s \text{ of } Pu=0\}$ for all $l \geq k$. One can try the same story for other homogeneous bundles (rather than $\mathcal{O}(k)$). Will get systems of p.d.e.. Which ones?

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FORMAL NEIGHBOURHOODS, SUPERMANIFOLDS & RELATIVISED ALGEBRAS

formal neighbourhoods have been around for some time in Twistor Theory. Because of their successful use in the Witten et. al. description of Yang Mills fields, there has been some hope that they or their analogues might play some useful role in the curved case. Since then, of course, $P \times P^*$ does not exist, one has no space into which the space Λ of complex null lines in a curved M can naturally be embedded. Much work has thus been done on generalizing the notion of a formal neighbourhood of an embedded space. Two approaches have been followed: the first seeks an extension

$$0 \rightarrow \Theta_A \rightarrow \hat{\Theta}_A \rightarrow N \rightarrow 0$$

of what would be the normal bundle of an embedding by the tangent space of Λ (cf MGE + CleB [1]) whilst the second specifies that locally the algebra of functions on a formal neighbourhood be, eg., $\mathcal{O}[t]/t^{k+1}$ and then defines a sheaf of algebras globally by studying transition functions (using "non-abelian" sheaf cohomology). MGE[1]

A similar problem is encountered when one defines & studies supermanifolds - these are of current (?) interest in Physics & of computational interest in Twistor Theory. This problem has been studied by Batchelor[1] & Green[1], using the transition function approach.

There is a way of extending & "abelianizing" these concepts & problems. What one obtains is a relativised version of algebras, much like the relativised version of vector spaces - vector bundles.

1 AUGMENTED ALGEBRAS

Let (X, \mathcal{O}_X) be a ringed space, where \mathcal{O}_X is a sheaf K -algebras (K a field). All algebras are assumed to have a unity. The basic idea is to extend \mathcal{O}_X , usually by nilpotent elements. To this end, a sheaf of K -algebras \mathcal{Q} over X will be said to be augmented over \mathcal{O}_X if there is an epimorphism of sheaves of K -algebras, ε , and an exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{Q} \xrightarrow{\varepsilon} \mathcal{O}_X \rightarrow 0$$

It is immediate that \mathcal{J} is a sheaf of ideals in \mathcal{Q} - the augmentation ideal sheaf. One considers $\mathcal{Q}^{(p)} \stackrel{\text{def}}{=} \mathcal{Q}/\mathcal{J}^{p+1}$ ($p \geq 0$), $\mathcal{J}^{(p)} = \mathcal{J}^p/\mathcal{J}^{p+1}$ ($p \geq 1$), and obtains a series of exact sequences:

$$(1) \quad 0 \rightarrow f^{(p+1)} \rightarrow \mathcal{Q}^{(p+1)} \rightarrow \mathcal{Q}^{(p)} \rightarrow 0$$

Naturally, $\mathcal{Q}^{(p)}$ is a sheaf of K -algebras augmented over \mathcal{O}_X . For example, let $X \hookrightarrow Y$ be an embedding, take $\mathcal{Q} = \mathcal{O}_Y|_X$. Then $\mathcal{Q}^{(p)} = \mathcal{O}_{(p)}$, the p^{th} formal neighbourhood of X in Y .

(1) provides a decomposition of \mathcal{Q} , and one asks how to specify the extensions (1) so that \mathcal{Q} might be reconstructed from \mathcal{O}_X and the $f^{(p)}$. Indeed - given $\mathcal{Q}^{(p)}$, one asks for the obstruction and choice in constructing $\mathcal{Q}^{(p+1)}$. What follows is a theory of extensions like (1). Notice that $(f^{(p+1)})^2 = 0$ in $\mathcal{Q}^{(p+1)}$; such extensions are called singular extensions of sheaves of K -algebras. Let

$$(2) \quad 0 \rightarrow f \rightarrow \mathcal{B} \rightarrow \mathcal{Q} \rightarrow 0$$

be such an object. If also one has $0 \rightarrow f \rightarrow \mathcal{B}' \rightarrow \mathcal{Q} \rightarrow 0$, we will say that these are equivalent singular extensions if \exists isomorphism λ s.t.

$$\begin{array}{ccccccc} 0 & \rightarrow & f & \rightarrow & \mathcal{B} & \rightarrow & \mathcal{Q} \rightarrow 0 \\ & & \parallel & & \parallel & & \\ 0 & \rightarrow & f & \rightarrow & \mathcal{B}' & \rightarrow & \mathcal{Q} \rightarrow 0 \end{array}$$

commutes.

2 SINGULAR EXTENSIONS OF SHEAVES OF ALGEBRAS

The plan is to classify extensions (2) up to equivalence. This is done in two steps - local & global.

Definitions: Let \mathcal{Q}^{op} be the sheaf of algebras \mathcal{Q} , but with the multiplication reversed, and define the sheaf of K -algebras $\mathcal{Q}^e = \mathcal{Q} \otimes_K \mathcal{Q}^{\text{op}}$. There is a natural epimorphism $\mathcal{Q}^e \xrightarrow{\pi} \mathcal{Q}$ and one obtains an exact sequence (defining f)

$$\begin{aligned} 0 \rightarrow f \rightarrow \mathcal{Q}^e &\xrightarrow{\pi} \mathcal{Q} \rightarrow 0 \\ a \otimes b &\mapsto ab \end{aligned}$$

f is natural a sheaf of \mathcal{Q}^e modules, and so is \mathcal{Q} . One makes similar definitions given a K -algebra A and an A (hence A^e) bimodule (hence module) I . Noting that A is an A^e module, one has:

LEMMA Singular extensions of algebras:

$$0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$$

are classified by $\text{Ext}_{A^e}^2(A, I) \cong \text{Ext}_{A^e}^1(J, I)$, that is, by the second Hochschild cohomology of A with coefficients in I .

(Proof: standard theory - see e.g. Cartan & Eilenberg [1])

This lemma specifies how local extensions are classified; globally one has:

THEOREM If \mathcal{F} an \mathcal{F} -affine cover $\{U_\alpha\}$ for X (i.e. a cover such that $H^k(U_\alpha, \mathcal{F}) = 0$, $k \neq 0$), then singular extensions of sheaves of algebras

$$0 \rightarrow \mathcal{F} \rightarrow B \rightarrow \mathcal{O} \rightarrow 0$$

are classified (up to equivalence) by $\text{Ext}_{\mathcal{O}^e}^1(X; \mathcal{F}, \mathcal{F})$.

Note: the existence of the \mathcal{F} -affine cover is a device to preserve the "local triviality" of \mathcal{O} in B

Before proving the theorem, here is its crucial

COROLLARY There is an exact sequence

$$0 \rightarrow H^1(\text{Hom}_{\mathcal{O}^e}(\mathcal{F}, \mathcal{F})) \rightarrow \text{Ext}_{\mathcal{O}^e}^1(X; \mathcal{F}, \mathcal{F}) \xrightarrow{\delta} \Gamma(\text{Ext}_{\mathcal{O}^e}^1(\mathcal{F}, \mathcal{F})) \xrightarrow{\delta} H^2(\text{Hom}_{\mathcal{O}^e}(\mathcal{F}, \mathcal{F}))$$

(cf Godement [1]), and an identification

$$\text{Hom}_{\mathcal{O}^e}(\mathcal{F}, \mathcal{F}) \cong \text{Derivations } (\mathcal{O}, \mathcal{F})$$

This sequence enables us to link local + global data thus

- (i) if $[I]$ $\in \text{Ext}_{\mathcal{O}^e}^2(\mathcal{F}, \mathcal{F})$ is a singular extension of algebras, then $\mathcal{O}[I]$ gives, via the lemma, the local extension of algebras
- (ii) $H^1(\text{Der } (\mathcal{O}, \mathcal{F}))$ classifies those extensions locally yielding the same extension of algebras
- (iii) If $\gamma \in \Gamma(\text{Ext}_{\mathcal{O}^e}^1(\mathcal{F}, \mathcal{F}))$ specifies the local extension desired, then $\delta \gamma \in H^2(\text{Der } (\mathcal{F}, \mathcal{F}))$ is the obstruction to using it

to construct a singular extension of sheaves of algebras.

EXAMPLE FORMAL NEIGHBOURHOODS

if N is a p -dimensional vector bundle on X (to mimic the normal bundle of an embedding) and $\mathcal{J}(P) \cong \mathcal{O}_P(N^*)$ then if $\Omega^{(k)}$ has been constructed, one has

$$\text{Der}(\Omega^{(k)}, \mathcal{J}(k+1)) \simeq \begin{cases} \mathcal{O}_X \otimes N^* & (k=0) \\ (\mathcal{O}_X + \mathcal{O}_X^k) \otimes \mathcal{O}_X^{k+1} N^* & (k \geq 1) \end{cases}$$

This & the corollary yields & extends the results of MGE & CLeB.

Note: As it stands, the above description does not apply directly to the case of superalgebras. A sheaf Ω of superalgebras is a sheaf of augmented K -algebras which admits a \mathbb{Z}_2 grading - i.e. a direct sum decomposition $\Omega = \Omega_0 \oplus \Omega_1$. A singular extension of sheaves of graded algebras is then obviously defined, with the proviso that \mathcal{J} be \mathbb{Z}_2 graded. If in the above one replaces the various functors involved (Hom, Ext etc.) by their \mathbb{Z}_2 graded preserving versions, then the lemma, theorem & corollary hold.

EXAMPLE SUPERMANIFOLDS

if $\mathcal{F} \downarrow X$ is a vector bundle over X , one might take $\mathcal{J}^{(k)} = \Lambda^k \mathcal{F}$ and specify the local extension of algebras by requiring locally

$$\Omega^{(k+1)} \simeq \bigoplus_{l=0}^{k+1} \Lambda^l \mathcal{F}$$

(thus $\Omega^{(k)}$ will be graded commutative - an equivalent requirement is that $\Omega^{(k)}$ be graded commutative and that $K^{(k)}$ be the ideal generated by $\Omega_1^{(k)}$, where $K^{(k)}$ is the augmentation kernel of $\Omega^{(k)}$ over (\mathcal{O}_X) .) This construction yields the usual graded commutative supermanifolds of Kostant [1]. Grading $\mathcal{J}^{(k)}$ odd or even according as k is odd or even, one obtains

$$\begin{aligned} \text{Der}_{\mathbb{Z}_2}(\Omega^{(k)}, \mathcal{J}(k+1)) &\simeq \begin{cases} \text{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{J}(k+1)) & (k+1) \text{ odd} \\ \text{Der}_{\mathcal{O}}(\Omega, \mathcal{J}(k+1)) & (k+1) \text{ even} \end{cases} \end{aligned}$$

This, together with the amended theorem yields the results of Batchelor [1] & Green [1] in the cases $(X, \Omega_X) = (X, E_X)$ (smooth category, when the sheaves $\text{Der}(\Omega^{(k)}, g(k+1))$ are all fine so globally $\Omega^{(k)} \cong \bigoplus_{i=0}^k \Lambda^i \mathbb{Z}$) and the \mathbb{Z}_2 holomorphic category (resp.).

Sketch of proof of theorem

On the set U_α , the lemma yields a $1 \leftrightarrow 1$ correspondence between equivalent extensions

$$(3) \quad 0 \rightarrow \mathcal{G}(U_\alpha) \rightarrow \mathcal{B}(U_\alpha) \rightarrow \mathcal{A}(U_\alpha) \rightarrow 0$$

and equivalent extensions of $\Omega^e(U_\alpha)$ modules

$$(4) \quad 0 \rightarrow \mathcal{G}(U_\alpha) \rightarrow \hat{\mathcal{G}}(U_\alpha) \rightarrow \mathcal{G}(U_\alpha) \rightarrow 0$$

The theorem is proved by showing first that patching maps to construct \mathcal{B} globally, which must preserve products naturally yield patching maps to construct $\hat{\mathcal{G}}$ globally; these must preserve the local extensions and the Ω^e module structure. This establishes that given a singular extension of sheaves of K -algebras (up to equivalence class),

$$(5) \quad 0 \rightarrow \mathcal{G} \rightarrow \mathcal{B} \rightarrow \mathcal{A} \rightarrow 0$$

one has (up to equivalence class) an extension of sheaves of Ω^e modules

$$(6) \quad 0 \rightarrow \mathcal{G} \rightarrow \hat{\mathcal{G}} \rightarrow \mathcal{G} \rightarrow 0$$

Second, one shows the reverse to be true. Since extensions (6) are classified by $\text{Ext}_{\Omega^e}^1(\mathcal{G}, \mathcal{G})$, this proves the theorem.

Comments: the transition functions for \mathcal{A} & \mathcal{G} determine those for \mathcal{B} , and one recovers directly the form of transition function given e.g. by MGE in TN 17 or in his preprint with CLeB. These enable one to describe directly the Bockstein maps S in long exact sequences of cohomology arising from the sequences (1). This should help in again describing the map S in the sequence

$$0 \rightarrow H^1(A, \Omega_{(3)}) \rightarrow H^1(A, \Omega_{(2)}) \xrightarrow{S} H^2(A, \Omega_{(-3, -3)}) \cong \ker J^\alpha \rightarrow \nabla^\alpha J_\alpha$$

↑ ↑ ↑
EM field Gauge field current of
 gauge field

and its generalization to gauge fields. It should now also be possible to describe the obstruction to extension of formal neighbourhoods on curved ambitwistor space directly. Indeed, since the local extension to a first formal neighbourhood is trivial, its corresponding element in $\Gamma(\text{Ext}^1(f, f))$ is zero so that elements of $H^1(\text{Der}(O, f))$ directly give the possible extensions. In the flat case, $H^1(A, \text{Der}(O, O(-1, -1))) \cong H^1(A, \oplus(-1, -1)) \cong \mathbb{C}$, and there is a natural choice, which does give the usual result. Indeed one can actually describe the element. Let $(Z^A, W_A) = ([w^A, \pi_{A'}], [\xi^A, \eta_A])$, let $P_\pi + P_\eta$ be, resp. the generators (locally) of $H^1(A, \mathcal{O}(-2, 0))$ and $H^1(A, \mathcal{O}(0, -2))$. Then $\gamma = P_\pi \pi^{A'} \frac{\partial}{\partial \xi^{A'}} + P_\eta \eta^A \frac{\partial}{\partial w^A}$ (which can be extended to all of A) is a representative of the class of the element.

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31

Homogeneous Bundles and deBruin's "Einstein Bundle"

In [1] CleBrun introduces E , the Einstein Bundle, over curved ambitwistor space A and shows that if E admits a section (globally) then the curved space M associated with A is conformally Einstein. Stalkwise, E is defined by:

$$E_{\gamma} = \{ f \in \mathcal{O}[I] |_{\gamma} \mid D^2 f + \frac{1}{2} R(x, x) f = 0 \}$$

for x tangent to γ , $D \equiv x \lrcorner \nabla$, $Dx = 0$. The properties of E depend crucially on the operator $D^2 + \frac{1}{2} R(x, x)$, which may seem to be rather "rigged". Actually, it arises naturally from a generalization to the curved case of the homogeneous bundles of [2]. In [1], CleB gives natural definitions of the analogues of $(1|0, 0|0)$ and $(0|0, 0|-1)$ and defines also a bundle F , as the cohomology of the monad

$$(1|0, 0|0) \xrightarrow{j} V \xrightarrow{r} (0|0, 0|1)$$

where V is defined, stalkwise, by covariant constant sections up T of sections of the local twistor bundle, I , on M . A moment's thought makes one realise that F is actually the curved analogue of $(0|0, 1|0)$! There are, similarly, generalizations of the homogeneous bundles on $F_{1|2|3}$. The techniques of [2] (which are mainly of a homological nature) extend happily. One of these [3] is the generalized DeRham sequence, examples of which are

$$\begin{aligned} 0 &\rightarrow \mu^{-1}(0|0, 1|n) \rightarrow (0|0|1|n) \xrightarrow{\delta^2} (0|2|-1|n) \rightarrow 0 & -(1) \\ 0 &\rightarrow \mu^{-1}(-n|-1, 0|0) \rightarrow (-n|-1|0|0) \xrightarrow{\delta^2} (-n|1|-2|0) \rightarrow 0 & -(2) \end{aligned}$$

for $n=1$, $-\delta^2$ is precisely $D^2 + \frac{1}{2} R(x, x)$! This makes $E \cong (0|0, 1|1)$ as claimed by deBruin (+ manifestly conformally invariant). $E' \cong (-1|-1, 0|0)$ gives the primed conformal weight version of CleB's work. Lastly, for $n \leq -1$, (1) & (2) are principal in the determination of choice / obstruction in finding formal neighbourhoods of A . [4]. For $n \neq 1$, up T , D becomes \tilde{D} (thorn), with spin weighted quantities to act on.

[1] Ambitwistors & Einstein's Equations; CleB preprint.

[2] The Generalized Penrose Ward Transform; MGE, Proc. Camb. Phil. Soc., JAN 1985.

[3] Duality for Homog. Bundles on Twistor Space; MGE, to appear in Jour L.M.S.

[4] RB, this TN1.

Rob Barton

Differential Geometry in Six Dimensions
by L.P. Hughston

Twistors are useful and illuminating in the analysis of manifolds of dimension six. This is, roughly, on account of the fact that twistors are the spinors for the group $O(6, \mathbb{C})$. Thus twistors play a role in the geometry of six-dimensional spaces similar in many respects to the role played by two-component spinors in the geometry of four-manifolds. Whether these considerations are of any physical interest remains to be seen — my purposes here are primarily geometrical and I shall summarize a number of results in outline form.

Conventions: i, j, k etc. = $0, 1, 2, 3, 4, 5$

α, β, γ etc. = $0, 1, 2, 3$

point: $X^i = X^{\alpha\beta}$ (skew) (abstract index convention)

metric: $g_{ij} = \epsilon_{\alpha\beta\gamma\delta}$ note $\epsilon_{\alpha\beta\gamma\delta} = \epsilon_{\gamma\delta\alpha\beta}$

vector field $V^i(X) = V^{\alpha\beta}(X)$ (skew)

null vector field $V^{\alpha\beta} = P^{[\alpha} Q^{\beta]}$, where $P^\alpha(X)$ and $Q^\alpha(X)$ are 'spinor fields'.

$$\Omega_{ij} V^i V^j = 0 \iff V^{\alpha\beta} = P^{[\alpha} Q^{\beta]}$$

two-forms $F^{ij} = F^{\alpha\beta\rho\sigma} = -F^{\rho\sigma\alpha\beta}$

$$\approx E_\beta^\alpha \text{ with } E_\alpha^\alpha = 0$$

three-forms $F^{ijk} = F^{[\alpha\beta][\rho\sigma]\gamma}$

$$\approx \phi^{\alpha\beta} \oplus \psi_{\alpha\beta} \quad \phi^{\alpha\beta} = \phi^{(\alpha\beta)}$$

self-dual 3-forms $F^{ijk} \sim \phi^{\alpha\beta}$ $\psi_{\alpha\beta} = \psi_{(\alpha\beta)}$

anti-self-dual 3-forms $F^{ijk} \sim \psi_{\alpha\beta}$

curvature tensor $R_{ijkl} \sim \psi_{\rho\sigma}^{\alpha\beta} \oplus \Phi_{\mu\nu\gamma} + 1$

105 components

'conformal spinor' $\psi_{\rho\sigma}^{\alpha\beta} = \psi_{(\rho\sigma)}^{\alpha\beta}$ ($\psi_{\alpha\sigma}^{\alpha\beta} = 0$) 84 comp

'Ricci spinor' $R_{ij} - \frac{1}{6} g_{\mu\nu} R$

$\approx \Phi_{\mu\nu\gamma}$ (symmetry) 20 com

'Vacuum' Bianchi identities: $\nabla_{\xi\alpha} \psi_{\rho\sigma}^{\alpha\beta} = 0$, $\nabla^{\gamma\rho} \psi_{\rho\sigma}^{\alpha\beta} = 0$.

Ricci identities:

$$\nabla_{[i} \nabla_{j]} \tilde{\zeta}_k = R_{ijk}^{\ell} \tilde{\zeta}_{\ell}. \quad \text{define } \square_{\beta}^{\alpha} \sim \nabla_{[i} \nabla_{j]} (\square_{\alpha}^{\alpha} = 0)$$

$$\square_{\beta}^{\alpha} \tilde{\zeta}_{\gamma} = \psi_{\beta \gamma}^{\alpha \delta} \tilde{\zeta}_{\delta} + \Phi_{\beta \gamma}^{\alpha \delta} \tilde{\zeta}_{\delta} + \delta_{\gamma}^{\alpha} \Lambda \tilde{\zeta}_{\beta}$$

$$\square_{\beta}^{\alpha} \tilde{\zeta}_{\gamma} = -\psi_{\beta \gamma}^{\alpha \delta} \tilde{\zeta}_{\delta} - \Phi_{\beta \gamma}^{\alpha \delta} \tilde{\zeta}_{\delta} - \delta_{\beta}^{\alpha} \Lambda \tilde{\zeta}_{\gamma}$$

$$\text{where } \Phi_{\beta \gamma}^{\alpha \delta} = \Phi_{[\beta \gamma]}^{[\alpha \delta]} = \Phi_{[\beta \gamma]} [\rho \sigma] \varepsilon^{\alpha \delta \rho \sigma}$$

Note that these formulae are actually somewhat simpler in form than their four-dimensional analogues!

'Maxwellian' equations: $F^{;ik} = F^{[i;k]} \sim \phi^{\alpha \beta} \oplus \psi_{\alpha \beta}$

$$\text{Set } \nabla^{[i} F^{k]l} = 0 \text{ and } \nabla_{;i} F^{;jk} = 0.$$

These are equivalent to:

$$\nabla_{\alpha \beta} \phi^{\beta \gamma} = 0 \quad \text{and} \quad \nabla^{\alpha \beta} \psi_{\beta \gamma} = 0$$

In flat 6-space these imply $\square \phi^{\beta \gamma} = 0$ and $\square \psi_{\beta \gamma} = 0$ where $\square = \nabla_i \nabla^i$.

General solution of $\nabla_{\alpha \beta} \phi^{\beta \gamma} = 0$ in flat space:

$$\phi^{\alpha \beta}(x^{\rho \sigma}) = \oint z^{\alpha} z^{\beta} f(X_{\rho \sigma} Z^{\sigma}, Z^{\sigma}) \mathcal{D}^3 Z$$

where $\mathcal{D}^3 Z = \varepsilon_{\alpha \beta \gamma \delta} Z^{\alpha} dZ^{\beta} dZ^{\gamma} dZ^{\delta}$ and $F(W_{\rho}, Z^{\sigma})$ is homogeneous of degree -6 , defined on a suitable region of the space $W_{\alpha} Z^{\alpha} = 0$. Note that the pair $\{W_{\alpha}, Z^{\alpha}\}$ is a spinor for the group $O(8, \mathbb{C})$, i.e. is in effect a 'twistor' for the flat six-space.

(For further discussion of this formula see my article in the I. Robinson Festschrift, W. Rindler & A. Trautman, eds.)

Algebraic classification of symmetric spinor fields is a more intricate matter than in four dimensions. Reality conditions aside, a field $\phi^{\alpha\beta}$ can at each point be one of four essentially distinct types. The most degenerate of these (which I shall call 'null') is when $\phi^{\alpha\beta}$ is of the form $P^\alpha P^\beta$ for some spinor field $P^\alpha(X)$.

- 1 Lemma If $\nabla_{\alpha\beta}\phi^{\alpha\beta} = 0$ and $\phi^{\alpha\beta} = P^\alpha P^\beta$ then $P^\alpha(X)$ satisfies $(P^\alpha \nabla_{\alpha\beta} P^{[\gamma}) P^{\delta]} = 0$. (*)
 - 2 Remark This condition is analogous to the g.s.f. condition $(\partial^A \nabla_{A'B} \partial^{[B}) \partial^{C]} = 0$ for a spinor field in four dimensions.
 - 3 Problem Suppose $P^\alpha(X)$ satisfies (*) as above. Does there necessarily exist a field $\phi^{\alpha\beta} = \phi P^\alpha P^\beta$ such that $\nabla_{\alpha\beta} \phi^{\beta\gamma} = 0$ for a suitable choice of the scalar $\phi(X)$?
- Solutions of (*) can be generated by consideration of analytic varieties of appropriate codimension:
- 4 Theorem Suppose $F_r(W_\alpha, Z^\alpha)$ $r=1,2,3$ is a triple of holomorphic functions, homogeneous of some degree, defined on regions of the quadric $W_\alpha Z^\alpha = 0$. Then the variety $F_r = 0$ determines a spinor field $P^\alpha(X)$ according to the scheme

$$F_r(X_{\alpha\beta} P^\beta(X), P^\alpha(X)) = 0,$$

and $P^\alpha(X)$ satisfies $(P^\alpha \nabla_{\alpha\beta} P^{[\gamma}) P^{\delta]} = 0$.

5. Problem Does every analytic $P^\alpha(X)$ satisfying (*) arise in this way?

6 Lemma Suppose a curved six-dimensional space satisfies $R_{ij} = 0$ and has a degenerate Weyl spinor to the extent that

$$\gamma^{\alpha\beta}_{\delta\delta} = P^\alpha P^\beta Q_{\alpha\beta}. \quad (**)$$

for some $P^\alpha, Q_{\alpha\beta}$. Then $(P^\alpha \nabla_{\alpha\beta} P^\delta) P^\delta = 0$.

7 Problem Does the converse hold, in the sense that if P^α satisfies (*) and $R_{ij} = 0$ is the Weyl spinor necessarily of the form (**)?

8 Definition (Robinson & Trautman) In a manifold of n dimensions (signature unimportant) let $k^a(x)$ ($a = 1 \dots n$) be a vector field which is conformally geodesic, i.e. if \hat{g}_{ab} is the metric of \mathcal{M} then for suitable $\Omega(x)$ we have $g_{ab} = \Omega^2 \hat{g}_{ab}$ such that $k^a \nabla_a k_b = 0$ where ∇_a is the connection associated with g_{ab} and indices are raised & lowered with g_{ab} . Then k^a is shear-free if

$$\mathcal{L}_{k^a} k_{[a} g_{b]}{}^c k_{d]} = \phi k_{[a} g_{b]}{}^c k_{d]}$$

for some ϕ , or equivalently

$$\nabla_{(a} k_{b)} = \gamma^a g_{ab} + \xi_{(a} k_{b)}$$

for some γ^a, ξ_a .

9 Remark If \mathcal{M} is space-time then this definition reduces to the 'standard' ones if k^a is null or timelike.

- 10 Lemma. Suppose the spinor fields A^α and B^β each satisfy (*). Then the vector field

$$K^{\alpha\beta} = A^{[\alpha} B^{\beta]}$$

is geodesic and shearfree in the sense noted above.

- 11 Remark To show $K^{\alpha\beta}$ is geodesic is straightforward enough: Since A^α and B^α satisfy (*) we have

$$A^\alpha \nabla_{\alpha\beta} A^\beta = \lambda_\beta A^\beta \quad \text{and} \quad B^\alpha \nabla_{\alpha\beta} B^\beta = \mu_\beta B^\beta$$

for some λ_β, μ_β . Thus

$$A^\alpha B^\beta \nabla_{\alpha\beta} A^\rho = \lambda A^\rho \quad \text{and} \quad A^\alpha B^\beta \nabla_{\alpha\beta} B^\rho = \mu B^\rho$$

for suitable λ, μ . Whence

$$A^\alpha B^\beta \nabla_{\alpha\beta} A^{[\rho} B^{\sigma]} = (\lambda + \mu) A^{[\rho} B^{\sigma]}. \quad \square$$

To show $A^{[\alpha} B^{\beta]}$ is shearfree is somewhat more intricate.

- 12 Problem Show that the converse to Lemma 10 does not hold.

13. Remark. Reality conditions: for signature +---- we impose the 'usual' twistor conjugation rules, i.e.
 $\bar{Z}^\alpha = \bar{Z}_\alpha$ with a Hermitian correlation of signature +-+-.
For signature +---- we impose 'Atiyah' conjugation,
i.e. $\bar{Z}^\alpha = \bar{Z}^\alpha$ component by component.

- 14 Problem In a real six-dimensional curved space-time of signature +---- determine all 'null' solutions of the vacuum equations, i.e. for which $\nabla_{\alpha\beta} \psi^{\alpha\beta} = Z^\alpha Z^\beta \bar{Z}_\alpha \bar{Z}_\beta$,
... or $Z^\alpha \bar{Z}_\alpha = 0$.

Deformations Of Ambitwistor Space.

According to a theorem of C.R. LeBrun, the space of null geodesics, N , of a 4-dimensional complex manifold M with holomorphic conformal structure $[g_{ab}]$ and connection ∇ determines (M, g_{ab}, ∇) (and, of course, via versa) and this correspondence is preserved under small deformations of the complex structure of N . Furthermore ∇ is torsion free iff the symplectic potential, Θ , on T^*M descends to N . (C.R. LeBrun D.Phil Thesis).

This theorem implies that 1st order deformations of N are in 1-1 correspondence with 1st order variations in $(M, [g], \nabla)$, and further that, if ∇ is torsion free, deformations of N preserving Θ correspond to variations in $[g_{ab}]$ alone. Such 1st order deformations seem a good place to start if we are interested in how field equations on (M, g_{ab}) are encoded into the C-str. of N .

$X_{ij} \in H^1(N, \Omega(TN))$ determines an infinitesimal deformation of N . For X_{ij} to preserve Θ it must preserve $\omega = d\theta$, the symplectic form on N . $dX_{ij} \omega = 0 \Rightarrow X_{ij} \lrcorner \omega + dH_{ij} = 0$ for some Hamiltonian $H_{ij} \in H^1(N, \Omega(N))$. The further condition that $d_{X_{ij}} \Theta = 0$ implies that $H_{ij} \in H^1(N, \Omega(1,1))$. (This follows from $\Theta = z \cdot \frac{\partial}{\partial z} \lrcorner \omega$, $z \cdot \frac{\partial}{\partial z}$ is defined in the curved case as $\frac{\partial}{\partial t} \lrcorner$ descends from $S \otimes S^* M$ to N).

It turns out that $H^1(N, \Omega(1,1)) \xrightarrow{\alpha} \frac{\text{trace free } h_{ab} = h_{ab} - h_{ab} \nabla \cdot \gamma}{h_{ab} = \nabla(\gamma \nabla \cdot \gamma) - \frac{1}{4} g_{ab} \nabla \cdot \gamma}$

This is exactly right for variations in the conformal structure.

This can be easily proved using the long exact sequence in cohomology derived from the short exact sequence of sheaves:

$$0 \rightarrow \Omega_N^{(1,1)} \xrightarrow{p^*} \Omega_{S \oplus S'M}^{(1,1)} \xrightarrow{\downarrow} \Omega_{S \oplus S'M}^{(2,2)} \xrightarrow{V(g)} 0.$$

Where p is the projection $p: S \oplus S'M \rightarrow N$ and $V = T\tilde{\pi}^A \tilde{\pi}^B \tilde{\pi}^C T\pi^D T\pi^E$

is the null geodesic spray on $S \oplus S'M$. The long exact sequence also shows $H^2(N, \Omega^{(1,1)}) = 0$, so there are no obstructions to the deformations I am considering.

The map α is given by taking $H_{ij} \in H^1(N, \Omega^{(1,1)})$, lifting to $S \oplus S'M$, so that $p^* H_{ij} = H_{ij}(x, T\pi^A, \tilde{T}\pi^A)$ and splitting over each fibre of $\gamma: S \oplus S'M \rightarrow M$, $H_{ij} = H_i - H_j$. So $h = VH_i$ is hom deg $(2,2)$ and global in $T\pi$ and $\tilde{T}\pi$ so $h = h_{ABCD} \tilde{\pi}^A \tilde{\pi}^B T\pi^C T\pi^D$.

We can check that the deformation really does give the variation h_{ab} in the cont. str. of (M, g_{ab}) by assuming that to 1st order the deformation of N doesn't alter the location of the celestial quadrics in N , merely altering the tangent vectors from one to the next. i.e. lift $X_{H_{ij}}$ to $S \oplus S'M$ then split $X_{H_{ij}}$,

$X_{H_{ij}} = X_{H_i} - X_{H_j}$. (X_f means the hamiltonian vector field with Hamiltonian f w.r.t. the symplectic form on T^*M) $p_{\#} \cong S \oplus S'M / \frac{\partial}{\partial \tilde{\pi}^A} - \frac{\partial}{\partial \tilde{\pi}^B}$

Then the variation of the null geodesic spray is given by:

$$SV = [\tilde{X}_{H_i}, V] = -[\tilde{X}_{H_i}, X_{P^2}] = -X_{[H_i, P^2]} = X_{V(H_i)} = X_h.$$

So we see that the hamiltonian for $V + SV$ is $P^a P^b (g_{ab} + h_{ab})$.

So we see that g_{ab} varies by h_{ab} as one would hope.

This is all very well, but what about the field equations? If we define the operator $\hat{H}^E = \pi^{A'} \nabla_{AA'} \partial/\partial \pi_A$ and $\tilde{\hat{H}}^E = \tilde{\pi}^A \nabla_{AA'} \partial/\partial \tilde{\pi}_A$ then h_{ab} satisfies the linearized vacuum equations iff $(\hat{H}^E \tilde{\hat{H}}^E + \tilde{\hat{H}}^E \hat{H}^E) h = 0$ when $h = h_{ab} \tilde{\pi}^A \tilde{\pi}^B \pi^{A'} \pi^{B'}$ as above. This will descend to a condition on H_{ij} on N if $\nabla \tilde{\hat{H}}^E - \tilde{\hat{H}}^E \nabla = 0$. However this is only the case when (M, g_{ab}) is flat.

When M is Minkowski space $\tilde{\hat{H}}^E$ descends to N to become:

$$\tilde{\hat{H}}^E = i(2 + z \cdot \frac{\partial}{\partial z}) \tilde{\pi}^A \partial/\partial \omega_A .$$

Rather than derive the appropriate H_{ij} in the flat case it is easier to make an educated guess. Let $h_{ab} = h_{ab}^+ + h_{ab}^-$ where h_{ab}^+ is a potential for an A.S.O. vacuum linearized field and h_{ab}^- for an S.O. one. Let $\tilde{g}(z)$ and $\tilde{g}(w)$ be corresponding hom. deg. 2 twistor/dual twistor functions for these fields. Then it is easily seen that :

$$H_{ij} = \tilde{\pi}^A \partial/\partial \omega_A \tilde{g}(z) + \pi^{A'} \partial \tilde{g}(w)/\partial \omega^{A'} \quad (\textcircled{R})$$

gives rise to the appropriate deformation. Furthermore, it has the correct conformal weight for linearized gravity.

Remarks:

Since $\tilde{\hat{H}}^E$ doesn't descend to N in the curved case it seems unlikely that it will be possible to exponentiate this

deformation in a well defined way such that the resulting (M, g_{ab}) would be conformal to vacuum. The form of the Hamiltonian is only distinguished by knowing twistor coordinates on N , after deforming with \star / we want know our z 's and w 's to 2nd order.

However if $\tilde{g}(w) = 0$, the Hamiltonian \mathfrak{H} will preserve the cotangent bundle structure of N over twistor space and the form of the Hamiltonian is distinguished to all orders. This yields the non-linear graviton construction.

Another form of the Hamiltonian of interest is:

$$H_{ij} = z^k \tilde{g}_{ik}(w) + w^k \tilde{g}_{ik}(z) \quad g_{ik}(w) \in H^1(\mathcal{O}_x(1)) \text{ etc.}$$

Then the corresponding linearized field satisfies the linearized Bach equations, $\nabla_{(A}^{\Lambda} \nabla_{B)}^{\Lambda} \Psi_{ABCD} = 0$. It would be interesting to know if it goes all such.

One can also find Hamiltonians which deform N in such a way as to preserve the spacetime geometry, but when N becomes the space of either: (A) null geodesics charged with respect to some electromagnetic field or (B) null geodesics together with a charged phase. The Hamiltonians are:

$$(A) \quad H_{ij} = f_{ij}(z) + \tilde{f}_{ij}(w) \quad f_{ij}(z) \in H^1(P\pi, \mathcal{O}) \text{ etc.}$$

$$(B) \quad H_{ij} = z \cdot w (f_{ij}(z) + \tilde{f}_{ij}(w)).$$

In case (B) one is merely deforming $\mathcal{O}(1)$.

Outlook: The interesting question is what does this all mean for full G.R.? As noticed in the above remarks, it seems unlikely that exponentiating the Hamiltonian above (*) will yield a space-time conformal to vacuum.

However, the above description of N for a linearized field on Minkowski space allows us to perform a straightforward analysis of the structure and evolution of hypersurface twistor spaces. It turns out that the necessary ingredients of this description survive in curved spacetime, at least if one reformulates the Čech description given here into a Dolbeault one.

In particular, the form $\pi^A' d\pi_A'$ plays a prominent role as a Dolbeault analogue of $\pi^A \pi^B' \partial^z \tilde{g}(w) / \partial w^A \partial \bar{w}^B$, and controls the evolution.

Unfortunately some of the details are still obscure and, in any case, there isn't enough room in this twistor newsletter for a fuller exposition. More will appear in the next TIN.

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