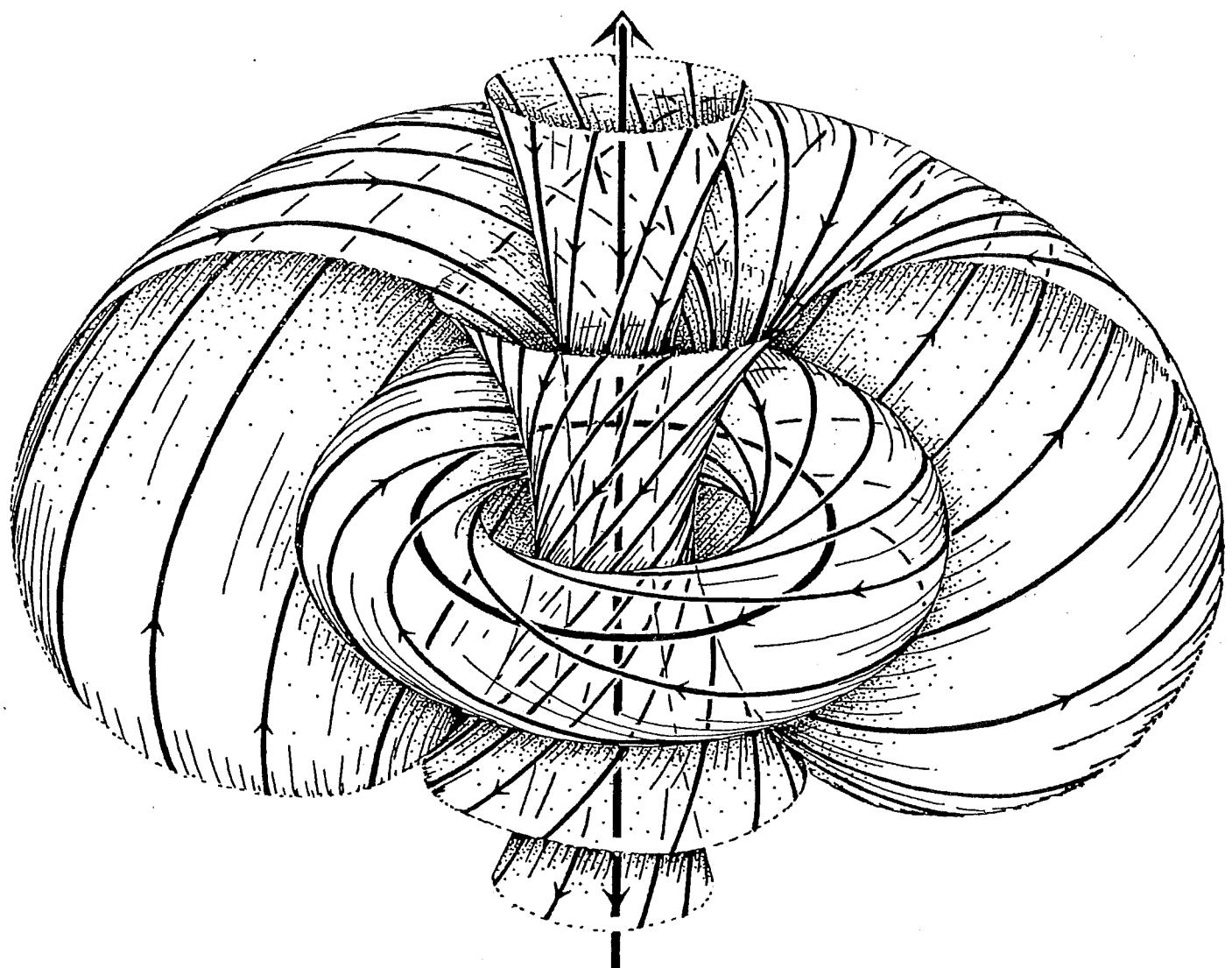


Twistor Newsletter (nº 20:11, Sept., 1985)



Mathematical Institute, Oxford, England.

LH

Embedding 2-surfaces in \mathbb{CM} .

A result of 2-surface twistor theory — apparently of the status of a “folk-lore theorem” — is that any spacelike 2-surface $S \subset M$ for which the space of 2-surface twistors is 4-dimensional (e.g. a generic S^2) can be embedded in complex conformally flat 4-space in such a way that σ and σ' remain unchanged (except for the standard scalings) whereas $\bar{\sigma}$ and $\bar{\sigma}'$ get replaced by new quantities $\tilde{\sigma}$ and $\tilde{\sigma}'$.

This embedding (or perhaps immersion) is to be achieved as follows: We have (“natural” immersion)

$$S \hookrightarrow \mathbb{CM}^\#(S) \quad (1)$$

where the points of $\mathbb{CM}^\#(S)$ are the 2-dimensional linear subspaces of $T^*(S)$ (or, equivalently, of $T(S)$).

To obtain (1), first identify each pair (p, λ_A) — for which $p \in S$ and λ_A is a spin-covector at p in M — with the element of $T^*(S)$ which carries the 2-surface twistor $\{\omega^A\}$ to $\omega^A \lambda_A$ (at p). Then p itself gets identified with the linear span of all the λ_A at p .

Now it turns out that the 2-surf. twistors on S can be identified with the restrictions to S of the solutions of the twistor equation

$$\nabla_{A'}^{(A} \omega^{B)} = 0 \quad (2)$$

in $\mathbb{CM}^\#(S)$ (which is a (conformally flat) complex manifold,

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so (2) is meaningful). To see this, note first that the solutions of (2) are precisely the spin-vector fields in $\mathbb{CM}^\#(\mathcal{S})$ for which $W^A \lambda_A$ is const. over each β -plane in $\mathbb{CM}^\#(\mathcal{S})$ — where λ_A is such that it is covariantly constant along the β -plans and $\lambda^A \xi^{A'}$ is tangent to it for all $\xi^{A'}$. This follows from the vanishing of

$$\lambda^A \xi^{A'} \nabla_{AA'} (\lambda_B W^B) = \xi^{A'} \lambda^A \lambda_B \nabla_{AA'} W^B = -\xi^{A'} \lambda_A \lambda_B \nabla_{A'}^{(A} W^{B)} . \quad (3)$$

Second, note that the same equation holds at \mathcal{S} in M for those directions for which $\lambda^A \xi^{A'}$ is tangent to \mathcal{S} by virtue of the 2-surface twistor equation on \mathcal{S} . Third, note that two neighbouring points of \mathcal{S} are null separated iff their images in $\mathbb{CM}^\#(\mathcal{S})$ are null separated; for if neighbouring points $p, p' \in \mathcal{S}$ satisfy $(pp')^{AA'} \lambda_A = 0$, then (p, λ_A) and (p', λ_A) both annihilate the same $\{W^A\}$, namely one given by $W^A \lambda_A = 0$ at p and \therefore at p' since $\lambda^A \xi^{A'} \nabla_{AA'} (W^B \lambda_B) = 0$.

Thus, the complex conformal metric of \mathcal{S} is preserved by its embedding in $\mathbb{CM}^\#(\mathcal{S})$. Now the spin-vector fields w^A in $\mathbb{CM}^\#(\mathcal{S})$ for which $(w^A)_A$ is const. along the $\lambda^A \xi^{A'}$ directions in $\mathbb{CM}^\#(\mathcal{S})$ induce spin-vector fields on \mathcal{S} which are constant along those same null directions at \mathcal{S} when they happen to be tangent to \mathcal{S} in M , which, by the above, gives the 2-surface twistor equation.

In $\mathbb{CM}^\#(\mathcal{S})$, the 2-surface twistor eqn. takes the standard form

$$\delta' w^0 = \sigma' \omega', \quad \delta \omega' = \sigma w^0 \quad (4)$$

with resp. to a spinor basis associated with the tangent 2-plane element, and with resp. to some complex flat metric for $\mathbb{CM}(\mathcal{S})$. These σ, σ' must scale in the standard way under conformal, and dyad, rescalings — which amounts to:

$$\sigma \mapsto \Gamma^2 \tilde{\Gamma}' \Gamma'^{-1} \sigma, \quad \sigma' \mapsto \Gamma'^2 \tilde{\Gamma} \Gamma'^{-1} \sigma' \quad (5)$$

under

$$0^A \mapsto \Gamma^A, \quad 1^A \mapsto \Gamma'^A, \quad \partial^A \mapsto \tilde{\Gamma}^A, \quad \tilde{\partial}^A \mapsto \tilde{\Gamma}'^A$$

so with normalized dyads, $\epsilon_{AB} \mapsto (\Gamma \Gamma')^{-1} \epsilon_{AB}, \quad \epsilon_{A'B'} \mapsto (\tilde{\Gamma} \tilde{\Gamma}')^{-1} \epsilon_{A'B'}$. Comparing (4) in $\mathbb{CM}^\#(\mathcal{S})$ with (4) in M , we find that the corresponding σ, σ' must scale, as in (5), between the two as required. More can be done using these ideas — later!

Twistors and Minimal Surfaces.

It has been known since the work of Weierstrass (1866) that the equations of a minimal surface in R^3 can be solved in terms of a single holomorphic function of a single variable. This can be cast into a result from twistor theory in three dimensions (Hitchin, 1982), since a minimal surface in R^3 corresponds to the real part of a null holomorphic curve in C^3 , which corresponds to a holomorphic curve in TP^1 , the twistor space associated with R^3 or C^3 . One can re-derive these results in a simple way by considering the problem of finding minimal 2-surfaces in four dimensions, using "normal" twistor theory. One can solve the minimal 2-surface equations in four dimensions in terms of two holomorphic functions of one complex variable. Looking for solutions confined to a "t=0" hyperplane leads immediately to the Weierstrass construction.

As usual, let projective twistor space be given homogeneous coordinates

$$z^\alpha = [\omega^A, \pi_A] \quad (1)$$

and consider the curve in P^3 defined by

$$\omega^A = f^A[\pi_A], \quad (2)$$

where f is homogeneous of degree +1 in π_A . Now consider the space-time points satisfying

$$ix^{AA'}\pi_{A'} = f^A[\pi_A]; \quad x^{AA'} = -i\partial f^A/\partial\pi_{A'}. \quad (3a,b)$$

Note that this only makes sense if f is homogeneous of degree +1. Also, by taking the exterior derivative of

$$\pi_{A'}\partial f^A/\partial\pi_{A'} = f^A \quad (4)$$

we see that

$$\frac{\partial^2 f^A}{\partial\pi_{A'}\partial\pi_{B'}} d\pi_{B'} = \pi^{A'}\Psi^A \quad (5)$$

for some differential form Ψ^A . On taking the derivative of (3b) and using (5) it follows that $x^a(\pi)$ is a null curve. If we introduce a coordinate

$$\zeta = \pi_0/\pi_1. \quad (6)$$

then we see that $x^a(\zeta)$ is also holomorphic in ζ . Thus the curve $x^a(\zeta) = x^a + iy^a$ is a complex null holomorphic curve in C^4 . Its real part x^a is

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therefore a minimal 2-surface S in Minkowski space-time. (see, e.g. Lawson, 1980: the real part of a null holomorphic curve is a minimal 2-surface in all dimensions). The coordinate ζ is related to isothermal coordinates (u,v) for S by $\zeta = u+iv$, and the metric of S is in the form

$$ds^2 = f(u,v)[du^2 + dv^2]. \quad (7)$$

Now consider explicit coordinates. Let

$$r^A[\pi_B] = \pi_1 \cdot F^A[\zeta] \quad (8)$$

where $F^A[\zeta] = f^A(\zeta, 1) = [f(\zeta), g(\zeta)]$. The conditions (3a,b) become

$$ix^{A0} = F^A_{,\zeta}. \quad (9)$$

$$ix^{A1} = F^A - \zeta F^A_{,\zeta}. \quad (10)$$

In terms of the usual vector-spinor correspondence, this becomes (with ' denoting $,\zeta = \partial/\partial\zeta$)

$$\sqrt{2} t = -if' - ig + i\zeta g' \quad (11)$$

$$\sqrt{2} x = -ig' - if + i\zeta f' \quad (12)$$

$$\sqrt{2} y = -g' + f - \zeta f' \quad (13)$$

$$\sqrt{2} z = -if' + ig - i\zeta g' \quad (14)$$

The real parts of (11) - (14) define the minimal 2-surface in Minkowski space-time. The variables X^a satisfy

$$X^a_{,uu} + X^a_{,vv} = 0, \quad (15)$$

$$X^a_{,u} X_{a,u} = X^a_{,v} X_{a,v}, \quad (16)$$

$$X^a_{,u} X_{a,v} = 0 \quad (17)$$

with respect to the Minkowski metric $\eta_{ab} = \text{DIAG}(+1, -1, -1, -1)$. Now since $x^a(\zeta)$ is null with respect to η_{ab} , then $p^a(\zeta) = (it, x, y, z)$ is null with respect to the Euclidean metric on \mathbb{R}^4 . Therefore the imaginary part of (11) and the real parts of (12)-(14) define a minimal 2-surface in Euclidean 4-space.

Now consider a surface confined to the hypersurface $t = 0$. Then we must impose the additional condition:

$$f' = \zeta g' - g. \quad (18)$$

Write $g = G'$, so that $f = \zeta G' - 2G$. Then (12)-(14) reduce to

$$ix/\sqrt{2} = 1/2(1-\zeta^2)G'' + \zeta G' - G \quad (19)$$

$$iy/\sqrt{2} = -1/2(1+\zeta^2)G'' + i\zeta G' - iG \quad (20)$$

$$iz/\sqrt{2} = \zeta G'' - G' \quad (21)$$

giving the Weierstrass representation of a minimal surface in Euclidean 3-space. Minimal surfaces in Minkowskian 3-space can be obtained by taking the real part of two of (19)-(21) and the imaginary part of the third.

Two other special cases are obtained by taking either f or g to be zero in (11)-(14). In the Minkowskian case taking $f=0$ leads to a surface confined to the null hyperplane $T+Z=0$, while $g=0$ generates a surface in the null hyperplane $T-Z=0$.

It is straightforward to show that any null holomorphic curve in \mathbb{CM} can be parametrized by (9)-(10). Firstly, let $x^0(\xi)$ be the given curve, parametrized by ξ . Since $x^0(\xi)$ is null,

$$x^0(\xi), \xi = \alpha^A(\xi)\beta^A(\xi) \quad (22)$$

for some spinors $\alpha^A(\xi)$, $\beta^A(\xi)$ holomorphic in ξ . Now define ζ as a holomorphic function of ξ by

$$\zeta = -\beta^0/\beta^1 \quad (23)$$

Now define F^A (up to an arbitrary constant) by

$$-iF^A, \zeta = x^{A0}(\xi[\zeta]) \quad (24)$$

Then (22) implies

$$x^{A1}, \zeta = i[\zeta F^A, \zeta - F^A], \zeta \quad (25)$$

integration of which gives (10) with the arbitrary constant in F^A fixed. It follows that null holomorphic curves in \mathbb{CM} are in 1-1 correspondence with holomorphic curves on Twistor space defined by (2).

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William Shaw.

A Suggested Further Modification to the Quasi-Local Formula

In view of the anomalous expression for the quasi-local mass for small contorted ellipsoids found by N.M.J.W. (TN 19, p.7), it seems that one must seek a further modification beyond that given by the determinant factor as suggested in TN 18. An appealing feature of the original form had always been the existence of the Tod expression $-\frac{i}{2\pi G} \oint \Pi_0 \Pi_1 \mathcal{S}$ (\mathcal{S} being the surface-area 2-form). The determinant modification

$$-\frac{i}{4\pi G} \oint \left\{ (\Phi_1 - \Phi_{01})(\omega^0)^2 + (\Phi_2 - \Phi_{11} - 1)2\omega^0\omega^1 + (\Phi_3 - \Phi_{21})(\omega^1)^2 \right\} \eta \mathcal{S}$$

(with $\eta \propto \begin{vmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^1 & - & - & - \\ \omega^2 & - & - & - \\ \omega^3 & - & - & - \end{vmatrix}$) leads, by integration by parts,

to a "Tod form" $-\frac{i}{2\pi G} \oint (\widehat{\Pi}_0 \widehat{\Pi}_1 + \omega^0 \omega^1 \mathcal{Z} \mathcal{Z}' \mathcal{S}) \mathcal{S}$

where

$$\widehat{\Pi}_{01} = \mathcal{Z} \Pi_{01} + i \omega^1 \mathcal{Z} \mathcal{Z}' \mathcal{S}, \quad \widehat{\Pi}_{11} = \mathcal{Z} \Pi_{11} + i \omega^0 \mathcal{Z} \mathcal{Z}' \mathcal{S}$$

with $\mathcal{Z}^2 = \eta$. These modifications to Π_{01} , Π_{11} seem natural enough as "complex conformal factor" terms, but the extra term $\omega^0 \omega^1 \mathcal{Z} \mathcal{Z}' \mathcal{S}$ has an awkward appearance.

Though the motivation is a little flimsy, it would seem to be worth trying $-\frac{i}{2\pi G} \oint \widehat{\Pi}_0 \widehat{\Pi}_1 \mathcal{S}$ as a possible modified definition for $A_{\alpha\beta} Z^\alpha Z^\beta$. The good results to date seem to be unaffected by this change.

(Thanks to K.P.T., N.M.J.W.) Ref: S&S-T Vol. 2.
for comments.

Roger Penrose

Higher-Dimensional 2-Surface Twistors

Generalizations of the 2-surface twistor concept to higher dimensions can be achieved in two different directions. In the first place we can keep the surface S two-dimensional, but allow the ambient space to become higher-dimensional. In the second place we can allow S itself to become higher-dimensional. In any case, the spinors and twistors need to be those for a higher-dimensional ambient space-time M . For simplicity, I shall consider M to be 6-dim. and S to be either 2-dim. (case A) or 4-dim (case B), but higher generalizations can also be readily achieved.

Here the reduced spinors for M are 4-dim, and the twistors for M , if M happens to be conformally flat, would be 8-dim. (i.e. reduced spinors for the pseudo-orthog. group for which the points of M correspond to generators of the null cone; see Appendix to Penrose-Rindler, S. & S-T, vol. 2 — when it comes out in a few months — for the relevant n-dimensional spinor and twistor concepts). For the moment at least, everything is complex, so the signature is irrelevant.

We need to see how to describe, in spinor terms, the projection operator E which takes us down from the 6-space to the 2-space $\langle \begin{smallmatrix} \text{tangential in case A} \\ \text{normal in case B} \end{smallmatrix} \rangle$ and its orthogonal complement $F = I - E$ which takes us from the 6-space to the 4-space $\langle \begin{smallmatrix} \text{normal in case A} \\ \text{tangential in case B} \end{smallmatrix} \rangle$. We use "twistor-type"

description of vectors in M , where l.c. Greek indices are 4-dimensional ("twistor-like") spinor indices, skew pairs of which represent tangent vectors in M . The tangent space (case A) or normal space (case B) to S at a point p (S assumed

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non-null) contains two (complex) null directions u, v . Now null vectors in M correspond to simple bivectors in the (twistor-like) spin-space of M :

$$U^{\alpha\beta} = a^{[\alpha} b^{\beta]} \quad , \quad V^{\alpha\beta} = c^{[\alpha} d^{\beta]}$$

$$U_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} a^\gamma b^\delta \quad , \quad V_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} c^\gamma d^\delta$$

and take $a^\alpha b^\beta c^\gamma d^\delta \epsilon_{\alpha\beta\gamma\delta} = 4$. The projector E is then given by

$$E^{\alpha\beta}_{\gamma\delta} = \frac{1}{2} U^{\alpha\beta} V_{\gamma\delta} + \frac{1}{2} V^{\alpha\beta} U_{\gamma\delta}$$

and

$$F^{\alpha\beta}_{\gamma\delta} = \frac{1}{6} \{^{\alpha\beta}_{\gamma\delta} \} - E^{\alpha\beta}_{\gamma\delta}$$

It is convenient to consider corresponding projectors in the spin-space:

$$e_\beta^\alpha = U^{\alpha\gamma} V_{\beta\gamma} \quad , \quad f_\beta^\alpha = V^{\alpha\gamma} U_{\beta\gamma}$$

which decompose the spin-space:

$$\{^{\alpha\beta}_{\gamma\delta} \} = e_\beta^\alpha e_\delta^\beta + f_\beta^\alpha f_\delta^\beta \quad (A)$$

where $e_\beta^\alpha e_\delta^\beta = e_\delta^\alpha$, $f_\beta^\alpha f_\delta^\beta = f_\delta^\alpha$, $e_\beta^\alpha f_\delta^\beta = 0 = f_\beta^\alpha e_\delta^\beta$. Note also

$$e^{\alpha\beta}_{\gamma\delta} e^{\gamma\delta}_{\gamma\delta} = \frac{1}{2} U^{\alpha\beta} V_{\gamma\delta} \quad , \quad f^{\alpha\beta}_{\gamma\delta} f^{\gamma\delta}_{\gamma\delta} = \frac{1}{2} V^{\alpha\beta} U_{\gamma\delta} \quad (B)$$

The corresponding decomposition of the vector space is given by

case A: \rightarrow $\underbrace{\text{normal}}$

$$\{^{\alpha\beta}_{\gamma\delta} \} = 2 e^{\alpha\beta}_{[\gamma\delta]}$$

$\underbrace{\text{tangential}}$

$$+ e^{\alpha\beta}_{[\gamma\delta]} e^{\gamma\delta}_{[\gamma\delta]} + f^{\alpha\beta}_{[\gamma\delta]} f^{\gamma\delta}_{[\gamma\delta]}$$

(C)

case B: \rightarrow $\underbrace{\text{tangential}}$

$\underbrace{\text{normal}}$

case A Let \mathcal{S} be "spacelike" with S^2 topology. The twistor equation in (conf. flat) 6-space can be written

$$\nabla_{\alpha\beta} \omega^\delta = \oint_{[\alpha \pi_\beta]}^\delta \mathcal{D} \quad (D)$$

By (C), $e^L e^T$ and $f^L f^T$ project tangentially to \mathcal{S} and we can extract the tangential parts of (D) as

$$e_j^\lambda f_\mu f_\nu \nabla_{\alpha\beta} \omega^\delta = 0, \quad f_\delta^\lambda e_\mu^\alpha e_\nu^\beta \nabla_{\alpha\beta} \omega^\delta = 0$$

i.e. (by (B)) $e_j^\lambda V^{\alpha\beta} \nabla_{\alpha\beta} \omega^\delta = 0, \quad f_\delta^\lambda U^{\alpha\beta} \nabla_{\alpha\beta} \omega^\delta = 0$

i.e. $e_\lambda^\mu V^{\alpha\beta} \nabla_{\alpha\beta} (\omega^\delta e_\lambda^\lambda) = \omega^\delta f_\lambda^\lambda e_\nu^\mu V^{\alpha\beta} \nabla_{\alpha\beta} e_\nu^\lambda,$ (E)
 $\underbrace{\phantom{e_\lambda^\mu V^{\alpha\beta} \nabla_{\alpha\beta}}}_{\mathfrak{J}' \text{ analogue}}$ $\underbrace{\phantom{V^{\alpha\beta} \nabla_{\alpha\beta}}}_{\mathfrak{W}^0 \text{ analogue}}$ $\underbrace{\phantom{V^{\alpha\beta} \nabla_{\alpha\beta}}}_{\mathfrak{W}' \text{ analogue}}$ $\underbrace{}_{\mathfrak{T}' \text{ analogue}}$

$$f_\lambda^\mu U^{\alpha\beta} \nabla_{\alpha\beta} (\omega^\delta f_\lambda^\lambda) = \omega^\delta f_\lambda^\lambda f_\nu^\mu U^{\alpha\beta} \nabla_{\alpha\beta} f_\nu^\lambda$$

Similar use of the Atiyah-Singer theorem to that used for the original S^2 in 4-space applies here also. (Look at a canonical case, with its adjoint, and then deform.) In general we get an 8-dim space of solutions of (E)

case B Let \mathcal{S} become a 4-dim space-time M of embedding class 2 (or 1), M being now a conformally flat 6-space. For example \mathcal{S} could be the Schwarzschild solution M_s . The restriction of (D) to M turns out to take the form

$$\nabla_{(A}^{A'} \omega_{B)} = \tilde{\nabla}_{AB}^{A'B'} \tilde{\omega}_{B'}, \quad \nabla_{(A'}^{A} \tilde{\omega}_{B')} = \tilde{\nabla}_{A'B'}^{AB} \omega_B$$

where $\nabla_{AA'}$ and $\nabla_{AA'}$ are analogues of \mathfrak{J} and \mathfrak{J}' , which differ from $\nabla_{AA'}$ by "gauge" dependent terms - like $\alpha, \beta, \delta, \epsilon$ of the spin-coeff. formalism. Strictly ω_A and $\tilde{\omega}_{A'}$ are "boost-weighted" spinors and the boost weight referring to the perpendicular frame transformations. (Thanks to TNB, LPH, KPT & I.G.) Rosenfeld

Asymptotically anti-de Sitter Space-Times

1. The Angular Momentum Twistor

I will first briefly review the work of Ashtekar and Magnon, [1]. A space-time $(\tilde{M}, \hat{g}_{ab})$ is said to be asymptotically anti-de Sitter if there exists a manifold M with a metric g_{ab} and a diffeomorphism from \tilde{M} to $M - \partial M$ such that

- (i) $\exists \omega$ on M such that $g_{ab} = \omega^2 \hat{g}_{ab}$ on \tilde{M} ;
- (ii) $\mathcal{F} = \partial M$ is topologically $S^2 \times \mathbb{R}$ and $\omega = 0$ on \mathcal{F} ;
- (iii) \hat{g}_{ab} satisfies $R_{ab} - \frac{1}{2} R \hat{g}_{ab} + \lambda \hat{g}_{ab} = -8\pi G \hat{T}_{ab}$ with $\lambda < 0$ and where $\omega^{-4} \hat{T}_{ab}$ admits a smooth limit as $\omega \rightarrow 0$;
- (iv) $B_{ab} \equiv \omega^{-1} B_{ab} \stackrel{\omega \rightarrow 0}{\approx} 0$, where B_{ab} is the magnetic part of the Weyl tensor. Colored of M and the symbol " $\stackrel{\omega \rightarrow 0}{\approx}$ " means "in the limit as $\omega \rightarrow 0$ ".

Notes:

- (a) In [1] they assumed the rather weaker condition that $\omega^{-8} \hat{T}_{ab}$ admits a smooth limit as $\omega \rightarrow 0$. The scaling chosen in (iii) leaves the conservation equation $\nabla^a T_{ab} = 0$ conformally invariant, see [2] page 371.
- (b) Condition (iii) and the equations for the change in the spinor decomposition of the curvatures under conformal scalings imply that:

(1) If $S\alpha = \nabla_a \beta^a$, then $S^\alpha S_\alpha \stackrel{?}{=} \lambda/3$;

(2) $C_{abcd} \stackrel{?}{=} 0$.

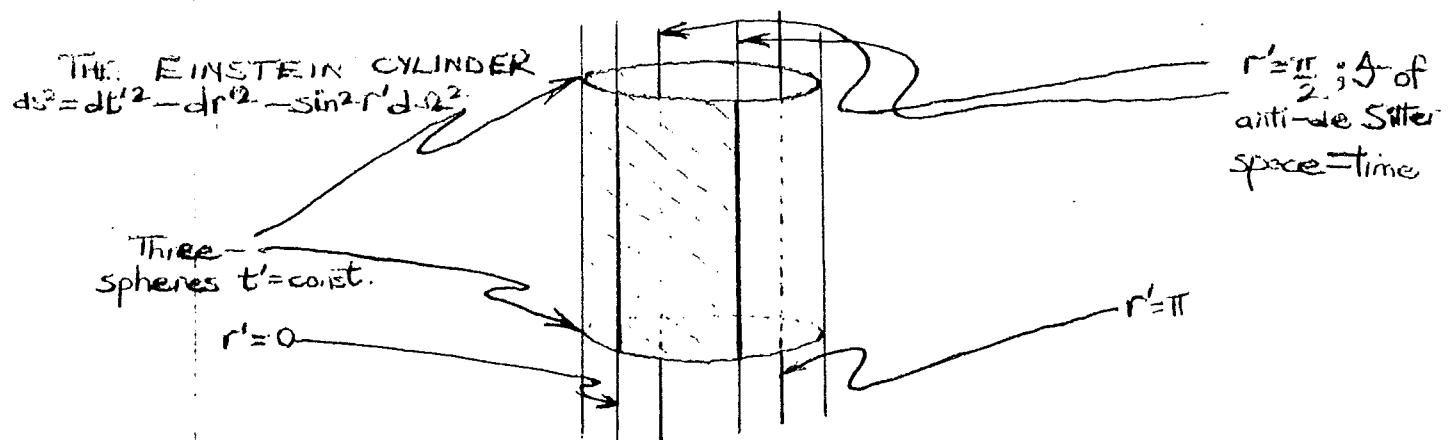
Thus \mathcal{I} is timelike. We may further choose β^a so that $\nabla_a S_b \stackrel{?}{=} 0$.

(c) Condition (ii) is equivalent to the Cotton-York tensor of \mathcal{I} vanishing and thus \mathcal{I} is conformally flat. It is also equivalent to the condition

$${}^3D_{[a}V_{b]c} = 0$$

where $V_{abc} = (\Phi_{abc} - \Lambda g_{ab}) - E_{abc}$ and 3D_a denotes the intrinsic covariant derivative of \mathcal{I} .

This, together with the condition $E_{abc} \stackrel{?}{=} 0$, (which is trivially satisfied from (b)(2)), implies that \mathcal{I} is embeddable in conformally flat space-times with the same first and second fundamental forms, (see reference [3]), and thus three surface twistors exist on \mathcal{I} . We may therefore think of \mathcal{I} as the conformal infinity of anti-de Sitter spacetime which can be embedded in the Einstein static universe as the time axis multiplied by the two-sphere, equator of the three-sphere cross-sections of constant time; (see over).



It follows that the conformal group of \mathcal{F} is the anti-de Sitter group $(O(2,3))$ and there are 10 conformal Killing vectors on \mathcal{F} , (with respect to the intrinsic metric). Hawking [4] has shown that (iv) is equivalent to gravitational radiation obeying a reflective boundary condition on \mathcal{F} .

Given a cross-section C of \mathcal{F} and a conformal Killing vector ξ^α on \mathcal{F} defines a conserved quantity by

$$Q_\xi[C] = -\frac{1}{8\pi G} \oint E_{ab} \xi^a dS^b. \quad (\text{see } [1])$$

This expression is conformally invariant. We may show that the flux $F_\xi(\Delta)$ across any region Δ of \mathcal{F} bounded by two cross-sections is given by

$$F_\xi(\Delta) = \int_{\Delta} [\lim_{n \rightarrow 0} s^{2n} T_{\alpha}^{\beta}] \xi^{\alpha} \beta^{\beta} d\Sigma.$$

Note that if there is no matter near \mathcal{F} the flux vanishes and there is no Bondi leakage.

Schwarzschild anti-de Sitter space-times has metric

$$ds^2 = \left(1 - \frac{2MG}{r} + \alpha^2 r^2\right) dt^2 - \left(1 - \frac{2MG}{r} + \alpha^2 r^2\right)^{-1} dr^2 - r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $\alpha = \sqrt{(-\lambda/3)}$.

If $\xi^\alpha = \frac{1}{\alpha} \frac{\partial}{\partial t}$, (which with $r_2 = 1/\alpha$ is a unit vector at \mathcal{J} with respect to the rescaled metric), we obtain $Q_\xi [C] = M$ for any cross-section C of \mathcal{J} .

To show that the above expression for conserved quantities is really Penrose's expression for quasi-local mass and angular momentum $[S]$ (modified for asymptotically anti-de Sitter space-times by the subtraction of the cosmological constant term) we first need the following lemma.

Lemma

IF Σ is a hypersurface in a conformally flat space-time with normal ξ^α and $F_{ab} = 0$ then $\xi^\alpha = w^a \xi_b^\wedge w^b$ is a (null) conformal Killing vector on \mathcal{J} if $\bar{V}_A(w^B) = 0$.

The proof is simply by substitution of the solution of the twistor equation in a conformally flat space-time.

For a general conformal Killing vector ξ^α $\xi_\alpha = F_{ab} \xi_b^\wedge$ where

$$F_{ab} = \sum_{i=1}^2 C_A B_i^\wedge (W_A^\wedge W_B^\wedge)$$

and w_A, \tilde{w}_A are solutions to the twistor equation.

Then if we write $\mathfrak{z}^a = 2i w^A \bar{s}^A \bar{w}^B$ where $s^b = s^b/a$ and let t^a be the normal to the cross-section C of \mathcal{F} that lies in \mathcal{F} ,

$$\begin{aligned} -\frac{1}{8\pi G} \oint_C E_{ab} \mathfrak{z}^a ds^b &= -\frac{1}{8\pi G} \oint_C 2\varphi_{ABCD} \bar{s}_A^{C\bar{D}} \cdot 2i w^A \bar{s}_B^B t^{AB} ds \\ &= \frac{-i}{4\pi G} \oint_C \varphi_{ABCD} w^A w^C \bar{o}^B \bar{t}^D ds \end{aligned}$$

where $\varphi_{ABCD} = -2^{-1}\varphi_{ABCD}$. This is Penrose's expression [5].

We shall choose our momentum and angular momentum conformal Killing vectors by embedding \mathcal{F} as the boundary of conformally anti-de Sitter spacetime on the Einstein cylinder and then restricting solutions of the twistor equation in conformally flat space-time to it. This depends on how we embed Minkowski space-time on the Einstein cylinder with respect to anti-de Sitter space-time. We choose to do this symmetrically, (see diagram over).

With respect to the constant spinor basis (α^A, β^A) in Minkowski space the solution to the twistor equation is given by

$$\omega^A = \omega^A - i \times \alpha^A \pi_A,$$

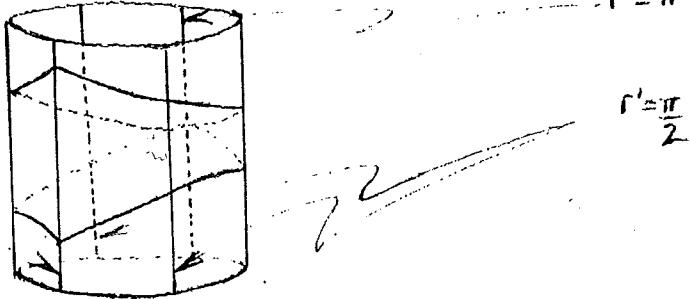
where $\omega^A = \omega^0 d^A + \omega^1 \beta^A$, $\pi_A = \pi^0 \alpha_A + \pi^1 \beta_A$ and $\omega^0, \omega^1, \pi^0, \pi^1$ are constants. Then with respect

to the null tetrad: $l^a = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right)$, $n^a = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial r} \right)$, $m^a = \frac{1}{\sqrt{2} \sin r} \left(\frac{\partial}{\partial \theta} - i \frac{\partial}{\partial \phi} \right)$ on the Einstein

cylinder we find that $\omega^A = \omega^0 \alpha^A + \omega^1 \beta^A$ where $\omega^0 = \sqrt{2} \cos \frac{1}{2}(t+r) \left(\omega^0 e^{i\phi_0} + \omega^1 e^{i\phi_1} \right) + i \sin \frac{1}{2}(t+r) \left(-\pi^0 e^{i\phi_2} \sin \frac{1}{2}\theta + \pi^1 e^{i\phi_3} \cos \frac{1}{2}\theta \right)$, $\omega^1 = \sqrt{2} \cos \frac{1}{2}(t-r) \left(-\omega^0 e^{-i\phi_2} \sin \frac{1}{2}\theta + \omega^1 e^{-i\phi_3} \cos \frac{1}{2}\theta \right) + i \sin \frac{1}{2}(t-r) \left(\pi^0 e^{i\phi_0} + \pi^1 e^{i\phi_1} \right)$.

Since the twistor equation is conformally invariant this is also a solution of the twistor equation on the Einstein cylinder

THE EMBEDDING OF
SCHWARZSCHILD SPACE-TIME
ON THE EINSTEIN
CYLINDER
 $r' \neq 0$



From these expressions we may calculate the conformal Killing vector $\xi^a = 2i(\omega^A - \pi^A) \zeta^a$

which lies in $r' = \frac{\pi}{2}$ and we find that

$$-\frac{i}{4\pi G} \oint_{r'=\pi/2} \omega^A \omega^B dS^{AB} = A_{\alpha\beta} Z^\alpha Z^\beta$$

where $Z^\alpha = (\omega^A, \pi_A)$ and $A_{\alpha\beta}$ is of the form

$$A_{\alpha\beta} = \begin{pmatrix} 2\Phi_{AB} & P_A^{B'} \\ P_A^{A'} & \bar{\Phi}_{AB'} \end{pmatrix}.$$

In particular

$$P^\alpha = -\frac{1}{8\pi G} \oint_C E_{ab} (\gamma^c, \frac{\eta_c + \bar{\eta}_c}{2}, \frac{\eta_c - \bar{\eta}_c}{2i}, \beta^c) dS^b$$

where $\gamma^\alpha = \frac{\partial}{\partial t'}, \eta^\alpha = e^{i\phi} \left(\sin t' \sin \frac{\theta}{2}, -\cos t' \cos \frac{\theta}{2}, -i \cos t' \frac{\theta}{2} \right),$
 $\beta^\alpha = \cos \theta \sin t' \frac{\partial}{\partial t'}, + \sin \theta \cos t' \frac{\partial}{\partial \theta}.$

For Schwarzschild - anti-de Sitter space-time
in the coordinates described earlier we find
that $t' = at$ and thus $P^\alpha = (M, 0, 0, 0), \Phi_{AB} = 0$
and we recover M as the mass.

The angular-momentum twistor $A_{\alpha\beta}$ obeys
the hermiticity property

$$A_{\alpha\beta} I^{\beta\alpha} = A_{\alpha\beta} I^{\alpha\beta}$$

with respect to the infinity twistor

$$I^{\alpha\beta} = \begin{pmatrix} -\frac{i}{2} \epsilon^{AB} & 0 \\ 0 & \epsilon^{AB'} \end{pmatrix}.$$

2. A positive energy theorem

We first need to look at the work of Gibbons et. al. [6]. Consider a spacelike hypersurface Σ in an asymptotically anti-de Sitter space-time, (with possibly an inner boundary on a {past} {future} apparent horizon H), which asymptotically approaches the $t=0$ cut of \mathcal{I} . Define a 'supercovariant' derivative on the 4-spinor (α^A, β^A) by

$$\hat{\nabla}_{MM'}\alpha^A = \nabla_{MM'}\alpha^A + \frac{a}{\sqrt{2}}\epsilon_{MA}B^{M'},$$

$$\hat{\nabla}_{MM'}\beta^A = \nabla_{MM'}\beta^A + \frac{a}{\sqrt{2}}\epsilon_{MA'}\alpha^M.$$

Then if D_A is the projection of the 4-dimensional connection into Σ and α_A, β_A satisfy the 'supercovariant' Witten equation on Σ i.e.

$$\hat{D}_{AA'}\alpha^A \equiv D_{AA'}\alpha^A + \frac{3a}{2\sqrt{2}}\beta^{A'} = 0,$$

$$\hat{D}_{AA'}\beta^{A'} \equiv D_{AA'}\beta^{A'} + \frac{3a}{2\sqrt{2}}\alpha_A = 0,$$

We may show that

$$\begin{aligned} -D_m[t^{AB}(\bar{\alpha}_A \hat{D}^m \alpha_B + \bar{\beta}_A \hat{D}^m \beta_B)] \\ = -t^{AB}[\hat{D}_m \alpha_A (\overline{\hat{D}^m \alpha_B}) + \hat{D}_m \beta_B (\overline{\hat{D}^m \beta_A})] + 4\pi G T_{ab} \xi^a \xi^b \end{aligned}$$

where $\xi^a = \alpha^A \bar{\alpha}^{A'} + \beta^A \bar{\beta}^{A'}$. Thus if T_{ab} satisfies the dominant energy condition then the R.H.S.

is non-negative. We may choose our spinors α^A, β^A to obey boundary conditions on an inner apparent horizon H such that, at H : $t^{AA'}(\alpha_A \bar{D}^m \alpha_A + \beta_A \bar{D}^m \beta_{A'}) N_{m,n} = 0$, where N^m is the normal to H lying in Σ .

If we use Green's Theorem on the above identity we therefore obtain

$$-\oint_{t=0} t^{AA'} (\alpha_A \bar{D}^m \alpha_A + \beta_A \bar{D}^m \beta_{A'}) S_m dS \geq 0. (*)$$

This integral is finite, providing that, as our spinors α_A, β_A' tend to \mathcal{G} ,

$$\bar{D}_{mm'} \alpha_A \rightarrow 0$$

$$\bar{D}_{mm'} \beta_{A'} \rightarrow 0.$$

In the original paper it was proposed that the boundary term could be written as

$$4\pi G [P^{(0)}_{AB} (\alpha_A \dot{\alpha}_A + \beta_A \dot{\beta}_A) - \lambda^{AB} \alpha_A \beta_B - \bar{\lambda}^{AB} \alpha_A' \beta_{B'}]$$

(credited as a private communication from D.R. Friedmann)

where $\dot{\alpha}_A, \dot{\beta}_A$ are the superconformity constant spinors that are the limits of α_A, β_A' as they tend to \mathcal{G} ,

(when divided by some suitable power of the conformal factor). For Schwarzschild anti-de

Sitter spacetime the component $P^{(0)} \equiv \frac{P^{(00)} + P^{(11)}}{\sqrt{2}}$

is proportional (via a positive constant of proportionality !!) to the mass parameter M .

In general this inequality obviously implies $p^{\text{so}} \geq 0$ and so if we identify this term with the mass, as was done in [6], we have shown that the mass is positive.

But, as shall be described, if we evaluate the above integral explicitly we obtain Penrose's expression and we find that the above inequality contains much information about A_{exp} .

We first write the metric in terms of coordinates (u, s, θ, ϕ) as

$$ds^2 = \frac{1}{\sin s} (du^2 - 2 \frac{ds}{a} ds \cos s d\theta^2) + O(s^{-1}) \times (\text{products of two differentials except } ds^2).$$

The $u = \text{const}$ surfaces are outgoing null hypersurfaces from \mathcal{I}_+ (with conformal factor $\omega = \sin s$, \mathcal{I}_+ is the hypersurface $s = \omega$), and s is a (non-affine) parameter along each null geodesic generator. The null geodesic generators are labelled by $x^2 = 0$ and $x^3 = \phi$. We shall choose our null tetrad by defining

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$$\ell^\alpha = \frac{dx}{\sin \zeta}, \quad \ell^\alpha = -a \sin \zeta \frac{\partial}{\partial z},$$

n^α is chosen to be the ingoing null vector orthogonal to the two-spheres $S=\text{const.}$ and m^α, n^α lie in these two spheres. m^α may be defined so that $\epsilon \cdot \hat{e} = 0$. We thus have

$$n^\alpha = \sin \zeta \left(\frac{\partial}{\partial u} + a b \frac{\partial}{\partial z} + X^k \frac{\partial}{\partial x^k} \right),$$

$$m^\alpha = \frac{\alpha - \beta k}{\sqrt{2}} \frac{\partial}{\partial x^k}.$$

For anti-de Sitter space-time ($\zeta = \frac{1}{2}$, $X^k = 0$), $\mathcal{G}^2 = \tan \zeta$ and $\mathcal{G}^3 = -i \tan \zeta / \sin \zeta$ and these are therefore our boundary conditions on the functions defining our tetrad. Also since $\lim_{\zeta \rightarrow 0} \mathcal{G}^A \Gamma_{ab} \rightarrow \text{finite limit}$ we have $\Phi_{ij} = O(\zeta^4)$, $\Delta = -\frac{\lambda}{6} = O(\zeta^4)$, $\Psi_i = O(\zeta^3)$. We may therefore solve for the spin coefficients as power series expansions in ζ and then we may calculate α_A, β_A similarly. We find that

$$\alpha^A = S^{-\frac{1}{2}} \sum_{i=0}^{\infty} \alpha_i^A S^i$$

$$\beta^A = S^{-\frac{1}{2}} \sum_{i=0}^{\infty} \beta_i^A S^i$$

where $\beta_0^A = \sqrt{2} \alpha_0^A S_A^A$, (this is really an identity in the rescaled space-time with, (on $S=0$), $\ell^\alpha = -a \frac{\partial}{\partial z}$, $n^\alpha = \frac{\partial}{\partial u} + a \frac{\partial}{\partial z}$ and $\beta^\alpha = \frac{1}{2} (\ell^\alpha - n^\alpha)$).

The integral (*) becomes

$$\begin{aligned} & -\frac{\sqrt{2}}{\alpha^3} \oint_{t=0, s=0} [\psi_1^{(3)} \alpha_0^\alpha \beta_0^\alpha + \psi_2^{(3)} [\alpha_0^\alpha \tilde{\beta}_0^\alpha + \alpha_0^\alpha \tilde{\beta}_0^\alpha] + \psi_3^{(3)} \alpha_0^\alpha \tilde{\beta}_0^\alpha] ds \\ & = -\frac{i}{\alpha^3} \oint_{t=0, s=0} [\psi_1^{(3)} \omega^\alpha \tilde{\omega}^\alpha + \psi_2^{(3)} (\omega^\alpha \tilde{\omega}^\alpha + \omega^\alpha \tilde{\omega}^\alpha) + \psi_3^{(3)} \omega^\alpha \tilde{\omega}^\alpha] ds \\ & = 4\pi G A_{\alpha\beta} \bar{Z}^\alpha \bar{Z}^\beta \end{aligned}$$

where $\omega^\alpha = \alpha_0^\alpha$, $\tilde{\omega}^\alpha = -\sqrt{2}i \beta_0^\alpha$ since $\omega^\alpha, \tilde{\omega}^\alpha$ are found to be two-surface twistors on the $t=0$ cross-section of \mathcal{S} . But from earlier work we have expressions for $\omega^\alpha, \tilde{\omega}^\alpha$ at $t=0$ in terms of our twistor coordinates $(\alpha^A, \bar{\alpha}^A)$, and the relationship between α_0^α and β_0^α gives us that

$$\bar{Z}^\alpha = 2 I^{\alpha\beta} \bar{Z}_\beta.$$

We have therefore proved that

$$A_{\alpha\beta} \bar{Z}^\alpha I^{\beta\gamma} \bar{Z}_\gamma \geq 0.$$

In particular this implies that P^α is timelike and future-pointing. From the angular momentum twistor $A_{\alpha\beta}$ we may calculate the associated mass m_P as

$$m_P^2 = -\frac{1}{2} A_{\alpha\beta} \bar{A}^{\alpha\beta} P_\alpha P_\beta - \Phi_{AB} \bar{\Phi}^{AB} - \bar{\Phi}_{AB} \bar{\Phi}^{AB}.$$

The above inequality implies that m_P^2 is non-negative and also provides a further

inequality relating components of P_{AB} and Φ_{AB} .

An alternative definition of mass is

$$M_B^4 = \text{Tr}_{\text{AdS}} A_{\alpha\beta} = 4 \cdot e^{i\pi/8} A_{01} A_{02} A_{03} A_{04}$$

and this other inequality also implies that

$$M_B^4 \text{ is non-negative, } P_A^\alpha P_\alpha^\beta = m_P^2 = M_B^2 \text{ which}$$

$\Phi_{AB} = 0$ but in general it is not possible,

to decide which mass is larger.

I would like to thank Dr. Paul Tod
for considerable assistance in this work
and Professor Penrose and William Shaw
for useful discussions. I hope to publish
a more detailed version of this work in
the near future.

Ron Kelly

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H-Space from a Different Direction

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It is the purpose of this note to point out an approach alternative to the use of the "good cut" equation, in order to study and obtain H-spaces [1].

We remind the reader that the conventional method of obtaining an H-space is by solving the "good cut" equation

$$\delta^2 Z = \sigma^0(z, \zeta, \bar{\zeta}). \quad (1)$$

Provided σ^0 is not "too large", the deformation theory of complex structures ensures a solution of the form

$$Z = Z(z^a, \zeta, \bar{\zeta}) \quad (2)$$

where z^a (the H-space coordinates) are four complex constants of integration and Z is regular in the neighborhood of the real sphere $\zeta = \bar{\zeta}$. The H-space metric can be obtained directly in terms of gradients (with respect to z^a) and δ and $\tilde{\delta}$ derivatives of Z [1,2].

The alternative method is based on an "intrinsic" coordinate system or rather a family (S^2) of intrinsic coordinate systems. If we define, from [2],

$$\begin{aligned} u &= Z(z^a, \zeta, \bar{\zeta}) \\ \omega &= \delta Z(z^a, \zeta, \bar{\zeta}) \\ \tilde{\omega} &= \tilde{\delta} Z(z^a, \zeta, \bar{\zeta}) \\ r &= \delta \tilde{\delta} Z(z^a, \zeta, \bar{\zeta}) \end{aligned} \quad (3a)$$

or

$$\theta^i = \theta^i(z^a, \zeta, \bar{\zeta}) \text{ with } \theta^i = (u, \omega, \tilde{\omega}, r) \quad (3b)$$

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then the θ^i are the intrinsic coordinates with (3) being the coordinate transformation (parametrized by $\zeta, \tilde{\zeta}$) between θ^i and z^a . Using the inversion of (3) namely

$$z^a = z^a(\theta^i, \zeta, \tilde{\zeta}),$$

we can calculate

$$\tilde{\delta}^2 Z = \tilde{\lambda}(z^a, \zeta, \tilde{\zeta}) = \tilde{\Lambda}(\theta^i, \zeta, \tilde{\zeta}). \quad (4)$$

It turns out that from knowledge of $\tilde{\Lambda}(\theta^i, \zeta, \tilde{\zeta})$ one can calculate [3] directly (and easily) the H-space metric in the θ^i coordinate system without any use or need of (2). The idea, then, is to find a differential equation for the direct determination of $\tilde{\Lambda}$ without using the "good cut" equation. We claim [4] this equation to be

$$\tilde{\delta}^2 \tilde{\Lambda} = \tilde{\delta}^2 \sigma^0. \quad (5)$$

The $\tilde{\delta}$ and $\tilde{\delta}'$ operators on this equation are to be understood as total derivatives in the sense that for $\Phi(\theta^i, \zeta, \tilde{\zeta})$ we have

$$\tilde{\delta}\Phi = \tilde{\delta}'\Phi + \frac{\partial\Phi}{\partial u} \cdot \omega + \frac{\partial\Phi}{\partial \omega} \cdot \sigma^0 + \frac{\partial\Phi}{\partial \tilde{\omega}} \cdot r + \frac{\partial\Phi}{\partial r} \cdot (\tilde{\delta}\sigma^0 - 2\omega) \quad (6)$$

where we have used

$$\tilde{\delta}u = \omega, \quad \tilde{\delta}\omega = \tilde{\delta}^2 u = \sigma^0, \quad \tilde{\delta}\tilde{\omega} = \tilde{\delta}\tilde{\delta}u = r, \quad \tilde{\delta}r = \tilde{\delta}\sigma^0 - 2\omega$$

and $\tilde{\delta}'$ is the partial derivative with respect to ζ . In other words, (5) is a second order partial differential equation in the six variables $\theta^i, \zeta, \tilde{\zeta}$, being "driven" by the free data $\tilde{\delta}^2 \sigma^0$.

We conclude with three final comments.

- (1) Note the different meanings of $\tilde{\Lambda}$. It describes the "acceleration" of the cut $Z(z^a, \zeta, \tilde{\zeta})$ in the $\tilde{\zeta}$ direction rather than in the ζ direction of the "good cut" equation. In addition it is the asymptotic (right) shear of the light cone from z^a [3]. It is also closely related to the infinitesimal holonomy operator of the H-space [4].

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(2) Equation (5) is (mildly) analogous to an alternative description of the Sparling equation

$$\bar{\delta}G = -GA. \quad (7)$$

Letting $\bar{H} = G^{-1} \cdot \bar{\delta}G$ we have

$$\bar{\delta}\bar{H} + [\bar{H}, A] + \bar{\delta}A = 0. \quad (8)$$

Directly and simply from \bar{H} , the self-dual Yang-Mills fields can be obtained [5].

(3) The "good cut" equations and Eq. (4) can be generalized to real space-times. The problem is then to find generalizations of (5) equivalent to the real Einstein equations.

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A Theorem on Null Fields in Six Dimensions

by L.P. Hughston

In what follows I shall outline a rather striking result, holding in six dimensions, which can be regarded as a generalization of Robinson's theorem on null electromagnetic fields in four dimensions.

By a 'massless field' in six dimensions I mean a symmetric spinor field $\phi^{\alpha\beta\dots\gamma}$ which satisfies $\nabla_{\alpha\delta}\phi^{\alpha\beta\dots\gamma} = 0$; to be a 'totally null' field it must satisfy $\phi^{\alpha\beta\dots[\delta}P^{\gamma]} = 0$ for some spinor P^α .

Lemma. Suppose $\phi^{\alpha\beta\dots\gamma}$ satisfies these conditions; then P^α must satisfy

$$(P^\alpha \nabla_{\alpha\beta} P^{[\delta}) P^{\gamma]} = 0. \quad (1)$$

Proof. It will be easily seen that $\phi^{\alpha\beta\dots\gamma}$ is totally null iff \exists a scalar ψ such that

$$\phi^{\alpha\beta\dots\gamma} = e^\psi P^\alpha P^\beta \dots P^\gamma. \quad (2)$$

The zero rest mass condition then implies, and indeed is equivalent to:

$$P^\beta P^\gamma \dots P^\delta \nabla_{\alpha\beta} \psi + P^\gamma \dots P^\delta \nabla_{\alpha\beta} P^\beta + (n-1) P^\beta (\nabla_{\alpha\beta} P^{[\delta}) \dots P^{\gamma]} = 0, \quad (3)$$

where n is the valence of $\phi^{\alpha\beta\dots\gamma}$. If one multiplies (3) by P^ε and skews over γ and ε the condition (1) follows at once ($n \geq 2$). \square

Now I shall establish, locally, a result which is essentially a converse to this lemma.

Theorem. Let P^α be a holomorphic spinor field satisfying (1) on a region of a six-dimensional complex manifold endowed with a non-degenerate holomorphic metric tensor and a Riemann-compatible holomorphic connection. Then locally there exists a totally null massless field of valence n , with principal spinor P^α , providing that

$$(n-2) P^\alpha P^\beta \nabla_{\alpha\beta} P^{[\alpha} P^{\gamma]} = 0, \quad (4)$$

where $\Psi_{\alpha\beta}^{\alpha\gamma}$ is the Weyl spinor (conformal curvature spinor).

Proof. (1) is equivalent to the existence of a spinor A_α such that $P^\alpha \nabla_{\alpha\beta} P^\gamma = A_\beta P^\gamma$,

$$\text{whence } P^\beta \nabla_{\alpha\beta} \psi + \nabla_{\alpha\beta} P^\beta - (n-1) A_\alpha = 0, \quad (6)$$

as follows from (3). Now consider an equation of the form

$$P^\beta \nabla_{\alpha\beta} \psi + A_\alpha = 0 \quad (7)$$

with ψ unknown, A_α specified, and P^β satisfying (1). Such an equation

admits solutions, locally, by Frobenius' theorem, iff A_α satisfies

$$P^\alpha \nabla_\alpha [{}_\beta A_\gamma] = -L_{[\beta} A_{\gamma]} . \quad (8)$$

(To see the necessity of (8) operate on (7) with $P^\beta \nabla_\beta P^\alpha$, and skew over α and σ ; (8) then follows by use of (5)).

We wish to see whether there exists a scalar ψ such that (6) holds; thus we examine the expression

$$P^\alpha \nabla_\alpha [{}_\beta A_\gamma] - L_{[\beta} A_{\gamma]} := I_{\beta\gamma} \quad (9)$$

$$\text{with } A_\alpha = \nabla_{\alpha\beta} P^\beta - (n-1) L_\alpha . \quad (10)$$

A straightforward calculation gives

$$I_{\beta\gamma} = -(n-2) P^\alpha \nabla_\alpha [{}_\beta L_\gamma] . \quad (11)$$

To arrive at (11) use is made of the Ricci identity

$$\nabla_\alpha [{}_\beta \nabla_\gamma] \delta P^\delta = R \epsilon_{\alpha\beta\gamma\delta} P^\delta , \quad (12)$$

where R is the scalar curvature; furthermore we require the simple identity

$$(\nabla_\delta [{}_\beta P^\alpha])(\nabla_\gamma]_\alpha P^\delta) = 0 . \quad (13)$$

Now we wish to examine the expression appearing in (11). Suppose we operate on (5) with $P^\beta \nabla_\beta \gamma$, skewing over γ and β . A short calculation gives

$$P^\beta P^\alpha \underbrace{\nabla_\beta \gamma \nabla_\alpha P^\gamma}_{(14)} = (P^\beta \nabla_\beta [{}_\gamma L_\gamma]) P^\gamma ; \quad (14)$$

but the vanishing of the left side of this equation is, by another Ricci identity, equivalent to

$$P^\alpha P^\beta \psi_{\alpha\beta}^{[\sigma-\rho\tau]} = 0 . \quad (15)$$

Therefore the vanishing of $I_{\beta\gamma}$, the desired integrability condition, is equivalent to (4). \square

Note that for $n=2$, the case corresponding to the classical Robinson theorem in dimension four, no restrictions are imposed on the curvature beyond those already implied by (1); these conditions, incidentally, are $P^\alpha P^\beta [{}^\mu \psi^{\nu\rho}] [\epsilon^{\sigma\tau} P^\delta] P^\delta = 0$, as follows from (14) directly by skew-symmetrization with P^ϵ over σ and τ . In flat space, given a solution of (1), null fields of any valence can be constructed: these may be generated via a contour integral formula with a holomorphic function showing an appropriately simple pole structure.

Gratitude is expressed to Lionel Mason and Ben Jeffreys, both of whom in discussion and correspondence made contributions to these results.

A New Proof of Robinson's Theorem
by L.P. Hughston

In 1976 P. Sommers published an elegant simplified proof of Robinson's theorem (1959) on shearfree congruences. In what follows I shall outline a proof even more straightforward than Sommers'.

Theorem (Robinson). Suppose \mathcal{M} is a complex manifold of dimension four with a holomorphic metric g_{ab} . Let K^A be a spinor field defined on an open set $U \subset \mathcal{M}$ satisfying

$$K^A K^B \nabla_{AA'} K_B = 0. \quad (1)$$

Then for each point $p \in U$ there exists a neighborhood $V \subset U$ such that there exists a scalar ψ on V with $\nabla_{AA'} \phi^{AB} = 0$, where

$$\phi^{AB} = e^\psi K^A K^B. \quad (2)$$

Proof. Note that $\nabla_{AA'} \phi^{AB} = 0$ is equivalent, by (2), to

$$K^B K^A \nabla_{AA'} \psi + K^B \nabla_{AA'} K^A + K^A \nabla_{AA'} K^B = 0. \quad (3)$$

Since (1) is equivalent to the existence of a spinor $\lambda_{A'}$ such that

$$K^A \nabla_{AA'} K^B = \lambda_{A'} K^B \quad (4)$$

it follows by insertion of (4) in (3) that we seek a scalar ψ such that

$$K^A \nabla_{AA'} \psi + \nabla_{AA'} K^A + \lambda_{A'} = 0. \quad (5)$$

Now consider an equation of the form

$$K^A \nabla_{AA'} \psi + \alpha_{A'} = 0, \quad (6)$$

where K^A satisfies (4). As a lemma we require the fact that if $\alpha_{A'}$ is specified then there exist solutions of (6) locally if and only if $\alpha_{A'}$ satisfies

$$K^A \nabla_{AA'} \alpha^{A'} = \lambda_{A'} \alpha^{A'}. \quad (7)$$

The proof of this lemma follows as an application of the Frobenius theorem. (To see how (7) arises as a necessary condition we transvect (6) with $K^B \nabla_B^{A'}$ to obtain

$$K^B \nabla_B^{A'} (K^A \nabla_{AA'} \psi) + K^B \nabla_B^{A'} \alpha_{A'} = 0;$$

whence,

$$(K^B \nabla_B^{A'} K^A) \nabla_{AA'} \psi + K^B K^A \nabla_B^{A'} \nabla_{AA'} \psi = K^B \nabla_B^{A'} \alpha^{A'},$$

which by (4) gives $\lambda^{A'} K^A \nabla_{AA'} \psi = K^A \nabla_{AA'} \lambda^{A'}$, which by use of (6) gives (7). Frobenius' theorem shows that (7) is also sufficient.)

We wish to determine whether there exists a scalar ψ such that (5) is satisfied. Thus we must examine the expression

$$I := K^A \nabla_{AA'} \lambda^{A'} - \lambda^{A'} \nabla_A K^A$$

$$\text{with } \lambda^{A'} = \nabla_A^{A'} K^A + \lambda^{A'}.$$

We have:

$$\begin{aligned} I &= K^A \nabla_{AA'} (\nabla_B^{A'} K^B + \lambda^{A'}) - \lambda^{A'} (\nabla_A^{A'} K^A + \lambda^{A'}) \\ &= K^A \nabla_{AA'} \nabla_B^{A'} K^B + (K^A \nabla_{AA'} \lambda^{A'} + \lambda^{A'} \nabla_{AA'} K^A) \\ &= -K^A \nabla_{BA'} \nabla_A^{A'} K^B + 2K^A \nabla_{A'(A} \nabla_B^{A') B} K^B + (\nabla_{AA'} K^A \lambda^{A'}) \\ &= -[\nabla_{BA'} (K^A \nabla_A^{A'} K^B) - (\nabla_{BA'} K^A) (\nabla_A^{A'} K^B)] \\ &\quad + 2K^A \square_{AB} K^B + \nabla_{AA'} K^A \lambda^{A'}. \end{aligned}$$

But $K^A \square_{AB} K^B$ vanishes for any K^A since $\square_{AB} K^B = -3 \perp K_A$.

Furthermore $(\nabla_{A'B} K^A)(\nabla_A^{A'} K^B) = 0$ for any K^A . Thus:

$$\begin{aligned} I &= -\nabla_{BA'} (K^A \nabla_A^{A'} K^B) + \nabla_{AA'} K^A \lambda^{A'} \\ &= -\nabla_{BA'} (\lambda^{A'} K^B) + \nabla_{AA'} K^A \lambda^{A'} \\ &= 0. \end{aligned}$$

Since I vanishes the integrability condition for ψ is satisfied, and the theorem is proved. \square

Theorem (Sommers - Bell - Szekeres generalization of Robinson's result). Suppose, in the venue as above, K^A satisfies (1). Furthermore let K^A be a p -fold principal spinor ($p \geq 1$) of a massless field of valence $p+q$. Then:

$$(p-2-3q) \Psi_{ABCD} K^A K^B K^C = 0,$$

where Ψ_{ABCD} is the Weyl spinor.

The proof follows essentially the same line of reasoning as in the first theorem. The Goldberg-Sachs theorem follows if we note that a spinor satisfying (1) is automatically a 1-fold principal spinor of the Weyl spinor, and that in a vacuum the Weyl spinor satisfies the zero-rest-mass equations.

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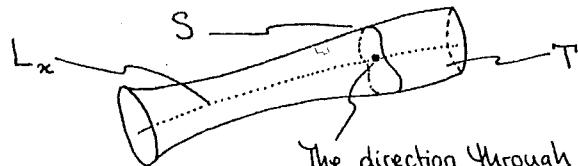
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31

A Twistor Description of Null Self-Dual Maxwell Fields

Robinson's Theorem [2]: After complexification, Robinson's Theorem (an important motivation in the birth of twistors) states that if M is a complex conformal manifold and $E \subset TM$ is an integrable distribution of α -planes, then E may be locally defined by a closed self-dual 2-form i.e. a self-dual Maxwell field. This is often proved with the aid of spinors (e.g. [3]): E may be defined by a spinor field α^A and integrability is the condition $\alpha^A \bar{\alpha}^B \nabla_{AA'} \alpha_{B'} = 0$ or, equivalently, $\alpha^A \nabla_{AA'} \alpha_{B'} = \lambda_A \alpha^B$. A curvature computation then shows that $\alpha^A \nabla_{AA'} \psi + \nabla_{AA'} \alpha^A + \lambda_A = 0$ may be locally solved for ψ whence $\nabla_{AA'} (\psi \alpha^A \bar{\alpha}^B) = 0$, as required. Without spinors, however, the theorem is immediate from Frobenius' integrability criterion: a simple 2-form defining a distribution of α -planes is, by definition, necessarily self-dual and the integrability of E is equivalent to being able to choose this form to be closed. In fact, more precisely, suppose E is integrable and let S denote the space of leaves (locally). Then there is a 1-1 correspondence between 2-forms on S and closed 2-forms on M defining E (given by pull-back under $\pi: M \rightarrow S$). Finally, since a null (\equiv simple) 2-form defines a plane distribution it follows that a null self-dual Maxwell field is exactly specified by E , a congruence of α -surfaces, and a 2-form ω on S , the 2-dimensional parameter space for the congruence.

A twistor description: Suppose now that M is conformally flat so that it has a 3-dimensional twistor space T parameterizing the α -surfaces. Then a self-dual null Maxwell field gives S as above, a submanifold of T :



The direction through $x \in M$ of the α -surface defined by E

As noted above, the Maxwell field specifies a 2-form ω on S . The usual Penrose transform, however, identifies the space of all self-dual Maxwell fields with the cohomology $H^1(T, \kappa)$ where κ is the canonical bundle Ω^3 . Hence there should be a natural homomorphism

$$T(S, \Omega^2) \rightarrow H^1(T, \kappa) \quad (*).$$

It is somewhat easier to see what is going on in "real" twistor theory:

"Real" twistors: The usual twistor correspondence between $\mathrm{Gr}_2(\mathbb{C}^4)$ and \mathbb{CP}_3 has a real form, namely the correspondence between $\mathrm{TRIM} \equiv \mathrm{Gr}_2(\mathbb{R}^4)$ and $\mathrm{TRP} \equiv \mathbb{RP}_3$. TRIM has a conformal metric of signature $(+, +, -, -)$ and TRP is the space of one family of totally null 2-planes in TRIM , the " α -planes". Victor Guillen has recently been investigating this "black-and-white" version of twistor theory and it seems that most twistor constructions have an

(often simpler) black-and-white analogue. The Penrose transform is replaced by the Gelfand-Radon transform e.g.

$$T(\mathbb{R}P, \kappa) \cong \{w \in T(\mathbb{R}M, \Omega^2_+) \text{ s.t. } dw = 0\}$$

which is simpler in that functions have replaced cohomology. Although the transform can be defined locally (i.e. in a neighbourhood of a line in $\mathbb{R}P$), it is then never an isomorphism. VG has shown, however, that globally it is an isomorphism (for all helicities and so on) and suggests that the crucial property of $\mathbb{R}M$ is that it is a Zoll manifold (all its null geodesics are closed). Suppose now that w is a null self-dual Maxwell field on $\mathbb{R}M$ (it will have singularities but ignore such technicalities). Then, as for the holomorphic case, one obtains a surface $S \hookrightarrow \mathbb{R}P$ and a 2-form on it, which may also be denoted by w . The analogue of $(*)$ is thus

$$T(S, \Omega^2) \rightarrow T(\mathbb{R}P, \kappa) \quad \mathbb{R}(*)$$

A function f on $\mathbb{R}P$ may be integrated over S against w . The linear functional

$$f \mapsto \int_S f w$$

therefore defines a distribution-valued 3-form supported on S . This is VG's identification of $\mathbb{R}(*)$. S determines the singularities of w .

Ordinary twistors: As a divisor, S gives rise to a line bundle \mathcal{E} over T , a section $s \in T(T, \mathcal{E})$ defining S , and therefore an exact sequence:

$$0 \rightarrow \mathcal{E}^* \xrightarrow{\times s} \mathcal{O}_T \rightarrow \mathcal{O}_S \rightarrow 0 \quad (**).$$

Moreover, $L|_S = N$ the normal bundle of S in T and the sequence may therefore be rewritten:

$$0 \rightarrow \mathcal{O}_T \rightarrow \mathcal{E} \rightarrow N \rightarrow 0.$$

Finally, noting that $N \otimes \kappa|_S = \Omega^2$ on S , it may be rewritten

$$0 \rightarrow \kappa \rightarrow \kappa \otimes \mathcal{E} \rightarrow \Omega^2_S \rightarrow 0$$

and $(*)$ is given by the connecting homomorphism of the corresponding long exact sequence:

$$T(S, \Omega^2) \rightarrow H^1(T, \kappa) \xrightarrow{\times s} H^1(T, \kappa \otimes \mathcal{E}).$$

Note also that the null fields obtained in this way are exactly those annihilated by $\times s$: an invariant way of saying that the cohomology class has a "simple pole" along S ($\mathbb{R}P$'s original identification of null fields [1]).

Googly photon? : The usual improved way to regard a Maxwell field is as a connection on a line-bundle. A null self-dual Maxwell field is then flat along an integrable distribution of α -planes and the covariant constant sections along the leaves give rise to a line-bundle on S . The residual information of the original connection (normal to the leaves) equips this line bundle on S with a connection of its own. This is surely the googly photon for a null self dual field. This may give clues as to what to do

about the general googly photon and also suggests trying to combine this construction with the usual Ward twisted photon construction in an attempt to describe "half algebraically special" Maxwell fields. A similar construction for the non-linear version (i.e. Yang-Mills bundles) gives vector bundles with connection flat along a congruence of α -surfaces. These are very special indeed (stronger than null and self-dual).

Higher dimensions: The Frobenius approach to the Robinson theorem evidently extends to higher (even) dimensions. LPH has also shown (lecture, Oxford, 30/4/85) how a spinor method works in dimension 6 and presumably a general spinor proof is available. In any case, a simple closed self-dual n -form on a conformal $2n$ -fold gives rise to a congruence of α - n -folds (by definition orthogonal to self-dual n -forms) parameterized by an n -dimensional space S , and an n -form w on S . Conversely, every such arises in this way. Higher dimensional twistors only exist in an obvious way for conformally flat space Q_{2n} , the complex quadric of dimension $2n$. This has a twistor space Z_n , consisting of one system of P_n 's lying thereon (the α - P_n 's). Z_n has dimension $n(n+1)/2$. Letting M denote an open subset of Q_n and T the corresponding subset Z_n , then (subject to mild topological restrictions on M) MFA has shown (lecture, Oxford, 7/11/83) that

$$H^{n(n-1)/2}(T, \kappa) \cong \{w \in T(M, \Omega_+^n) \text{ s.t. } dw = 0\}.$$

The homomorphism

$$T(S, \Omega^n) \rightarrow H^{n(n-1)/2}(T, \kappa)$$

is given by composing a series of connecting homomorphisms or by a spectral sequence construction induced by the appropriate Koszul complex instead of (**). There is a corresponding black-and-white version for real split quadrics. Note that $n(n-1)/2$ is the codimension of S in T as one would expect.

Many thanks to VG for much interesting conversation.

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The Penrose Transform for Complex Homogeneous Spaces

Introduction: The original integral formulae for massless fields are now seen as part of a general machine known as the Penrose transform (e.g. [4]). This machine has undergone several modifications and refinements. The aim of this note is to provide a user's guide to a state-of-the-art machine for complex homogeneous spaces. Thus, the emphasis is on how to perform computations rather than the theory (based on the representation theory of complex Lie groups for which [8] is an excellent reference). A new aspect is the systematic use of the BGG (Bernstein-Gelfand-Gelfand) resolution [2,9] in differential operator form [7]. This replaces the relative deRham complex used in [4]. It previously appeared in [6] (cf. [5]) but in a special form where it was derived from the relative deRham complex (by a process now recognized [7] as the Taft-Zuckerman translation functor [11]).

Complex Homogeneous Spaces: Let G be a simply connected complex semi-simple Lie group (e.g. $SL(4, \mathbb{C})$) for ordinary twistor theory ($= Spin(6, \mathbb{C})$). Roughly speaking, a complex subgroup $P \leq G$ is called parabolic if G/P is compact. One can classify as follows. Replace Lie group by Lie algebra and recall [8] Cartan's classification of the simple ones: fix a Cartan subalgebra and a Borel (= minimal parabolic) subalgebra containing it (no loss since other choices are obtained by suitable conjugation). The resulting simple positive roots together with the Cartan matrix describing relative angles allow one to reconstruct the Lie algebra. This is conveniently rephrased by listing the allowed Dynkin diagrams ($A_1, B_2, C_2, D_2, E_6, E_7, E_8, F_4, G_2$) where each node corresponds to a simple positive root. E.g. $sl(4, \mathbb{C}) = \bullet\bullet\bullet$. A similar story applies for the corresponding Lie groups. A parabolic $P \leq G$ may now be specified by adding to a minimal such (i.e. a Borel) on the Lie algebra level a finite collection of simple root spaces and forming the Lie subalgebra so generated. In summary, the parabolic subgroups $P \leq G$ may be classified on the Dynkin diagram for G by crosses through some of the nodes, namely those for which the corresponding negative root space is not to be added to the underlying Borel. These crosses may be placed arbitrarily (Ex. 16.5.6 of [8]). We adopt the same notation for the homogeneous space G/P .

Examples: Compactified complexified Minkowski space $= \mathbb{M} = \mathbb{F}_2 = \bullet\star\bullet$
 Projective twistor space $= \mathbb{P} = \mathbb{F}_1 = \bullet\bullet\star$, Dual projective twistor space $= \mathbb{P}^* = \mathbb{F}_3 = \star\bullet\bullet$
 Ambitwistor space $= \star\bullet\star$, Projective primed spin bundle over $\mathbb{M} = \mathbb{F}_{1,2} = \bullet\star\star$
 $\mathbb{F}_{2,3,5}(\mathbb{C}^7) = \bullet\star\bullet\star\star$ and similar notation for the flag manifolds in [4].
 $SO(2n+2, \mathbb{C})$ acts on \mathbb{M}_{2n} the complex non-singular quadric of dimension \mathbb{C}^{2n} i.e. $2n$ -dimensional "Minkowski space" (or $\mathbb{C}S^{2n}$). $\mathbb{M}_{2n} = \star\bullet\bullet\cdots\bullet\star$ (n+1 nodes). Its "twistor space" (i.e. the space of $\alpha \cdot P_n$'s in \mathbb{M}_{2n} (lecture MFA Oxford 7/11/83) i.e. the manifold of pure spinors [10]) is $\mathbb{Z}_n = \bullet\bullet\bullet\cdots\bullet\star$. Similarly, its "dual

"twistor space" is $Z_n^* = \bullet \dots \bullet \rightarrow \leftarrow$. Notice (i) triality for $n=3$ ($M_6 \cong Z_3$), (ii) reduction to ordinary twistors for $n=2$. Ambitwistors = $\bullet \dots \bullet \rightarrow \leftarrow$ but for $n \geq 3$, the space of null geodesics in M_{2n} is $\bullet \rightarrow \dots \rightarrow \leftarrow$. As usual, $SO(2n+2, \mathbb{C})$ is best replaced by $Spin(2n+2, \mathbb{C})$ its universal (2-sheeted) cover. Similarly, $\mathbb{C}S^{2n+1} = M_{2n+1} = \bullet \rightarrow \dots \rightarrow \leftarrow$ ($n+1$ nodes) with twistors = $\bullet \dots \bullet \rightarrow \leftarrow$.

Correspondences: This notation is evidently well suited to describing correspondences generated by two parabolics of G . Typical examples:

$$\begin{array}{ccc} \begin{array}{c} Y \\ \searrow \mu \\ X \end{array} & \xrightarrow{\quad} & \begin{array}{c} F_{1,2} \\ P_1 \searrow \mu \nearrow P_2 \\ M \end{array} \end{array}$$

[In general, the fibres are given by erasing from the diagram for the correspondence space the nodes which have crosses after projection (note $\bullet \times = \{\text{pt}\} \times P_i = P_i = \bullet$.)]

$$\begin{array}{ccc} \begin{array}{c} \bullet \rightarrow \dots \rightarrow \leftarrow \\ \searrow \mu \\ \bullet \rightarrow \dots \rightarrow \leftarrow \end{array} & \xrightarrow{\quad} & \begin{array}{c} Z_{n-1} \searrow \mu \nearrow P_n \\ M_{2n} \end{array} \end{array}$$

Homogeneous Bundles: The holomorphic homogeneous vector bundles on G/P are, by definition, induced by finite dimensional complex representations of P . We shall now describe the irreducible such representations starting with the case $P=G$ (again, [8] provides an excellent reference but our presentation is slightly non-standard). A weight for a simple Lie algebra is an assignment of an integer to each node of the corresponding Dynkin diagram, e.g. $\overset{r}{\bullet} \overset{2}{\longrightarrow} \overset{r}{\bullet}$ or $\overset{r}{\bullet} \overset{2}{\longrightarrow} \overset{r}{\bullet} \overset{r}{\longrightarrow} \overset{r}{\bullet}$. Let Λ = the set of weights. The case where each integer is 1 has a special rôle: denote this weight by δ . Theorem (Cartan-Weyl Theorem of "highest weight"): The irreducible representations of G are classified by the dominant weights (those weights for which all the integers are non-negative) \square [Of course the real content of the theorem is in saying how to obtain the weight of a given representation and, conversely, how to construct a representation with prescribed weight] Examples:

$SL(4, \mathbb{C})$: $\overset{r}{\bullet} \overset{2}{\longrightarrow} \overset{r}{\bullet} = (0, r, 2+r, p+2+r)$ in the notation of [4] (note that since we are considering $SL(4, \mathbb{C})$ rather than $GL(4, \mathbb{C})$ the first entry of (a, b, c, d) for $a \leq b \leq c \leq d$ may be normalized to zero by tensoring with $(-a, -a, -a, -a) = \det^{-a}$)

$$= \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \quad \text{as Young tableau.} \quad \begin{array}{l} \text{Self rep}^{\#}: \overset{1}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet} \\ \text{Adjoint rep}^{\#}: \overset{1}{\bullet} \overset{0}{\longrightarrow} \overset{1}{\bullet} \end{array}$$

$SO(2n+2, \mathbb{C})$ $n \geq 3$: Self rep $^{\#}$: $\overset{1}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet}$, Adjoint rep $^{\#}$: $\overset{0}{\bullet} \overset{1}{\longrightarrow} \overset{0}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet}$. A spinor rep $^{\#}$: $\overset{0}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet} \overset{0}{\longrightarrow} \overset{1}{\bullet}$, The other spinor representation: $\overset{0}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet} \overset{1}{\longrightarrow} \overset{0}{\bullet}$.

G_L : Self representation: $\overset{1}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet}$, Adjoint rep $^{\#}$: $\overset{2}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet} \overset{0}{\longrightarrow} \overset{0}{\bullet}$

There are more rules for taking the dual of a representation and so on. Irreducible representations of parabolics are similarly described by those elements of Λ which have non-negative integers for those nodes which do not have crosses through them. Use the same notation for the corresponding homogeneous bundle. Examples: $\overset{0}{\bullet} \overset{0}{\longrightarrow} \overset{1}{\bullet} = \mathcal{O}(k)$ on P , $\overset{p}{\bullet} \overset{2}{\longrightarrow} \overset{r}{\bullet} = \mathcal{O}_{(AB \dots D)(E'F' \dots H')} (p+2+r) = \mathcal{O}^{(AB \dots D)(E'F' \dots H')} (g)$ on M .

$p \quad r \quad$ conformal weight

- = tangent bundle on P_4 . = cotangent bundle on P_4
 = tangent bundle on $G_{r_3}(\mathbb{P}^7)$. = cotangent bundle on $G_{r_3}(\mathbb{P}^7)$.
 = $(0, u | t+u | st+t+u, r+s+t+u | qt+r+s+t+u, p+q+r+s+t+u)$ on $\mathbb{F}_{2,3,5}(\mathbb{P}^7)$ (notation of [4]).
 = tangent bundle on M_{2n} . = cotangent bundle on M_{2n}
 = dual primed spin bundle $\Omega^{A'}_{\pm}$ on M_{2n} . If n is even: $\Omega_{A'}^{\pm} = \Omega^{A''}_{\pm}$ swap these if n odd.
 = $\mathcal{O}(k)$, the line bundle of conformal densities of weight k on M_{2n}
 = Ω_+^4 , the bundle of self dual 4-forms on M_8 (note that $\Omega_+^4 \neq \mathcal{O}_{(A'B')}(-3)$)
 but is rather the highest weight part thereof: $\dim \Omega_+^4 = 35$ but $\dim \mathcal{O}_{(A'B')}(-3) = 36$.
 = $\mathcal{O}(k)$ or Σ_n . $\mathcal{O}(-2n) = \pi$, the canonical bundle on Σ_n .
 = canonical bundle on \mathbb{M}_3 . = tangent bundle on M_3 .

Even for maximal parabolics (one cross on the Dynkin diagram), the (co)tangent bundle may be reducible (but not decomposable). E.g. has tangent bundle (not of [4]). For non-maximal P this always happens, e.g. cotangent bundle of ambitwistor $= \Omega^{A''}_{\pm} + \Omega^{B''}_{\pm}$ (see [4]). But, this is okay: the BGG resolution will eliminate the need to consider cotangent bundles. Irreducible bundles are more natural e.g. = $\mathcal{O}(g, p)$ on ambitwistor space.

Cohomology and the Weyl Group: Fix a Dynkin diagram with weight lattice Λ . For each node define σ , a permutation of Λ as follows. Case of a short root (there is an arrow in the Dynkin diagram pointing towards this node): Reverse the sign of the integer attached to the node and, to neighbouring nodes, add the integer for the chosen node. Case of a long root (all others): Reverse sign and add to neighbours with multiplicity corresponding to the number of connecting lines. Examples: Short $\xrightarrow{\sigma} \Omega^{A''}_{\pm}$, Long $\xrightarrow{\sigma} \Omega^{A''}_{\pm}$, $\xrightarrow{\sigma} \Omega^{A''}_{\pm}$, $\xrightarrow{\sigma} \Omega^{A''}_{\pm}$. Note that always $\sigma^2 = 1$. Such permutations are called simple reflections. The Weyl group is the group of permutations of Λ generated by all these σ . It is a finite group and will be denoted by W . An element of W conjugate in W to a simple reflection is called a reflection (and will also satisfy $\sigma^2 = 1$). It is often convenient to modify the action of W on Λ by translating the origin to $-\delta$: $w.\lambda = w(\lambda + \delta) - \delta$. This is called the affine action of the Weyl group and features strongly in the BBW (Bott-Borel-Weil) Theorem: Suppose given a parabolic subgroup $P \leq G$ and an irreducible representation of P with weight $\lambda \in \Lambda$. Let λ also denote the sheaf of germs of holomorphic sections of the corresponding homogeneous bundle on G/P and enquire as to $H^k(G/P, \lambda)$. There are two cases: (i) There is an element $w \in W$ such that $w.\lambda$ is dominant (in which case w is unique), (ii) There is no such w . Case (i) is usually referred to as λ being regular, case (ii) as singular. For $w \in W$ define the length of w , $l(w) =$ least number of simple reflections needed to write w as a product thereof. The BBW Theorem states that for λ regular $H^{l(w)}(G/P, \lambda) = w.\lambda$ as a representation of G where w is the unique element of W for which $w.\lambda$ is dominant and all other cohomology vanishes whereas for λ singular all cohomology vanishes. In the Penrose transform the BBW Theorem is used along the fibres of ν to compute direct images: The fibres are themselves complex homogeneous spaces and the corresponding Weyl group acts in the obvious manner as in the following examples: Consider $\nu: \bullet \rightarrow \bullet \rightarrow \bullet$ (usual twistor example); BBW \Rightarrow

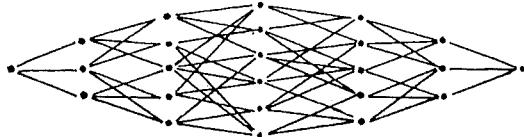
$\nu_*(\overset{P}{\begin{smallmatrix} p & q & r \\ \times & \times & \times \end{smallmatrix}}) = \overset{P}{\begin{smallmatrix} p & q & r \\ \times & \times & \times \end{smallmatrix}}$ if $r \geq 0$, $\nu'_*(\overset{P}{\begin{smallmatrix} p & q & r \\ \times & \times & \times \end{smallmatrix}}) = \overset{P}{\begin{smallmatrix} p & q+r+1 & -r-2 \\ \times & \times & \times \end{smallmatrix}}$ if $r \leq -2$ and all other direct images vanish. E.g. $\lambda = \overset{P}{\begin{smallmatrix} p & q & r \\ \times & \times & \times \end{smallmatrix}}$ for $r \leq -2 \Rightarrow \lambda + \delta = \overset{P+1}{\begin{smallmatrix} p+1 & q+r+1 & r+1 \\ \times & \times & \times \end{smallmatrix}} \Rightarrow \sigma(\lambda + \delta) = \overset{P+1}{\begin{smallmatrix} p+1 & q+r+2 & -r-1 \\ \times & \times & \times \end{smallmatrix}}$ (where σ is reflection w.r.t. the right hand node) $\Rightarrow \sigma(\lambda) = \sigma(\lambda + \delta) - \delta = \overset{P}{\begin{smallmatrix} p+2+r+1 & -r-2 \\ \times & \times & \times \end{smallmatrix}}$ q.e.d.. In more usual notation this result reads $\nu'_*(\overset{\text{homogeneity}}{\mathcal{O}}_{(AB\dots D)}(r, q)) = (\overset{\text{conformal weight}}{\mathcal{O}}_{(AB\dots D)}(E/F/\dots H'))(-q-1)$. As a more exotic example consider: $\nu: \overset{1}{\begin{smallmatrix} 2 & -3 & -1 & 2 & 0 & -6 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}} \rightarrow \overset{2}{\begin{smallmatrix} 3 & -2 & 0 & 3 & 1 & -5 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}} \rightarrow \overset{2}{\begin{smallmatrix} 1 & 2 & -2 & 3 & 1 & -5 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}} \rightarrow \overset{2}{\begin{smallmatrix} 1 & 2 & -2 & 3 & -4 & 5 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}} \rightarrow \overset{2}{\begin{smallmatrix} 1 & 2 & -2 & -1 & 4 & 1 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}}$ and the bundle $\overset{1}{\begin{smallmatrix} 2 & -3 & -1 & 2 & 0 & -6 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}}$. BBW gives: $\overset{1}{\begin{smallmatrix} 2 & 1 & 2 & -2 & 3 & 1 & -5 \\ \times & \times & \times & \times & \times & \times & \times \end{smallmatrix}} \rightarrow \overset{2}{\begin{smallmatrix} 1 & 2 & -2 & 3 & -4 & 5 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}} \rightarrow \overset{2}{\begin{smallmatrix} 1 & 2 & -2 & -1 & 4 & 1 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}}$, where each step is by using a simple reflection with least total number. Subtracting δ gives $\nu^5(\overset{1}{\begin{smallmatrix} 2 & -3 & -1 & 2 & 0 & -6 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}}) = \overset{1}{\begin{smallmatrix} 0 & 1 & -4 & 0 & 1 & 0 \\ \times & \times & \times & \times & \times & \times \end{smallmatrix}}$; others vanish.

Lepowsky's BGG Resolution: This is usually phrased in terms of Verma modules but is equivalent (dually) to the following construction with homogeneous differential operators. Suppose $P \leq G$ as usual and $\lambda \in \Lambda$ dominant. Then λ describes a finite dimensional irreducible representation of G . Let $\mathbb{C}(\lambda)$ denote the constant sheaf on G/P whose fibre is this finite dimensional vector space. On the other hand λ describes a finite dimensional representation of P and hence a homogeneous vector bundle on G/P whose sheaf of sections may be denoted $\mathcal{O}(\lambda)$. There is an obvious inclusion: $\mathbb{C}(\lambda) \hookrightarrow \mathcal{O}(\lambda)$. The BGG construction extends this to a resolution by completely reducible homogeneous bundles linked by homogeneous differential operators. Let W act affinely on λ . Of the resulting weights, some will describe representations of P (if they are non-negative on the nodes without crosses on the Dynkin diagram). These are the required homogeneous bundles. Order them by length of the corresponding Weyl group element: two such of adjacent lengths are joined by an invariant differential operator if and only if the Weyl group elements differ by a reflection (not necessarily simple). This is best illustrated by some examples:

$P_1 = \mathbb{X}$: $0 \rightarrow \mathbb{C}(\overset{k}{\times}) \rightarrow \mathcal{O}(\overset{k}{\times}) \rightarrow \mathcal{O}(-k-2) \rightarrow 0$ more often written as $0 \rightarrow \mathbb{C}^k \mathbb{C}^2 \rightarrow \mathcal{O}(k) \xrightarrow{\delta^{k+1}} \mathcal{O}(-k-2) \rightarrow 0$. When $k=0$ this is $0 \rightarrow \mathbb{C} \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow 0$ the holomorphic deRham sequence. $M = \overset{p}{\begin{smallmatrix} p & q & r \\ \times & \times & \times \end{smallmatrix}}$: For any $p, q, r \geq 0$ obtain $0 \rightarrow \mathbb{C}(\overset{p}{\begin{smallmatrix} p & q & r \\ \times & \times & \times \end{smallmatrix}}) \rightarrow \mathcal{O}(\overset{p}{\begin{smallmatrix} p & q & r \\ \times & \times & \times \end{smallmatrix}}) \rightarrow \mathcal{O}(\overset{p+q+1}{\begin{smallmatrix} p+q+1 & q+r+1 \\ -q-2 & \times \end{smallmatrix}}) \rightarrow \mathcal{O}(\overset{p}{\begin{smallmatrix} p & q+p+q+2 \\ -p-2-3 & \times \end{smallmatrix}}) \oplus \mathcal{O}(\overset{p}{\begin{smallmatrix} p+q+2 & q \\ -q-r-3 & \times \end{smallmatrix}}) \rightarrow \mathcal{O}(\overset{p+r+1}{\begin{smallmatrix} p+r+1 & p+2+1 \\ -p-2-r-4 & \times \end{smallmatrix}}) \rightarrow \mathcal{O}(\overset{p}{\begin{smallmatrix} p & p \\ -p-2-r-4 & \times \end{smallmatrix}}) = 0$.

Again, for $p=q=r=0$ this is the holomorphic deRham complex with two-forms split into (anti-)self-dual parts: $0 \rightarrow \mathbb{C} \rightarrow \Omega^0 \rightarrow \Omega^1 \xrightarrow{\Omega^2} \Omega^2 \xrightarrow{-\Omega^2} -\Omega^2 \rightarrow \Omega^4 \rightarrow 0$. For $p=q=0$ this is NPB's generalized deRham sequence [3]. It relates the field versus potential/modulo gauge description of massless fields on M and is therefore of fundamental importance in twistor theory. The general case is used extensively in [6] for the same reason. Notice that the pattern of the BGG resolution $\cdots \leftarrow \cdot \rightarrow \cdots$ is independent of original choice of λ . In fact, this pattern may be recognized as describing the standard cell structure of M (dots representing cells of even dimensions, lines recording attaching maps). This is not a coincidence! This always happens and is known as the Bruhat decomposition. When $\lambda = 0$ (i.e. the resolution starts with \mathbb{C}) the BGG resolution may be realized by those holomorphic forms orthogonal to such typical cells. The BGG resolution is often far more efficient than the deRham resolution. Consider, for example the case

of $F_{1,2,3}(t^4) = \times \times \times$: without going into any detail, the general pattern is:



This is the classical BGG case when P is minimal (i.e. a Borel). The general (Lepowsky) case may be deduced from this and the BBW theorem for $G/B \rightarrow G/P$. In this case all irreducible bundles are line bundles so the dimension pattern for the BGG resolution is $1 \ 3 \ 5 \ 6 \ 5 \ 3 \ 1$ ($\#W = 24 (= 4!)$) so already much more efficient than the deRham resolution of \mathbb{C} where the dimensions are $1 \ 6 \ 15 \ 20 \ 15 \ 6 \ 1$. When the tangent bundle on G/P is irreducible, the BGG resolution of \mathbb{C} coincides with the deRham resolution, so for general λ on such a P one may reasonably maintain the terminology of "generalized deRham sequence". As a final example one can write down the deRham sequence on $M_6 = X$ as follows:

$$0 \rightarrow \mathbb{C} \rightarrow \overset{\circ}{\times} \xrightarrow{\circ} \overset{-2}{\times} \xrightarrow{\circ} \overset{-3}{\times} \xrightarrow{\circ} \overset{1}{\times} \xrightarrow{\oplus} \overset{-4}{\times} \xrightarrow{\circ} \overset{-5}{\times} \xrightarrow{\circ} \overset{1}{\times} \xrightarrow{\oplus} \overset{-6}{\times} \xrightarrow{\circ} \overset{-6}{\times} \xrightarrow{\circ} 0$$

The Penrose Transform: We assume familiarity with the general approach to constructing the Penrose transform for a given correspondence say $\begin{matrix} Y \\ \downarrow \mu \\ X \end{matrix}$ as described, for example, in [4]. Thus, one supposes U to be a suitably shaped open subset of X and the Penrose transform interprets the cohomology $H^k(\mu^{-1}(U), \mathcal{O}(V))$ in terms of differential equations on U determined by choice of the vector bundle V on Z . To simplify notation we shall remove U from explicit mention throughout. There are four steps in constructing the transform (1) $H^k(Z, \mathcal{O}(V)) = H^k(Y, \mu^{-1}\mathcal{O}(V))$ (technical conditions here); (2) Use the relative deRham complex $0 \rightarrow \mu^{-1}\mathcal{O}(V) \rightarrow \Omega^1_\mu(V)$ to compute $H^k(Y, \mu^{-1}\mathcal{O}(V))$ in terms of analytic cohomology $H^k(Y, \Omega^P_\mu(V))$; (3) Use the BBW theorem (and, usually degenerate, Leray spectral sequence) to interpret $H^k(Y, \Omega^P_\mu(V))$ down on X ; often this is the end result but sometimes: (4) reinterpret the equations on X (e.g. potential/gauge = field). For complex homogeneous spaces as discussed earlier one may replace (2) by (2)': Use the BGG resolution along the fibres of μ . Step (3) is then easier in general. Step (4) now uses the BGG sequences on X . This approach is also useful in the "curved" case (cf. [1] and MGE in this TN). It is about time to see this machine in action:

Examples: $H^k(\overset{\circ}{\times} \xrightarrow{\circ} \overset{-5}{\times})$ (This is $H^1(\mathcal{O}(4|-1,0,0))$ discussed in [5]): The BGG resolution along the fibres of $\overset{\circ}{\times} \xrightarrow{\circ} \overset{-5}{\times}$ is $\overset{\circ}{\times} \xrightarrow{\circ} \overset{-5}{\times} \rightarrow \overset{2}{\times} \xrightarrow{\circ} \overset{-3}{\times} \rightarrow \overset{1}{\times} \xrightarrow{\circ} \overset{-4}{\times} \xrightarrow{\circ} \overset{-2}{\times}$. Taking direct images under $\overset{\circ}{\times} \xrightarrow{\circ} \overset{-5}{\times}$ using BBW gives a spectral sequence converging to $H^{p+q}(\overset{\circ}{\times} \xrightarrow{\circ} \overset{-5}{\times})$ whose E_1 -level is $\overset{0}{\times} \xrightarrow{\circ} \overset{-3}{\times} \xrightarrow{\circ} \overset{2}{\times} \xrightarrow{\circ} \overset{-5}{\times} \xrightarrow{\circ} \overset{1}{\times} \xrightarrow{\circ} \overset{-5}{\times}$ i.e. $\mathcal{O}_{(AB'C')}^1 \xrightarrow{\nabla_A^B \nabla_B^C} \mathcal{O}_{(AB'C')}^2 (-2) \xrightarrow{\nabla_{BC}^A} \mathcal{O}_A^1 (-4)$

Thus $H^1(\overset{\circ}{\times} \xrightarrow{\circ} \overset{-5}{\times}) = \{ \text{Sol}'s \text{ of } \nabla_A^B \nabla_B^C \phi_{AB'C'} = 0 \text{ for } \phi_{AB'C'} = \phi_{(AB'C')} \text{ of weight zero} \}$ as in [5]. $H^k(\overset{\circ}{\times} \xrightarrow{\circ} \overset{-5}{\times})$: BGG resolution: $\overset{\circ}{\times} \xrightarrow{\circ} \overset{-5}{\times} \rightarrow \overset{2}{\times} \xrightarrow{\circ} \overset{-2}{\times} \rightarrow \overset{0}{\times} \xrightarrow{\circ} \overset{-4}{\times} \xrightarrow{\circ} \overset{-2}{\times}$ so BBW gives

$$\text{E}_1\text{-level: } \begin{array}{ccccccc} & 1 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 1 & -2 & 1 & -4 & 3 & & \end{array}$$

However, the BGG resolution of $\begin{array}{ccc} 1 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 2 & -2 & 1 \end{array}$ on $\begin{array}{c} * \\ \longrightarrow \\ * \end{array}$ is:

$$\begin{array}{ccccccc} & 1 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 1 & -2 & 1 & -4 & 3 & & \end{array} \rightarrow \begin{array}{ccccccc} & 1 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 1 & -5 & 2 & -5 & 1 & & \end{array}$$

$$\text{So } H^1(\begin{array}{ccc} 1 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 2 & -2 & 1 \end{array}) = \frac{\ker: \begin{array}{ccc} 2 & -2 & 1 \\ \downarrow & \downarrow & \downarrow \\ 0 & 4 & -3 \end{array}}{\text{im: } \begin{array}{ccc} 1 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 2 & -2 & 1 \end{array}} = \ker (\begin{array}{ccc} 1 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 2 & -2 & 1 \end{array}) = \ker (\nabla_A^\Lambda \nabla_B^\Lambda: \mathcal{O}_{(ABC)} \rightarrow \mathcal{O}_{C(A'B')}(-2))$$

In other words, this is a potential/gauge description of the "dual" field of $H^1(\begin{array}{ccc} 0 & 1 & 0 \\ \downarrow & \downarrow & \downarrow \\ 2 & -2 & 1 \end{array})$.

Thus we have the twistor transform: $H^1(\begin{array}{ccc} 0 & 1 & 0 \\ \downarrow & \downarrow & \downarrow \\ 2 & -2 & 1 \end{array}) \Rightarrow H^1(\begin{array}{ccc} 0 & 1 & 0 \\ \downarrow & \downarrow & \downarrow \\ 1 & -5 & 2 \end{array})$.

$$\underline{H^k(\begin{array}{ccc} 0 & 0 & *^{-4} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array})} \text{ BGG: } \begin{array}{ccccccc} & 1 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 1 & -4 & -2 & 1 & -4 & -3 & -2 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \text{ BBW: } \begin{array}{ccccccc} & 2 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\text{Thus } H^3(\begin{array}{ccc} 0 & 0 & *^{-4} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) = \ker: \begin{array}{ccc} 2 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \text{ or, using obvious notation, } H^3(\Omega(-4)) = \{ \text{Sol}^k \text{ of } \square \phi = 0 \text{ for } \phi \in T(\Omega(-2)) \}$$

$$\text{In general } H^{(2)}(Z_n, \Omega(2-2n)) = \{ \text{Sol}^k \text{ of } \square \phi = 0 \text{ where } \phi \in T(M_{2n}, \Omega(1-n)) \} \text{ (see LPH in TN's 9, 14, \& 17). Again there is the twistor transform so } \{ \square \phi = 0 \} \text{ can also be realized as } H^2(\begin{array}{ccc} 0 & 0 & *^{-2} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) \text{ [or } H^2(\begin{array}{ccc} 0 & 0 & *^{-2} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array})].$$

$$\underline{H^k(\begin{array}{ccc} 0 & 0 & *^{-6} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array})} \text{ BGG: } \begin{array}{ccccccc} & 1 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 1 & -6 & -2 & 1 & -6 & -5 & -4 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \text{ BBW: } \begin{array}{ccccccc} & 4 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\text{Thus } H^3(\begin{array}{ccc} 0 & 0 & *^{-6} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) = \ker: \begin{array}{ccc} 4 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \text{ From }$$

our earlier identification of the deRham sequence

$$\text{This may be rephrased: } H^3(\Omega^6) = \ker: \Omega^3 \xrightarrow{d} \Omega^4 \text{ i.e. anti-self-dual Maxwell fields}$$

$$\underline{H^k(\begin{array}{ccc} 0 & 0 & *^0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array})} \text{ BGG: } \begin{array}{ccccccc} & 1 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 1 & -2 & 1 & -3 & 1 & -4 & 2 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \text{ BBW: } \begin{array}{ccccccc} & 0 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

and from the deRham sequence on M_6 it follows

$$\text{that } H^2(\begin{array}{ccc} 0 & 0 & *^0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) = \frac{\ker: \Omega^2 \rightarrow \Omega^3}{\text{im: } \Omega^1 \rightarrow \Omega^2} = \ker \Omega^3 \xrightarrow{d} \Omega^4.$$

$$\text{Generally } H^3(\begin{array}{ccc} 0 & 0 & *^{-n-4} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) = H^2(\begin{array}{ccc} 0 & 0 & *^{n-2} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) = \ker: \begin{array}{ccc} -n-2 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ n-1 & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} -n-3 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ n-1 & 0 & 0 \end{array} \text{ with similar patterns}$$

in higher dimensions (as indicated in MFA lecture 7/11/83). Notice that $H^{n-1}(Z_n, \Omega)$ always represents anti-self-dual Maxwell fields whereas $H^{(2)}(Z_n, \Omega)$ gives self-dual fields for n even and anti-self-dual fields for n odd. Consequently the twistor transform is from Z_n to itself for n odd (there are good reasons for this).

$$\underline{H^k(\begin{array}{ccc} 0 & 0 & *^6 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array})} \text{ BGG: } \begin{array}{ccccccc} & 1 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 1 & -6 & -2 & 1 & -6 & -4 & -2 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \text{ So BBW: } \begin{array}{ccccccc} & 2 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\text{Thus } H^3(\begin{array}{ccc} 0 & 0 & *^6 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) = \ker: \begin{array}{ccc} 2 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \text{ or, in other words, }$$

$$H^3(\begin{array}{ccc} 0 & 0 & *^6 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) = \{ w \in T(M_5, \Omega^3) \text{ st. } dw = 0 \}.$$

$$\underline{H^k(\begin{array}{ccc} 0 & 0 & *^0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array})} \text{ BGG: } \begin{array}{ccccccc} & 1 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 1 & -2 & 1 & -3 & 2 & -2 & 0 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \text{ So BBW: } \begin{array}{ccccccc} & 0 & 0 & 0 & & & \\ & \downarrow & \downarrow & \downarrow & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\text{Thus } H^2(\begin{array}{ccc} 0 & 0 & *^0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) = \frac{\text{im: } \begin{array}{ccc} 1 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}}{\text{im: } T(\Omega^1) \rightarrow T(\Omega^2)} = \frac{T(\Omega^2)}{T(\Omega^1) \rightarrow T(\Omega^2)}$$

i.e. This is a potential/gauge description of $\{ w \in T(\Omega^3) \text{ st. } dw = 0 \}$. Thus there is a twistor transform as usual $H^2(\begin{array}{ccc} 0 & 0 & *^0 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}) \Rightarrow H^3(\begin{array}{ccc} 0 & 0 & *^6 \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array})$.

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Conformally Invariant Differential Operators on Spin Bundles

Let M be a 4-dimensional complex Riemannian manifold. With usual conventions, if $\phi_{AB\dots D E' F' \dots H'} = \phi_{(AB\dots D)(E' F' \dots H')}$ has conformal weight d then [6]:

$$\hat{\nabla}_{P'} \phi_{AB\dots D E' F' \dots H'} = \nabla_{P'} \phi_{AB\dots D} + (d-m-n) \sum_{(P', E' F' \dots H')} \eta_{(P', E' F' \dots H')}^m \eta_{(P', E' F' \dots H')}^{n+1} \phi_{(P', E' F' \dots H')Q},$$

$$+ \sum_{(P', E' F' \dots H')} \frac{(n-d-1)m}{m+1} \eta_{(P', E' F' \dots H')}^m \phi_{(P', E' F' \dots H')Q} + \frac{(d+2)mn}{(m+1)(n+1)} \eta_{(P', E' F' \dots H')}^{m+1} \eta_{(P', E' F' \dots H')Q}.$$

The usual factors of Ω have been absorbed into the definition of ϕ . This seemingly complicated formula is a consequence of a rather more elegant formula on $G =$ the total space of the Cartesian product of the projective spin-bundles over M . The metric connection ∇_a on M can be horizontally lifted to G . Let $\mathbb{F}(\frac{x}{\lambda})$ denote the line bundle over G defined by $\mathcal{O}(\mathbb{F}(\frac{x}{\lambda})) \ni f \iff f(x, \lambda \gamma_A, \mu \pi_A) = \lambda^p \mu^q f(x, \gamma_A, \pi_A)$ and f has conformal weight g . Then, for such f : $\hat{\nabla}_a f = \nabla_a f + \eta_A \eta_{BA'} \frac{\partial f}{\partial \gamma_B} + \pi_{A'} \eta_{AB'} \frac{\partial f}{\partial \pi_B} + g \eta_A f$.

The previous formula may be induced by direct image under $\nu: G \rightarrow M$. In other words, if f is globally defined in γ and π then it is necessarily polynomial: $f(x, \gamma_A, \pi_A) = \phi_{AB\dots D E' F' \dots H'}^{\frac{AB\dots D E' F' \dots H'}{2}} \eta_A \eta_B \dots \eta_D \pi_E \pi_F \dots \pi_H$. Many calculations and constructions are more easily performed on G . In the flat case ($M = \mathbb{M} = \mathbb{F}_2(\mathbb{T}')$) G is the space \mathbb{G} of full flags $\mathbb{F}_{1,2,3}(\mathbb{T}')$ and a homogeneous space of the form semi-simple/Borel. Such are generally easier to deal with and, for example, $BG, \mathbb{G}G$ show [3] that all homogeneous differential operators on \mathbb{G} are compositions of ones in the BGG resolutions (in [2] with this notation). Although the homogeneous operators for \mathbb{M} (exhibited in [4]) cannot all be derived directly from those on \mathbb{G} they are certainly related (and can be derived indirectly). Just as in [4] where these operators for M have conformally invariant analogues, it appears that all homogeneous differential operators on \mathbb{G} have conformally invariant analogues on G . These arise by adding curvature correction terms.

Examples: (1) Introduce $\nabla = \eta^A \pi^B \nabla_{AB}$: $\mathcal{O}(\mathbb{F}(\frac{x}{\lambda})) \rightarrow \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2))$ and let $\eta^A \pi^B \eta_{AB}$. There, by *, $\hat{\nabla}f = \nabla f + g \eta^A f$ so, in particular, $\nabla: \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2)) \rightarrow \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2 \frac{x}{\lambda}^2))$ is conformally invariant. (2) Iterating \mathbb{G} for $g=1$ gives: $\hat{\nabla}^2 f = \nabla [\nabla f + \eta^A f] - \eta [\nabla^2 f + \eta^A f] = \nabla^2 f + [\nabla \eta - \eta^2] f$ but, letting $\Phi = \Phi^{ab} \eta_A \eta_B \pi_A \pi_B \in \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2))$, $\hat{\Phi} = \Phi - \nabla \eta + \eta^2$ so $\nabla^2 + \Phi: \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2)) \rightarrow \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2 \frac{x}{\lambda}^2))$ is conformally invariant (case $p=r=0$ is in [1]). (3) $\nabla^3 + 4\Phi \nabla + 2(\nabla \Phi): \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2)) \rightarrow \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^3 \frac{x}{\lambda}^4))$ is invariant. (E) If $f \in \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2))$ define $\tilde{f} \in \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2))$ by $\pi^A \tilde{f} = \frac{\partial f}{\partial \pi_A}$ and similarly $\tilde{f}^{p+1}: \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2)) \rightarrow \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^{p+1}))$ for any $p \geq 0$. (A) $\eta^A \frac{\partial}{\partial \pi_A}: \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2)) \rightarrow \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^3))$; A version of this on just one spin bundle is a curved antitwistor operator [5].

Recognizing ∇ as the null geodesic spray and following RJB in [1] define $\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2)$ on A , the space of null geodesics by $\mu^{-1}(\mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2)) = \ker: \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^2 \frac{x}{\lambda}^2)) \rightarrow \mathcal{O}(\mathbb{F}(\frac{x}{\lambda}^{p+q+1} \frac{x}{\lambda}^{q+1} \frac{x}{\lambda}^{r+1}))$, $\mu: G \rightarrow A$. Note that $g \geq 0$ by definition. The notation is arranged to generalize [2] and it is clear how to Penrose transform in this curved case: answer exactly as for flat case but using the conformal versions [4] of homogeneous operators. Question: Relate to Cartan connection and recent work of RJB.

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4 1

CONFORMALLY INVARIANT DIFFERENTIAL OPERATORS FOR CURVED SPACE

Mike Eastwood and John Rice, in [1], outline an elegant algebraic technique for the construction of conformally invariant differential operators on conformally flat spaces. This extends via the local twistor connection (or Cartan's conformal connection) to the curved case. The resulting method is entirely constructive, producing the necessary curvature correction terms explicitly. This note reviews the flat space construction, indicates the curvature modifications & provides several examples. For an exposition of the notation, theory etc. of [2].

CONFORMALLY FLAT INVARIANT OPERATORS (the ideas in the following are due to MGE & JR). Let G be a complex, semi-simple Lie group, \mathfrak{p} a simply connected parabolic subgroup, so that the quotient $M = G/\mathfrak{p}$ is defined. Homogeneous bundles on M are induced by representations of \mathfrak{p} , equivalently \mathfrak{t} , where $\mathfrak{t} \subset \mathfrak{p}$ are the Lie Algebras of G/\mathfrak{p} . For ease of our exposition, restrict attention to invariant operators between irreducible homogeneous bundles: examples (in the notation of [2]) are $\overset{1-2-0}{\longleftrightarrow} = \mathcal{O}_{\mathbb{A}^1}$ on M , $\overset{5-3-0}{\longleftrightarrow} = \mathcal{O}(p, q)$ on A etc. Such bundles are determined by selecting a dominant integral weight, λ say, for \mathfrak{t} . If $M(\lambda)$ is the finite dimensional representation (of \mathfrak{t}) associated to such a weight, then $\mathcal{O}(\lambda)$ (or the numbered Dynkin diagram giving λ) represents the bundle induced by the contragredient representation, $M(\lambda)^*$ (the dual induction is simply a matter of convenience). Associated to $\mathcal{O}(\lambda)$ are the various jet bundles $J^+(\lambda)$ (whose stalk at a point consists of the germs of sections of $\mathcal{O}(\lambda)$ at the point, factored out by those which vanish to order n). There are obvious surjections $J^{+t}(\lambda) \rightarrow J^+(\lambda)$, and $J^{-t}(\lambda)$ denotes the inverse limit of these. ($J^{-t}(\lambda)$ may be thought of in terms of formal power series). These bundles are obviously homogeneous bundles under the obvious action of \mathfrak{t} induced on germs by the action of \mathfrak{t} on $\mathcal{O}(\lambda)$. Now a DIFFERENTIAL OPERATOR $D: \mathcal{O}(\lambda) \rightarrow \mathcal{O}(\mu)$ is simply a morphism of sheaves $J^{-t}(\lambda) \xrightarrow{D} \mathcal{O}(\mu)$ which factors through the projection $J^{-t}(\lambda) \rightarrow J^+(\mu)$ for some finite t . An invariant differential operator is further a morphism of sheaves of \mathfrak{t} -modules - i.e. the map \tilde{D} preserves the action of \mathfrak{t} . The crucial observation of Eastwood & Rice is the identification of the modules inducing $J^{-t}(\lambda) \otimes J^+(\lambda)$ as homogeneous bundles. Let $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{t})$ denote the universal enveloping algebras of \mathfrak{g} & \mathfrak{t} . Then $J^{-t}(\lambda)$ is induced by the dual of the (generalized) Verma module [3]: $V(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{t})} M(\lambda)$. Under the natural filtration $\mathcal{U} = \mathcal{U}^0(\mathfrak{g}) \subset \mathcal{U}^1(\mathfrak{g}) \subset \dots \subset \mathcal{U}^t(\mathfrak{g})$, $V(\lambda)$ acquires a filtration $M(\lambda) \cong V^0(\lambda) \subset V^1(\lambda) \subset \dots \subset V(\lambda)$ (which is preserved by \mathfrak{t} but not by \mathfrak{g} , of course). $J^+(\lambda)$ is induced by the dual of $V^+(\lambda)$. Since a differential operator is to be determined locally, it is clear that an invariant differential operator corresponds dually to a \mathfrak{t} -module homomorphism $M(\mu) \rightarrow V(\lambda)$. As a \mathfrak{t} -module, $M(\mu)$ is generated by a "highest weight vector," which is characterized, up to scale, by the requirements that it be annihilated by elements of the maximal nilpotent subalgebra of \mathfrak{t} (when $\mathfrak{g} = sl(n, \mathbb{C})$ these correspond to the superdiagonal matrices), and have weight μ (which describes the scalars by which it is acted on by "diagonal" elements of \mathfrak{t}). It follows that a \mathfrak{t} -module homomorphism of $M(\mu) \rightarrow V(\lambda)$ is determined by a maximal vector in $V(\lambda)$, of weight μ (if such exists). So the problem of finding invariant operators is equivalent to finding such maximal vectors, which is a well established problem in representation theory ([4], [5]). It is clearly equivalent to finding a \mathfrak{g} -homomorphism from $V(\mu) \rightarrow V(\lambda)$, both of which are generated by maximal vectors under the action of $\mathcal{U}(\mathfrak{g})$. It is clear that the centre of $\mathcal{U}(\mathfrak{g})$ must act by scalars on $V(\mu) \otimes V(\lambda)$ (since it must necessarily act by a scalar set on the generators). Furthermore, these scalars must be the same for $V(\mu) \otimes V(\lambda)$ if there is a \mathfrak{g} -homomorphism between them. In the parlance of representation theory, $V(\mu) \otimes V(\lambda)$ must have the same INFINITESIMAL CHARACTER [14], [6]. This is easily checked in practise, using HARISH-CHANDRA's theorem (cf [4]: infinitesimal character is sometimes called central character): thus out

RULE 1: A necessary condition for the existence of an invariant differential operator $D: \mathcal{O}(\lambda) \rightarrow \mathcal{O}(\mu)$ is that there exist an element w of $W(\mathfrak{g})$ (the Weyl group of \mathfrak{g}) with $\mu = w \cdot \lambda$ (cf [2]) (\cdot indicates the translation (by $\frac{1}{2}$ sum +ve roots) action of $W(\mathfrak{g})$ on the weight lattice).

EXAMPLE (i) Begin with $\lambda = \overset{0-0-0}{\longleftrightarrow}$; then $\overset{2-3-0}{\longleftrightarrow} = w \cdot \lambda$ with $w = \sigma_1 \sigma_3$ (numbering $\overset{2-1-3}{\longleftrightarrow}$) for σ_i the simple reflections. ([cf [2]])
 Indeed, one may verify that the following array contains all the weights (dominant for $\overset{0-0-0}{\longleftrightarrow}$) of the same central character as $\overset{0-0-0}{\longleftrightarrow}$:

$$\begin{array}{ccccc} \overset{0-0-0}{\longleftrightarrow} & \overset{1-2-1}{\longleftrightarrow} & \overset{2-3-0}{\longleftrightarrow} & \overset{1-4-1}{\longleftrightarrow} & \overset{0-4-0}{\longleftrightarrow} \\ \sigma_1 \sigma_3 & \sigma_1 & \sigma_1 \sigma_3 & \sigma_1 & \sigma_1 \end{array} \quad (1)$$

As homogeneous bundles, these are, more familiarly,

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$$\Omega_m \quad \Omega^1 \quad \Omega^2 \quad \Omega^3 \quad \Omega^4 \quad -(1')$$

Any univariant operator with domain one of these must have as range one of these also. \square

The criterion given in the rules is not generally sufficient. If \mathfrak{t} is Borel (i.e. in the case of $sl(n, \mathbb{C})$ it consists of the upper triangular matrices, essentially - or in the notation of [2], the Dynkin diagram for \mathfrak{t} has every node crossed through), then the following rule is necessary & sufficient: ([4], [3] cont 2.6)

RULE 1'
(\mathfrak{t} Borel)

\exists non-zero op-homomorphism from $V(\mu) \rightarrow V(\lambda)$ iff $\mu = \sigma_{i_n} \cdot \sigma_{i_{n-1}} \cdots \sigma_i \cdot \lambda$
with $\sigma_{i_1}, \dots, \sigma_{i_n}$ simple reflections in $W(\mathfrak{o})$ and such that at each stage
 $((\sigma_{i_n} \cdot \sigma_{i_{n-1}} \cdots \sigma_{i_1} \cdot \lambda) + p)$ has the integer over the i^{th} node non-negative
(p is the distinguished weight given by 1's over each node in the Dynkin diagram of \mathfrak{o})

EXAMPLE (ii)

$\star\star\star$ is the space of projectified primed & unprimed spinors over M (often denote $\bar{\Gamma}_{123}$). Consider $\star\star\star = \lambda$. Then adding $\star\star\star$ obtain $\star\star\star$; $\star\star\star$ followed by σ_2 gives $\star\star\star$ followed by $\star\star\star$. σ_1 then gives $\star\star\star$ & subtracting $\star\star\star$ gives $\star\star\star$. At each stage, numbering nodes $\overset{2}{\star} \overset{1}{\star} \overset{3}{\star}$, before applying σ_i , the i^{th} node integer was non-negative, so the RULE yields an invariant differential operator on (the bundle pulled back to $\star\star\star = \bar{\Gamma}_{123}$) ($\star\star\star \xrightarrow{P} \star\star\star$)

$$p^* \mathcal{O}[-1] \longrightarrow p^* \mathcal{O}[-3]$$

It turns out that this operator is just $\nabla^2 \delta \delta'$, where $\nabla^A \eta^B \nabla_{BA} = \nabla$ (the null geodesic spray) and δ, δ' are defined respectively by $\nabla_A \delta f = - \partial / \partial x^A f$ and $\eta_A \delta f = - \partial / \partial x^A f$.

It is particularly significant that this operator does not pass down to an invariant operator on $\star\star\star$, i.e. on $\mathcal{O}[-1]$ on M - clearly if f is constant on the fibres of p , $\nabla^2 \delta \delta' f = 0$. \square

More generally, to determine if a possible operator actually exists, it is useful to recall the notion of a symbol of a differential operator. The kernel of the projection $J^+(\lambda) \rightarrow J^{+1}(\lambda)$ is $\mathcal{O}^+ \Omega_M^1 \otimes \mathcal{O}(\lambda)$ and the composition $\mathcal{O}^+ \Omega_M^1 \otimes \mathcal{O}(\lambda) \rightarrow J^+(\lambda) \xrightarrow{p} \mathcal{O}(\mu)$, & say, is the symbol of D . Dually, this corresponds to $M(\mu) \rightarrow V^+(\lambda) \rightarrow V^+(\lambda) / V^{+1}(\lambda)$, the last term of which induces $\mathcal{O}^+ \Omega_M^1 \otimes \mathcal{O}(\lambda)$. This immediately yields

RULE 2

A further necessary condition for the existence of an invariant differential operator $D: \mathcal{O}(\lambda) \rightarrow \mathcal{O}(\mu)$ is that μ be a weight of $\mathcal{O}^+ M \otimes M(\lambda)$ where M induces Ω_M^1 . Thus $\mathcal{O}(\mu)$ is a non-trivial component of $\mathcal{O}^+ \Omega_M^1 \otimes \mathcal{O}(\lambda)$ and the symbol of the operator is simply projection onto this component.

Splitting the filtration of $\mathcal{U}(\mathfrak{o})$ by choosing a basis for \mathfrak{o} & using the Poincaré-Birkhoff-Witt theorem [4], the highest weight vector in $M(\mu)$ maps, under the dual symbol $\star\star\star$, to a weight vector, β say, in $V^+(\lambda)$. If the symbol is to lift to an invariant operator then this must be maximal in $V(\lambda)$ (i.e. annihilated by the elements of the maximal nilpotent subalgebra of \mathfrak{o}). So

RULE 3

Use the symbol & the highest vector of $M(\mu)$ to determine a weight vector in $V^+(\lambda)$ hence $V(\lambda)$. If this is maximal, then there exists an invariant differential operator from $\mathcal{O}(\lambda) \rightarrow \mathcal{O}(\mu)$.

This rule completes the task of finding invariant differential operators on M . Combined with RULE 2, it in fact determines the operator explicitly.

EXAMPLE (iii) Note that $\Omega_M^1 \simeq \overset{1-2}{\star} \overset{-1}{\star}$; $\mathcal{O}^2 \Omega_M^1 \simeq \overset{2-4}{\star} \overset{2}{\star} \oplus \overset{0-2}{\star} \overset{0}{\star}$; $\mathcal{O}^3 \Omega_M^1 \simeq \overset{3-6}{\star} \overset{3}{\star} \oplus \overset{1-4}{\star} \overset{-1}{\star}$; $\mathcal{O}^4 \Omega_M^1 \simeq \overset{4-8}{\star} \overset{4}{\star} \oplus \overset{2-6}{\star} \overset{2}{\star} \oplus \overset{0-4}{\star} \overset{0}{\star}$.

RULE 2 shows that there are at most three invariant operators from $\Omega_M^1 \simeq \overset{2-4}{\star} \overset{2}{\star}$, namely
(a) $\overset{0-2}{\star} \overset{0}{\star} \xrightarrow{Y_1} \overset{1-2}{\star} \overset{-1}{\star}$; (b) $\overset{0-2}{\star} \overset{0}{\star} \xrightarrow{Y_2} \overset{1-4}{\star} \overset{1}{\star}$ or (c) $\overset{0-2}{\star} \overset{0}{\star} \xrightarrow{Y_3} \overset{0-4}{\star} \overset{0}{\star}$.

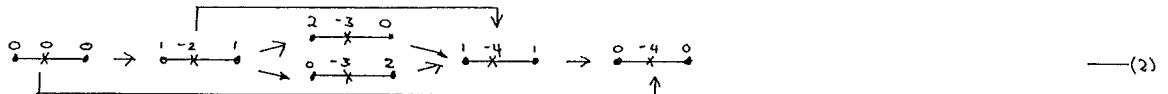
In order to check RULE 3, it is necessary to be more explicit about the generators of \mathfrak{o} , i.e. $sl(4, \mathbb{C})$. Label the generators of $sl(4, \mathbb{C})$ as in fig 1. By this I mean that the generator y_1 , for example, has a 1 in the position indicated by y_1 and zeros elsewhere. Let $h_1 = H_2 - H_3$, $h_2 = H_1 - H_2$ & $h_3 = H_3 - H_4$. Then $sl(4, \mathbb{C})$ has generators x_i, z_i (raising operators) & y_i (lowering operators) & h_i .

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To specify the weight vectors corresponding to the symbols (a) - (c) above, let $\alpha \in V(\mathfrak{so}_n)$ be a highest weight vector. Then the corresponding vectors are (a) $y_1\alpha$, (b) $y_3(y_4y_4 - y_2y_3)\alpha$ & (c) $(y_1y_4 - y_2y_3)^2\alpha$. Of these, only (b) is not maximal. ((a) & (c) are: check by seeing that they are annihilated by x_2 and the X_i 's.) Thus the only invariant operators from \mathcal{O}_{λ} are ∇_α (i.e. exterior differentiation) and \square^2 . One may check that the only invariant operators between bundles in (1) are.

FIGURE 1

$$\begin{bmatrix} H_1 & X_1 & x_2 & x_4 \\ Y_1 & H_2 & X_2 & X_3 \\ Y_2 & Y_3 & H_3 & X_2 \\ Y_4 & Y_2 & Y_2 & H_4 \end{bmatrix}$$



The three rules given above suffice to compute all invariant operators. There is, however, an elegant shortcut, enabling one to generate new operators from old. This is an application of an algebraic TRANSLATION PRINCIPLE. So far, discussion has been limited to operators on irreducible homogeneous bundles. Let F be a finite dimensional irreducible \mathfrak{g} -module. Regard, by restriction, as a \mathfrak{g} -module: it induces (dually) a homogeneous bundle $\mathcal{O}(F)$ and it is clear that $\mathcal{J}^\infty(\mathcal{O}(F))$ is dually induced by $V(F) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g})} F$. More generally, $\mathcal{J}^\infty(\mathcal{O}(F) \otimes \mathcal{O}(\lambda))$ is dually induced by $V(F \otimes M(\lambda))$ and one has $F \otimes V(\lambda) \cong V(F \otimes M(\lambda))$ as \mathfrak{g} -modules. F is not likely to be \mathfrak{g} -irreducible - it will be given as a \mathfrak{g} -module by means of a composition series of irreducible \mathfrak{g} -modules F_i . Then, tensoring this with $M(\lambda)$, one obtains a composition series for $F \otimes M(\lambda)$. Now use the fact that V is an exact functor and that the induced modules are projective (cf [6]) to deduce that one has a direct sum decomposition

$$F \otimes V(\lambda) \cong V(F \otimes M(\lambda)) \cong \bigoplus_i V(F_i(\lambda))$$

where $F_i(\lambda)$ are the irreducible \mathfrak{g} -modules from $M(\lambda) \otimes F_i$. Thus, if \exists a \mathfrak{g} -module homomorphism $V_{\mu_i} \xrightarrow{\cong} V(\lambda)$, then one has

$$\bigoplus_i V(F_i(\mu_i)) \longrightarrow F \otimes V(\mu_i) \xrightarrow{\text{id} \otimes D} F \otimes V(\lambda) \longrightarrow \bigoplus_i V(F_i(\lambda))$$

and hence (recalling the restrictions of infinitesimal character) several possible \mathfrak{g} -module homomorphisms. Under suitable circumstances, most of these will be zero or the identity: in the end only one new differential operator will arise (this because the above construction may be shown to be an equivalence between suitable categories). The upshot of this abstract nonsense is a nice calculable result:

RULE 4:

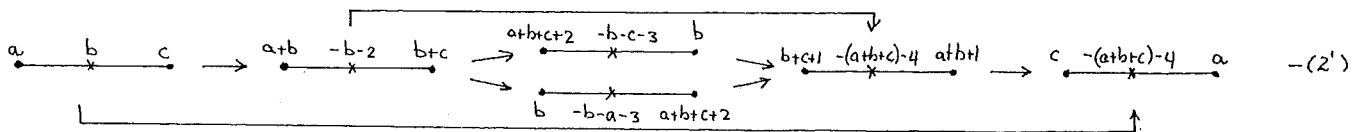
Let F be an irreducible \mathfrak{g} -module of weight χ . Suppose that λ is also dominant for \mathfrak{g} and that $\mu_i = w_i \cdot \lambda$ are dominant for \mathfrak{g} (w_i ($i=1,2$) $\in W(\mathfrak{g})$). Then \exists an invariant differential operator:

$$\mathcal{O}(\mu_1) \rightarrow \mathcal{O}(\mu_2)$$

iff there exists an invariant differential operator:

$$\mathcal{O}(w_1 \cdot (\lambda + \xi)) \rightarrow \mathcal{O}(w_2 \cdot (\lambda + \xi))$$

Example (iv) In (2) take $\lambda = \begin{smallmatrix} a & b & c \\ \bullet & \times & \bullet \end{smallmatrix}$, $\chi = \begin{smallmatrix} a & b & c \\ \bullet & \times & \bullet \end{smallmatrix}$ and run through all possible w_i , with $a, b, c \geq 0$ to obtain the following complete set of invariant differential operators between bundles of non-singular infinitesimal character:



(for more details on the translation principle, cf [1], [6], [8], [15]). (These sequences appear in [16])

EXAMPLE (v) To illustrate a "singular" operator, i.e. one based on bundles of singular infinitesimal character, take $\lambda = \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}$ and obtain $\square : \overset{\circ}{\times} \xrightarrow{\circ} \overset{\circ}{\times} \xrightarrow{\circ} \overset{\circ}{\times}$, i.e. $\square : \mathcal{O}[-1] \rightarrow \mathcal{O}[-3]$. There is a general theory which extends this example to $\square^{k+2} : \overset{\circ}{\times} \xrightarrow{\circ} \overset{\circ}{\times} \xrightarrow{\circ} \overset{\circ}{\times}$ ($k \geq 0$). Example of this is $\square^2 : \mathcal{O} \rightarrow S^4 \cong \mathcal{O}[-4]$, originally found in [18].

To make contact with the more usual definitions of conformal invariance of [10] or [17], let $f \in \Gamma(\mathcal{O}[-1])$ on M . f may be thought of as a function \hat{f} on $G = SL(4, \mathbb{C})$ satisfying $f(p) = \pi(p^{-1})\hat{f}$, where $\pi(p^{-1})$ is the transform on \mathbb{C} had by regarding \mathbb{C} as a weight space for p , of weight $\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}$. Under the element of \mathbb{C} corresponding to conformal rescaling, $f \mapsto \sqrt{z}f = \hat{f}$ and $y_1 \mapsto \sqrt{z}y_1 = \hat{y}_1$. Restricting to $SO(4, \mathbb{C}) \subset SL(4, \mathbb{C})$, $\hat{y}_1, \hat{f}, y_1, f$ generate $\hat{\nabla}_{AA'}\hat{f}$ and $\nabla_{AA'}f$ respectively. From $\hat{y}_1, \hat{f} = z^{-2}y_1, (z^{-1}f) = z^{-3}\{y_1, f\} - (y_1, \ln z)f$ one recovers

$$\hat{\nabla}_{AA'}\hat{f} = \{y_1, f\} - f(\nabla_{AA'}\ln z)/z^3$$

CONFORMALLY INVARIANT OPERATORS: CURVED CASE: In the curved case, of course, the homogeneous structure $M = G/\mathbb{C}$ is not available. This structure enters the above reasoning only at the point of identifying the tangent bundle (and hence the jet bundles) as induced bundles. In the curved case there exists instead a P -principal bundle ly over M - defined once a conformal structure on M is given [7][8] - which is a bundle of restricted local twistor frames (restricted to preserve the "structure" implied by the sequence $O_A' \rightarrow \mathcal{V}^* \rightarrow \mathcal{O}^A$). Additionally, there is defined a P -homomorphism $\omega : \mathcal{O}^P \rightarrow \Gamma(Tly)$ (injective) with the property that for $u \in \mathbb{C}$, $v \in \mathcal{O}^P$, $\omega[u, v] = [\omega(u), \omega(v)]$. This means that, as in the flat case, the tangent bundle of M can be identified as the bundle induced by the representation of \mathbb{C} on \mathcal{O}^P/\mathbb{C} , and jet bundles are correspondingly induced dually by the induced \mathcal{O} -modules $V(\lambda)$ etc. Then ω gives vector fields $\omega(y_i)$ etc. on ly and one interprets these, as in the flat case, as inducing invariant differential operators. All that remains is to identify these operators in terms of the Levi-Civita connection given a choice of conformal factor. Details will be found in [8], but the basic principle is as follows. A choice of conformal factor (on M , locally) determines an $SO(4, \mathbb{C})$ principal bundle on M which lies inside ly as a submanifold. This gives, via the Levi-Civita connection, an $\tilde{\omega}$ on ly [details in [7], [97]] which differs from ω , but which induces the Levi-Civita connection via $\tilde{\omega}(y_i)$ and the action of $SO(4, \mathbb{C})$. Expressions involving $\tilde{\omega}(y_1 y_4 - y_2 y_3)$, for instance, have an immediate interpretation in terms of the Levi-Civita connection. (This corresponds to having a local twistor transport [10] with $P_{AA'BB'}$ set to zero). $\tilde{\omega}$ does not obey certain curvature conditions (required of ω) so must be "adjusted" so to do. This corresponds, in LTT, to introducing the term in $P_{AA'BB'}$. $\omega = \tilde{\omega}$ on \mathbb{C} , but if $\mathcal{O}^P = \mathbb{C} \oplus \mathbb{C}$, then $\tilde{\omega}|_{\mathbb{C}}$ needs changing. If $\mathbb{C} = \mathbb{C} \oplus \mathbb{C}$, then $\tilde{\omega}$ is changed only by adding in pieces corresponding to $\tilde{\omega}(u) = \omega(u)$. Concretely, if we abuse notation & write $\tilde{\omega}(y_i) = \tilde{y}_i$, $\omega(y_i) = y_i$, we get

$$y_i = \tilde{y}_i + \tilde{y}_i \perp P \quad (3)$$

where P is a one form on ly with values in $\omega(\mathbb{C})$. P really is just $P_{AA'BB'}$. So to obtain curved space invariant operators, one writes formulae of the flat case in terms of y_i & re-interprets in terms of \tilde{y}_i using (3).

EXAMPLE (vi) In the notation of [2] & [11] work on the curved version of $\times \rightarrow \times$ (projectified framed & unprimed spin bundles) Set $\nabla =$ null geodesic spray and take $P_{11} = P_{AA'BB'} \tilde{\pi}^A \tilde{\pi}^B \pi^A \pi^B$, $P_{12} = P_{AA'BB'} \tilde{\pi}^A \tilde{\pi}^B \pi^A \pi^B$ ($\pi_B \cdot \pi^B = 1$) etc. (in Newman-Penrose style). Then $y_1 = \tilde{y}_1 + P_{11} x_1 + P_{12} x_2 + P_{13} x_3 + P_{14} x_4$. Consider the invariant operator $\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \rightarrow \begin{smallmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}$ corresponding to $y_1^\alpha \alpha$ for $\alpha \in V(\overset{\circ}{\times} \overset{\circ}{\times})$. Then $y_1^\alpha \alpha = (\tilde{y}_1 + P_{11} x_1) \tilde{y}_1 \alpha = \tilde{y}_1 \alpha + P_{11} \alpha$, inducing $\nabla^2 + P_{11} : \begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \rightarrow \begin{smallmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}$. This is just the operator occurring in the definition of the LeBrun Einstein bundle [12][13] (See below). It induces the operator $\nabla_{AB}^{(A} \nabla^{B)} + \Phi_{AB}^{AB} : \mathcal{O}[1] \rightarrow \mathcal{O}_{(AB)}^{(AB)}[-1]$, which is conformally invariant. (cf [11] for some similar examples)

EXAMPLE (vii) $\square : \mathcal{O}[-1] \rightarrow \mathcal{O}[-3]$ is induced by $(y_1 y_4 - y_2 y_3) \alpha$ for α highest in $V(\overset{\circ}{\times} \overset{\circ}{\times})$. Let $\tilde{y}_1 \perp P = P_{11} x_1$; then $(y_1 y_4 - y_2 y_3) \alpha = \{(\tilde{y}_1 \tilde{y}_4 - \tilde{y}_2 \tilde{y}_3) - (P_{14} - P_{23})\} \alpha$. It is easily checked that $P_{14} - P_{23} = -R/6$ where R is the scalar curvature, and one obtains the familiar conformally invariant operator $(\square + R/6)$.

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EXAMPLE (viii)

The invariant operator $\overset{1-2}{\xrightarrow{\times}} \overset{2}{\xrightarrow{\times}} \overset{3-4}{\xrightarrow{\times}}$ is induced by $(-y_3 + y_1 Y_2)(-2y_3 + y_1 Y_2)\alpha$; a short calculation gives this in terms of \tilde{y}_i as $(-\tilde{y}_3 + \tilde{y}_1 Y_2)(-2\tilde{y}_3 + \tilde{y}_1 Y_2)\alpha + \{P_{11} Y_2^2 \alpha - 2P_{31} Y_2 \alpha + 2P_{33} \alpha\}$; noting that $Y_2 P_{11}$ is $-2P_{13}$ and $Y_2^2 P_{11} = 2P_{33}$, $\{ \}$ is recognised as the highest weight in $\overset{3-4}{\xrightarrow{\times}}$ from decomposing the tensor product $\overset{1-2}{\xrightarrow{\times}} \otimes \overset{2}{\xrightarrow{\times}} \overset{3-4}{\xrightarrow{\times}}$ and the invariant operator which results is $\lambda_{(BC)}^{AB} \rightarrow \nabla_{(A}^A \nabla_{B'}^{B'} \lambda_{C)}^{AB} + \Phi_{(AB}^{AB} \lambda_{C)}^{AB}$. (as might be expected) \square

There are several other applications of the space \mathfrak{g} and Cartan Connection ω . One may ask if ω extends to a homomorphism of Lie Algebras on parabolics other than $\mathfrak{x} \rightarrow \mathfrak{x}$. It always does for $\mathfrak{x} \rightarrow \mathfrak{x}$, for example. When this happens (a neighbourhood of the origin in) the corresponding group acts on \mathfrak{g} and the quotient space is defined, with a principal bundle structure: \mathfrak{g}/P , so induced bundles are well defined. For instance, $\mathfrak{x} \rightarrow \mathfrak{x}$ leads to (curved) ambitwistor space, and the induced bundle $\overset{1}{\xrightarrow{\times}} \overset{2}{\xrightarrow{\times}}$ is the LeBrun Einstein bundle. One can perform a curved space Penrose transform on it, obtaining the resolution (cf example vi)

$$0 \rightarrow \overset{0}{\xrightarrow{\times}} \overset{1}{\xrightarrow{\times}} \overset{0}{\xrightarrow{\times}} \rightarrow \overset{0}{\xrightarrow{\times}} \overset{1}{\xrightarrow{\times}} \overset{0}{\xrightarrow{\times}} \rightarrow \overset{2-3}{\xrightarrow{\times}} \overset{2}{\xrightarrow{\times}} \overset{2}{\xrightarrow{\times}} \rightarrow 0$$

so that $H^0(A, \overset{0}{\xrightarrow{\times}} \overset{1}{\xrightarrow{\times}} \overset{0}{\xrightarrow{\times}}) \cong \ker \nabla_{(A}^A \nabla_{B'}^{B')} + \Phi_{AB}^{AB} : \Omega^0[-1] \rightarrow \Omega_{AB}^{AB}[-1]$, whence LeBrun's argument.

The Cartan connection ω is a homomorphism of Lie Algebras on $\mathfrak{x} \rightarrow \mathfrak{x}$ only if the curvature Ψ_{ABCD}^{ABCD} vanishes. When this happens, the corresponding quotient space is \mathfrak{g}/P , the non-linear graviton space. The curved analogues of $\mathcal{O}(k)$ etc. are well defined as induced bundles and solutions of 2-R.M. equations (of the right helicity) are obtained: e.g. for $\overset{k-2}{\xrightarrow{\times}} \sim \mathcal{O}(-2-k)$, one obtains solutions of $\nabla_A^A \Psi_{B...D}^{B...D} = 0 (k \geq 1)$ or $(\Box + R) \varphi = 0$. Cf [20]. Investigations of formal neighbourhoods of lines in $P\mathbb{P}$ can be done easily in the formalism.

There is, thus, a Curved Analogue of (2'); it is not however a complex as in the flat case, i.e. compositions of the short arrows do not give zero. The simple reason is that in the flat case, the y_i 's commute, giving the complex property. In the curved case, the y_i 's do not commute & curvature terms result (this formalism makes it easy to compute these). Choosing the normal Cartan Connection as a base minimises this. One would hope to use this non-commutativity to somehow characterize all conformal invariants.

Would also expect relations with recent work of Kobayashi & Newman [19].

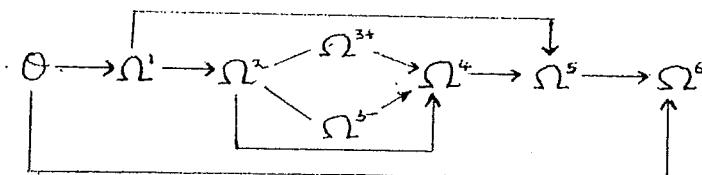
Details to follow in [8]: Many thanks to Mike Eastwood for discussions.

Rob Baston

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Note added: Conformally invariant differential operators in six dimensions

On $\mathbb{C}S^6$,



gives a complete list of the conformally invariant differential operators between the indicated bundles, and from which all operators between bundles of non-singular infinitesimal character may be generated via the translation principle.

Rajesh Gover

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The Fefferman-Graham Conformal Invariant

In [1] it is shown that, on an arbitrary holomorphic spin 4-manifold, there are natural conformally invariant differential operators:

$$\begin{array}{ccccccc} \mathcal{O}\left(\begin{smallmatrix} p & q & r \\ -q & -r & -p \end{smallmatrix}\right) & \rightarrow & \mathcal{O}\left(\begin{smallmatrix} p+2 & q+r+1 \\ -q-2 & -r-1 & -p \end{smallmatrix}\right) & \rightarrow & \mathcal{O}\left(\begin{smallmatrix} p+2+r+2 \\ -q-2-3 & -r-3 & -p \end{smallmatrix}\right) & \rightarrow & \mathcal{O}\left(\begin{smallmatrix} p+r+1 & p+q+1 \\ -q-2-r-4 & -p & -p-q-r-4 \end{smallmatrix}\right) & \rightarrow & \mathcal{O}\left(\begin{smallmatrix} p & p \\ -p & -p-q-r-4 & -p \end{smallmatrix}\right) \\ & & \downarrow & & & & & & \\ & & \text{The diagram commutes save for this operator} & & & & & & \end{array}$$

(cf. RJB & MGE, this TIN) where $\mathcal{O}\left(\begin{smallmatrix} p & q & r \\ -q & -r & -p \end{smallmatrix}\right) = \mathcal{O}_{(AB\ldots D)(E'F'\ldots H')}\{p+q+r\}$. As a typical example ($p=1, q=r=0$) with the notation of [3]: $\begin{smallmatrix} 2 & 2 & 1 \\ -2 & -2 & -1 \end{smallmatrix} = \mathcal{O}_{(AB)}\{1\} \rightarrow \mathcal{O}_{(A'B'C')}\{-1\} = \begin{smallmatrix} 0 & -4 \\ -4 & 3 \end{smallmatrix}$, $\phi_{ABC'} \mapsto \nabla_A^R \nabla_B^S \phi_{C'} + \tilde{\Psi}_{(A'B'C')AB}^R \phi_{C'}$

On (compactified) Minkowski space there are no others and the sequence (without the long arrow) is exact. On a curved space, except for the case $p=q=r=0$ (the deRham sequence), the composition of two consecutive operators involves (conformal) curvature. For example, composing

$$\begin{smallmatrix} 1 & 0 & 0 \\ -1 & -2 & -1 \end{smallmatrix} = \mathcal{O}_A\{1\} \xrightarrow{\nabla_{AB}^R} \mathcal{O}_{(AB)B'}\xrightarrow{\nabla_{B'}^R} \mathcal{O}_{(ABC)} = \begin{smallmatrix} 3 & -3 & 0 \\ -3 & 0 & 3 \end{smallmatrix}$$

gives the anti-self-dual Weyl curvature $\phi^R \mapsto \tilde{\Psi}_{ABCD}^R \phi^D$. Since this is merely a tensor it acts more generally: $\begin{smallmatrix} p & q & r \\ -p & -q & -r \end{smallmatrix} = \mathcal{O}_{(AB\ldots D)(E'F'\ldots H')}\{q\} \xrightarrow{\tilde{\Psi}_{A'EFG}} \mathcal{O}_{(BC\ldots G)(E'F'\ldots H')}\{q+3\} = \begin{smallmatrix} p+2 & q+3 & r \\ -p & -q & -r \end{smallmatrix}$

and a conformally invariant zeroth order operator is obtained even though it is not strictly speaking a composition of two in the list in [1]. It is remarkable that the same thing seems to happen for higher order operators too provided a judicious adjustment of constants is made.

Some heavy calculation is required to check this thoroughly. In this relaxed way an algebra of invariant operators may be generated by the "basic" ones (i.e. those in [1]).

Example: $\mathcal{O}_A\{1\} \rightarrow \mathcal{O}_{(AB)A'}\{1\} \rightarrow \mathcal{O}_{(B'C'D')}\{-1\}$
 $\phi_A \mapsto -\frac{1}{2} \tilde{\Psi}_{A'B'C'D'}^R \nabla^{AA'} \phi_A + (\nabla^{AA'} \tilde{\Psi}_{A'B'C'D'}) \phi_A$ is an

invariant operator obtained directly as a combination of basic ones but by altering constants it is clear that $\tilde{\Psi}_{A'B'C'D'}^R \nabla^{AA'} \phi_{AB\ldots D} - (w+1)(\nabla^{AA'} \tilde{\Psi}_{A'B'C'D'}) \phi_{AB\ldots D}$ is conformally invariant when acting on $\phi_{AB\ldots D}$ of weight w . In particular, $i[\tilde{\Psi}_{A'B'C'D'}^R \nabla^{AA'} \tilde{\Psi}_{ABCD}^R - \tilde{\Psi}_{ABCD}^R \nabla^{AA'} \tilde{\Psi}_{A'B'C'D'}]$ is a conformally invariant tensor (like but \neq duPlessis tensor). It's real on a real spacetime and gives a 3rd order scalar invariant by taking discriminant for primed/unprimed spinor indices.

A more exotic example: Changing constants of ($p=q=1, r=0$):

$$\mathcal{O}\left(\begin{smallmatrix} 4 & -4 & 0 \\ -4 & 0 & 0 \end{smallmatrix}\right) \rightarrow \mathcal{O}\left(\begin{smallmatrix} 2 & -6 & 0 \\ -2 & 0 & 0 \end{smallmatrix}\right) \rightarrow \mathcal{O}\left(\begin{smallmatrix} 0 & -6 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$$

gives: $\mathcal{O}\left(\begin{smallmatrix} 4 & -4 & 0 \\ -4 & 0 & 0 \end{smallmatrix}\right) = \mathcal{O}_{(ABCD)}$ $\rightarrow \mathcal{O}\{-6\} = \mathcal{O}\left(\begin{smallmatrix} 0 & -6 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$

$$\phi_{ABCD}^R \mapsto 2\tilde{\Psi}_{ABCD}^R \square \phi_{ABCD}^R + 12(\nabla^{AE} \tilde{\Psi}_{EBCD}^R)(\nabla^D \phi_{ABCD}) + (\nabla^{EE'} \tilde{\Psi}_{EBCD}^R)(\nabla_{EE'} \phi_{ABCD}) + (2\square \tilde{\Psi}_{ABCD}^R + 32\Lambda \tilde{\Psi}_{ABCD}^R)\phi_{ABCD},$$

a conformally invariant operator! In particular, putting $\phi_{ABCD} = \tilde{\Psi}_{ABCD}^R$ gives:

$$2\square \tilde{\Psi}^2 + 12(\nabla^{AE} \tilde{\Psi}_{EBCD}^R)(\nabla^D \tilde{\Psi}_{ABCD}^R) - 3(\nabla^{EE'} \tilde{\Psi}_{EBCD}^R)(\nabla_{EE'} \tilde{\Psi}_{ABCD}^R) + 32\Lambda \tilde{\Psi}^2$$

as a conformally invariant scalar (of weight -6). Save for adding a multiple of $\tilde{\Psi}^3$ this is the Fefferman-Graham 4th order invariant [2] (after 2-day calculation).

Question: Do all conformally invariant operators or tensors arise in this fashion?

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Thanks: to RP and Robert Bryant.

Michael Eastwood.

DEFORMATIONS OF AMBITWISTOR SPACE AND VANISHING BACH TENSORS

In [1], Lionel Mason considered, after the "plate tectonics" fashion, deformations of A preserving the contact structure. As an example of the "new notation" and the Bernstein-Gelfand-Gelfand sequences [2] I want to consider those infinitesimal deformations which have the property that the perturbation of the Bach tensor vanishes. Setting $\mathcal{U} \subset M$ to be "nice" (e.g. Stein), with $\mathcal{U}_A, \mathcal{U}_{PT}, \mathcal{U}_{PT^*}$ the corresponding spaces, the deformations of [1] are elements of $H^1(\mathcal{U}_A, \mathcal{O}(1,1))$; the Penrose transform of this is easily computed:

$$0 \rightarrow \overset{!}{\underset{\uparrow}{\bullet}} \xrightarrow{\quad} \overset{2}{\underset{\uparrow}{\bullet}} \xrightarrow{\quad} \overset{2}{\underset{\uparrow}{\bullet}} \rightarrow H^1(\mathcal{U}_A, \overset{!}{\underset{\uparrow}{\bullet}}) \rightarrow 0 \quad (1)$$

vectors trace free metrics contact preserving deformations

which looks like a potential modulo gauge description for H^1 (though the potentials satisfy no equations). (1) is part of a generalized de Rham (i.e. Bernstein-Gelfand-Gelfand) sequence on M :

$$0 \rightarrow \overset{!}{\underset{\uparrow}{\bullet}} \xrightarrow{\quad} \overset{1}{\underset{\uparrow}{\bullet}} \xrightarrow{\quad} \overset{2}{\underset{\uparrow}{\bullet}} \xrightarrow{\quad} \overset{4}{\underset{\uparrow}{\bullet}} \xrightarrow{\quad} \overset{2}{\underset{\uparrow}{\bullet}} \xrightarrow{\quad} \overset{1}{\underset{\uparrow}{\bullet}} \rightarrow 0 \quad (2)$$

trace free twistors
[Conformal Killing
Vectors.]

Ψ_{ab} h_{ab} $\Psi_{ABCD}^c \oplus \bar{\Psi}_{ABCD}^c$ B_{ab} $\nabla^a B_{ab}$

The composition $h_{ab} \rightarrow \overset{4}{\underset{\uparrow}{\bullet}} \rightarrow \overset{2}{\underset{\uparrow}{\bullet}}$ produces the perturbation of the Bach tensor. Suppose this vanishes; then (2) tells us immediately that there exists an h_{ab}^+ with $\Psi_{ABCD}^c = \nabla_A^\alpha \nabla_B^\beta h_{CD}^+$ and $0 = \nabla_A^\alpha \nabla_B^\beta h_{CD}^+$ and a similar h_{ab}^- (producing $\bar{\Psi}_{ABCD}^c$) with $h_{ab} = h_{ab}^+ + h_{ab}^-$; that is, the perturbation of A splits into a sum of self and antiselfdual parts iff the Bach tensor vanishes.

In fact, (2) tells us more, since it contains information about the Penrose transforms of $H^1(\mathcal{U}_{PT}, \overset{!}{\underset{\uparrow}{\bullet}})$ and $H^1(\mathcal{U}_{PT^*}, \overset{!}{\underset{\uparrow}{\bullet}})$. Here, $\overset{!}{\underset{\uparrow}{\bullet}}$ and $\overset{!}{\underset{\uparrow}{\bullet}}$ are the tangent bundles of PT and PT^* , so we are deforming H_{PT} and H_{PT^*} :

$$\begin{aligned} H^1(\mathcal{U}_{PT}, \overset{!}{\underset{\uparrow}{\bullet}}) &\cong \ker (\overset{2}{\underset{\uparrow}{\bullet}} \rightarrow \overset{4}{\underset{\uparrow}{\bullet}}) \text{ modulo } (\overset{1}{\underset{\uparrow}{\bullet}} \rightarrow \overset{2}{\underset{\uparrow}{\bullet}}) \ni h_{ab}^- \\ H^1(\mathcal{U}_{PT^*}, \overset{!}{\underset{\uparrow}{\bullet}}) &\cong \ker (\overset{2}{\underset{\uparrow}{\bullet}} \rightarrow \overset{0}{\underset{\uparrow}{\bullet}}) \text{ modulo } (\overset{1}{\underset{\uparrow}{\bullet}} \rightarrow \overset{2}{\underset{\uparrow}{\bullet}}) \ni h_{ab}^+ \end{aligned} \quad (3)$$

Thus:

A contact preserving infinitesimal deformation of A gives a perturbation of M with vanishing Bach tensor iff it is the restriction to $A \hookrightarrow PT \times PT^*$ of a pair of infinitesimal perturbations of PT and PT^*

This is surely the start of a gravitational W1YG construction [3], since the vanishing of the Bach tensor is half the condition for conformality to Einstein. Even more can be said if one takes the "twistor transforms" of (3), obtaining

$$H^1(\mathcal{U}_{PT}, \overset{-6}{\underset{\uparrow}{\bullet}}) \cong \ker \overset{4}{\underset{\uparrow}{\bullet}} \rightarrow \overset{2}{\underset{\uparrow}{\bullet}} \quad \text{and} \quad H^1(\mathcal{U}_{PT^*}, \overset{0}{\underset{\uparrow}{\bullet}}) \cong \ker \overset{4}{\underset{\uparrow}{\bullet}} \rightarrow \overset{2}{\underset{\uparrow}{\bullet}} \quad (4)$$

Since the existence of sections of $\overset{!}{\underset{\uparrow}{\bullet}}$ and $\overset{!}{\underset{\uparrow}{\bullet}}$ on $\widetilde{\mathcal{U}}_{PT}, \widetilde{\mathcal{U}}_{PT^*}$ corresponds to Einstein half flat spaces, it seems very likely that Einstein perturbations of A come from a section on A of $\overset{!}{\underset{\uparrow}{\bullet}}$ (the Februn Einstein Bundle, [4][5])

(which has a canonical section in the flat case) with $\mathcal{O}(-6,0)$ and $\mathcal{O}(0,-6)$ twistor functions (i.e. H^1 's) since the isomorphisms in (4) remain unchanged when \mathcal{U}_A replaces \mathcal{U}_{PT} , \mathcal{U}_{PT^*} and $\overset{!}{\underset{\uparrow}{\bullet}} + \overset{!}{\underset{\uparrow}{\bullet}}$ are used as sheaves. Question is, how much of this survives in the curved case?

Many thanks to MGE.

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L.J. Mason

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On the topology of families of manifolds and the 'pullback mechanism'

Let $f: Y \rightarrow Z$ be a differentiable map between paracompact differentiable manifolds; let f have maximal rank and let Z be connected. Then for each point z in Z , $f^{-1}(z)$ is a differentiable manifold of fixed dimension n say, and we can regard $(f^{-1}(z))_{z \in Z}$ as a smooth family of such manifolds. In this article, I discuss notions of homology and cohomology for a family of differentiable manifolds defined in the above way.

A natural notion of de Rham cohomology of the family, for example, is provided by the cohomology of the f -relative differential forms on Y in the following way. We have, on Y , the exact sequence of sheaves of f -relative differential forms:

$$\dots \rightarrow \mathcal{E}_f^{p-1} \xrightarrow{df} \mathcal{E}_f^p \rightarrow \mathcal{E}_f^{p+1} \rightarrow \dots$$

(see refs. 3 or 4 for definitions) which gives rise to the complex of \mathcal{E}_Z -modules:

$$\dots \rightarrow f_* \mathcal{E}_f^{p-1} \rightarrow f_* \mathcal{E}_f^p \rightarrow f_* \mathcal{E}_f^{p+1} \rightarrow \dots$$

where \mathcal{E}_Z denotes the sheaf of smooth functions on Z . Define the p -th de Rham cohomology sheaf of the family f as $\mathcal{H}^p(f) = H^p(f_* \mathcal{E}_f^p)$. Note that when Z is a one-point space, $\mathcal{H}^p(f)$ is identical to the p -th de Rham cohomology group of Y .

Next we shall define the singular homology sheaves of the family f . Let $\Delta^p \subset \mathbb{R}^p$ be the standard geometric simplex of dimension p . A differentiable f -relative singular p -simplex in Y is a map $s: U \times \Delta^p \rightarrow Y$ where U is an open subset of Z , $s(z, \Delta^p) \subset f^{-1}(z)$ for each point z in U and s is the restriction of a differentiable map defined on the product of U with some neighbourhood of Δ^p in \mathbb{R}^p . Now if V is an open subset of Z , we let $\mathcal{C}_p(f)(V)$ be the $\mathcal{E}_Z(V)$ -module of locally finite linear combinations $\sum a_k s_k$, where $s_k: U_k \times \Delta^p \rightarrow$

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$f^{-1}(V)$ is an f -relative p -simplex in V and U_k is an open subset of V , a_k being a smooth function on V which is identically zero on some neighbourhood in V of $V - U_k$ for each k . Taking into account the obvious restriction maps, we see that $\mathcal{C}_p(f)(V)$ is the group of sections over V of a sheaf to be denoted $\mathcal{C}_p(f)$ and called the sheaf of f -relative p -chains in Y . Now it is possible to define, just as in singular homology theory, a boundary operator b_f with the property that $b_f^2 = 0$. Thus we have a complex of sheaves of \mathcal{E}_Z -modules on Z :

$$\dots \rightarrow \mathcal{C}_{p+1}(f) \xrightarrow{b_f} \mathcal{C}_p(f) \xrightarrow{b_f} \mathcal{C}_{p-1}(f) \rightarrow \dots$$

Define the p -th singular homology sheaf of the family f as $\mathcal{H}_p(f) = h_p(\mathcal{C}_*(f))$. If Z is a one-point space then $\mathcal{H}_p(f)$ is identical to the p -th singular homology group of Y (with real coefficients).

To derive relations between $\mathcal{H}_p(f)$ and $\mathcal{H}^p(f)$ (which one hopes would be analogous to the duality between H^p and H_p for a single differentiable manifold) it is possible to adapt the beautiful arguments of Weil (ref. 7 or 2). Thus one shows that Y admits an f -simple cover $\mathcal{U} = (U_i)_{i \in I}$; that is, \mathcal{U} is a locally finite cover of Y by relatively compact open sets such that for each non-empty intersection $U_\sigma = \bigcap_{i \in \sigma} U_i$, there is a diffeomorphism $U_\sigma \rightarrow Z_\sigma \times V_\sigma$ where Z_σ is an open subset of Z , V_σ is a contractible space and the following diagram is commutative:

$$\begin{array}{ccc}
 U_\sigma & \longrightarrow & Z_\sigma \times V_\sigma \\
 f \searrow & & \swarrow \text{projection } (z, v) \mapsto z \\
 & Z_\sigma &
 \end{array}$$

Let N be the nerve of \mathcal{U} , i.e. the set of all (necessarily finite) subsets σ of I for which $U_\sigma \neq \emptyset$. The dimension of σ is p if σ contains $p+1$ elements of I . The homology over \mathbb{Z} of N is defined as follows. Let

$$\mathcal{C}_p(N) = \bigoplus \mathcal{E}_\sigma$$

where the sum is over the p -dimensional σ 's in N and \mathcal{E}_σ is the extension by zero of the sheaf $\mathcal{E}_z|_{Z_\sigma}$. [Thus a section of $\mathcal{E}_\sigma(V)$ is a smooth function on V which is identically zero on some neighbourhood in V of $V - Z_\sigma$.] As in simplicial homology theory, there is a map $\partial: \mathcal{C}_p(N) \rightarrow \mathcal{C}_{p-1}(N)$ such that $\partial^2 = 0$ which enables one to define the simplicial homology sheaves of the nerve of the cover $\mathcal{H}_p(N) = h_p(\mathcal{C}_*(N))$. One also defines

$$\mathcal{S}^p(N) = \text{Hom}_{\mathbb{Z}}(\mathcal{C}_p(N), \mathcal{E}_z) = \prod \text{Hom}_{\mathbb{Z}}(\mathcal{E}_\sigma, \mathcal{E}_z).$$

Note that $\mathcal{E}_\sigma^\vee = \text{Hom}(\mathcal{E}_\sigma, \mathcal{E})$ is not equal to \mathcal{E}_σ . Rather, it is $\iota_*(\mathcal{E}_z|_{Z_\sigma})$ where $\iota: Z_\sigma \rightarrow Z$ is the inclusion. [So a section u of $\mathcal{E}_\sigma^\vee(V)$ is a function on V such that $u|_{V - Z_\sigma}$ is zero and such that $u|_{Z_\sigma}$ is smooth.] There is an induced map $\delta: \mathcal{S}^p(N) \rightarrow \mathcal{S}^{p+1}(N)$ which enables one to define the simplicial cohomology sheaves of the nerve of the cover, $\mathcal{H}^p(N) = h^p(\mathcal{S}^*)$.

Now one can prove that

- (i) $\mathcal{H}^p(f)$ is canonically isomorphic to $\mathcal{H}^p(N)$;
- (ii) $\mathcal{H}_p(f)$ is canonically isomorphic to $\mathcal{H}_p(N)$.

(See refs. 7 or 2.) Next, if \mathcal{U}_z is the family of open subsets $U_i \cap f^{-1}(z)$ of $f^{-1}(z)$ (z being any point of Z) we write N_z for the nerve of the cover \mathcal{U}_z . Then we have that

- (iii) $\mathcal{H}_p(N)_z$ is canonically isomorphic to $H_p(N_z, \mathcal{E}_z)$ where \mathcal{E}_z denotes the ring of germs of smooth functions at z . Since \mathcal{U}_z is a simple cover of $f^{-1}(z)$, we have, from (ii) and (iii), our first main result.

Theorem A The stalk $\mathcal{H}_p(f)_z$ of the p -th singular homology sheaf of the family at the point z in Z is canonically isomorphic to the homology $H_p(f^{-1}(z), \mathcal{E}_z)$ of the fibre of f at z .

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Furthermore, if \mathcal{Z}_p and \mathcal{B}_{p-1} are sheaves on Z defined by the exactness of the sequence $0 \rightarrow \mathcal{Z}_p \rightarrow \mathcal{C}_p \xrightarrow{\delta} \mathcal{B}_{p-1} \rightarrow 0$, then (iv) $\mathcal{Z}_p(N)$, $\mathcal{C}_p(N)$ and $\mathcal{B}_{p-1}(N)$ are projective \mathcal{E}_z -modules.

This technical condition enables one to follow through the standard universal coefficient theorem argument (ref.6) to prove the following.

Theorem B The duality between $\mathcal{H}_p(f)$ and $\mathcal{H}^p(f)$ is expressed by the exactness of the following sequence of \mathcal{E}_z -modules:

$$0 \rightarrow \operatorname{Ext}_z^1(\mathcal{H}_{p-1}(f), \mathcal{E}_z) \rightarrow \mathcal{H}^p(f) \rightarrow \operatorname{Hom}_z(\mathcal{H}_p(f), \mathcal{E}_z) \rightarrow 0$$

With these results, we can construct a useful generalization of the pullback mechanism of twistor theory. For this, we suppose that f , Y and Z are as above and additionally f is a holomorphic map between the complex manifolds Y and Z . Let E be a holomorphic vector bundle on Z . We want to know the relationship between $H^p(Y, f^{-1}\mathcal{O}(E))$ and the analytic cohomology of E . As one might expect, the answer is in general a spectral sequence

$$E_1^{p,q} = H^q(Z, \mathcal{H}^q(f) \otimes \mathcal{E}^{0,p}(E)) \Rightarrow H^{p+q}(Y, f^{-1}\mathcal{O}(E))$$

where the differential $E_1^{p,q} \rightarrow E_1^{p+1,q}$ is induced from the $\bar{\partial}_E$ operator of the Dolbeault resolution of E , $0 \rightarrow \mathcal{O}(E) \rightarrow \mathcal{E}^{0,*}(E)$. Theorems A and B are useful in the calculation of the E_1 term of this spectral sequence in terms of the homology of the fibres of f .

Applications include the twistor transform (ref. 5), sourced massless fields (ref. 1) and (in principle) the twistor description of massless fields on peculiarly shaped regions of Minkowski space.

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Super Yang-Mills

by Alex Pilato

Define a d-operator on functions on super Minkowski space by

$$d\hat{f} := dx^a \frac{\partial f}{\partial x^a} + \theta^a_j \frac{\partial f}{\partial \theta^a_j} + \tilde{\theta}^{a'k} \frac{\partial f}{\partial \tilde{\theta}^{a'k}}$$

and consider the following "supersymmetry transformations"

$$x^a \mapsto x^a - \varepsilon_j^a \tilde{\theta}^{a'j} + \theta^a_k \tilde{\varepsilon}^{a'k}, \quad \theta^a_j \mapsto \theta^a_j + \varepsilon^a_j, \quad \tilde{\theta}^{a'k} \mapsto \tilde{\theta}^{a'k} + \tilde{\varepsilon}^{a'k}$$

where ε^a_j and $\tilde{\varepsilon}^{a'k}$ are constant anticommuting spinors satisfying $\theta^a_j \varepsilon^a_j = 0$ and $\tilde{\theta}^{a'k} \tilde{\varepsilon}^{a'k} = 0$ respectively.

Although d is invariant under these transformations, the differential operators $\partial/\partial x^a$, $\partial/\partial \theta^a_j$ and $\partial/\partial \tilde{\theta}^{a'k}$ are not. However, it is easy to rearrange terms and rewrite d as

$$d := (dx^a + \theta^a_j d\tilde{\theta}^{a'j} + \tilde{\theta}^{a'k} d\theta^a_k) \partial_a + d\theta^a_j \partial^j + d\tilde{\theta}^{a'k} \partial_{a'k}$$

where the differential operators

$$\partial_a = \partial/\partial x^a, \quad \partial^j = \partial/\partial \theta^a_j + \tilde{\theta}^{a'k} \partial/\partial x^a, \quad \partial_{a'k} = \partial/\partial \tilde{\theta}^{a'k} + \theta^a_k \partial/\partial x^a \quad -(1)$$

are now indeed invariant under the above transformations. They also satisfy the following (anti-)commutation relations which define a superalgebra

$$\begin{aligned} [\partial_a, \partial_b] &= [\partial_a, \partial_{B'}^j] = [\partial_a, \tilde{\partial}_{B'k}] = 0 \\ \{\partial_a^j, \partial_{B'}^k\} &= \{\partial_{B'j}, \tilde{\partial}_{B'k}\} = 0 \\ \{\partial_a^j, \tilde{\partial}_{B'k}\} &= 2\delta_{jk}^a \partial_a \end{aligned} \quad -(2)$$

where $[,]$ and $\{ , \}$ denote a commutator and anticommutator respectively.

Consider now a 1-form or 'potential'

$$\begin{aligned} \varphi &= (dx^a + \theta^a_j d\tilde{\theta}^{a'j} + \tilde{\theta}^{a'k} d\theta^a_k) \Phi_a + d\theta^a_j \Psi^j + d\tilde{\theta}^{a'k} \tilde{\Psi}^{a'k} \\ &:= d\tilde{x}^A \varphi_A \quad \text{where } \varphi_A := (\Phi_a, \Psi^j, \tilde{\Psi}^{a'k}) \end{aligned}$$

and define a connection $\nabla = d + \varphi$ and a curvature or 'field strength' $F = [\nabla, \nabla]$ where $[A, B] = AB - (-1)^{|A||B|} BA$. We can write the curvature more explicitly as $F = dx^A \wedge d\tilde{x}^B F_{AB}$ where

$$F_{AB} = \begin{bmatrix} f_{AB} & F_{AB}^k & F_{aB}^{a'k} \\ F_{A'b}^j & F_{A'B'}^k & F_{AB'}^k \\ F_{A'jB} & F_{A'jB'}^k & F_{A'jB'}^k \end{bmatrix}.$$

(Note that the definition of the wedge ' \wedge ' should be clear from that of ' $[,]$.)

In Minkowski space, a connection is automatically integrable on null lines. In super Minkowski space, integrability on the ($1/N$ -dimensional) super null lines (see [1] and [5]) imposes differential equations on the potentials φ_A . Such equations demand that F takes the form

$$F_{AB} = \begin{bmatrix} \varepsilon_{AB} f_{A'B'} + \varepsilon_{A'B'} f_{AB} & \varepsilon_{AB} \tilde{W}_{A'k} & \varepsilon_{A'B'} W_{Ak} \\ \varepsilon_{AB} \tilde{W}_{B'j} & \varepsilon_{AB} \tilde{W}^{jk} & 0 \\ \varepsilon_{A'B'} W_{B'j} & 0 & \varepsilon_{A'B'} W_{jk} \end{bmatrix} \quad -(3)$$

where $W_{jk} = -W_{kj}$, $\tilde{W}^{jk} = -\tilde{W}^{kj}$; $f_{AB} = f_{BA}$, $f_{A'B'} = \tilde{f}_{B'A'}$.

Note that F automatically splits $F_{AB} = F^{(+)}_{AB} + F^{(-)}_{AB}$, $-(4)$

where

$$F^{(+)}_{AB} = \varepsilon_{AB} \begin{bmatrix} \tilde{f}_{A'B'} & \tilde{W}_{A'j}^k & 0 \\ \tilde{W}_{B'j}^k & \tilde{W}_{jk} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \& \quad F^{(-)}_{AB} = \varepsilon_{A'B'} \begin{bmatrix} f_{AB} & 0 & W_{Ak} \\ 0 & 0 & 0 \\ W_{Bj} & 0 & W_{jk} \end{bmatrix},$$

so that we can define a *-operator by

$$*F = F^{(+)} - F^{(-)}. \quad -(5)$$

Not all the components of F are independent. Equations between them are imposed by the super Bianchi identities [4]. We denote them by $[\nabla, F] = 0$,

because they are just the super Jacobi identities for ∇ .

We now define the super Yang-Mills equations by $[\nabla, *F] = 0$ where F is of the form (3) and satisfies the super Bianchi identities. By virtue of (4), this is equivalent to either $[\nabla, F^{(+)}) = 0$ or $[\nabla, F^{(-)}) = 0$.

In this latter form, for $N=1$ it can be shown (rewriting the Bianchi identities of Sønnes with F^c replacing F) that our definition agrees with that in [3], $[\nabla_A, W^A] = 0$, arrived at by physical considerations.

We can also define A.S.D.Yang-Mills by $F^c = 0$. For $N=1$ this is equivalent to $W_A = 0$.

Theorem:

(Take $N=1$ throughout.)

(i) The general solution of Witten's (integrability on super null lines) equations

$$\begin{aligned} \partial_C \Psi_B + \frac{1}{2} \{ \Psi_A, \Psi_B \} &= 0 \\ \partial_{C'} \tilde{\Psi}_B + \frac{1}{2} \{ \tilde{\Psi}_A, \tilde{\Psi}_B \} &= 0 \\ \partial_A \tilde{\Psi}_B + \partial_{A'} \Psi_B + \{ \Psi_A, \tilde{\Psi}_B \} &= 2 \Phi_A \end{aligned}$$

with the gauge condition

$$\Theta^A \Psi_A + \tilde{\Theta}^{A'} \tilde{\Psi}_{A'} = 0,$$

is given by

$$\begin{aligned} \Psi_A &= \tilde{\Theta}^B \tilde{\Phi}_{AB} + \frac{\Theta^B \tilde{\Theta}^B}{3} \tilde{\epsilon}_{AB} \tilde{W}_B - \frac{\tilde{\Theta}^2}{4} \tilde{W}_A - \frac{\tilde{\Theta}^2 \Theta^B}{4} (\tilde{f}_{AB} + \epsilon_{AB} \tilde{g}) + \frac{\tilde{\Theta}^2 \Theta^2}{12} \nabla_{AC} \tilde{W}^C \\ \tilde{\Psi}_{A'} &= \Theta^B \tilde{\Phi}_{A'B} - \frac{\Theta^B \tilde{\Theta}^B}{3} \tilde{\epsilon}_{A'B} \tilde{W}_A - \frac{\Theta^2}{4} \tilde{W}_{A'} - \frac{\Theta^2 \Theta^B}{4} (\tilde{f}_{A'B} + \epsilon_{A'B} \tilde{g}) + \frac{\Theta^2 \Theta^2}{12} \nabla_{A'C} \tilde{W}^C \\ \Phi_{AA'} &= \tilde{\Phi}_{AA'} - \frac{\tilde{\Theta}^B}{3} \epsilon_{AB} \tilde{W}_A - \frac{\Theta^B}{3} \epsilon_{AB} \tilde{W}_{A'} + \frac{\Theta^B \tilde{\Theta}^B}{72} [\tilde{f}_{AB} \epsilon_{A'B} - \tilde{f}_{A'B} \epsilon_{AB} + \epsilon_{A'B} \epsilon_{AB}] \\ &\quad + \Theta^2 \tilde{\Theta}^B \left[\frac{5}{24} \epsilon_{AB} (\partial_{AC} \tilde{W}^C + [\tilde{\Phi}_{AC}, \tilde{W}^C]) - \frac{1}{12} \partial_A (\tilde{A}' \tilde{W}_B) \right] \\ &\quad + \tilde{\Theta}^2 \Theta^B \left[\frac{5}{24} \epsilon_{AB} (\partial_{A'C} \tilde{W}^C + [\tilde{\Phi}_{A'C}, \tilde{W}^C]) - \frac{1}{12} \partial_{A'} (\tilde{A} \tilde{W}_B) \right] \\ &\quad + \frac{\Theta^2 \tilde{\Theta}^2}{16} \left[(\partial_{A'} \tilde{f}_{AB} + [\tilde{\Phi}_{A'}, \tilde{f}_{AB}]) + (\partial_A \tilde{f}_{A'B} + [\tilde{\Phi}_A, \tilde{f}_{A'B}]) \right. \\ &\quad \left. + \frac{2}{3} \{ \tilde{W}_A, \tilde{W}_{A'} \} + 2 [\tilde{\Phi}_{AA'}, \tilde{g}] \right] \end{aligned}$$

where $\tilde{f}_{A'B'} = 2 \partial_C (\tilde{\Phi}_{AB}) + [\tilde{\Phi}_C(A', \tilde{\Phi}_B)]$ & $f_{AB} = 2 \partial_C (\Phi_{AB}) + [\tilde{\Phi}_C(A, \tilde{\Phi}_B)]$.

(ii) $W_A = \partial_{AA'} \Psi^{A'} + \tilde{\partial}_{A'} \tilde{\Phi}_A + [\tilde{\Phi}_{A'A}, \tilde{\Psi}^{A'}]$

is given by

$$\begin{aligned} W_A &= \tilde{W}_A + \Theta^B (2 \tilde{f}_{AB} + \epsilon_{AB} \tilde{g}) - \frac{\Theta^B \tilde{\Theta}^B}{2} [\epsilon_{AB} (\partial_{B'C} \tilde{W}^C + [\tilde{\Phi}_{B'C}, \tilde{W}^C]) - \nabla_{B'C} \tilde{W}_B] \\ &\quad - \Theta^2 (\partial_{AA'} \tilde{W}^{A'} + [\tilde{\Phi}_{AA'}, \tilde{W}^{A'}]) \\ &\quad + \Theta^2 \tilde{\Theta}^B \left[\frac{3}{8} \partial_{B'} \tilde{f}_{AB} + \frac{3}{8} [\tilde{\Phi}_{B'}, \tilde{f}_{AB}] + \frac{5}{8} \partial_A \tilde{f}_{A'B} + \frac{5}{8} [\tilde{\Phi}_A, \tilde{f}_{A'B}] \right. \\ &\quad \left. - \frac{1}{2} (\partial_{AB'} \tilde{g} + [\tilde{\Phi}_{AB'}, \tilde{g}]) + \frac{1}{3} \{ \tilde{W}_A, \tilde{W}_{A'} \} \right] \end{aligned}$$

so that the A.S.D.equations. are

$$\tilde{f}_{AB} = 0 \quad \& \quad \partial_{AA'} \tilde{W}^{A'} + [\tilde{\Phi}_{AA'}, \tilde{W}^{A'}] = 0.$$

(iii) The full super Yang-Mills equations are

$$\tilde{g} = 0$$

$$\partial_{AA'} \tilde{W}^{A'} + [\tilde{\Phi}_{AA'}, \tilde{W}^{A'}] = 0$$

$$\partial_{AA'} \tilde{W}^{A'} + [\tilde{\Phi}_{AA'}, \tilde{W}^{A'}] = 0$$

$$\partial \tilde{*} \tilde{F}_{ab} + [\tilde{\Phi}^b, \tilde{*} \tilde{F}_{ab}] = \{ \tilde{W}_a, \tilde{W}_b \}$$

where $\tilde{*} \tilde{F}_{ab} := \epsilon_{ab} \tilde{f}_{ab} + \epsilon_{a'b'} \tilde{f}_{a'b'}$.

(Proof omitted for obvious reasons.)

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For $N > 1$, it is still possible to obtain, without much effort, classical field equations implied by the super Yang-Mills (super) field equations. To obtain an equivalence the algebra gets out of hand and some clever inductive method is required. Using recursive relations generated by a kind of Euler homogeneity operator, H.H.L&S [2] showed that integrability on super null lines for $N=3$ is equivalent to a system of equations which they claim are the super Yang-Mills equations. Adopting our definition of super Yang-Mills the claim is indeed true in the abelian case. But in the general non-abelian case $[\nabla, F^\leftrightarrow] = 0$ implies the vanishing of certain (anti-)commutators of the equations in [2], i.e. it implies stronger equations.

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Conformally Invariant Differential Operators on Minkowski Space

By Michael G. Eastwood and John W. Rice

Abstract: We determine all conformally invariant differential operators on \mathbb{M} (= complexified compactified Minkowski space) where this is taken to mean homogeneous for the action of $GL(\mathbb{P})$. All operators in this list admit curved analogues e.g. $\square + R/6$ or the operator $\square^2 + \dots : \Omega^2 \rightarrow \Omega^4$ described in the following article:-

A Conformally Invariant Maxwell Gauge Phys Lett 107A By Michael Eastwood and Singer (1985) 73-74

Abstract: We formulate a conformally invariant gauge fixing for Maxwell fields: $\nabla_b [\nabla^b \nabla^a - 2R^{ab} + \frac{2}{3}Rg^{ab}] \Phi_a = 0$. If the background spacetime is vacuum then this gauge is weaker than Lorenz gauge.

On a different approach to supermanifolds

by Alex Pilato

§0 Introduction

Kostant's supermanifolds [8] consist of an underlying manifold together with an enlarged sheaf of functions. Locally it is required to have the form $\Lambda^* \mathbb{R}^N$ so that we have the following terminating series for a superfunction

$$f(x, \vartheta^j) = f_{\sigma_1 \dots \sigma_N} \vartheta^{j_1} \dots \vartheta^{j_N} \quad (\sigma_j = 0 \text{ or } 1)$$

where ϑ^j , $j=1, \dots, N$ are anticommuting labels. Since the coefficients in the above series are classical functions there is nothing mysterious here. The only problem with this approach is that it does not seem to do what physicists want.

An alternative approach to supermanifolds was started by Rogers [12]. Here one considers functions on Super Euclidean space $E_L^{m,n} = (\Lambda_0)^m \times (\Lambda_1)^n$ (of dimension $(m+n+1)$) where Λ_0 and Λ_1 are respectively the even and odd parts of $\Lambda := \Lambda^* \mathbb{R}^L$. Further one requires that these functions take values on a \mathbb{Z}_2 -graded Grassmann algebra $\Lambda^\infty = \bigwedge I_K$ where $I_K = \Lambda^k + \dots + \Lambda^L$ is an ideal in $\Lambda = \Lambda^0 + \Lambda^1 + \dots + \Lambda^L$; and the grading of Λ^∞ is given by setting $\Lambda_0^{(k)} = \sum_{i_1 < i_2 < \dots < i_k} \Lambda^{i_1 i_2 \dots i_k}$ and $\Lambda_1^{(k)} = \sum_{i_1 < i_2 < \dots < i_k} \Lambda^{i_1 i_2 \dots i_k}$. There is then an algebraic isomorphism [2] between functions on $U \subset E_L^{m,n}$ taking values in Λ^∞ and $\sum_{p=0}^{m+n} \Lambda^p(V) \otimes C^\infty(\varepsilon(U), \Lambda^{k-p})$

where V is an n -dimensional vector space and ε is the augmentation map $\varepsilon: E_L^{m,n} \rightarrow \mathbb{R}^m$.

There are many disadvantages to Rogers' approach. The arbitrariness in the choice of the number L (usually taken to be very large or even infinite) is one of them. Another disadvantage is that the coefficients in the expansion of a superfunction are not classical functions since they take values in either the even or the odd part of the Grassmann algebra. The argument in favour of this approach would be that it is supposed to provide a framework for "doing supersymmetry".

We will suggest a third approach motivated by certain observations concerning supertwistor and superambitwistor functions and by the work of A. Kock on 'Synthetic Differential Geometry' [7].

Kock considers the real line segment, R , with its commutative ring structure relative to a fixed choice of end points, and observes that this basic structure does not depend on having the real numbers, \mathbb{R} , as a mathematical model for R . He then considers $R[\varepsilon]/\varepsilon^2 = \{R \times R$ with the 'ring of dual numbers multiplication $(\phi, \psi) \cdot (\phi_2, \psi_2) = (\phi\psi, \phi\psi_2 + \phi_2\psi)$ ' and notes that this ring structure is that of functions with unique power series expansion on a set $D = \{\varepsilon \in R \text{ s.t. } \varepsilon^2 = 0\} \subset R$. Starting from these axioms he develops a 'synthetic theory of differential geometry'.

We define a third type of superfunction so that locally it looks like $R[\varepsilon_1, \dots, \varepsilon_N]/\varepsilon_1^2, \dots, \varepsilon_N^2$. For $N=1$, the ring structure of the above functions agrees with that of Kostant and Rogers (on $E_L^{m,1}$) superfunctions. However it is easy to check that in general the ring structures will not agree. The resulting supermanifold may be thought of as a sort of generalization of a thickened manifold [3].

§1 Supertwistors

Clearly in the new approach we can no longer have anticommuting coordinates. However we can still have nilpotent ones—i.e. quantities with the property that their squares vanish—in the superspaces. We claim that only the latter are needed and appear naturally in the various superspaces of "physical" interest.

Contact with the superfields in physics is established by writing the nilpotent coordinates as a product of two anticommuting 'quantities', say $\epsilon_j = \alpha_j \theta^A$, and grouping the α -labels together in the coefficients in an appropriate order. These anticommuting quantities must be regarded as markers or formal devices useful for doing calculations of interest in physics.

Denote, using abstract indices [9] arbitrary elements of super Minkowski space M_{EN} , supertwistor space T_{EN} and dual supertwistor space T_{EN}^* respectively by $(x^\alpha, \Theta^j, \bar{\Theta}^{j'})$, (z^α, f^j) and (w_α, ψ_k) . $\Theta^j, \bar{\Theta}^{j'}$ and f^j, ψ_k must not be thought of as anticommuting coordinates but as the fermion part of a supervector, supertwistor and dual supertwistor respectively with the property, indicated by a hidden label, that they anticommute. If one now wants to choose a frame and represent the above elements in coordinates with respect to this frame we need a different notation for the indices. For example, in the classical twistor case it is customary to choose a frame $A_\alpha, B_\alpha, C_\alpha, D_\alpha$ for T^* and to define coordinates on T

$$(Z^a = A_\alpha Z^\alpha, Z^b = B_\alpha Z^\alpha, Z^c = C_\alpha Z^\alpha, Z^d = D_\alpha Z^\alpha) \equiv Z^a$$

Analogously, by choosing a frame with N fermion anticommuting elements $A_\alpha, B_\alpha, C_\alpha, D_\alpha; \alpha_j, \dots, h_j$ for T_{EN} one can define coordinates on T_{EN} by

$(Z^a = A_\alpha Z^\alpha, Z^b = B_\alpha Z^\alpha, Z^c = C_\alpha Z^\alpha, Z^d = D_\alpha Z^\alpha; f^i = a_j f^j, \dots, h_j f^j) \equiv (Z^a, f^i)$. Note that all the actual coordinates above are "perfectly commuting quantities". However, the last N coordinates have the property that their squares vanish. One can carry out a similar formal construction for super Minkowski space and denote the coordinates by the $4/4N$ -tuple $(x^\alpha, \Theta^j, \bar{\Theta}^{j'}, \psi_k)$.

We will write the new type of superfunctions on T_{EN} and T_{EN}^* as

$$f(Z^a, f^i) = f_{\sigma_1 \dots \sigma_N}(Z^\alpha)(f^i)^{\sigma_1} \dots (f^N)^{\sigma_N} \quad (\sigma_j = 0 \text{ or } 1) \quad (1)$$

$$g(w_\alpha, \psi_k) = g_{\mu_1 \dots \mu_N}(w_\alpha)(\psi_1)^{\mu_1} \dots (\psi_N)^{\mu_N} \quad (\mu_k = 0 \text{ or } 1) \quad (2)$$

The supertwistor contour integral formulae of Ferber [5] remain essentially unchanged by the new definition. We will briefly recall them below.

The incidence relations are

$$\left\{ \begin{array}{l} \omega^a = (x^{AA'} - \Theta^a_j \bar{\Theta}^{j'}) \Pi_{A'} \\ f^j = \bar{\Theta}^{A'j} \Pi_{A'} \end{array} \right\} \quad \& \quad \left\{ \begin{array}{l} \xi^{A'} = (-x^{AA'} - \Theta^A_j \bar{\Theta}^{j'}) \gamma_A \\ \psi_k = \Theta^A_k \gamma_A \end{array} \right\}$$

Supertwistors and dual supertwistors satisfying the above incidence relations for fixed $x, \Theta, \bar{\Theta}$ form a surface denoted by $L_{\alpha, \Theta, \bar{\Theta}}$ and $L_{\alpha, \Theta, \bar{\Theta}}^*$ respectively

Let $f(Z^a, f^i)$ be 'homogeneous of degree $-n-2$ '. By this we mean 'take $f_{\sigma_1 \dots \sigma_N}(Z^\alpha)$ in (1) to be homogeneous of degree $-n-2-|\sigma|$ where $|\sigma| = \sum \sigma_j$ '. Then the following contour integral formula over any closed curve T on $L_{\alpha, \Theta, \bar{\Theta}}$,

$$\oint_T \Pi_{A'_1} \dots \Pi_{A'_n} f(\omega^a, \Pi_{A'}, f^i) \Big|_{L_{\alpha, \Theta, \bar{\Theta}}} \Pi_{E'} d\Pi^{E'}, \quad (3)$$

gives rise to an antichiral superfield $\Psi_{A'_1 \dots A'_n} = \Psi_{A'_1 \dots A'_n} (x^{AA'} - \Theta^A_j \bar{\Theta}^{j'})$

Similarly, take $g(w_\alpha, \psi_k)$ to be homogeneous of degree $-n-2$ to mean that $g_{\mu_1 \dots \mu_N}(w_\alpha)$ is homogeneous of degree $-n-2-|\mu|$ where $|\mu| = \sum \mu_k$. One then gets a chiral superfield

$$\Phi_{A'_1 \dots A'_n} (-x^{AA'} - \Theta^A_j \bar{\Theta}^{j'}, \Theta^B_k \psi_k) = \oint_{T^*} g(\xi^{A'}, \gamma^a, \psi_k) \Big|_{L_{\alpha, \Theta, \bar{\Theta}}^*} \gamma_E d\gamma^E \quad (4)$$

Substituting (1) and (2) in the integrals (3) and (4) one obtains the expansion of an antichiral and chiral superfield respectively in terms of the nilpotent coordinates Θ^A_j and Θ^B_k . For example, ignoring the external indices we have for $N=1$

$$\Psi(x^{AA'}, \bar{\Theta}^{A'}) = \phi(x^a) + \psi_B(x^a) \bar{\Theta}^B + \chi(x^a) \bar{\Theta}^2$$

where $x^{AA'} = x^{AA'} - \Theta^A_j \bar{\Theta}^{j'}$.

If one now wishes to return to the anticommuting physics notation, set $\tilde{\theta}^A = \theta_A^\alpha \tilde{\theta}^\alpha$ and $\tilde{\theta}^2 = \theta_A^\alpha \tilde{\theta}^\alpha \theta_B^\beta \tilde{\theta}^\beta = -\theta_A^\alpha \theta_B^\beta \tilde{\theta}^\beta \tilde{\theta}^\alpha$ so that

$$\begin{aligned}\Psi &= \phi + \psi_A' \tilde{\theta}_A^\alpha \tilde{\theta}^\alpha - \chi_{A'B'} \tilde{\theta}_A^\alpha \tilde{\theta}_B^\beta \tilde{\theta}^\beta \\ &= \phi + \psi_A' \tilde{\theta}^\alpha - \chi_{A'B'} \tilde{\theta}^\alpha \tilde{\theta}^\beta\end{aligned}$$

where $\psi_A' = \psi_A' \tilde{\theta}_A^\alpha$ and $\chi_{A'B'} = \chi_{A'B'} \tilde{\theta}_A^\alpha \tilde{\theta}_B^\beta$

Thus we see that the new definition of a superfunction fixes a unique way of writing a superfield using anticommuting labels which agrees with that used in physics. Rogers' superfunctions do not fix the order of the Grassmann algebra generators in the coefficients, and therefore there is an arbitrariness in the sign of each term.

One can also rewrite the superambitwistor correspondence in terms of nilpotent (homogeneous) coordinates;

$$[z^*, f^i, w_\alpha, \psi_b] \text{ s.t. } z^* w_\alpha = 2 f^j \psi_j \text{ & } (f^i)^2 = 0, (\psi_b)^2 = 0.$$

Let F be a 'function' on the flag superspace with coordinates $(x^a, \theta_1^\alpha, \tilde{\theta}_1^\beta, \gamma_b, \pi_A)$. It can be shown that $F = f(x^a - \theta_1^\alpha \tilde{\theta}_1^\beta, \tilde{\theta}_1^\beta \pi_A, -x^a \theta_1^\alpha \tilde{\theta}_1^\beta \gamma_b, \theta_1^\alpha \gamma_b)$ iff

$$\gamma_A^\alpha \pi_A^\beta \partial_{AB} F = 0, \quad \gamma_A^\alpha \partial_A^\beta F = 0, \quad \pi_A^\beta \partial_A^\alpha F = 0$$

where $\partial_{AB} = \partial/\partial x^{A+B}$, $\partial_A^\beta = \partial/\partial \theta_A^\beta + \theta^{B\beta} \partial/\partial x^{A+B}$, $\partial_A^\beta = \partial/\partial \tilde{\theta}_A^\beta - \theta_A^\beta \partial/\partial x^{A+B}$.

The operators ∂_{AB} , ∂_A^β & ∂_A^β satisfy the following commutation relations

$$[\partial_{AB}, \partial_A^\beta] = 0 = [\partial_{AB}, \partial_B^\beta], \quad [\partial_A^\beta, \partial_B^\beta] = [\partial_A^\beta, \partial_A^\beta] = 0 \text{ & } [\partial_A^\beta, \partial_B^\beta] = -2 \delta_{ij}^\beta \partial_{AB}.$$

(c.f. (1) and (2) in [10])

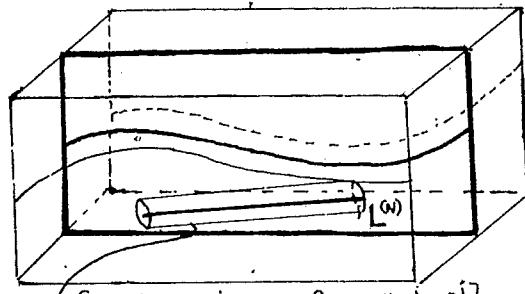
§2 Elementary States and the Scalar Product

$P:$



$$\{A_i Z^i = 0\} \cap \{B_i Z^i = 0\}$$

$P_{DN}:$



$$\{A_i Z^i - a_j f^j = 0\} \cap \{B_i Z^i - b_j f^j = 0\}$$

Elementary states based on the thickened line $L^{(n)}$ are a typical example of this new type of superfunction. In fact in either Kostant or Rogers' superfunction formalism it is not possible to give a suitable definition of super elementary states consistent with certain properties.

Analogously to the classical case [4] a choice of $U(2) \times U(2/N) \subset U(2, 2/N)$ stabilizes $L^{(n)}$. It is convenient to choose coordinates so that $L^{(n)} = \{Z^0 - f^1 = 0, Z^1 - f^2 = 0\}$ and to expand each elementary state, for example

$$\frac{1}{(Z^0 - f^1)(Z^1 - f^2)} = \frac{1}{Z^0 Z^1} + \frac{f^1}{(Z^0)^2 (Z^1)} + \frac{f^2}{(Z^0)(Z^1)^2} + \frac{f^1 f^2}{(Z^0)^2 (Z^1)^2}.$$

Then it is easy to check, by discarding coboundary terms, that the action of

$$\left[\begin{array}{c|c} * & * \\ \hline 0 & * \\ \hline 0 & * \end{array} \right]^N$$

is preserved.

In the supersymmetry literature (see for example [11]) it is often stated that Berezin Integration [1]

$$\int a d\mathcal{F} = 0, \quad \int f d\mathcal{F} = b \quad (a, b = \text{non-zero 'constants'})$$

can be derived by imposing linearity and invariance under the following so-called 'supersymmetry transformation' $\theta \mapsto \theta + \varepsilon$, where θ and ε are said to be 'anticommuting' but the relation between them is normally left unexplained.

In fact whether Θ is anticommuting or not is irrelevant to the derivation. What is essential is that $\Theta^2 = 0$.

Let $\varepsilon, \theta \in D = \{x \in R \text{ s.t. } x^2 = 0\} \subset R$ where R is some commutative ring (e.g. $\mathbb{R} \cup \{\text{nilpotent quantities}\}$). Then $\theta + \varepsilon \in D$ iff $\theta \cdot \varepsilon = 0$ [7].

We shall retain Berezin's notation and impose linearity to get

$$\int f(\theta) d\theta = \int (f_0 + \theta f_1) d\theta = I_0 + I_1. \quad (5)$$

If one then equates the r.h.s. of (5) with

$$\int f(\theta + \varepsilon) d(\theta + \varepsilon) = \int (f_0 + \theta f_1 + \varepsilon f_2) d\theta = I_0 + I_1 + \varepsilon I_0.$$

one sees that I_0 is forced to vanish. Defining $I_i = f_i$, 'Berezin Integration' then agrees with Kock's definition of derivatives in synthetic differential geometry.

We are now able to write a neat formula for the scalar product of superfields.

$$\langle g | f \rangle := \int \Delta Z W f(z^\alpha, \psi_i) g(w_\mu, \psi_i) (z^\alpha w_\mu - \delta^\alpha_\mu \psi_i)_{n-N} d\delta^N d\psi_N \dots d\delta^1 d\psi_1 \quad (6)$$

$$= \sum_{|\sigma|=0}^N (-1)^{|\sigma|} \int \Delta Z W \delta_{\sigma_1 \dots \sigma_N}(z^\alpha) g_{\mu_1 \dots \mu_N}(w_\mu) \delta^\mu_\sigma (z^\alpha w_\mu - x)_{n+1-\sigma} \quad (6')$$

where $x = \delta^\alpha_\mu \psi_i$ (so that $x^{N+1} = 0$), $\delta^\mu_\sigma = \delta^{\mu_1}_{\sigma_1} \dots \delta^{\mu_N}_{\sigma_N}$, $\Delta Z W = d^4 Z \wedge d^4 W$ and $(\)_m$ is the well known propagator (see, for example, [6] for definition).

The proof is simply by substitution of (1) and (2) in (6) above. The crucial point is that once Berezin Integration (i.e. synthetic differentiation) is performed, a large number of terms vanish after the Z and W integrations essentially because they are of the wrong homogeneity. This can be seen by expanding $(z^\alpha w_\mu - x)_m$ as a terminating power series in x and thus transforming all integrals into projective ones.

If we write f^1, \dots, f^N and ψ_1, \dots, ψ_N as products of anticommuting quantities, e.g. $f^1 = \alpha_{ij} f^i$, $\psi_k = \beta_{jk} \psi_j$, it can be shown, taking great care with signs, that the above formula holds.

Note that (6') is the sum of $N+1$ ordinary twistor scalar products with inhomogeneous propagators. Clearly the extension of complicated twistor diagrams to super elementary states leads naturally to the use of inhomogeneous twistor integrals. The properties of our nilpotent coordinates are not incompatible with those of the quantities appearing in recent work by Andrew Hodges [6] on twistor diagrams. The use of inhomogeneous twistor integrals allows him to eliminate the infrared divergence of the Möller scattering problem. This is reminiscent of the fact that super Feynman diagrams are 'more finite' than the conventional ones.

Many thanks to Mike Eastwood who suggested looking at Kock's papers.
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Quaternionic Geometry & The Future Tube

It is hardly traditional in twistor theory to think of the future tube $C\mathbb{M}^+$ as a Riemannian manifold. But, as I shall point out here, it has an absolutely natural quaternionic Kähler metric associated with the conformal geometry of real Minkowski space. Moreover, this phenomenon is not an isolated accident; any real-analytic Lorentzian 4-manifold is the special boundary of a quaternionic Kähler 8-manifold.

First, let's recall that the future tube is the set of complex linear 2-planes in T such that the restriction of a hermitian form of signature +--- is positive definite, while real compactified Minkowski space consists of those 2-planes which are totally null. This arrangement is totally natural in that every local conformal transformation of real Minkowski space corresponds to an element of $SU(2,2)$. But this makes $C\mathbb{M}^+$ into the Riemannian symmetric space $SU(2,2)/S(U(2)\times U(2))$; the Riemannian metric is precisely the one corresponding to the norm

$$\| \cdot \|^2 : \text{Hom}(P, P^\perp) \rightarrow \mathbb{R}$$

defined by

$$\| A \|^2 := - \text{tr } A^* A,$$

where the 2-plane $P \subset T$ is positive definite with respect to the twistor inner product, and we remember that the tangent space of $G_2(T)$ at P is precisely $\text{Hom}(P, T/P)$. Thus, any local conformal transformation of M extends to $C\mathbb{M}^+$ as a Riemannian isometry.

Now $U(2,2)/U(2) \times U(2)$ isn't just any old symmetric space; it is a quaternionic Kähler manifold [1], [2]. This means that the holonomy group is a subgroup of $sp(k) \times sp(1)/\mathbb{Z}_2$; i.e parallel transport around any loop induces a linear transformation that amounts to left multiplication by a quaternionic matrix followed by right multiplication by a quaternionic scalar--which is somewhat weaker than being quaternionic linear precisely because the quaternions aren't commutative. Manifolds satisfying this condition are always Einstein; the scalar curvature is -1.

(CM^+ also happens to be a Kähler manifold--its holonomy is contained in $SU(4)$. This will not concern us here; suffice it to say that this property is not generally a consequence of being quaternionic Kähler.)

One of the most compelling properties of quaternionic Kähler manifolds is the fact that they have twistor spaces [1]; these arise as sphere bundles over them and generalize the Hopf map $CP_{2k+1} \rightarrow HP_k$, the quaternionic projective spaces being the quintessential examples of quaternionic Kähler manifolds when equipped with metrics analogous to the Fubini-study metric on CP_m .

The punch-line is that the twistor space of CM^+ is ambitwistor space! More precisely, it is

$$A^+ = \{([z], [w]) \in PT \times PT^* \mid z \cdot w = 0, z \cdot \bar{z} > 0, \bar{w} \cdot w > 0\},$$

which fibres non-holomorphically over CM^+ by $([z], [w]) \mapsto \text{span}\{z, \bar{w}\}$.

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("Complex conjugation" here is, as usual, via the +---hermitian inner product on T_x .) Geometrically, A^+ is the set of null lines avoiding the closure of $CM^+ \cup CM^-$.

Now the above picture has an immediate generalization. Suppose that M is any real-analytic Lorentzian 4-manifold with civilized complexification \hat{M} ; let \hat{N} be the ambitwistor space of \hat{M} . Complex conjugation $\hat{M} \rightarrow \hat{M}$ induces an anti-holomorphic involution $\sigma: \hat{N} \rightarrow \hat{N}$ whose fixed-point set is the real 5-fold N of Lorentzian null geodesics in M . For each $x \in M$, the sky $O_x \hat{N}$ is a copy of $P_1 \times P_1$, and contains a 3-parameter family of P_1 's near the diagonal. As x varies, this sweeps out a 7-complex-parameter family of curves. But this family is not complete; it consists of P_1 's with normal bundle $2\mathcal{O}(1) + \mathcal{O} + \mathcal{O}(2)$, which has $H^1 = 0$, $\dim H^0 = 8$. Hence the complete analytic family generated by these curves has dimension 8. I claim that the generic element of this complete family is transverse to the contact form of \hat{N} and has normal bundle $4\mathcal{O}(1)$; for in the flat case the family consists of sections of $\mathbf{A}\mathbb{C}P_3 \times P_3^*$ by quadrics $P_1 \times P_1 \subset \mathbb{C}P_3 \times P_3^*$ in general position, from which the assertion follows by stability arguments. Let \hat{X} be the family of these generic curves in N , and let $x \in \hat{X}$ be the real 8-manifold of curves fixed by σ . The subset of curves avoiding the fixed-point set N will consist of two connected components x^+ and x^- , which are the analogues of CM^+ and CM^- . A converse of Salamon's twistor construction then builds quaternionic Kähler metrics on x^\pm by utilizing the contact form of \hat{N} .

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What is not completely clear is the role to be played by x^+ in the study of curved space-time. One application might be an approximate notion of positive frequency; for although x^+ is not generally a complex manifold, there is (via Penrose transforms $M \leadsto N \leadsto x^+$) nonetheless a well-defined notion of analytic continuation to x^+ . There may also be a place for \hat{X} in the theory of formal neighbourhoods of N , since in the flat case \hat{X} is constructed from $P_3 X P_3^*$.

The present construction is not, it would seem, a creature of only one dimension and signature, but rather has kindred in every swamp and hidey-hole. Details will appear elsewhere.

Claude LeBrun

Acknowledgment: It is a pleasure to thank Simon Salamon and Nigel Hitchin for stimulating discussions and raw ingredients.

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On the Density of Twistor Elementary States
By Mike Eastwood and Alex Pilato

Abstract: $U(p,q)$ may be represented on $H^{p-1}(\bar{P}^+, \mathcal{O}(k))$ for \bar{P}^+ an appropriate open subset of $\mathbb{C}\mathbb{P}_{p+q-1}$. The twistor elementary states are the $U(p) \times U(q)$ -finite vectors. We show that $H^{p-1}(\bar{P}^+, \mathcal{O}(k))$ has a natural Fréchet topology in which these states are dense. Using this we show that a certain Hermitian product defined on $H^{p-1}(\bar{P}^+, \mathcal{O}(k))$ is positive definite and hence complete a twistor construction of a family of unitary representations of $U(p,q)$, namely the ladder representations.

COMPLEX, QUATERNIONIC KÄHLER MANIFOLDS

A nondegenerate symmetric form g_{ab} on a complex vector space of dimension $4k$ may always be written as $g_{ab} = \epsilon_{AB} \epsilon_{A'B'}$ for nondegenerate skew forms ϵ_{AB} and $\epsilon_{A'B'}$ of ranks $2k$ and 2 respectively. Such a choice determines a subgroup $Sp(k) \times_{\mathbb{Z}_2} Sp(1) \leq SO(4k)$. These are complex Lie groups ($SO(4k)$ means $SO(4k, \mathbb{C})$ etc.). $Sp(k) \times Sp(1)$ double-covers this subgroup, a familiar case being $k=1$ where $Sp(1) = SL(2, \mathbb{C})$ and there is equality $Sp(1) \times Sp(1) \xrightarrow{\cong 2:1} SO(4)$. Just as in 4 dimensions where twistor theory begins with this observation (i.e. with spinors), a twistor theory in $4k$ dimensions may be based on the more general observation above. In the real case, i.e. starting with $SO(4k, \mathbb{R})$, this is Simon Salamon's twistor theory (Quaternionic manifolds, D.Phil thesis, Oxford 1980. Quaternionic Kähler manifolds, Inv. Math. 67 (1982) 143-171. Quaternionic manifolds, in: Symposia Mathematica #26, Academic Press 1982) of quaternionic Kähler manifolds and hyperKähler manifolds. As always, the real construction is based on the Newlander-Nirenberg integrability theorem for complex structures whereas the alternative complex construction is based on complex differential geometry. These two are essentially equivalent. Thus, there is nothing new in this article whose aim is to explain the complex case using notation which parallels the usual spinor notation in case $k=1$. There is no real loss in restricting $k=2$.

Suppose M is a complex Riemannian manifold of dimension 8 with metric g_{ab} . The Riemannian curvature has Young tableau symmetries \boxplus under $SL(8)$ and under $SO(8) < SL(8)$. This representation splits [All splitting formulae here have been deduced from Freudenthal's formula (p.122 in J.E. Humphreys, Introduction to Lie algebras and representation theory, Springer GTM #9 (1972). For notation see RJB & MGF in this TIN]:

| | | | | | | |
|------------------|-----|----------|-----|----------|-----|-----------|
| $\boxplus =$ | $=$ | \oplus | | \oplus | | |
| curvature: R | $=$ | W | $+$ | Φ | $+$ | Λ |
| dimension: 336 | $=$ | 300 | $+$ | 35 | $+$ | 1 |

giving the familiar splitting of R into Weyl, trace free Ricci, and scalar curvatures. Suppose further that the structure group can be reduced to $Sp(2) \times_{\mathbb{Z}_2} Sp(1)$ and moreover that one can lift to a principal $Sp(2) \times Sp(1)$ (there is a \mathbb{Z}_2 obstruction created exactly like the Stiefel-Whitney class in case $k=1$). In more concrete terms this means that one can write the cotangent bundle as

$$\mathcal{O}_a = \mathcal{O}_{AA'} \equiv \mathcal{O}_A \otimes \mathcal{O}_{A'}$$

for vector bundles \mathcal{O}_A and $\mathcal{O}_{A'}$ of ranks $2k$ and 2 respectively and that

$\Lambda^2 \mathcal{O}_A$ and $\Lambda^2 \mathcal{O}_{A'}$ come equipped with sections ε_{AB} and $\varepsilon_{A'B'}$ such that

$$g_{ab} = \varepsilon_{AB} \varepsilon_{A'B'}$$

Then one can further split the curvature under $\text{Sp}(2) \times \text{Sp}(1) < \text{SO}(8)$:

| | |
|-------|--|
| | $= \overset{\circ}{\leftarrow} \overset{2}{\leftarrow} \overset{4}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{4}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{4}{\cdot} \oplus \overset{2}{\leftarrow} \overset{1}{\leftarrow} \overset{2}{\cdot} \oplus \overset{2}{\leftarrow} \overset{0}{\leftarrow} \overset{2}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{2}{\cdot} \oplus \overset{4}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot} \oplus \overset{2}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot}$ |
| dims: | 300 = 70 + 25 + 5 + 105 + 30 + 15 + 35 + 14 + 1 |
| | $= \overset{2}{\leftarrow} \overset{0}{\leftarrow} \overset{2}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{0}{\cdot}$ |
| dims: | 35 = 30 + 5 |

In order to proceed further, however, one must suppose that the holonomy also reduces to $\text{Sp}(2) \times \text{Sp}(1)$ i.e. that \mathcal{O}_A and $\mathcal{O}_{A'}$ are equipped with connections which preserve the ε 's and which together induce the Levi Civita connection on \mathcal{O}_A (i.e. also torsion free). Such manifolds are called complex quaternionic Kähler and their curvature is severely restricted as follows. The adjoint representation of $\text{Sp}(2) \times \text{Sp}(1)$ is $\overset{\circ}{\leftarrow} \overset{2}{\leftarrow} \oplus \overset{2}{\leftarrow} \overset{0}{\cdot}$ and occurs as only part of the two-forms:

$$\square = \overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\leftarrow} = \overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{2}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{2}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot}$$

dimensions: 28 = 15 + 3 + 10

By virtue of the interchange symmetry ($R_{abcd} = R_{cdab}$) one must have

$$R \in \mathbb{O}^2(\overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\leftarrow}) = \mathbb{O}^2(\overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{2}{\cdot}) \oplus (\overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{2}{\cdot}) \oplus (\overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot}) \oplus \mathbb{O}^2(\overset{2}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot}) \\ = (\overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{2}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{2}{\cdot}) \oplus (\overset{2}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot}) \oplus (\overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{0}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot})$$

But R must also be totally skew ($R_{[abcd]} = 0$) whereas 4-forms split:

$$\square = \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\leftarrow} \overset{1}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\leftarrow} = \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{4}{\cdot} \oplus \overset{2}{\leftarrow} \overset{0}{\leftarrow} \overset{2}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{2}{\cdot} \oplus \overset{0}{\leftarrow} \overset{2}{\leftarrow} \overset{0}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{0}{\cdot} \oplus \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot}$$

It is elementary linear algebra to check that all the possible homomorphisms between the pieces of $\mathbb{O}^2(\overset{\circ}{\leftarrow} \overset{1}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\leftarrow}) \rightarrow \square$ are non-zero.

Thus, $R = \Psi + \Lambda$ where $\Psi \in \overset{4}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot}$ is part of the Weyl curvature and $\Lambda \in \overset{\circ}{\leftarrow} \overset{0}{\leftarrow} \overset{0}{\cdot}$ is the scalar curvature. In particular, $\bar{\Psi} = 0$ i.e. M is Einstein with cosmological constant. To be more explicit and suggestive:

$$R_{abcd} = \bar{\Psi}_{ABCD} \varepsilon_{A'B'} \varepsilon_{C'D'} + 2\Lambda (\varepsilon_{AC} \varepsilon_{BD} \varepsilon_{A'C'} \varepsilon_{B'D'} - \varepsilon_{AD} \varepsilon_{BC} \varepsilon_{A'D'} \varepsilon_{B'C'})$$

The curvature on \mathcal{O}_A is also $\Psi + \Lambda$ and the curvature on $\mathcal{O}_{A'}$ is just Λ . At this point all standard twistor constructions proceed with no change. A twistor space for M is defined as the space of integral curves of the distribution $\pi'^* \mathcal{O}_{A'}$ on the total space of $\mathcal{O}_{A'}$. Since the curvature of ∇ is just Λ this is always integrable. If further $\Lambda = 0$ then $\mathcal{O}_{A'}$ is flat and the twistor space fibres over $\mathcal{O}_{A'}$ with $\varepsilon_{AB} \pi'^* \mathcal{O}_A \pi'^* \mathcal{O}_B$ a symplectic form on the (4-dimensional) fibres. All other fauna exist e.g. α -planes, Ward correspondence (for bundles flat along α -planes), Penrose transform, ... Many thanks to Devendra Kapadia for useful chat.

Michael Eastwood.

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A Note on Background-Coupled Massless Fields: Formal Neighborhoods, a Contour Integral, and the Fierz-Pauli and Buchdahl Conditions

Recall that if an anti-self-dual electromagnetic field is described by a Ward line bundle \mathcal{L} over a region U in PT, the space of massless fields of charge q and helicity s over the corresponding region in CM is

$$\mathcal{F}_{q,s} \simeq H^1(U, \mathcal{L}^q \otimes \mathcal{O}(-2-2s)).$$

It is known that, if $s > 0$ and f is a representative twistor function in the group on the right, the massless field is given by an integral of the form

$$\frac{1}{2\pi i} \oint_{\Gamma} \pi_1 \dots \pi_n e^{g(x, \pi)} f(\epsilon \times \pi, \pi) d\pi. \quad (*)$$

Two questions arise:

- 1) Give a contour integral for $s = -\frac{1}{2}$.
- 2) How does the twistor description have anything to do with the Fierz-Pauli conditions?

In this note, we give an answer to the first, and show that it seems to have something to do with the second, the link being the insidious and ever-more-pervasive theory of formal neighborhoods in twistor theory. Similar analyses will hold for Yang-Mills fields and the non-linear graviton. The idea is as follows.

To evaluate the field due to an element f of $H^1(\dots)$ at x^a for $s > 0$, we do the following:

- 1) Restrict f to the line L_x in PT
- 2) Trivialize \mathcal{L} over L_x . (This amounts to choosing a gauge at x^a .)
- 3) Use this trivialization to regard $f|_{L_x}$ as an element of $H^1(L_x, \mathcal{O}(-2-2s))$
- 4) Do the usual contour integral (Serre duality) on $L_x = P_1$ to get the field.

As we shall see, this agrees with (*).

For $s = -\frac{1}{2}$, replace the above with:

- 1') Restrict f to the first formal neighborhood of L_x (call this F_x)
- 2') Trivialize \mathcal{L} over the first formal neighborhood (F_x)
- 3') Use this trivialization to regard $f|_{F_x}$ as an element of " $H^1(F_x, \mathcal{O}(-1))$ " (the quote marks are because this is unorthodox notation)
- 4') Do the usual integral with one $\partial/\partial \omega^A$ derivative.

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I will be very informal with formal neighborhoods. (See MG's previous TN articles and Griffiths & Harris for a rigorous treatment.) For the present purposes, it suffices to know that a function defined on the first formal neighborhood L_x of L_x is a Taylor series in a neighborhood of L_x in which one sets to zero all terms of higher than first order off L_x . That is, if

$$\tilde{\omega}^A(x^\alpha) = \omega^A - i x^{\alpha_1} \pi_{A_1}$$

then the functions are of the form

$$f(\pi_{A_1}) + \tilde{\omega}^A f_A(\pi_{A_1}) + \text{nothing else} \dots$$

Geometry of the Bundles

We call the line bundle on $\mathbb{C}M$ for which the electromagnetic potential is a connection the em bundle; its fibre at x^a the em space at x^a . The Ward bundle \mathcal{L} is the em bundle translated into twistor terms, and elements of the em space at x^a are sections of the Ward bundle over L_x .

To derive some formulae from this, let $U = U_1 \cup U_2$, and the fibre coordinate of \mathcal{L} over U_j be ζ_j . The patching is

$$\zeta_1 = \zeta_2 \exp -ie F(z^\alpha)$$

where e is the unit of electric charge and F is the twistor function representing the electromagnetic field.

We know there are splitting functions $g_j(x^a, \pi_{A_1})$ holomorphic over U_j with

$$F(ix\pi, \pi) = g_1(x, \pi) - g_2(x, \pi).$$

The splitting functions are unique up to

$$g_j(x, \pi) \mapsto g_j(x, \pi) + \alpha(x) \quad (**)$$

and satisfy

$$\pi^{A_1'} \nabla_{A_1'} g_j = \pi^{A_1'} (ie \Phi_{A_1}). \quad (***)$$

Note that the effect of $(**)$ on $(***)$ is a gauge transformation. The choice of splitting is the choice of gauge at x^a .

Let a section $\zeta_j(\pi_{A_1})$ of \mathcal{L} over L_x be given. Then

$$\zeta_j(\pi_{A_1}) = \zeta_2(\pi_{A_1}) \exp -ie (g_1(x, \pi) - g_2(x, \pi))$$

whence

$$\zeta_1 \exp ieq_1(x, \pi) = \zeta_2 \exp ieq_2(x, \pi).$$

The left-hand side is holomorphic over U_1 , the right-hand side over U_2 , and both are homogeneous of degree zero. Therefore both must be equal to some constant, say ζ_0 . Then

$$\zeta_j(\pi_{A'}) \mapsto (\zeta_0, \pi_{A'})$$

provides a trivialization of \mathcal{L} over L_x and ζ_0 is the point in the em space at x^a corresponding to the section $\zeta_j(\pi_{A'})$. Note again that the freedom (**) leads to a gauge transform, i.e.

$$\zeta_0 \mapsto \zeta_0 \exp ie\alpha(x).$$

Similarly, a trivialization of the q^{th} power of \mathcal{L} is

$$\zeta_j^q(\pi_{A'}) \mapsto (\zeta_0^q, \pi_{A'}).$$

Now let

$$f_j(z^\alpha) = f_2(z^\alpha) \exp -ieq^F$$

represent an element of $H^1(U, \mathcal{L}^q \otimes O(-2-2s))$ (f_j with respect to the bundle coordinate ζ_j). Then (1-4) amount to

$$\frac{1}{2\pi i} \oint \pi_{A'} \dots \pi_B e^{ieq g_j(x, \pi)} f_j(ix\pi, \pi) d\pi.$$

as promised. This is the usual formula.

The Case $s = -\frac{1}{2}$

Let $\zeta_j(\omega^A, \pi_{A'})$ be a section of \mathcal{L} over F_x which we wish to trivialize, so

$$\omega^A = ix^{AA'} \pi_{A'} + \eta^A$$

where we keep no terms of order η^2 or higher. Choose an arbitrary $\alpha_{A'}$.

Then

$$y^{AA'} = x^{AA'} + \eta^A \alpha^{A'}/i\pi\alpha$$

is a line through $(\omega^A, \pi_{A'})$, infinitesimally separated from L_x .

 $P\pi$

CM

We know that, for fixed η^A ,

$$\zeta_j \exp ieq_j(y, \pi) = \zeta_j \exp ieq_j(x, \pi)$$

is a complex number which is an element of the em space at y^a . We may use the connection Φ_α on CM to move this element to x^a , and we get

$$[\zeta_j \exp ieq_j(y, \pi)] [1 - ie \frac{\eta^A \alpha^{A'}}{i\pi \cdot \alpha} \Phi_{AA'}].$$

This is holomorphic on $F_x \cap U_j$. To see this, expand in η^A .

$$\begin{aligned} & \left\{ \zeta_j \exp ieq_j(x, \pi) + \frac{\eta^A \alpha^{A'}}{i\pi \cdot \alpha} \nabla_{AA'} g_j(x, \pi) \right\} [1 - ie \frac{\eta^A \alpha^{A'}}{i\pi \cdot \alpha} \Phi_{AA'}] \\ &= [\zeta_j \exp ieq_j(x, \pi)] [1 + ie \frac{\eta^A \alpha^{A'}}{i\pi \cdot \alpha} (\nabla_{AA'} g_j(x, \pi) - \Phi_{AA'})]. \end{aligned}$$

This is clearly holomorphic over U_j so long as $\pi_{A'} \neq \alpha_{A'}$. As $\pi_{A'} \rightarrow \alpha_{A'}$, however, (***) insures it is regular. Thus we may take

$$\zeta_0 = \zeta_j [\exp ieq_j(x, \pi)] [1 + ie \frac{\eta^A \pi^{A'}}{i\pi \cdot \alpha} (\nabla_{AA'} g_j(x, \pi) - \Phi_{AA'})]$$

and similarly for ζ^q .

Then (1'-4') give

$$\frac{1}{2\pi i} \oint \frac{\partial}{\partial \omega^A} \left\{ f_j(\omega^4, \pi_{A'}) [1 + ie \frac{\eta^A \alpha^{A'}}{i\pi \cdot \alpha} (\nabla_{AA'} g_j - \Phi_{AA'})] e^{ieq_j g_j(x, \pi)} \right\} d\pi.$$

One can verify directly that this satisfies the correct field equation, and agrees with the evaluation of the field by potentials as in Eastwood, Penrose & Wells.

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Discussion

The motivation for the form of this construction was to reduce the problem to something we know how to solve: evaluation of $s = -\frac{1}{2}$ fields when no electromagnetic field is present, i.e. when \mathcal{L} is trivial. Trivializing \mathcal{L} over F_x gives us enough of the usual structure to do this: we need one $\partial/\partial\omega^A$ derivative, which points off L_x ; we can compute it in F_x .

\mathcal{L} does not trivialize to second order off L_x . This seems to be a reflection of the Fierz-Pauli conditions, as it is what prevents one from finding $s = -1$ massless charged fields.

In the gravitational case, one can trivialize the curved twistor space to second order off the line in question, in the following sense. There is a formal-neighborhood biholomorphism between the second formal neighborhood of a line in curved twistor space and that of a line in flat twistor space. One can then get formulae for $s = -\frac{1}{2}$ (one $\partial/\partial\omega$) and $s = -1$ (two $\partial/\partial\omega$'s) fields. The obstruction to trivialization to third order would seem to be the Buchdahl conditions..

Roughly speaking, the order to which one can trivialize comes about in the following way. The line bundle is given by a cohomology element with coefficients in O , and the non-linear graviton by one with coefficients in Θ . In order to "detect" the line bundle or the deformation near a line, one must be able to take enough $\partial/\partial\omega$ derivatives of O or Θ to construct the field

$$\varphi_{AB} \quad \text{or} \quad \psi_{ABCD}$$

at x^a : two for O , three for Θ .

That this seems to fit in, in at least a general sense, with the pattern of extensions to formal neighborhoods in other twistor and related work is remarkable but not wholly understood (by me).

- Adam Helfer

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An Elementary Relativistic Model for Quark Confinement

by L.P. Hughston

Consider the problem of a pair of quarks bound together in such a way that the distance between them never exceeds some given quantity, say q_0 . In a non-relativistic setting the problem is not particularly difficult; in the relativistic case, however, the problem will not seem so easy, and insofar as I am aware the problem has not been treated yet, as such, in an exact way.

It turns out, however, that essentially the same methods that were used in my relativistic oscillator construction (Proc. Roy. Soc. A 382, 459 (1982)) can be applied; in what follows free use will be made of the ideas and notation contained therein. The results are of considerable interest.

Suppose we have a pair of spin zero particles of masses m_1 and m_2 . We treat the particles essentially as free fields, and thus their joint wave function $\Psi(x, y)$ satisfies $(\square_x + m_1^2)\Psi = 0$ and $(\square_y + m_2^2)\Psi = 0$; the idea now is to impose a constraint of limited separation on this system. If the composite particle has a definite total mass M and a definite total spin s , then (after a Fourier transformation with respect to the total momentum variable) the wave equations above can be reduced to a single effective radial equation, viz.:

$$\left(-\frac{\partial^2}{\partial q^2} - \frac{2}{q} \frac{\partial}{\partial q} + \frac{s(s+1)}{q^2} \right) \phi = \lambda \phi ,$$

where ϕ is a function of $q = \sqrt{(\vec{x}^a p_a/M)^2 - \vec{x}^a \vec{x}_a}$, with $\vec{x}^a = x^a - y^a$; and

$$\lambda = \frac{1}{4} M^2 \left[1 - \frac{(m_1 + m_2)^2}{M^2} \right] \left[1 - \frac{(m_1 - m_2)^2}{M^2} \right].$$

The general solution of the radial equation non-singular at the origin is $A j_s(\lambda^{1/2} q)$, where j_s is the spherical

Bessel function of order s , and R is a constant. Thus we may impose confinement by requiring that the wave function vanish for $g \geq g_0$, whence as a boundary condition we obtain $j_s(\pi R^{1/2} g_0) = 0$.

Let us denote by $\beta_{s,n}$ the n^{th} root of $j_s(\pi \beta) = 0$. Then the allowable mass levels of the system are given by $\lambda = (\pi \beta_{s,n} / g_0)^2$ for $s = 0, 1, 2, \dots$ and $n = 1, 2, 3, \dots$. For information I tabulate below to three significant figures the first few values of $\beta_{s,n}$:

| $\beta_{s,n}$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ |
|---------------|---------|---------|---------|---------|---------|
| $s = 0$ | 1 | 2 | 3 | 4 | 5 |
| $s = 1$ | 1.430 | 2.459 | 3.471 | 4.478 | 5.482 |
| $s = 2$ | 1.835 | 2.895 | 3.927 | 4.939 | 5.949 |
| $s = 3$ | 2.224 | 3.316 | 4.360 | 5.387 | 6.405 |
| $s = 4$ | 2.605 | 3.726 | 4.787 | 5.826 | 6.852 |

It will be noted that the dependence of $\beta_{s,n}$ on both s and n is approximately linear, suggestive of Regge behaviour. Can this be made more precise? Suppose we set $g_0 = \alpha M$ where α is a constant with dimensions mass^{-2} . This is perhaps the simplest and most sensible assumption that can be made about the range of confinement. In the case of two particles of equal mass m , say, the spectrum is then given by

$$M^2 = (2m)^2 + \left(\frac{2\pi \beta_{s,n}}{\alpha M} \right)^2 ;$$

which does indeed generate approximately linear Regge trajectories. In the case $m \approx 0$ and $s = 0$, for example, we have $M^2 = 2\pi\alpha^{-1}n$.

The model described above is an elementary one; it does not entail the introduction of interactions; it is all set essentially in the context of free fields.

GEOMETRY OF NULL HYPERSURFACES

Let M be Minkowski space, with the usual coordinates (t, x^a) $a=1\dots 3$, and suppose \mathcal{H} is a null hypersurface^[1] in M satisfying $g = \bar{g}$. Also, let u be an affine parameter along the generators of \mathcal{H} such that $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial u}) = 1$.

Now, M is foliated by the surfaces $\{t = \text{constant}\}$; denote the surface $\{t = k\}$ by C_k . Then $C_k \cap \mathcal{H}$ is perpendicular to \mathcal{H} , and, in addition, one can choose u along the generators such that $u = k$ on each generator in $C_k \cap \mathcal{H}$.

Consider a small area element of $C_u \cap \mathcal{H}$, say γ_u . Dragging this area element up the generators of \mathcal{H} which pass through it, to C_{u+du} , one obtains γ_{u+du} . It is well known that

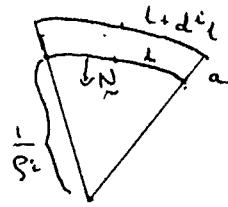
$$\gamma_{u+du} - \gamma_u = -2g\gamma_u du.$$

The purpose of this note is to relate this to some ideas in classical (i.e. surfaces in \mathbb{R}^3) differential geometry, and to give (from a slightly different viewpoint) the relations between g, σ and the geometry of \mathcal{H} .

If S is a surface in \mathbb{R}^3 , then a surface \bar{S} is parallel to S if its points are a constant distance along the normal from S .^[2] In other words, if ξ is the position vector of a point P , the point \bar{P} of \bar{S} corresponding to ξ is given by $\bar{\xi} = \xi - aN$, where N is the normal to S .

Now, consider a small circle in S , centred at P (use geodesic polar coordinates), radius l . As it is dragged out along the normal vector field, this circle will in general be deformed to an ellipse

as follows. There are two special radii of the curve, those given by the geodesics through P of maximum and minimum geodesic curvature. These become the semi-axes of the ellipse. If the maximum and minimum curvatures are ρ_1 and ρ_2 respectively, then it is easy to see that the semi-axes are given by $l + l\rho_i \cdot a$, by considering the following figure:



$$\frac{l+d^2l}{\frac{l}{\rho_1}+a} = l\rho_1$$

$$\Rightarrow d^2l = l\rho_1 a$$

Hence we see that the area of the ellipse on S is given by

$$\begin{aligned}\bar{A} &= \pi(l+d^2l)(l+d^2l) \\ &= \pi(l^2 + l(d^2l + d^2l) + d^2l d^2l) \\ &= \pi l^2(1 + a(\rho_1 + \rho_2)) + O(a^2) \\ &= A + 2\mu A \cdot a + O(a^2)\end{aligned}$$

where μ is defined as $\frac{1}{2}(\rho_1 + \rho_2)$, called the mean curvature.^[2]

The relevance of all this is as follows: if the sets $C_u \cap \mathcal{M}$ and $C_{u+a} \cap \mathcal{M}$ are projected on to C_0 , they form a pair of parallel surfaces, and the normal is given by projecting \mathbf{f}_u . (This is, in fact, $-\mathbf{N}$). Thus we find that

$$d\mathbf{f} = -2g\mathbf{f}_u du = 2\mu\mathbf{f}_u du$$

and so $\mathbf{f} = -\mu$,

(with $du = a$), that is, \mathbf{f} is precisely the (negative of the) mean curvature.

Now, if ρ_0 and σ_0 are the values of ρ and σ at $u=0$, we have^[3]

$$\rho = \frac{\rho_0 - u(\rho_0^2 - 1\sigma_0^2)}{1 - 2u\rho_0 + u^2(\rho_0^2 - 1\sigma_0^2)}$$

and setting $\bar{g} = -\mu$, $u = a$, we find

$$\begin{aligned}\mu &= \frac{\mu_0 + a(\bar{g}_0^2 - 1\sigma_0 l^2)}{1 + 2a\mu_0 + a^2(\bar{g}_0^2 - 1\sigma_0 l^2)} \\ &= \frac{\mu_0 + aK_0}{1 + 2a\mu_0 + a^2K_0}\end{aligned}$$

where K is Gaussian curvature.^[2]

and so we find that the Gaussian curvature of $C_0 \cap \mathcal{N}$ is given by

$$K_0 = \bar{g}_0^2 - 1\sigma_0 l^2.$$

In addition, K satisfies the relation^[2]

$$K = \frac{K_0}{1 + 2\mu_0 a + K_0 a^2}$$

and it is a simple exercise to show that $\bar{g}^2 - 1\sigma l^2$ also does, and so the relation

$$K = \bar{g}^2 - 1\sigma l^2$$

holds on any $C_0 \cap \mathcal{N}$.

Thus, we have seen that the mean and Gaussian curvature of a constant time slice of a null hypersurface are sufficient to specify \bar{g} and $1\sigma l$ on the surface, but more information is required for $\arg \sigma$.

Robert how.

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The Structure and Evolution of
Hypersurface Twistor Spaces.

In the last Twistor Newsletter (TN 19) I promised that I would show how the ideas for deforming ambitwistor space led to a description of the structure and evolution of hypersurface twistor spaces that generalize from the linearized theory, as discussed in that article, to curved space.

Recap. of article in TN19:

Infinitesimal deformations of \mathcal{A} , ambitwistor space, are represented by $H^1(\mathcal{A}, T\mathcal{A})$, $T\mathcal{A}$ representing the sheaf of sections of the tangent bundle of \mathcal{A} . Those that lead to M with linearized metric and no torsion are obtained from $h \in H^1(\mathcal{A}, O(1,1))$ by taking the hamiltonian vector field of h , X_h , w.r.t. the standard symplectic structure on \mathcal{A} , $dZ dW$, to obtain an element of $H^1(\mathcal{A}, T\mathcal{A})$. The deformations that give rise to a space-time M whose conformal metric is conformal to one satisfying the linearized field equations are obtained from hamiltonians h of the form:

$$h = Z^\alpha I_{\alpha\beta} \tilde{\partial}g/\partial W_\beta + W_\alpha I^{\alpha\beta} \tilde{\partial}g/\partial Z^\beta. \quad (*)$$

$= X\tilde{g} + \tilde{X}g$ where $X = Z^\alpha I_{\alpha\beta} \tilde{\partial}/\partial W_\beta = \pi^A \tilde{\partial}/\partial \tilde{\omega}^A$, π 's and ω 's are the usual spinorial twistor coordinates and the " \sim " denotes the corresponding dual twistor quantities. \tilde{X} is similarly defined. $g \equiv g(W)$ and $\tilde{g} \equiv g(Z)$, and $g(Z)$ is pulled back to \mathcal{A} from PT and is the element of $H^1(PT, O(2))$ corresponding to the A.S.D. part of the linearized field, similarly for $\tilde{g}(W)$. $I_{\alpha\beta}$ is the infinity twistor.

A point not mentioned is that the information of the (infinitesimally) deformed conformal factor is contained in the forms:

$$\delta i = f_{X_h} i, \text{ where } i = Z^\alpha I_{\alpha\beta} dZ^\beta$$

$$\text{and } \tilde{\delta} i = f_{X_h} \tilde{i}, \text{ where } \tilde{i} = W_\alpha I^{\alpha\beta} dW_\beta$$

These forms have the status of cohomology classes in $H^1(\mathcal{A}, \Omega^1)$. The condition that h is of the form *, is that the first of these forms vanish modulo dW , and the second modulo dZ . Unfortunately this condition cannot be implemented for finite deformations as we will no longer know our dZ 's from our dW 's.

It is interesting to note, however, that this implies that the form i survives to first order modulo dW , and \tilde{i} modulo dZ , so that i and \tilde{i} are well defined as forms modulo dW and dZ respectively on \mathcal{A} deformed to first order.

Furthermore, when $g(W)$ vanishes, then i is preserved to all orders, and becomes the form $\pi^A d\pi_A$, and the form \tilde{i} survives to all orders modulo dZ and when one takes its exterior derivative and convert it from a 2-form to a bivector using the symplectic structure on \mathcal{A} , it becomes the poisson bivector on the fibres of the nonlinear graviton:

$$\tilde{\partial}/\partial \tilde{\omega}^A \tilde{\partial}/\partial \omega_A.$$

For general vacuum space-times we will have to resort to hypersurface twistors in order to be able to distinguish our dZ 's from our dW 's.

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Hypersurface twistors in linearized theory:

I shall use the preceding ideas to derive a formula for the evolution of hypersurface twistors in linearized theory where the hypersurfaces are flat. The formula generalizes in exactly the same form to non flat hypersurfaces, but the proof is somewhat more lengthy (although more rigorous and useful).

So foliate M with surfaces of constant time t , and consider the hypersurface twistor spaces infinitesimally deformed by a linearized gravitational field whose A.S.D. and S.D. parts are given by the cohomology classes g and \tilde{g} as above.

When the hypersurface is flat, the hypersurface twistor space at time t , $P\mathcal{T}_t$, is the space of null geodesics lying in the hypersurface. So for each t , $P\mathcal{T}_t$ is a subset of A , and is a section of the fibration $A \rightarrow P\mathcal{T}$. This section can be described easily by use of the conjugation " \circ " on A which takes a null geodesic n to the null geodesic n° reflected in the hypersurface. Under such a reflection a twistor Z turns into a dual twistor $W = Z^\circ$ and two conjugations give the (-1) -identity. Clearly the null geodesics in the hypersurface, ie the hypersurface twistors, are left invariant by this conjugation. So the section of $A \rightarrow P\mathcal{T}$ is given by ambitwistors $(Z, W) = (Z, Z^\circ)$. Note that this section and conjugation are time dependant.

To find the infinitesimal deformation of $P\mathcal{T}_t$, we can restrict the deforming vector field, X_h , to the section, $A|_{W=Z^\circ}$, and push it down to $P\mathcal{T}$. This gives an element of $H^1(P\mathcal{T}, T\mathcal{T})$ for each t , corresponding to the infinitesimal deformation of each $P\mathcal{T}_t$ due to the linearized field.

Differentiating this family with respect to t yields the evolution at each t , $\epsilon(t)$, which is an element of $H^1(P\mathcal{T}, T\mathcal{T})$ for each t . Using the above ideas we obtain the formula:

$$\epsilon(t) = \left[\frac{\partial}{\partial w_\alpha} \left[Z^\alpha I_{\alpha\beta} \frac{\partial}{\partial w_\beta} \right] \tilde{g} \right] |_{[W = Z^\circ] \frac{\partial}{\partial Z^\alpha}} = \left[\frac{\partial X^\alpha \tilde{g}}{\partial w_\alpha} \right] |_{[W = Z^\circ] \frac{\partial}{\partial Z^\alpha}}$$

In this formula, we have taken a hamiltonian h' on A where, $h' \in H^1(A, O(2, 0))$ is given by: $h' = (Z^\alpha I_{\alpha\beta} \frac{\partial}{\partial w_\beta}) \tilde{g} = X^\alpha \tilde{g}$

Then we take the hamiltonian vector field, $X_{h'}$, restrict it to $W = Z^\circ$, and then push down to $P\mathcal{T}$.

Remarks: 1. The evolution of the C-str. is caused by the S.D. part of the field, ie the googly part, and if this vanishes, the C-str. doesn't evolve and all the $P\mathcal{T}_t$ are the same and are a linearized nonlinear graviton.

2. The form of the hamiltonian h' is that of the self-dual part of h above in (*) except in that h' is the derivative of h along the integral curves of the vector field X , in fact $h' = Xh$. The X derivative corresponds to the time derivative of h .

3. I have been a little sloppy about the identification of the $P\mathcal{T}_t$'s at different times, this is, however, correct up to infinitesimal ambiguities which are inessential here as they do not affect the cohomology class of $\epsilon(t)$. In the full nonlinear theory it is however important, and we shall require not just a cohomology class but also a fixed Dolbeault representative since the identification between the $P\mathcal{T}_t$'s at different times is fixed.

Structures on curved hypersurface twistor spaces:

To test to see whether the formula above works in full general relativity we need a little more structure than is immediately apparent. The conjugation is relatively straightforward being precisely the same as above, ie reflection of the null geodesic in the hypersurface. It is not, however, antiholomorphic in general in the sense that $f(W)$ restricted to $W = Z''$ will now be a function of both Z and W rather than just Z as it would be above. In fact it is not antiholomorphic for extrinsically curved hypersurfaces in flat space, but nevertheless the derivation above can be performed, at least for special Dolbeault representatives for h^* . Further we know that for the full theory we will require fixed representatives for $\epsilon(t)$.

Consider the complexification of a real Lorentzian space-time M , $\mathbb{C}M$, foliated by hypersurfaces of constant time, t , where t has been chosen such that: $(\nabla_a t)(\nabla_a t) = 2$, so that $T_a = \nabla_a t$ is the timelike normal of length $\sqrt{2}$.

The space of scaled complex null geodesics $\mathbb{C}N$ can be identified with the direct sum of the primed and unprimed cospin bundles restricted to $\mathbb{C}H_t$, the complexified hypersurface at time t . The real slice is the hypersurface twistor C.R. manifold, N , which is similarly identified with the primed cospin bundle restricted to the hypersurface H_t .

One can pull back all the spinor and tensor bundles to $\mathbb{C}N \& N$. Since the spin bundle is pulled back to itself it has a canonical section, $z_{A'}$, which at a point $n \in N$ or $\mathbb{C}N$ takes on the value of the spinor aligned along the null geodesic n and the phase of $z_{A'}$ corresponds to that of n . Denote the canonical section of the unprimed spin bundle by w_A . This can, of course, be identified with \bar{z}_A on N since we are working on a Lorentzian manifold. Using the connection on M we can extend the ordinary de Rham operator, d , to act on indexed quantities. This will also be denoted by d .

Denote the solder form by $dx^a = dx^{AA'}$ and define the spinor valued vector field $\partial/\partial z_{A'}$ by:

$\partial/\partial z_{A'} \lrcorner dx^{BB'} = 0$, $\partial/\partial z_{A'} \lrcorner dz_{B'} = \epsilon_{B'}^{A'}$, and $\partial/\partial z_{A'} \lrcorner dw_B = 0$.
Similarly ∇_a is defined by: $\nabla_a \lrcorner dx^b = \delta^b_a$, $\nabla_a \lrcorner dz_{A'} = 0 = \nabla_a \lrcorner dw_A$.

The twistor distribution is then defined on the spin bundle by $D = \{z^{A'} \nabla_{AA'}, \partial/\partial \bar{z}_{A'}\}$, and the C.R. structure on N is given by the intersection of this with the tangent bundle of N .

We then have the following twistorial structures canonically defined on A and N :

Euler vector field: $\Upsilon = z_{A'} \partial/\partial z_{A'} = z^\alpha \partial/\partial z^\alpha$.

Symplectic potential: $\theta = z_{A'} w_A dx^{AA'} = z^\alpha dw_\alpha$.

Symplectic form: $\Omega = d\theta = dz^\alpha \wedge dw_\alpha$.

"Infinity" forms: $\iota = z^{A'} dz_{A'} = z^\alpha \iota_{\alpha\beta} dz^\beta$.

$\tau = dz^{A'} dz_{A'} = \tau_{\alpha\beta} dz^\alpha \wedge dz^\beta$.

The entries on the right are the corresponding objects on flat twistor space. We also have the complex conjugates of the forms ι and τ and the vector field Υ .

We can also construct an analog of the vector field X above by defining it so that: $X \lrcorner \Omega = \iota$.

The forms ι and τ contain the information of the conformal factor and are analogous to Dolbeault representatives for the S.D. part of the Weyl curvature. This analogy can only be made precise in linearized theory, where:

$$\iota = z^A dz_A + \Gamma_{AB}^C z^B dx^A$$

Γ_{AB}^C is the S.D. part of the variation of the connection. If the linearized field satisfies the field equations then ι descends to A and defines a Dolbeault form in the equivalence class defined by the evolution hamiltonian h^* discussed above.

In the full theory ι and τ have various nice properties connected with the field equations. In A.S.D. spacetimes they play well known important roles, as mentioned above and in Lorentz real space-times, $d\iota$ restricted to the complex conjugate twistor distribution, D , vanishes iff Φ_{ab} , the trace free Ricci tensor, vanishes. Unfortunately since D is not integrable in the full theory it is not entirely clear what this means. Furthermore it can be seen that it bears a close relationship to the structures Claude LeBrun introduces in order to have a condition for $\Phi_{ab} = 0$ on A . This is not straightforward but I shall not go into it here.

The conjugation \sim is given by:

$$(x, z^A, w_A) \rightarrow (x, T_A^B w_B, T_A^B z_B).$$

When combined with the lorentzian conjugation this yields the standard euclidean conjugation \sim as defined below.

Evolution in the full theory:

We now have nearly all the ingredients required to check to see if the formula derived in linearized theory will generalize. That is: the conjugation, the evolution hamiltonian h^* given by ι , the symplectic structure Ω . There is however an extra subtlety. The symplectic structure used in the derivation above was that on TXT^* , this is essentially because h^* has homogeneity degree $(0,2)$ and so can only be represented as a function on $TXT^*|z \cdot w = 0$ not on the $O(1,1)$ bundle over A which is a factor space and one cannot take the hamiltonian vector field of a section of a line bundle. All we have here is the reduced one on the $O(1,1)$ bundle over A . This introduces extra complications and we must use a Poisson bivector produced by analogy from:

$$\frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial w_\alpha} = \frac{\partial}{\partial w_\alpha} \frac{\partial}{\partial z^\alpha} + \frac{\partial}{\partial \pi_A} \frac{\partial}{\partial \tilde{w}^A}$$

$\frac{\partial}{\partial \pi_A}$ can be modelled by $\frac{\partial}{\partial z_A}$ and $\frac{\partial}{\partial w^A}$ by:

$$\frac{\partial}{\partial w^A} = \frac{1}{z \cdot \bar{z}} \bar{z}^A \nabla_{AA} - \frac{1}{z \cdot \bar{z}} z^A \bar{z}^B \nabla_{BB}$$

Where \bar{z}^A is the euclidean conjugate of z^A , $\bar{z}^A = T_A^B z^B$.

A further irritation is that using a form as Hamiltonian to obtain a form valued vector field is not in general a well defined procedure. However, in this situation we are ok, since it is well defined in the presence of a complex structure if one is only interested in taking an antiholomorphic form and obtaining an antiholomorphic form valued holomorphic vector field.

We have:

$$2dz_A = (\Psi_{ABCD} + \Lambda \epsilon_D^{AB} \epsilon_C^{CD}) z^B dx^D + \Psi_{ABCD} z^B dx^C$$

And when restricted to a hypersurface:

$$2d\bar{z}_A = (\bar{\Sigma}_{ABCD} + \frac{1}{2} G_{AB} \epsilon_{AC} D - \frac{1}{2\sqrt{z}} G_{AB} \epsilon_{BC} - \frac{1}{2\sqrt{z}} G_{AB} \epsilon_{BD} \epsilon_{CA}) z^D dx^C$$

Using the fact that: $d^2x^{AB} = T^A_B d^2x^{AB}$
 Where: $d^2x^{AB} = dx^A dx^{BA}$, $\Xi_{ABCD} = \Psi_{ABCD} + \Phi_{AB}^{TA} (AB^T C^T D)$
 and: $G_{ab} = G_{ab}^{TT}$, $G_{(AB)} = G_{cB}^{TT} (A^T B)$

With all the above ingredients the over eager student can find: $\dot{\epsilon}(t) = \frac{1}{z \cdot \bar{z}} \Xi_{A^* B^* C^* D^*} z^A dx^C \bar{z}^B \partial/\partial z^D$

Whereas lie dragging the ∂_B operator on N along the null geodesic spray yields the actual evolution as:

$$\dot{\epsilon}(t) = \frac{1}{(z \cdot \bar{z})^2} \Psi_{A^* B^* C^* D^*} z^A dx^C \bar{z}^B \partial/\partial z^D$$

It can be seen that, as Dolbeault representatives, the two expressions above for $\dot{\epsilon}$ differ in that the expression calculated from the formula depends on Ξ , which can be calculated from initial data on the hypersurface, whereas the actual evolution depends on Ψ . If we demand that the two expressions are equal then we must put: $\Xi_{A^* B^* C^* D^*} = \Psi_{A^* B^* C^* D^*}$

However, this then forces: $\Phi_{AB}^{TA} (A^* B^* T^C C^* D^*) = 0$.

These are precisely the Einstein evolution equations.

So if we are presented with a hypersurface ambitwistor space with its twistorial structures and conjugation corresponding to a hypersurface with initial data satisfying the constraint equations, then with purely twistorial manipulations we can evolve the twistor structures such that the Einstein equations are satisfied. Note that $D_i = 0$ tells us how to propagate i . If the constraints are not satisfied we will not be able to propagate i consistently.

Remarks:

The above calculations do not rely on analyticity, and can be performed when the complexifications do not exist.

There are several features in the above calculations that are not entirely satisfactory, the use of the non antiholomorphic conjugation $''$, the obscure status of the forms i and τ , and the need to use the symplectic structure on $T^*T^*|_Z W = 0$. It appears, however, that when one comes to consider finite evolutions the conjugation $''$ will not play such a prominent role.

Two generalizations seem particularly desirable, firstly to finite evolutions, and secondly to fields with sources, ie electrovac. Only then will hypersurface twistors become a useful technique for studying G.R.. There are some indications that the first of these will be somewhat more elegant, however the work on this is far from complete.

I would like to thank my supervisor N.M.J. Woodhouse for his continuous attention, valuable criticism and comments.

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(This is a good introduction to calculations on the spin bundle, the various conjugations used except $''$, and the use of Dolbeault cohomology in twistor theory.)

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