

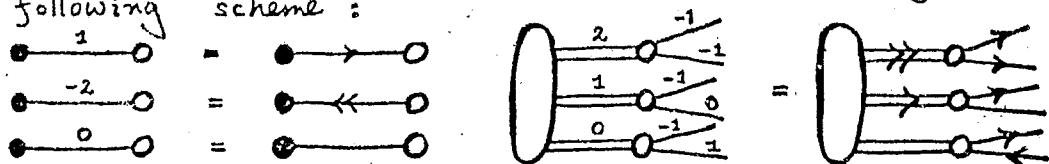
Twistor Newsletter (no 2: 10 June, 1976)

①

Twistor diagrams and duality diagrams

The purpose of this note is to point out how some interesting formal correspondences¹ can be drawn between certain classes of twistor diagrams and the 'duality diagrams' of dual-resonance theory². These interrelationships might act as a guide towards the eventual construction of amplitudes for hadronic (and possibly other) processes using twistor theory methods. It seems reasonably likely that more extensive (and perhaps less formal) bridges between twistor theory and dual-resonance theory can be built (e.g. considering dual string models.)

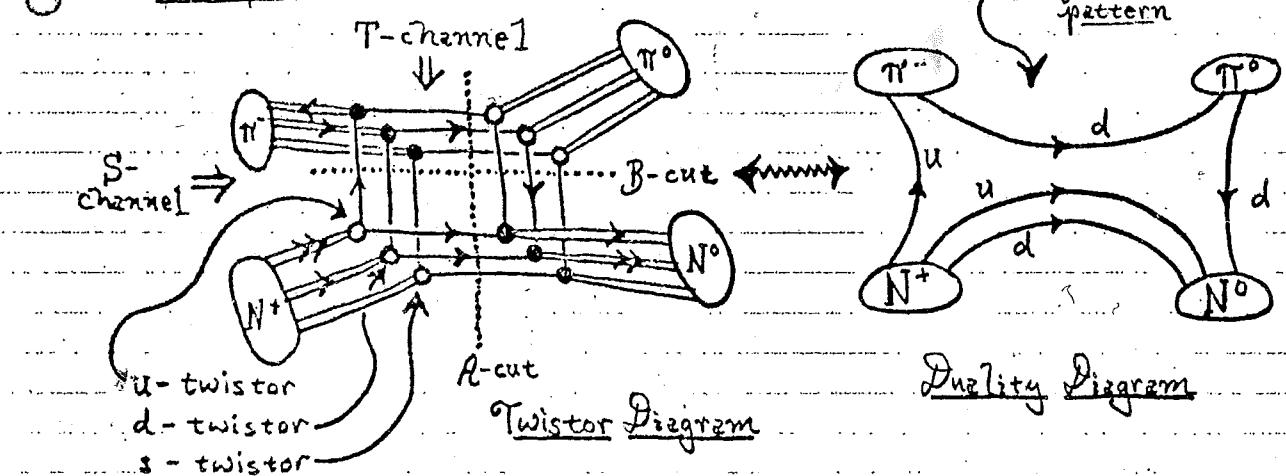
The standard conventions for twistor diagrams (as summarized in Quantum Gravity, p. 335) will be used here with but one minor modification³: for the sake of clarity and compactness, rather than putting integers on the lines in a diagram we'll employ "helicity flow" arrows (as described in the twistor Physics Reports article, p. 284-286) according to the following scheme:



and so forth, i.e. positive helicity flows away from a filled vertex, and towards an open vertex. The total helicity flowing into a vertex must equal the total flowing out.

Example: $N^+ \pi^- \rightarrow N^0 \pi^0$

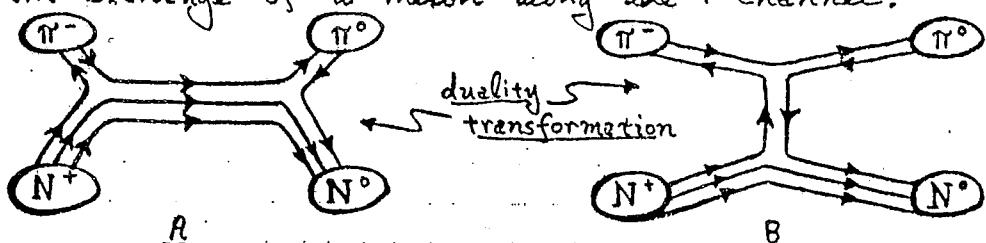
shows
quark flow
pattern



The duality diagram for $N^+ \pi^- \rightarrow N^0 \pi^0$ can be read in two different ways. These ways of reading the diagram are obtained by bunching together the quark strands in two different ways.

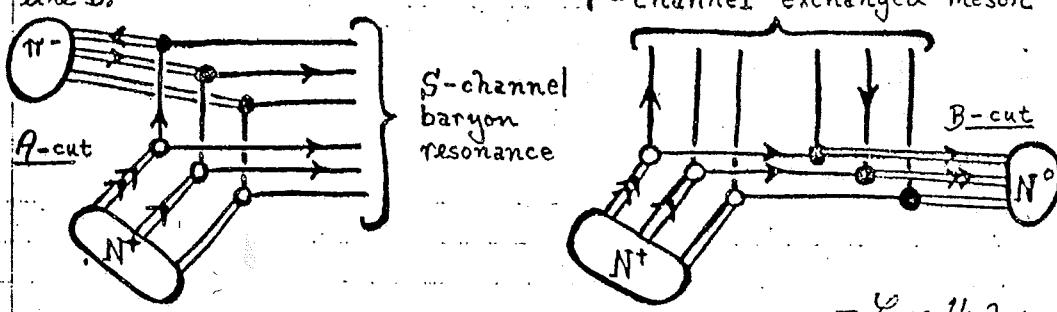
(2)

The duality diagram can be read as picture A, with the reaction going through a baryon resonance in the S-channel, or alternatively as picture B, with interaction resulting from exchange of a meson along the T-channel.



The process of going from one of these pictures to another is commonly referred to as a duality transformation. Dual theory says A and B depict two alternative ways of describing one and the same process, and the complete amplitude for the process can be found either by summing over all the S-channel resonances or by summing over all the T-channel meson exchanges. (Ordinary quantum mechanics, in contrast, says A and B represent distinct means of achieving the same end product, and therefore the amplitude is found by summing over the S-channel effects and the T-channel effects.) Dual interpretations of the twistor diagram are suggested analogously by cutting either along dotted line A or dotted line B.

T-channel exchanged meson



—Lene Hughston

- ¹ Grown out of conversations with M. Sheppard, and with J. Moussouris.
- ² A useful reference is the book Dual Theory (ed. M. Jacob, North-Holland, 1974) which is essentially a bound volume of several Physics Reports articles on the subject. Also useful (dating back more) are articles by Jacob and Mandelstem in the 1970 Brandeis Lecture volumes. Duality diagrams first appear in a flourish in 1969 : H. Harari, P.R. Letts, 22, 562 (1969); J. Rosner, P.R. Letts, 22, 689 (1969); D. Neville, P.R. Letts, 22, 494 (1969); and T. Matsukaze, et al., Prog. Theor. Phys. 42, 56 (1969). Extensions to higher order diagrams, which also fit well into the twistor framework, are due to K. Kikkawa, et al., P.R. 184, 1701 (1969). (Their 'quark loops' correspond to helicity loops in twistor diagrams.)
- ³ Suggested by R.P.. (Note how remarkably well the notation adapts itself to the subject matter at hand!)

(3)

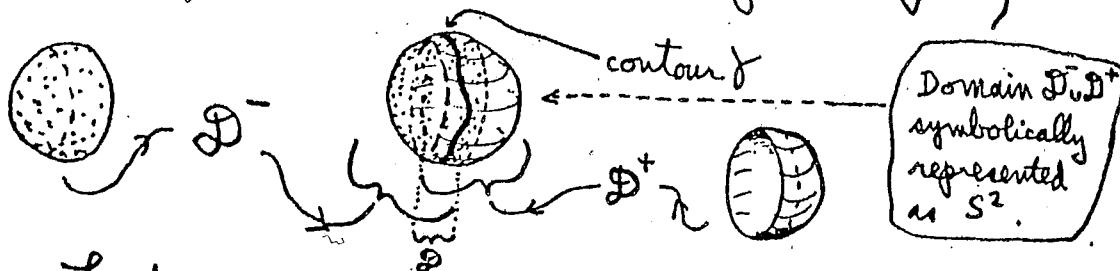
Twistor functions and sheaf Cohomology

Recall the following properties of a twistor function $f(Z^\alpha)$, to be used for generating a zero rest-mass field $\varphi_{A \dots L}$ (or $\varphi_{A \dots K}$) by means of a contour integral:

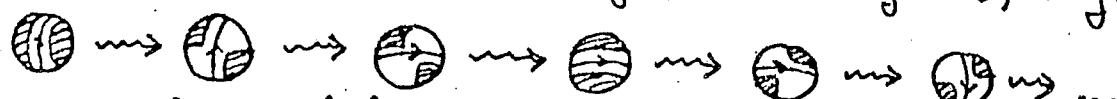
1. f is holomorphic on some domain D of twistor space ~~not invariant~~ under $SU(2,2)$ (nor under the Poincaré group, nor the Lorentz group)
2. there is a "gauge" freedom ~~in D~~ whereby

$$f \rightarrow f + h^- + h^+$$

where h^\pm is holomorphic on some extended domain D^\pm ($= D$) of twistor space in which the contour γ can be deformed to a point (to the "left" in D^- and to the "right" in D^+)



3. γ depends on the location of γ (and γ is not invariant under $SU(2,2)$ — nor Poincaré nor Lorentz — so φ is not invariant either)
4. by invoking γ , then moving γ , then invoking new γ , moving γ :



we can obtain a whole family of equivalent twistor functions, all giving the same field, the entire family being invariant under $SU(2,2)$

5. this seems a little nebulous, and, for example, how do we add two such families (to give $\varphi_{A \dots L} + \varphi'_{A' \dots L'}$) etc. etc. ?

(4)

A new viewpoint concerning twistor functions has been gradually emerging which makes good mathematical sense of all this: a twistor function is really to be viewed as a representative function (or cocycle) defining an element of a sheaf cohomology group. Now, the twistor theorist, when attacked by a purist for shoddiness in the domains, can counter-attack armed with his sheaf!

Thumbnail sketch of (relevant) sheaf cohomology theory *

First, let us recall how ordinary (Čech) cohomology works. Let X be a space (topological, at least). Cover X with open sets \mathcal{U}_i . We define a cochain (with respect to this covering) with coefficients in an additive abelian group G (say, the integers \mathbb{Z} , the reals \mathbb{R} , or the complex field \mathbb{C}) in terms of a collection of elements $f_i, f_{ij}, f_{ijk}, \dots \in G$, assigned to the various \mathcal{U}_i and their non-empty intersections:

f_i assigned to \mathcal{U}_i ; f_{ij} assigned to $\mathcal{U}_i \cap \mathcal{U}_j$; f_{ijk} to $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$, etc.

$$f_{ij} = -f_{ji}, \quad f_{ijk} = -f_{jik} = f_{jki} = \dots, \text{ i.e. } f_{i\dots l} = f_{[i\dots l]}.$$

Then:

$$\begin{aligned} 0\text{-cochain } \alpha &= (f_1, f_2, f_3, \dots) \\ 1\text{-cochain } \beta &= (f_{12}, f_{23}, f_{13}, \dots) \\ 2\text{-cochain } \gamma &= (f_{123}, f_{124}, \dots) \end{aligned} \quad \begin{array}{l} \text{(taking } \{\mathcal{U}_i\} \text{ countable, for notational convenience)} \\ \text{con. non-empty } \mathcal{U}_i \cap \mathcal{U}_j \\ \text{etc. } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \end{array}$$

Define coboundary operator δ as follows:

$$\delta \alpha = (f_2 - f_1, f_3 - f_2, f_3 - f_1, \dots) \quad \begin{array}{l} \text{(a 1-cochain)} \\ \text{etc. } f_{i_1} \text{ " } f_{i_2} \text{ " } f_{i_3} \text{ " } \dots \end{array}$$

$$\delta \beta = (f_{12} - f_{13} + f_{23}, f_{12} - f_{14} + f_{24}, \dots) \quad \begin{array}{l} \text{(a 2-cochain)} \\ \text{etc. } f_{i_1 i_2} \text{ " } f_{i_1 i_3} \text{ " } f_{i_2 i_3} \text{ " } \dots \end{array}$$

Then we have $\delta^2 = 0$. We call γ a cocycle if $\delta\gamma = 0$; we call γ a coboundary if $\gamma = \delta\beta$ for some β . Define the p th cohomology group by:

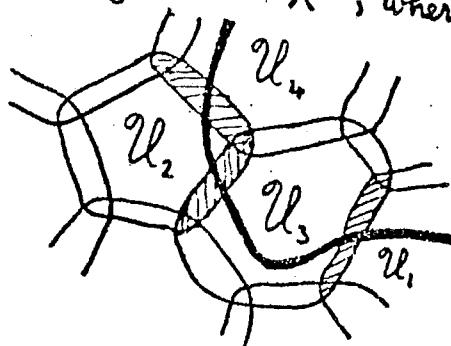
$$H^p(X, G) = \frac{\text{additive group of } p\text{-cocycles}}{\text{additive group of } p\text{-coboundaries.}}$$

*We are indebted to Michael Atiyah for edifying discussions.

Note: $H^p(X, \mathbb{G})$, as defined, depends on the covering $\{\mathcal{U}_i\}$.⁽⁵⁾

What we should do, to define $H^p(X, \mathbb{G})$, is to take the appropriate "limit" of all these $H_{\mathcal{U}_i}^p(X, \mathbb{G})$ for finer and finer coverings $\{\mathcal{U}_i\}$ of X . However (for X suitably non-pathological) we can always settle on a particular "sufficiently fine" covering $\{\mathcal{U}_i\}$ where, in effect, there is no "relevant topology" left on each \mathcal{U}_i or intersections thereof (i.e. all the $H^p(\mathcal{U}_i, \dots, \cap \mathcal{U}_k, \mathbb{G})$ vanish for all $p > 0$ — though as a definition of "sufficiently fine" this is a little circular). Then this $H_{\mathcal{U}_i}^p(X, \mathbb{G}) = H^p(X, \mathbb{G})$, so a limit need not be taken. I shall henceforth assume that such a "sufficiently" fine covering has been taken, and that it is countable and locally finite.

Now what does this definition have to do with the familiar "dual" relation to ordinary homology $H_p(X, \mathbb{G})$? How does δ assign values (elements of \mathbb{G}) in a linear fashion to p -cycles in X , where δ is some element of $H^p(X, \mathbb{G})$? Intuitively,



Add together the f_{ij} for shaded regions (centered by 1-cycle) and similarly for higher dimensional p -cycles.

If δ defined by (f_{12}, f_{23}, \dots) , $\delta(K) = f_{12} + f_{23} + f_{31} + \dots$

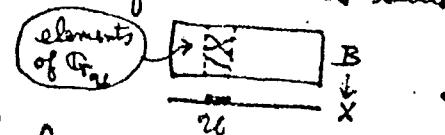
(correctly signed) f_i 's on regions entered by K .

What about SHEAF cohomology though? It's really a rather natural generalization of the above. But first we must rephrase the concept of a cochain slightly. Rather than thinking of f_i as simply an element of \mathbb{G} "assigned" to \mathcal{U}_i , and f_{ij} "assigned" to \mathcal{U}_{ij} , etc., we think of f_i as a function defined on \mathcal{U}_i which happens to take this constant value " f_i " $\in \mathbb{G}$, and we think of f_{ij} as a constant function on $\mathcal{U}_i \cap \mathcal{U}_j$ with values in \mathbb{G} , etc. (This has the incidental advantage that the requirement $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ is now unnecessary.) This is still ordinary

(Čech) cohomology. But the generalization to sheaf cohomology is now easily made: the functions f_i, f_{ij}, f_{ijk} ⁽⁶⁾ are now not required to be constant. (In fact, we can even allow that the group G , in which the values of the f 's reside, may vary from point to point in X — but I shall not consider such situations here.) We may require that the f 's are restricted in some way — in particular, for the purposes of the present applications we shall often require the f 's to be holomorphic (with, say, $G = \mathbb{C}$) and X being a complex manifold — or, we may consider other related classes of functions.

So, what's a sheaf? Actually, I shan't even bother with a formal definition (which can be found in, for example, R.C. Gunning & H. Rossi: "Analytic Functions of Several Complex Variables" Prentice-Hall 1965, or J. Morrow & K. Kodaira: "Complex Manifolds" Holt, Rinehart & Winston Inc. 1971). The essential point is that a sheaf is so defined that the Čech cohomology works just as well as before. In fact, a sheaf \mathcal{F} defines an additive group $\mathcal{G}_{\mathcal{U}}$ for each open set $\mathcal{U} \subset X$. For example, $\mathcal{G}_{\mathcal{U}}$ might be the additive group of all holomorphic functions on \mathcal{U} (taking X to be a complex manifold). In this case we get the sheaf, denoted \mathcal{O} , of germs of holomorphic functions on X . Slightly more generally, we might consider "twisted" holomorphic functions, i.e. functions whose values are not just ordinary complex numbers, but taken in some complex line bundle over X (thinks of "spin-weighted" functions, for example). An important example of such a twisted function would arise if X were taken to be projective twistor space PTT (or a suitable portion thereof) and the functions considered were to be homogeneous (and holomorphic) of some fixed degree n in the twistor variable. For each open set $\mathcal{U} \subset X$ we take $\mathcal{G}_{\mathcal{U}}$ to

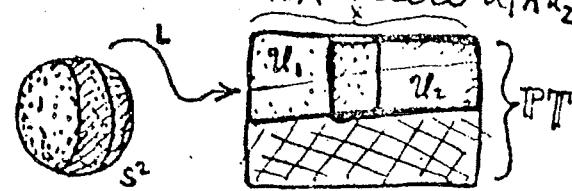
consist of all such twisted functions on \mathcal{U} , and the⁷ resulting sheaf, denoted $\mathcal{O}(n)$, is called the sheaf of germs of holomorphic functions twisted by n , on X . More generally we might consider functions whose values lie in some vector bundle B over X (e.g. we might consider tensor fields on X) and $\mathcal{G}_{\mathcal{U}}$ would consist of the cross-sections of the portion of B lying above \mathcal{U} :



Cochains are defined as before (with $f_i \in \mathcal{G}_{\mathcal{U}_i}$, $f_{ij} \in \mathcal{G}_{\mathcal{U}_i \cap \mathcal{U}_j}, \dots$) and the coboundary operator δ , just as before. Then we obtain the p^{th} cohomology groups of X , with coefficients in the sheaf \mathcal{I} , as: $H^p(X, \mathcal{I}) = \frac{p\text{-cochains with coefficients in } \mathcal{I}}{p\text{-coboundaries with coeffs. in } \mathcal{I}}$. As before, we would need to take the appropriate limit for finer and finer coverings $\{\mathcal{U}_i\}$ of X , but we can settle on one "sufficiently fine" covering if desired. Provided that \mathcal{I} is what's called a coherent analytic sheaf (and we are interested primarily in such sheaves — locally defined by n holomorphic functions modeled on holomorphic functions), then "sufficiently fine" can be taken to mean that each of $\mathcal{U}_i, \mathcal{U}_i \cap \mathcal{U}_j, \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k, \dots$ is a Stein manifold. In effect, a Stein manifold is a holomorphically convex open subset of \mathbb{C}^n (or a domain of holomorphy). If X is Stein and \mathcal{I} coherent, then $H^p(X, \mathcal{I}) = 0$, if $p > 0$. Note: \mathcal{O} and $\mathcal{O}(n)$ are coherent. End of thumbnail sketch.

Twistor functions as elements of $H^1(X, \mathcal{O}(n))$

Let X be some suitable portion of projective twistor space \mathbb{PT} , say some neighbourhood of a line in \mathbb{PT} (corr. some nbd. of a pt. in Mink. space), or say \mathbb{PT}^+ , or $\overline{\mathbb{PT}}^+$. Suppose we can cover X with two sets $\mathcal{U}_1, \mathcal{U}_2$ (each open in X) such that every projective line in X meets $\mathcal{U}_1 \cap \mathcal{U}_2$ in an annular region and where $\mathcal{U}_1 \cap \mathcal{U}_2$ corresponds to the domain of definition of some twistor function



$f(z^\alpha)$, homogeneous of degree n in the twistor Z^α .
 Then $f = f_{12}$ is a twisted function on $\mathcal{U}_1 \cap \mathcal{U}_2$. There
 are no other $\mathcal{U}_i \cap \mathcal{U}_j$'s, so f_{12} by itself defines a
 1-cochain β , with coefficients in $\mathcal{O}(n)$, for X . Clearly
 $\delta\beta = 0$, so β is a coboundary. The 1-coboundaries, for
 this covering, are functions of the form $h_2 - h_1$, where
 h_2 is holomorphic on \mathcal{U}_2 and h_1 on \mathcal{U}_1 . Calling
 $\mathcal{D} = \mathcal{U}_1 \cap \mathcal{U}_2$, $\mathcal{D}^- = \mathcal{U}_1$, $\mathcal{D}^+ = \mathcal{U}_2$, $h_1 = -h^-$, $h_2 = h^+$
 we observe that the "equivalence" between twistor functions
 under the "gauge" freedom of that we started out with
 is just the normal cohomological equivalence between
 1-cochains β, β' that their difference be a coboundary:
 $\beta' - \beta = \delta\alpha$, with $\alpha = (h_1, h_2)$. This suggests that we
 view the twistor function f as really defining us an
 element of $H^1(X, \mathcal{O}(n))$.

But this is all with respect to a particular covering
 of X , namely by $\{\mathcal{U}_1, \mathcal{U}_2\}$. Is this covering "fine" enough?
 There is actually a technical problem here. We cannot
 (normally) arrange that \mathcal{U}_1 and \mathcal{U}_2 are Stein manifolds
 (and in those exceptional cases when we can so arrange
 this, we would lose the invariance properties that we are
 striving for). It turns out, in fact, that this problem is not
 serious. One can show by direct construction (using
 the inverse twistor function — in the cases $n \leq -2$ at
 least, but probably in all cases) that such a covering
 by two such sets $\mathcal{U}_1, \mathcal{U}_2$ is sufficient in the case
 $X = \overline{TP\Gamma^+}$. Note that though X is invariant under $SU(2,2)$,
 in this case, the covering is not. However, the cohomology
 group $H^1(X, \mathcal{O}(n))$ is invariant. Let us illustrate this
 by adding two elements of $H^1(X, \mathcal{O}(n))$ one of which is

defined by a twistor function f , with respect to the covering $\{\mathcal{U}_1, \mathcal{U}_2\}$, and the other, by \hat{f} , with respect to $\{\widehat{\mathcal{U}}_1, \widehat{\mathcal{U}}_2\}$, the second covering being a rotated version of the first.

Schematically:  We define a representative cochain for the sum by taking both coverings together. Denote $\mathcal{U}_3 = \widehat{\mathcal{U}}_1$, $\mathcal{U}_4 = \widehat{\mathcal{U}}_2$ in the new combined covering $\{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4\}$. The 1-cocycle $\beta + \hat{\beta} = \widehat{\beta}$ is $(\widehat{f}_{12}, \widehat{f}_{13}, \widehat{f}_{23}, \widehat{f}_{14}, \widehat{f}_{24}, \widehat{f}_{34}) = (f, 0, 0, 0, 0, \hat{f})$. Because of the "direct construction" argument mentioned above, this 1-cocycle will be cohomologous to (i.e. differing by a coboundary from) a cocycle of the form $(\widehat{f}, 0, 0, 0, 0, 0)$, so we can refer it back to the original covering if desired.

But we need not do so if we prefer not to. We have a generalization of the concept of a twistor function, namely as a collection of functions on portions of twistor space, defining a 1-cocycle with respect to some covering. We can actually use such a cocycle directly, obtaining the required space-time field by means of a branched contour integral. I shall just illustrate this with an example. Suppose that X is covered by 3 open sets $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$, where on a projective line L in X we get the picture:

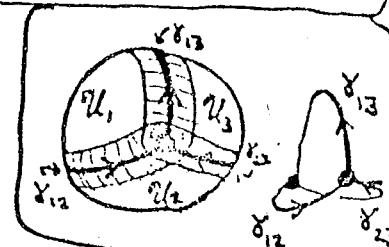
The 1-cocycle β is (f_{12}, f_{13}, f_{23}) . To

get the field φ ... we perform the sum of

three contour integrals with common end-points:

$$\text{in } \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 \quad \varphi = \int_{\gamma_{12}} f_{12} d^2 Z + \int_{\gamma_{13}} f_{13} d^2 Z + \int_{\gamma_{23}} f_{23} d^2 Z$$

$$L = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$$



It is a simple matter to check that the cocycle condition $f_{12} - f_{13} + f_{23} = 0$ ensures that the contour's end-points can be moved without affecting the result. This easily generalizes to coverings with N open sets.

Some applications (some knowledge of sheaf theory helpful!) ⑩

- (A) charge integrality in "twisted photon" description vs. vanishing charge in $\Phi_{AB} = \frac{1}{(2\pi i)^2} \oint \frac{\partial}{\partial w^A} \frac{\partial}{\partial \bar{w}^B} f(z^*) d^2\pi$ description (f hom. deg. 0).

We have the exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{x^{2\pi i}} \text{exp} \rightarrow \mathcal{O}^* \rightarrow 0$

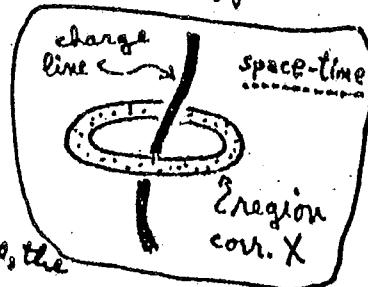
(cf. Morrow & Kodaira) from which we derive the long exact sequence:

$$\dots \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z}) \rightarrow \dots$$

{ ordinary integer cohomology } { $\Phi_{AB} = \oint \frac{\partial}{\partial w^A} \frac{\partial}{\partial \bar{w}^B} f$ } { "twisted photon" } { ordinary integer 2nd cohomology }

non-zero holomorphic functions taken multiplicatively

Choose X as region in \mathbb{PT} corr. to small space-time tube surrounding a charge world-line. Then topology of X is $S^2 \times S^2 \times \mathbb{R}^2$, so $H^1(X, \mathbb{Z}) = 0$, and $H^2(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus, the space $H^1(X, \mathcal{O}^*)$ effectively contains $H^1(X, \mathcal{O})$ and is strictly larger than $H^1(X, \mathcal{O})$ if the map to $\mathbb{Z} \oplus \mathbb{Z}$ is not only to the zero element. The image in the first \mathbb{Z} is always zero, but follows, for example, from examination of the "twistor quadrille" description (only works if the charge has integer value (i.e. lies in \mathbb{Z}) whereas the $H^1(X, \mathcal{O})$ description only works if the charge value is zero. (In contrast to this, it may be remarked that there is no restriction on the charge value if the -4-homogeneity $f(z^*)$ description $\tilde{\Phi}_{AB} = \frac{1}{(2\pi i)^2} \oint \pi_A \pi_B f(z^*) d^2\pi$ is used, this corresponding to $H^1(X, \mathcal{O}(-4))$.)



- (B) How to grow (or at least $\frac{1}{2}$ grow) new twistors from old.

There is another way of representing elements of $H^p(X, \mathcal{O})$, namely by taking the additive group of all $\bar{\partial}$ -closed $(0, p)$ -forms modulo the additive group of all $\bar{\partial}$ -exact $(0, p)$ -forms. (cf. Morrow & Kodaira)

By a (q, p) -form is meant a $(q+p)$ -form on X (not generally holomorphic) having q holomorphic differentials and p antiholomorphic differentials : $\alpha_{i_1 \dots i_q, j_1 \dots j_p} dz^{i_1} \wedge \dots \wedge dz^{i_q} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_p} = \alpha$. (11)

Then $\bar{\partial}\alpha = \frac{\partial \alpha}{\partial \bar{z}^k} \dots d\bar{z}^k \wedge dz^{i_1} \wedge \dots \wedge dz^{i_p}$ is a $\bar{\partial}$ -exact

$(q, p+1)$ -form whose vanishing is the condition for α to be $\bar{\partial}$ -closed. (We have $\bar{\partial}^2 = 0$.) Note that the $\bar{\partial}$ -closed $(0, 0)$ -forms are holomorphic functions. We can also "twist" such forms or functions, as before.

Applying this to $H^1(X, \mathcal{O}(n))$ we obtain the result that the helicity $\frac{1}{2}(n-2)$ massless fields can be described in terms of forms $g^\alpha(z; \bar{z}_\alpha) d\bar{z}_\alpha$ on X , modulo forms $\frac{\partial h(z; \bar{z}_\alpha)}{\partial \bar{z}_\alpha} d\bar{z}_\alpha$, where $\frac{\partial g^\alpha}{\partial \bar{z}_\beta} = \frac{\partial g^\beta}{\partial \bar{z}_\alpha}$, the functions g^α being homogeneous of degree n in \bar{z}^α and of degree -1 in \bar{z}_α (to balance the $d\bar{z}_\alpha$). In order that the form $g^\alpha d\bar{z}_\alpha$ be actually defined on the projective space (appropriately twisted) we require also $g^\alpha \bar{z}_\alpha = 0$. The massless field integral now becomes

$$\varphi_{A\dots D} = \frac{1}{(2\pi i)^4} \oint \frac{\partial}{\partial w^\alpha} \dots \frac{\partial}{\partial w^\beta} g^*(z; \bar{z}_\alpha) d\bar{z}_\alpha (I_{\alpha\beta} Z^\alpha dZ^\beta)$$

or

$$\varphi_{A'\dots D'} = \frac{1}{(2\pi i)^4} \oint \Pi_{A'} \dots \Pi_{D'} g^*(z; \bar{z}_\alpha) d\bar{z}_\alpha (I_{\alpha\beta} Z^\alpha dZ^\beta).$$

See N.M.J.W.'s article for a detailed discussion

There is presumably much freedom in the choice of g^* . I shall suppose that its singularities can be arranged so that the contour in the above integrals (which is initially S^2 , i.e., over the entire projective line corresponding to the space-time point in question) may be deformed, freeing \bar{z}_α from Z^α in the process, until $W_\alpha (= \bar{z}_\alpha = \text{"fixed" } \bar{z}_\alpha)$ passes through the line I . Writing $w = \sqrt{x}$, we get

$$g^* d\bar{z}_\alpha I_{\alpha\beta} Z^\beta dZ^\alpha = \frac{d\bar{v}}{\sqrt{x}} \frac{dz}{\sqrt{x}} = g dx_\alpha dz$$

$\uparrow W_\alpha = I_{\alpha\beta} X^\beta$

The function g can be taken to be holomorphic in z and γ , with $\gamma = 0$ and $\bar{\partial}_w g = 0$. Restricting to $\gamma = \Gamma x$ we get $\bar{g}^* x = 0 \Rightarrow \bar{g}^* = q x$ (some $q(z, \Gamma x)$, hom. degs. $(n, -2)$). (12)

Thus, the massless field integral can be written:

$$\Phi_{...} = \frac{1}{(2\pi i)^n} \oint_{S^2} \left\{ \frac{\bar{\partial}_w \alpha}{\pi} \right\} q^* \bar{x} dx \wedge \bar{z} dz$$

which is of the standard form for a 2-twistor integral. Thus we have conjured a new twistor out of thin air! This new twistor x is still only $\frac{1}{2}$ grown, though, since it appears as Γx . This can be stated as $[\bar{\partial}_x g] = 0$ — which implies the massless relation $\bar{z}^* \bar{\partial}_z \bar{\partial}_x g = 0$. In fact, the $\bar{\partial}$ -closed property $\bar{\partial}_w g = 0$ now disappears on $\gamma = \Gamma x$, $g \rightarrow g^*$, being incorporated in $[\bar{\partial}_x g] = 0$, $\bar{\partial}_x g = -2g$ (homogeneity relation).

~ Roger Penrose (with help from
G.A.J.S., A.S.H., N.M.J.W.
and M.F.A., particularly)

Some remarks on the 2-twistor description of electromagnetic and linearized gravitational potentials.
(This follows on from some remarks due to Geoff. Curtis and was inadvertently omitted from TN1.)

CPT considerations, as applied to 2-twistor functions, suggest that the photon and graviton, when they sit in the 2-twistor scheme, ought to be described by $(-2, -2)$ -homogeneous functions, $F(z, x), G(z, x)$, respectively, where

e.m. pot. $\rightarrow \Phi_{AA'} = \frac{1}{(2\pi i)^n} \oint (Z_A \frac{\partial}{\partial Z_A} - X_A \frac{\partial}{\partial X_A}) F d^2 z_n d^2 x$ } or could take therefore on x to give $\tilde{F}(z, y)$, $\tilde{g}(z, y)$

and

grav. pot. $\rightarrow h_{AA'BB'} = \frac{1}{(2\pi i)^n} \oint (Z_A X_B \frac{\partial}{\partial Z_A} \frac{\partial}{\partial X_B}) G d^2 z_n d^2 x$

We have Lorenz $\nabla^\alpha \Phi_\alpha = 0$ and DeDonder $\nabla^\alpha (h_{AB} - \frac{1}{2} h^a \circ g_{AB}) = 0$ gauges ~ Roger Penrose $\& h_{AB} = h_{BA}$.

It follows from the cohomological interpretation of twistor functions that zero rest-mass fields on Minkowski space can be represented by equivalence classes of 1-forms on twistor space. Here I shall show how one can go from these equivalence classes directly to the spacetime fields. This involves a slight reinterpretation of some well known calculations. The advantage of this approach is that there is no contour ambiguity involved.

Notation: $\partial(\bar{\partial})$ denotes the (anti-)holomorphic exterior derivative. Thus,

$$\partial(f^{\alpha\beta\dots}\lambda_\mu\dots d\bar{Z}_\alpha\wedge dZ^\lambda\dots) = \partial_{\bar{Z}^\lambda} f^{\alpha\beta\dots}\lambda_\mu\dots] d\bar{Z}_\alpha\wedge dZ^\lambda\dots \quad (1)$$

$$\partial(\pi^{AB\dots DE\dots} d\pi_A\wedge\dots d\bar{\pi}_D\wedge\dots) = \partial_{\bar{\pi}_D} [\pi^{AB\dots DE\dots}] d\pi_A\wedge\dots d\bar{\pi}_D\wedge\dots$$

$$\text{where } \partial_\alpha = \frac{\partial}{\partial Z^\alpha} \text{ and } \partial^A = \frac{\partial}{\partial \pi_A}. \quad (\kappa \text{ defined on spin-space}) \quad (2)$$

Z is the vector field $Z^\alpha \frac{\partial}{\partial Z^\alpha}$, θ is the 1-form $I_{\alpha\beta} Z^\alpha dZ^\beta = \pi^B d\pi_B$, and \lrcorner denotes the contraction between a vector field and a form. Note that $d = \partial + \bar{\partial}$, $\bar{\partial}\theta = 0$, $Y\lrcorner d\theta = 2\theta$ and $Y\lrcorner\theta = 0$.

Consider the set of all $(0,1)$ -forms α on the (nonprojective) \mathbb{H} -twistor space $T^+ = \{z^\alpha | z^\alpha \bar{z}_\alpha > 0\}$, satisfying

$$\bar{\partial}\alpha = 0, \quad Y\lrcorner\alpha = 0 \quad \text{and} \quad Y\lrcorner d\alpha = (-n-2)\alpha \quad (3)$$

where n is an integer. That is, $\alpha = f^\alpha d\bar{Z}_\alpha$ and

$$\frac{\partial \alpha}{\partial \bar{Z}_\beta} = \frac{\partial f^\alpha}{\partial \bar{Z}_\beta}, \quad \bar{Z}_\alpha f^\alpha = 0 \quad \text{and} \quad Z^\alpha \frac{\partial f^\beta}{\partial \bar{Z}_\alpha} = (-n-2) f^\beta. \quad (4)$$

Let H denote the set of equivalence classes of these forms under the equivalence relation $\alpha \sim \beta$ whenever $\alpha - \beta = \bar{\partial}h$ for some function $h = h(z, \bar{z})$ such that $Y\lrcorner d h = (-n-2)h$. Each element of H can be identified with a cohomology class of twistor functions, homogeneous of degree $-n-2$, and thus with a massless spin-1/2 field on Minkowski space. Two lemmas are needed to construct this field explicitly:

Lemma 1: Suppose that $\varepsilon(z)$ is holomorphic and homogeneous of degree n and that $[\alpha] \in H$. Then, if $\beta = \varepsilon \cdot \alpha \wedge \theta$, $Y\lrcorner d\beta = 0 = \bar{Y}\lrcorner d\beta$.

Proof: Since $\bar{\partial}\beta = 0$,

$$Y\lrcorner d\beta = Y\lrcorner d\beta = Y(\varepsilon) \alpha \wedge \theta + \varepsilon(Y\lrcorner d\alpha) \wedge \theta + \varepsilon(\alpha \wedge (Y\lrcorner d\theta))$$

$$= (n + (-n-2) + 2)d\wedge \theta = 0$$

$$Y \lrcorner d\varphi = \bar{Y} \lrcorner d\varphi = d(\bar{Y} \lrcorner \varphi) = 0$$

Lemma 2: Suppose that $h(\pi_A, \bar{\pi}_A)$ satisfies $\pi_A \frac{\partial h}{\partial \pi_A} = -2h$ and $\bar{\pi}_A \frac{\partial h}{\partial \pi_A} = 0$. Then $d(h\theta) = \bar{\partial}h \wedge \theta$.

Proof: $d(h\theta) = d(h\theta) + \bar{\partial}(h\theta) = d(h\theta) + (\bar{\partial}h)\wedge \theta$ and

$$\bar{\partial}(h\theta) = \frac{\partial}{\partial \pi_A} h(\pi, \bar{\pi}) d\pi_A \wedge (\pi^B d\pi_B) + h d\pi^B \wedge d\pi_B = 0.$$

Now, given a $(0,1)$ -form α satisfying the relations (3) and a point $x^{AA'}$ in the past tube of Minkowski space, put

$$\varphi_{B' \dots C'} = \oint \pi_{B'} \dots \pi_{C'} \alpha \wedge \theta, \quad (5)$$

the integral being taken over any 2-sphere in $X = \{z^A = (ix^{AA'}, \pi_A)\} \subset \mathbb{R}^+$ on which π_A varies over all (real) null directions through x . Then:

1) The integral is contour independent: any two such 2-spheres form the boundary of a 3-surface, everywhere tangent to some combination of Y and \bar{Y} . The integral of $d(\pi_{B'} \dots \pi_{C'} \alpha \wedge \theta)$ over this 3-surface vanishes since

$$Y \lrcorner d(\pi_{B'} \dots \pi_{C'} \alpha \wedge \theta) = 0 = \bar{Y} \lrcorner d(\pi_{B'} \dots \pi_{C'} \alpha \wedge \theta) \quad (\text{by Lemma 1}). \quad (6)$$

2) φ depends only on the equivalence class of α : if $\alpha = \bar{\partial}h$, then the integrand in eqn. (5) is $d(\pi_{B'} \dots \pi_{C'} h \theta)$ (by lemma 2) and the integral vanishes.

3) φ satisfies the zero rest mass field equation:

$$\begin{aligned} \frac{\partial}{\partial x^{AA'}} \varphi_{B' \dots C'} &= \oint \frac{\partial}{\partial x^{AA'}} \lrcorner (\bar{\partial} + \partial)(\pi_{B'} \dots \pi_{C'} \alpha \wedge \theta) \\ &= \oint \frac{\partial}{\partial x^{AA'}} \lrcorner \bar{\partial}(\pi_{B'} \dots \pi_{C'} \alpha \wedge \theta) \\ &= \oint \pi_A \pi_{B'} \dots \pi_{C'} \left(\frac{\partial f^X}{\partial \omega^A} d\bar{\omega}_X \right) \lrcorner (\pi^B d\pi_B) \end{aligned}$$

where $\omega = f^X d\pi_X$. Contracting over A' and B' gives zero.

4) φ is positive frequency: a similar argument gives $\frac{\partial}{\partial x^{AA'}} \varphi_{B' \dots C'} = 0$.

Some errata for TN 1:

P.1,6 refs. to Penrose & MacCallum
should be dated 1972

P.6 change sign in def. of I_2 ; in last
eqn. factor 2 should be y_2 .

Nicholas Woodhouse