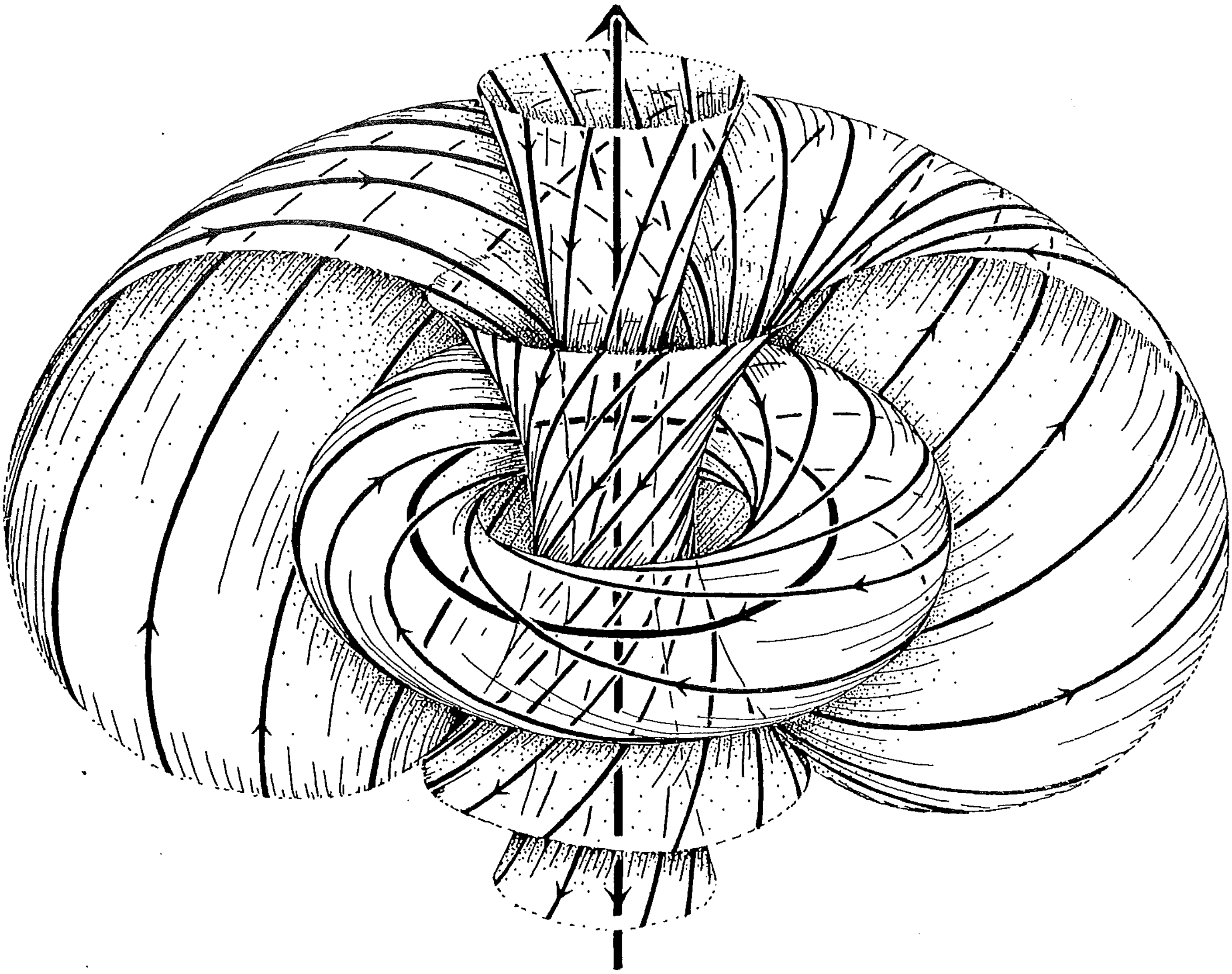
Twistor Newsletter (no 23: 16, Feb., 1987)



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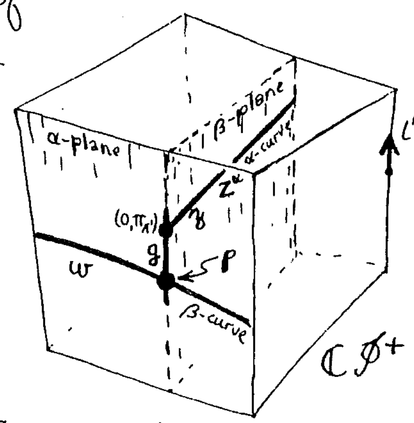
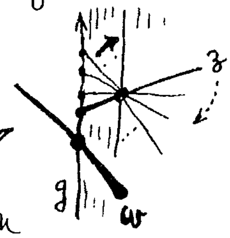
Local Twistor Transport at $\mathscr{I}^+$: an Approach to the Googly

This note reports on some work in progress concerning the googly graviton, and suggests a certain shift from the viewpoint I have been describing in some recent Twistor Newsletters (TN16 pp. 9, 10, TN17 pp. 7, 8, TN21 pp. 2,3,4). This shift results from an attempt to relate the structure of the googly twistor space to local twistor transport on $\mathbb{C}\mathscr{I}^+$. (Several of the ideas described here have resulted from discussions with V.D.T.)



The central idea is to use local twistor transport along β-curves on $\mathbb{C}\mathscr{I}^+$ to provide a family of "flat coordinate systems" for $\mathbb{T}$ ($\mathbb{T}$ = the non-projective blown-up twistor space, which for flat space twistors is the space of $\frac{1}{2}\Sigma] = Z^\alpha Z^\beta I_{\beta\gamma}$). Recall that apart from generators of $\mathbb{C}\mathscr{I}^+$, the null geodesics on $\mathbb{C}\mathscr{I}^+$ are of two kinds: β-curves (lying on α-planes on $\mathbb{C}\mathscr{I}^+$ and representing projective asymptotic dual twistors) and α-curves (lying on β-planes on $\mathbb{C}\mathscr{I}^+$ and representing projective asymptotic twistors). Let w be a β-curve. Then local twistor transport along w can be used to establish an identification between all the local twistor spaces at the various points of w. Let us call the resulting (flat) identified twistor space $\mathbb{T}(w)$. Now any α-curve z which does not lie too far from the non-singular region of w will meet a unique generator g of $\mathbb{C}\mathscr{I}^+$ intersecting w. Assigning an appropriate scaling to z (given by a $\pi_{A'}$ spinor parallelly propagated along z, for which $l_A\pi_{A'}$ is a tangent covector to z, the tangents to generators of $\mathbb{C}\mathscr{I}^+$ being $l^A\tilde{l}^{A'}$) we get an asymptotic twistor Z^α. With respect to any point of z, Z^α has the local twistor description $(0, \pi_{A'})$. This can be carried along g to the point $p = g \cap w$ by local twistor transport, and Z^α is thereby labelled by a unique element of $\mathbb{T}(w)$. Thus, the elements of $\mathbb{T}(w)$ provide "coordinates", locally, for the asymptotic twistor space $\mathscr{T}$. For points of $\mathscr{T}$ which do not lie on I ($\subset \mathscr{T}$), these provide smooth coordinates. However, the coordinates will not generally be smooth at points of I. The "blown-up" local twistors $\frac{1}{z}Z]$ ($\in \mathbb{T}^{(\alpha\beta)}(w)\, I_{\beta\gamma}$) have a better chance of being smooth "coordinates" locally at points of $\mathbb{I} \subset \mathcal{T}$, where $\mathcal{T}$ is the blown-up (non-projective) twistor space $\mathcal{T}$. (I do not yet know if — or under what circumstances — they are smooth. My main doubts would arise when $\frac{1}{z}$ approaches I by having z swing to the vertical, about some point not on g ⟿



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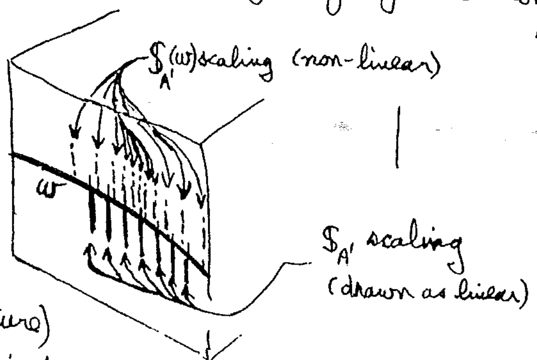
We need to be careful about the interpretation of $Z]$ in terms of $\mathbb{T}(w)$. The infinity twistor I is well-defined as a local twistor and is local-twistor constant over the whole of $\mathbb{C}\mathscr{I}^+$. (Indeed, it is constant, in any vacuum space-time, throughout the finite regions also.) The form that I takes does not depend upon the structure (news function, radiation field) of $\mathbb{C}\mathscr{I}^+$.

$$\boxed{I_{\alpha\beta} = \begin{bmatrix} 0 & -i\iota_A\tilde{\iota}^{B'} \\ i\tilde{\iota}^{A'}\iota_B & 0 \end{bmatrix} \text{ at } \mathbb{C}\mathscr{I}^+;\ \rho'=0.}$$

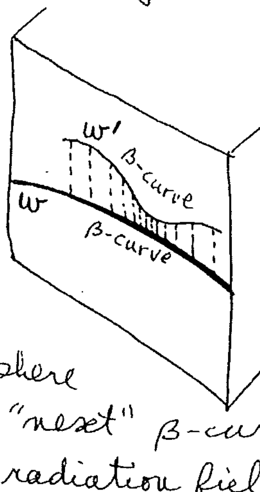
However, the "π-space" $\mathbb{S}_{A'}(w)$ of $\mathbb{T}(w)$ does not agree with the asymptotic "π-space" $\mathbb{S}_{A'}$ that is defined for $\mathbb{C}\mathscr{I}^+$ as a whole (cf. P&R. vol.2, p.411, 417; here $\mathbb{S}_{A'} = \mathbb{S}_{A'}(\mathscr{S})$ for any cut $\mathscr{S}$ of $\mathscr{I}^+$ — in the sense of 2-surface twistors). The difference between $\mathbb{S}_{A'}(w)$ and $\mathbb{S}_{A'}$ was extensively discussed, in effect, by G.B.-S. in his 1982 D.Phil. thesis, the elements of $\mathbb{S}_{A'}$ being his "π"s and the elements of $\mathbb{S}_{A'}(w)$, his "p"s (cf. also TN 18, p.24), though he was not concerned there with local twistor transport. In my Physics Reports (1972) article with M.A.H.MacC., the difference between local twistor transport of $Z]$ twistors ("twistors entirely on $\mathscr{I}$"), which gives $\mathbb{S}_{A'}(w)$, and a modified local twistor transport, which gives $\mathbb{S}_{A'}$ was pointed out (p.301). The modification results from supressing some of the curvature terms in $P_{ABA'B'}$ which appear in the local-twistor propagation of $\pi_{A'}$s. The suppressed terms are those of the form $P_{11A'B'}$ and $P_{AB1'1'}$, these being essentially the anti-self-dual and self-dual parts, N and $\tilde{N}$, of the news function, respectively:

$$P_{11A'B'} = N\tilde{\iota}_{A'}\tilde{\iota}_{B'}, \quad P_{AB1'1'} = \tilde{N}\iota_A\iota_B .$$

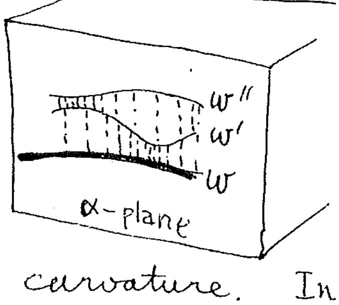
The relation between $\mathbb{S}_{A'}(w)$ and $\mathbb{S}_{A'}$ cannot extend globally unless this relation is linear. The presence of self-dual news function $\tilde{N} \neq 0$ implies that the relation is non-linear. It is not global because w is not a $\mathbb{CP}^1$, but enters the singular region of $\mathbb{C}\mathscr{I}^+$ (where $\tilde{N}$ blows up) and so is non-compact. (It is only $\tilde{N}$, and not N, which enters into the local twistor propagation along w. N would show up, on the other hand, if we were to propagate along α-curves.) In the figure ⟶ I have schematically indicated the non-linear relation between the $\mathbb{S}_{A'}(\omega)$ scalings and the $\mathbb{S}_{A'}$ scalings.



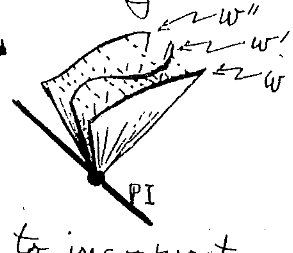
In P.&R. vol. 2 (pp. 374–377), a connection between the strong conformal geometry of $\mathscr{I}^+$ (angle and null-angle structure) and the product $\sqcup\sqcap$ of the infinity twistors was pointed out. In effect, we can interpret $\sqcap$ as providing a structure which converts tangent vectors in a β-plane to scalings up the generators. More correctly, it converts spinors λ^A (take $\lambda^A\tilde{\iota}^{A'}$ to be tangent to ω) to multiples of $\tilde{\iota}^{A'}$: $\lambda^A \mapsto \lambda^A(-i\iota_A\tilde{\iota}^{B'}) = \underbrace{(-i\lambda^A\iota_A)}_{\text{spinor part}}\tilde{\iota}^{B'}$.



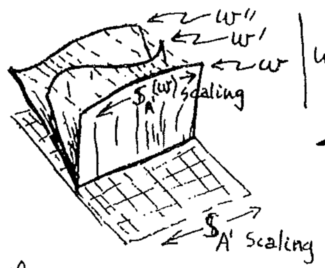
(To achieve the convertion of tangent vectors, we need a fixed scaling in this direction: ↗ in the fig.) The scaling up the generators allows us to locate the "next" β-curve ω'. Actually there is a family of such scalings, related by bilinear transformations in the Riemann sphere of $\mathbb{S}_{A'}(\omega)$. These give all the various possible "next" β-curves. The curvature $\tilde{\Psi}_4$ (self-dual gravitational radiation field) of each α-plane on $\mathbb{C}\mathscr{I}^+$ is measured by the derivative of $\tilde{N}$ up the generators, and it gives the relation between ω, a "next" β-curve ω' and a "next-but-one" β-curve ω''. The googly structure of asymptotic twistor space must find a way of encoding this curvature.



In terms of $\mathbb{P}\mathcal{T}$, the picture looks like ⟶



whereas in terms of $\mathbb{P}\mathcal{T}$, it looks like



It would seem that in order to incorporate the local structure afforded by $\sqcap$, which allows the $\mathbb{S}_{A'}(\omega)$ spaces to be described, rather than simply the global $\mathbb{S}_{A'}$ structure, we need something like the form $\bar{Z}d\bar{Z}$ (which is local) rather than $\bar{Z}\sqcap$ (which, as a structure on $\mathcal{T}$ or $\mathcal{T}$, requires the global space $\mathbb{S}_{A'}$). Recall that in TN 16, 17, 21 I considered the forms $\bar{Z}\sqcap\bar{Z}\sqcap\mathcal{D}Z$, $\bar{Z}\sqcap\bar{Z}\sqcap d^4Z$ and things like $f_{-6}\bar{Z}\sqcap\bar{Z}\sqcap\mathcal{D}Z$, where $\mathcal{D}Z = \frac{1}{6}\overline{Z d Z \wedge dZ \wedge dZ}$ and $d^4Z = \frac{1}{24}\overline{dZ \wedge dZ \wedge dZ \wedge dZ}$, the point being that $\bar{Z}\sqcap\bar{Z}\sqcap\mathcal{D}Z$

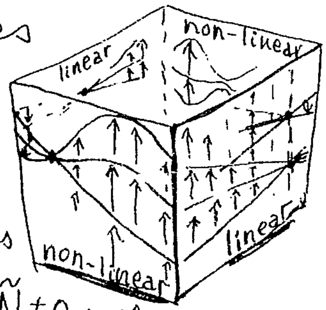
and $Z\rceil Z\rceil\, d^4Z$ are finite on $\mathbb{I}$, whereas $\mathcal{D}Z$ and d^4Z are not, and moreover $d(fZ\rceil Z\rceil\mathcal{D}Z)=0$ iff f has homogeneity degree -6. The present point of view is that we need the more local structure afforded by $Z dZ$, or say $dZ \wedge dZ$, in order to allow $\mathcal{S}_{A'}(\omega)$ to appear naturally, rather than just $\mathcal{S}_{A'}$. It is presumably significant, therefore, that the "form"

$$\Gamma Z = Z dZ \otimes \mathcal{D}Z$$

(which has Young symmetry $\boxplus$, the skew part "showing") is also finite on $\mathbb{I}$, and so is $Z dZ \otimes d^4Z$ (Young symm. $\boxplus$), whereas $dZ\wedge dZ \otimes \mathcal{D}Z$ and $dZ\wedge dZ \otimes d^4Z$ are not. Note that ΓZ is "almost" the blown-up twistor $ZZ\rceil$ itself — or, more exactly, $ZZ\rceil|||$ — where we snuff off the "legs" with dZs. I do not yet adequately see how to incorporate -6 functions in a natural way. Perhaps one needs some sort of invariant derivative which can act on $f_{-6}\Gamma Z$. (Simply "d" won't do, since it doesn't invariantly act on $\boxplus$ things. We got away with it before — more or less — because the "legs" of $Z\rceil Z\rceil\mathcal{D}Z$ were treated as constant, though this may well be inappropriate.) Note that $f\Gamma Z$ is Lie const. along Υ iff f is deg. -6. (Υ: Euler field)

To conclude, let us consider some relevant transformations of a flat twistor space $\mathbb{T}$. These correspond to: ⓐ supertranslations of $\mathbb{C}\mathscr{I}^+$ which are self dual (or should I call them anti-self-dual?) translations in the sense that they preserve the linear (or affine) structure of the β-planes on $\mathbb{C}\mathscr{I}^+$, and so preserve α-curves (whence they can act directly on $\mathbb{T}$); these are given by

$$Z \mapsto Z + \Phi \quad \text{where } \Phi \text{ is a function}$$



of $Z\rceil$ homogeneous of degree 1 — and: ⓑ transformations of $Z\rceil$ which send $\mathcal{S}_{A'}$ to things like $\mathcal{S}_{A'}(\omega)$ above, when $\tilde{N}\neq 0$; these are given by transformations of $\pi_{A'}$ preserving $\delta\pi = \pi_{A'}d\pi^{A'} = Z dZ$. The transformations ⓐ take the form

$$\begin{cases} \omega^A \mapsto \omega^A + \Omega^A(\pi_{A'}) \\ \pi_{A'} \mapsto \pi_{A'} \end{cases} \quad \Omega^A \text{ homogeneous deg. 1}$$

whereas ⓑ are generated by ⓐ and those of the form

$$\begin{cases} \omega^A \mapsto \omega^A \\ \pi_{A'} \mapsto \Pi_{A'}(\pi_{B'}) \end{cases}$$

where $\Pi_{A'}$ can be given by a generating function $G(\pi_{0'}, \Pi_{1'})$ homogeneous of degree 2, with $\pi_{1'} = \dfrac{\partial G}{\partial \pi_{0'}}$, $\Pi_{0'} = \dfrac{\partial G}{\partial \Pi_{1'}}$.

It is noteworthy that both ⓐ and ⓑ preserve both the forms $Z dZ$ and $\mathcal{D}Z$, so they certainly preserve ΓZ (and $dZ\wedge dZ$ and d^4Z and Υ).

Further work is in progress.

~ Roger Penrose

The Complex Structure of Deformed Twistor Space.

In their existence theorem for complex structures (cf. [1] or [2]), Kodaira and Spencer (KS) characterize deformed complex structures M_t w.r.t. a given fixed complex manifold M_0 by the condition

$$f \in \mathcal{O}(M_t) \text{ iff } (\bar\partial - \phi(t))f = 0 \quad , \qquad \text{—(1)}$$

where $f \in$ is a smooth function, $\bar\partial$ refers to the complex structure of M_0, and $\phi(t)$ is a vector-valued (0,1)-form (again, w.r.t. M_0). This approach was first used in twistor theory by GAJS ([3],[4]). In TN 22, LJM has further employed this approach to discuss the Kähler structure of asymptotic twistor space and pointed out an application to the googly graviton. I first considered a generalization of GAJS's treatment in my D. Phil. thesis as a possible way to shed light on the googly graviton in the non-asymptotic case. It may be useful to reconsider this.

As far as producing deformations of M_0 is concerned, the role played by (1) is through the Newlander-Nirenberg theorem ([5], [6]). On M_0, let (Z^α) be holomorphic coordinates. Then $\partial = \partial/\partial Z^\alpha \, dZ^\alpha$ and $\bar\partial = \partial/\partial \bar Z^{\alpha'} \, d\bar Z^{\alpha'}$ may be regarded as compact ways of writing what I'll call Cauchy-Riemann (C-R) vector fields: f is holomorphic if $\partial f/\partial \bar Z^{\alpha'} = 0$, which one may write as $\bar\partial f = 0$. Since (Z^α) are holomorphic coordinates, then for smooth f, $\bar\partial f$ is the (0,1) part of df i.e., $\bar\partial f$ in the "usual" sense. To deform M_0, define a deformed complex structure $J(t)$ on M_0 by specifying the splitting of the complex tangent bundle $\mathbb{C}T$ into (1,0) and (0,1) type: $\mathbb{C}T = T^{(1,0)}(t) \oplus T^{(0,1)}(t)$. For $T^{(0,1)}(t)$ to differ from $T^{(0,1)}(0)$, one will need to add elements of $T^{(1,0)}(0)$ to the generators of $T^{(0,1)}(0)$. Writing the deformed C-R vector fields compactly as vector-valued forms of type (0,1) (w.r.t. M_0) gives KS's $\bar\partial - \phi(t)$. Thus, one has $T^{(0,1)}(t) = \langle \bar\partial - \phi(t) \rangle$ and $T^{(1,0)}(t) = \langle \partial - \bar\phi(t) \rangle$. The integrability condition for this complex structure is KS's equation

$$\bar\partial \phi(t) = \tfrac{1}{2}[\phi(t), \phi(t)] \qquad \text{— (2)}$$

(which is just a compact way of writing the condition that the C-R vector fields Lie commute).

Now let $M_0 = \mathbb{T}$ and $(Z^\alpha) = (\omega^A, \pi_{A'})$ and put

$$\phi(t) = \phi^\alpha_{\beta'}(t) \, d\bar Z^{\beta'} \, \partial/\partial Z^\alpha \qquad \text{— (3)}$$

In addition to (2), the preservation of the usual NLG structures imposes conditions on $\phi(t)$. The preservation of the standard fibration requires holomorphicity of $\pi_{A'}$ i.e., $(\bar\partial - \phi(t))\pi_{A'} = 0$. The preservation of the Euler operator $\Upsilon = Z^\alpha \partial/\partial Z^\alpha$ and the two form $\mu = d\omega^0 \wedge d\omega^1$ requires the vanishing of their Lie derivatives along the C-R vector fields (i.e., the generators of $T^{(0,1)}(t)$ cf. [3], [4]). The holomorphic sections of the fibration invariant under Υ may be given as the zero set of the pair of functions $\omega^A - g^A(\pi_{B'}, \bar\pi_B)$ where

$$g^A(\lambda\pi_{B'}, \bar\lambda\bar\pi_B) = \lambda g^A(\pi_{B'}, \bar\pi_B) \qquad (\bar\partial - \phi(t))(\omega^A - g^A) = 0. \quad —(4)$$

Altogether, these conditions lead to

$$\phi(t) = \phi^A(t)\bar\pi^B d\bar\pi_B \, \partial/\partial\omega^A \quad —(5)$$

$$\Upsilon(\phi^A) = \phi^A \quad —(6)$$

$$\partial_A \phi^A = 0 \quad —(7)$$

$$\bar\partial_{B'}\phi^A = 0 \qquad \bar\pi_B \bar\partial^B \phi^A = -2\phi^A \quad —(8)$$

$(\partial_A = \partial/\partial\omega^A,\ \bar\partial_{A'} = \partial/\partial\bar\omega^{A'},\ \partial^{A'} = \partial/\partial\pi_{A'},\ \bar\partial^A = \partial/\partial\bar\pi_A)$, and the sections $\omega^A = g^A$ satisfy

$$\bar\partial^B g^A + \phi^A \bar\pi^B = 0$$

while $Z := -i g^A \bar\pi_A$ satisfies

$$\bar\partial^A \bar\partial^B Z = 2i\phi^A \bar\pi^B + i\bar\pi^B \bar\pi_C \bar\partial^A \phi^C \quad —(9)$$

$$\bar\partial^B Z = -ig^B + i\phi^C \bar\pi_C \bar\pi^B \quad —(10)$$

A simplification occurs when $\phi^A \propto \bar\pi^A$. With $\phi^A = -i\sigma(t)\bar\pi^A$, (8b) becomes $\bar\pi_A \bar\partial^A \sigma = -3\sigma$ while (7) becomes $\bar\pi^A \partial_A \sigma = 0$. Thus, σ depends upon ω^A only through $R = -i\omega^A \bar\pi_A$. This is GAJS's original version and (9) becomes the good cut equation $\bar\partial^A \bar\partial^B Z = \sigma \bar\pi^A \bar\pi^B$. In general, one does not seem to be able to interpret (9) as a generalized good cut equation on an auxillary space "$\mathbb{C}\mathcal{S}^+$". Thus, the complex structure of the deformed twistor space is prescribed by the splitting:

$$T^{(1,0)}(t) = \langle \partial/\partial\omega^A, \partial/\partial\pi_{A'} - \bar\phi^{B'}\pi^{A'}\partial/\partial\bar\omega^{B'} \rangle \quad —(11)$$

$$T^{(0,1)}(t) = \langle \partial/\partial\bar{\omega}^{A'}, \partial/\partial\bar{\pi}_A - \phi^B\bar{\pi}^A\partial/\partial\omega^B \rangle$$

One finds the dual splitting $\mathbb{C}T_\bullet = T_{(1,0)}(t) \oplus T_{(0,1)}(t)$ of 1-forms to be

$$T_{(1,0)}(t) = \langle d\omega^A + \phi^A\bar{\pi}^B d\bar{\pi}_B, d\pi_{A'} \rangle$$
$$T_{(0,1)}(t) = \langle d\bar{\omega}^{A'} + \bar{\phi}^{A'}\pi^{B'}d\pi_{B'}, d\bar{\pi}_A \rangle \qquad -(12)$$

Now the operators $\partial_t, \bar{\partial}_t$ on the space of smooth (p,q)-forms $\psi \in \mathcal{A}^{p,q}$ are defined as the projections of $d\psi$ into $\mathcal{A}^{p+1,q}$ and $\mathcal{A}^{p,q+1}$ respectively. Writing $df = \frac{\partial f}{\partial Z^\alpha}dZ^\alpha + \frac{\partial f}{\partial \bar{Z}^{\alpha'}}d\bar{Z}^{\alpha'}$ in terms of the given bases of $T_{(1,0)}(t)$ and $T_{(0,1)}(t)$ in (12), one finds

$$\partial_t f = \partial f - \bar{\phi}^{B'}\pi^{A'}d\pi_{A'}\,\partial f/\partial\bar{\omega}^{B'} + \phi^A\bar{\pi}^B d\bar{\pi}_B\,\partial f/\partial\omega^A$$
$$\bar{\partial}_t f = \bar{\partial} f - \phi^B\bar{\pi}^A d\bar{\pi}_A\,\partial f/\partial\omega^B + \bar{\phi}^{A'}\pi^{B'}d\pi_{B'}\,\partial f/\partial\bar{\omega}^{A'} \qquad -(13)$$

Note that f holomorphic $\Leftrightarrow$ $\bar{\partial}_t f = 0$ (of course)

$$\Leftrightarrow \quad \partial f/\partial\bar{\omega}^{A'} = 0$$
$$\frac{\partial f}{\partial\bar{\pi}_A} - \phi^B\bar{\pi}^A\,\partial f/\partial\omega^B = 0$$
$$\bar{\phi}^{A'}\pi^{B'}\,\partial f/\partial\bar{\omega}^{B'} = 0.$$

The last equation is redundant and the first two expressable as $(\bar{\partial} - \phi(t))f = 0$. It rapidly becomes tedious to compute ∂_t and $\bar{\partial}_t$ for larger values of p and q. Noting that

$$\xi^A := d\omega^A + \phi^A\bar{\pi}^B d\bar{\pi}_B = \partial_t\omega^A, \qquad -(14)$$

let $\psi = \psi_{A'}\bar{\xi}^{A'} + \psi^A d\bar{\pi}_A \in T_{(0,1)}(t)$. One may compute that

$$\partial_t\psi = \partial_A\psi_{B'}\,\xi^A\wedge\bar{\xi}^{B'} + (\partial^{A'}\psi_{B'} - \pi^{A'}\psi_{C'}\bar{\partial}_{B'}\bar{\phi}^{C'} - \pi^{A'}\bar{\phi}^{C'}\bar{\partial}_{C'}\psi_{B'})\,d\pi_{A'}\wedge\bar{\xi}^{B'}$$
$$+ (\partial^{A'}\psi^B - \pi^{A'}\psi_{C'}\bar{\partial}^B\bar{\phi}^{C'} - \pi^{A'}\bar{\phi}^{C'}\bar{\partial}_{C'}\psi^B)\,d\pi_{A'}\wedge d\bar{\pi}_B + \partial_A\psi^B\,\xi^A\wedge d\bar{\pi}_B$$
$$\bar{\partial}_t\psi = \bar{\partial}_{A'}\psi_{B'}\,\varepsilon^{A'B'}\,\bar{\xi}^{0'}\wedge\bar{\xi}^{1'} + (\bar{\partial}_{A'}\psi^B - \bar{\partial}^B\psi_{A'} + \phi^C\bar{\pi}^B\partial_C\psi_{A'})\,\bar{\xi}^{A'}\wedge d\bar{\pi}_B$$
$$+ (\bar{\pi}^B\phi^C\partial_C\psi^A - \bar{\partial}^B\psi^A)\,\varepsilon_{AB}\,d\bar{\pi}_0\wedge d\bar{\pi}_1 \qquad -(15)$$

Consider the function $\Sigma = \omega^A\bar{\pi}_A + \bar{\omega}^{A'}\pi_{A'}$. For small deformations, $\partial_t\bar{\partial}_t\Sigma$ will define a pseudo-Kähler metric of signature $(++--)$. Using preceeding formulae, one finds

$$\partial_t\bar{\partial}_t\Sigma = (\varepsilon_A{}^B - \bar{\pi}_C\bar{\pi}^B\partial_A\phi^C)\,\xi^A\wedge d\bar{\pi}_B$$

$$-\left(\pi^{A'}\pi_{C'}\bar{\partial}^{B}\bar{\phi}^{C'} + \bar{\pi}_{C}\bar{\pi}^{B}\partial^{A'}\phi^{C}\right) d\pi_{A'}\wedge d\bar{\pi}_{B} \qquad -(16)$$
$$+\left(\varepsilon_{B'}{}^{A'} - \pi^{A'}\pi_{C'}\bar{\partial}_{B'}\bar{\phi}^{C'}\right) d\pi_{A'}\wedge\bar{\xi}^{B'}$$

In the "asymptotic" case $\phi^A \propto \bar{\pi}^A$, one recovers LJM's results in TN 22 :

$$\partial_t\bar{\partial}_t\Sigma = \xi^A\wedge d\bar{\pi}_A + d\pi_{A'}\wedge\bar{\xi}^{A'} \qquad -(17).$$

$DZ^\alpha := (\xi^A, d\pi_{A'})$ is the frame for $T_{(0,0)}(t)$ employed throughout. It is not a holomorphic frame ($\bar{\partial}_t\xi^A = \bar{\partial}_t\partial_t\omega^A$ $= -(\bar{\pi}^C\partial_B\phi^A d\bar{\pi}_C\wedge\xi^B + \bar{\pi}^C\partial^{B'}\phi^A d\bar{\pi}_C\wedge d\pi_{B'}) = -\bar{\pi}^C\frac{\partial\phi^A}{\partial Z^\alpha}d\bar{\pi}_C\wedge DZ^\alpha$) and, in the non-asymptotic case, this frame is not trivially related to a pseudo-unitary frame (i.e., the components $g_{\alpha\beta'}$ of the Kähler metric are not constant w.r.t. this frame). The Kähler connection seems to be more complicated in general than in the asymptotic case as described by LJM.

In any event, this generalization may prove useful in studying the googly graviton.

Peter Law

[1] Morrow and Kodaira: "Complex Manifolds" . 1971

[2] Kodaira: "Complex Manifolds and Deformations of Complex Structures" (Springer-Verlag) 1986.

[3] K. P. Tod in TN 9

[4] MGE and KPT Math. Proc. Camb. Phil. Soc. 92 (317–330) 1982

[5] Newlander and Nirenberg Ann. Math. 65 (391–404) 1957.

[6] Folland and Kohn: "The Neumann Problem for the $\bar{\partial}$ Cauchy-Riemann Complex" (P.U.P.) 1972.

Cohomology on $\mathbb{A}$ and obstructions

Let $\mathbb{A} = \{ Z \cdot W = 0 \} \subset \mathbb{PT} \times \mathbb{PT}^*$ be the space of null geodesics in $\mathbb{C}M$ (compactified), commonly called Ambitwistor Space. Pick an infinity twistor $I_{\alpha\beta}$ & let $M \subset \mathbb{C}M$ be the complement of the light cone of the corresponding point ι. Define

$$V^W = \{ W_\alpha I^{\alpha\beta} = 0 \} \cap \mathbb{A}, \quad V_Z = \{ Z^\alpha I_{\alpha\beta} = 0 \} \cap \mathbb{A}$$

These are resp. the pre-images in $\mathbb{A}$ of the lines I in $\mathbb{P}^*$ & $\mathbb{P}$ under the evident projection. It is not hard to see that both are biholomorphic to $\mathbb{P}_3$ blown up along a line. They correspond to null geodesics in the light cone of ι lying in a β- or α-plane through ι, & so represent α-curves or β-curves in $\mathscr{I}$. That is, they are the asymptotic twistor spaces of $\mathscr{I}$. Then, evidently, $A := \mathbb{A} - (V^W \cup V_Z)$ corresponds to M and cohomology on A corresponds to solutions of differential equations on M. Generally, these are rather weak. For instance, (evaluating via the Penrose Transform):

$$H^1(A, \mathcal{O}) \cong \Omega^1_M / d\mathcal{O} = \text{closed 2-forms on } M$$

This space is acted on by $sl(4,\mathbb{C})$, interpretted as conformal Killing Vector fields on M, acting by Lie differentiation. Under this action, $H^1(A,\mathcal{O})$ is reducible: this is easy to see, for equations such as $d^*F = 0$ $(F \in \Omega^2)$ are conformally invariant, and so single out subspaces of $H^1(A,\mathcal{O})$.

None of this is surprising to a representation theorist: FACT 1 $H^1(A,\mathcal{O}) \cong H^2_{V^W \cup V_Z}(\mathbb{A},\mathcal{O})$ (by usual arguments). FACT 2 A very beautiful theorem of Beilinson and Bernstein [1] asserts that if $\mathcal{F}$ is positive (in some sense), if $X \cong G/P$ is complex homogeneous and $Y \subset X$ is closed then $H^*_Y(X,\mathcal{F})$ is irreducible as a $\mathcal{U}(\mathfrak{g})$ module iff Y is irreducible as a projective variety (i.e. not a union of closed, proper subvarieties).

This suggests we analyze cohomology on A more closely, relative to V^W and V_Z separately. The tool is the Mayer-Vietoris sequence for $\mathbb{A} - V^W = U_1$ and $\mathbb{A} - V_Z = U_2$. As part of this, we require $H^*(\mathbb{A} - V^W, \mathcal{O})$ (say) which is easily computed by taking direct images on $\mathbb{PT}^*$:

$$H^1(\mathbb{A} - V^W, \mathcal{O}) \cong H^1(\mathbb{PT}^* - \mathbb{L}_\iota, \mathcal{O}) \cong \ker d : \Omega^2_- \to \Omega^3 \quad \left\{\begin{matrix}\text{others vanish} \\ \text{or are just } \mathbb{C}\end{matrix}\right\}$$

The result is a decomposition into irreducibles (by [1], V^W, V_Z & $\mathbb{Q}_L$ being irreducible)

$$0 \to \begin{matrix} (\ker d : \Omega^2_- \to \Omega^3) \\ \oplus \\ (\ker d : \Omega^2_+ \to \Omega^3) \end{matrix} \to H^1(A,\mathcal{O}) \xrightarrow{\varphi} H^3_{\mathbb{Q}_L}(\mathbb{A},\mathcal{O}) \to 0$$

The last group is really $H^2(\mathbb{A} - \mathbb{Q}_L, \mathcal{O})$; $\mathbb{Q}_L = V^W \cap V_Z$ of course. But the relative cohomology group is easily identified. It is irreducible & one may compute its character directly to find

$$H^3_{\mathbb{Q}_L}(\mathbb{A},\mathcal{O}) \cong \ker d : \Omega^3 \mapsto \Omega^4$$

On M, if $[\omega] \in H^1(A,\mathcal{O})$ is represented by a 1 form ω, then $\varphi : \omega \mapsto d^* d\omega$.

Remarks :

1. This solves the extension problem for $H^1(A, \mathcal{O})$. For the kernel of φ consists of elements of $H^1(A, \mathcal{O})$ which extend to all of $\mathbb{P} - \mathbb{L}_I \times \mathbb{P}^* - \mathbb{L}_I$, as one easily sees by taking direct images & applying the Künneth formula. It follows that the obstruction to extending $[\omega] \in H^1(A, \mathcal{O})$ is its current, $d^*d\omega = J$. This is the Abelian case of the Witten-et-al description of non-self-dual Yang Mills fields (since $[\omega]$ exponentiates to a line bundle in $H^1(A, \mathcal{O}^*)$, trivial along quadrics). Actually, the calculation above gives an <u>intrinsic</u> characterization of such a field, independent of the embedding of A, and hence valid also on an arbitrary curved background spacetime. In a curved A, associated to a curved conformal manifold with $\mathscr{I}$, V^W and V_Z analogues are provided by the asymptotic twistor spaces $\mathbb{P}\mathscr{I}$ and $\mathbb{P}\mathscr{I}^*$ and the Mayer Vietoris argument may still work.

2. The calculation works for any $\mathcal{O}(k, k)$, $k > 0$, also. $\mathcal{O}(1,1)$ yields (another) proof of the twistor characterization of linearized conformal gravity.

There are evident & tantalizing links with two constructions. Firstly, $V^W \cup V_Z$ is the variety defined by the section $\overline{Z}dZ \bmod \frac{W}{dZ}$ of LeBrun's Einstein bundle. In the curved case, therefore, one might expect the Einstein bundle to come as an extension of the normal bundle of $V^W \cup V_Z$, with a section defining the variety. The associated spacetime would then be Einstein. Secondly, the group $H^3_{\mathbb{Q}_L}(A, -)$ is closely related to $H^2_{\mathbb{Q}_L}(V^W, -)$. RP. has suggested [2] that such groups should be related to deformations of $\mathbb{Q}_L$ sitting in V^W, and so to the googly question. $\mathbb{Q}_L$ in V^W is morally related to "fat" $\mathbb{I}$, and further work is therefore in progress.

[1] Beilinson + Bernstein CR Acad Sci (Paris) t 292 (1981) 15 – 18

[2] RP, TN 17 (1984) 7–8

Rob Baston

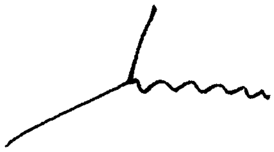
A New Programme for Twistor Diagram Theory

The "new programme" suggested in this note is not so much new as an attempted renewal of the original Penrose theory. The idea is that we should take seriously the proposal that twistor diagrams are analogous to Feynman diagrams. By "analogous" I mean that they should provide a complete prescription for evaluating quantum field-theoretic scattering amplitudes to any order (in principle, at least), and that the diagrams for any process should be given by some quite simple sort of combinatorial rule which can be interpreted as a summing over all possible intermediate states. It's not expected, however, that individual twistor diagrams should be *translations* of individual Feynman diagrams; quite the reverse. The whole idea is to avoid the difficulties of Feynman diagrams - their divergences and gauge-dependence. Nevertheless, the amplitudes calculated by twistor diagram theory must (essentially) agree with those specified by Feynman diagram theory.

A physicist could well take a more drastic line than this: unless twistor diagram theory can come up with such a complete prescription, there is little point to pursuing it; the theory as it currently stands has no predictive value. It merely offers a sort of description of some first-order amplitudes. Over the years the hope has been expressed that by studying twistor diagrams which correspond to first-order calculations in quantum field theory, new features will emerge which can be generalised to suggest a complete description. After some fifteen years of looking at first-order calculations perhaps it is time to have a stab at suggesting what these features might be.

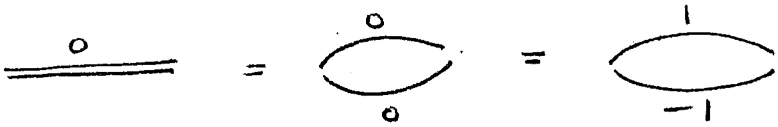
In what follows I shall discuss twistor diagrams disregarding all the mathematical difficulties associated with defining them, and carry on as if the "lines" of the twistor diagrams were well-understood objects (which of course they are not; for recent advances in understanding the "philosophy" of defining and evaluating twistor diagrams, see the article in this **TN** by SAH and MAS .) The present line of thought starts from looking at the form taken by these ill-understood "lines" of the twistor diagrams for first-order massless QED, and seeing whether we can discern any formal analogy with (as opposed to translation of) the structure of Feynman diagrams.

Now in the Feynman diagrams for QED every vertex looks like



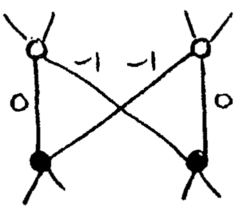
corresponding to the $\bar{\psi} A \psi$ term in the Lagrangian. The Feynman diagram calculus can be seen as the rule: Using such vertices alone, write down all possible diagrams which connect the given in- and out-states, then evaluate them and sum over them. Is there an analogy in the form of the QED twistor diagrams? If so, the vertices of twistor diagrams should likewise have some standard form. Do they?

Well, perhaps! The obvious starting-point is the rule of "four lines at a vertex" - which now becomes a essential part of the theory, not just a notational convention. But if we take this seriously then we have to believe that double lines can be considered as the composition, in some sense, of two single lines. Moreover this composition must satisfy e.g.

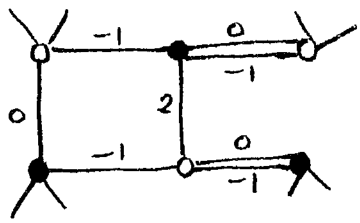


etc.

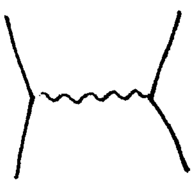
If we are prepared to believe that this can be given mathematical sense, then it turns out that we can indeed identify a feature of all the QED diagrams known, namely



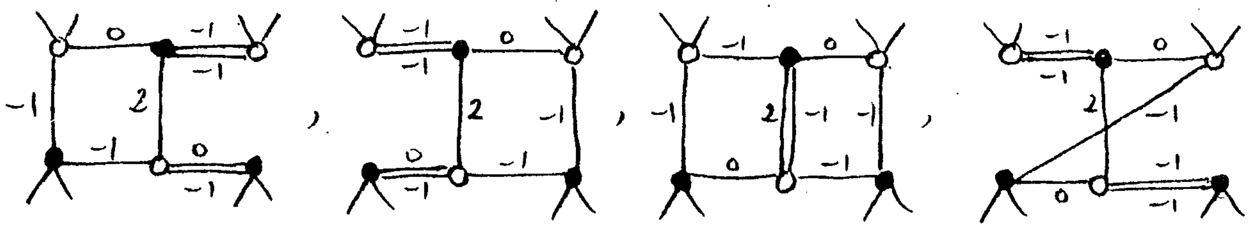
or



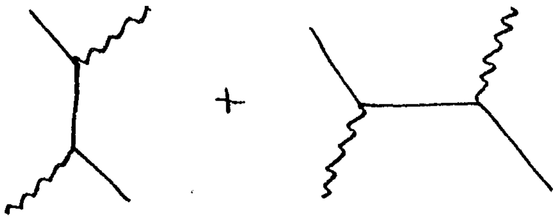
for Møller scattering of massless electrons, represented by Feynman diagram



and

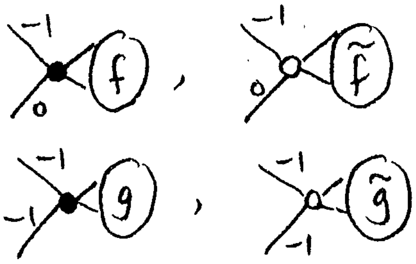


any one of which can serve to give the correct (gauge-invariant) amplitude for Compton scattering, represented in the Feynman diagram picture by the sum of the two (gauge-dependent) terms



In all these diagrams, the vertices have a common form:

provided we accept also the following rule: an *external* field may be attached thus to a vertex:



(electron)

(photon)

The picture this suggests is of an "interaction" generated not by the space-time quantum field operator

$$\bar{\psi} A \psi$$

but by the twistor quantum field operator

$$a_{-1}(Z^\alpha)a_{-1}(Z^\alpha)a_0(Z^\alpha)a_2(Z^\alpha)\,\tilde{a}_{-1}(W_\alpha)\tilde{a}_{-1}(W_\alpha)\tilde{a}_0(W_\alpha)a_2(W_\alpha)$$

where the lines are (anti-)commutators:

$$[a_0(Z^\alpha), \tilde{a}_0(W_\alpha)] \approx W \overset{0}{\text{———}} Z \quad \text{etc.}$$

(here the suffices indicate homogeneities according to the same convention as the labelling of twistor diagram lines).

The general idea is that electrons and photons are only clearly distinguished as *external* states; the internal dynamics is described by propagators not of particle states but of "twistor states" which are in some sense *square roots* of particle states - and *more fundamental* than particles. This also accords with the point of view advanced in **TN** 22 ("Elemental states"). This view also encapsulates the very striking and elegant feature of the first-order twistor diagrams which gives the present theory much of its interest: propagators in the Feynman diagrams *factorise* into two twistor diagram lines (doubled up, for on-shell, separated for off-shell).

However, all this only tells us that a Feynman diagram analogy is not absolutely *ruled out* by the form of the twistor diagram vertices. It does not suggest any coherent scheme for generating twistor diagrams by a combinatorial rule. Point one: these QED results don't bring in any idea of *summing over* different diagrams generated by these vertices. Point two: there are other simple diagrams which satisfy the vertex rule but which don't appear above, namely

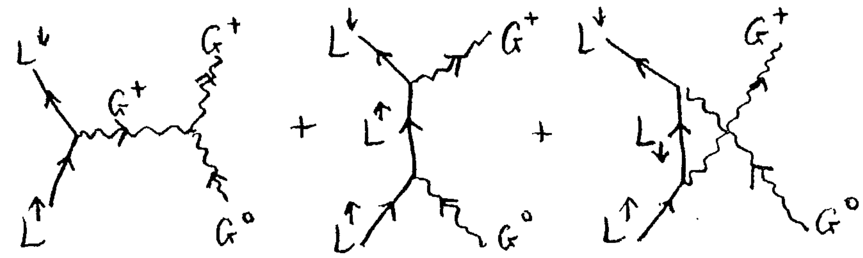
(1), (2)

Some new investigation began with the guess that if these mean anything at all, they must involve the interaction of "charged photons". We might therefore see whether they find a place in the SU(2) gauge theory - where the vector gauge fields are themselves charged.

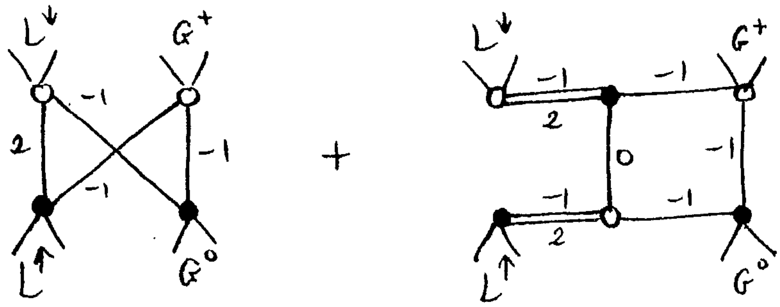
In what follows we shall study the standard model of SU(2) gauge theory with *unbroken symmetry.* In this theory we have leptons with spin 1/2 and isospin 1/2, and a spin-1 gauge field with local SU(2) symmetry. In the usual discussion the leptons are identified from the beginning with the neutrino and the left-handed part of the electron - but for discussing the pure SU(2) theory it is better to think of them only as up-lepton and down-lepton, $L^{\uparrow}$ and $L^{\downarrow}$. Similarly the gauge field becomes identifiable (when the symmetry is broken) with the appearance of physical W-bosons W^{+},W° W^{-} but I shall avoid a spurious identification with phenomenology and write G^{+}, G°, G^{-}.

In the SU(2) theory there are new first-order processes, arising from the capacity of the vector boson to carry isospin-charge. Thus in the analogy to Coulomb scattering we have contributions from *three* Feynman diagrams. For example:

$L^{\uparrow}G^{\circ} \to L^{\downarrow}G^{+}$ by:



It is a lengthy but comparatively straightforward process to show that this amplitude can be represented by a <u>sum</u> of the *two* twistor diagrams*



So we find that the diagram (1) does indeed have a place in the SU(2) theory. This suggests strongly that in looking for a scheme for twistor diagrams, we should no longer restrict ourselves to massless QED but extend to a theory combining at least U(1) and SU(2) gauge fields.

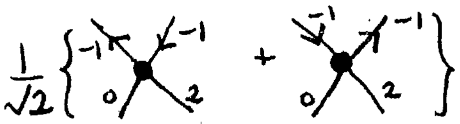
*There are many difficulties glossed over here. E.g. (a) the SU(2) vector fields, like U(1) photons, appear in these diagrams only in their separate left and right-handed parts. There is no natural way of adding these two parts; they behave like independent fields, independently conserved. Also (b) all the Feynman diagrams here are actually infra-red divergent, and the twistor diagrams correspondingly can only make sense via a regularisation with inhomogeneous propagators.

Note that just as in the U(1) Coulomb scattering case, the individual Feynman diagrams are gauge-dependent, although their sum is gauge--invariant. In contrast, the two twistor diagrams which now have to be summed are each separately (and manifestly) gauge-invariant. These sums agree - but it must be stressed that there is no correspondence at all between the individual terms.

However, we have not yet expressed the full gauge-invariance of the theory. We can go a stage further by studying all the possible LG $\rightarrow$ LG processes at once, instead of picking out the specific isospin components which appear in the amplitude just described. If we do this we find that the coefficients of the two twistor diagrams become, in general, the SU(2) invariants

$$\overset{in}{L}_I \overset{in}{G}_{JM} \overset{out}{L}_M \overset{out}{G}_{IJ} \, , \quad \overset{in}{L}_I \overset{in}{G}_{IJ} \overset{out}{L}_M \overset{out}{G}_{JM} \quad \text{respectively}$$

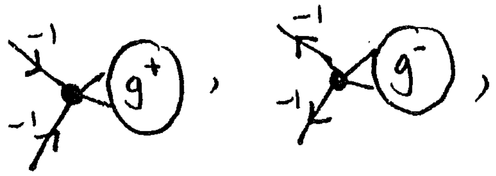
where the indices are SU(2) isospinor indices. Remarkably, these index contractions correspond precisely to *contracting along the (-1)-lines* in each diagram! This can be represented graphically in the diagram formalism by adopting the following rule:

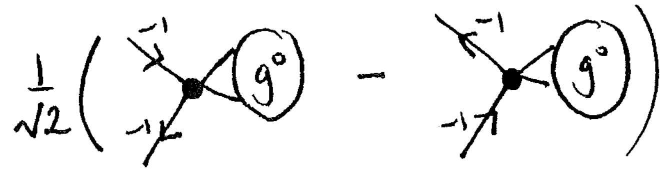
the vertices are written  , etc.

(i.e. they are isoscalars)

the external leptons appear as  etc.

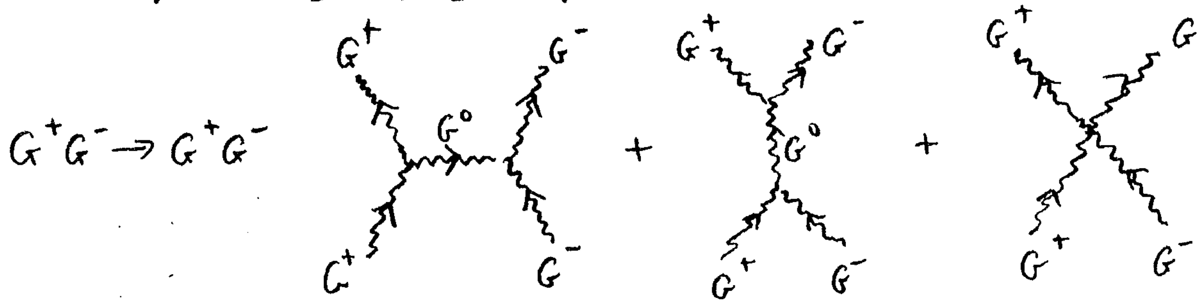
(according to isospin)

and the external G-bosons 

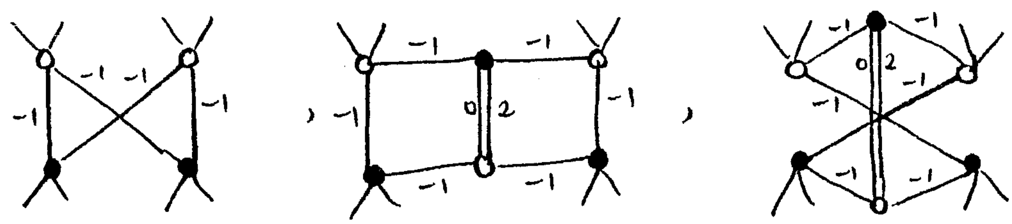
(according to isospin) 

There is now an easy way to write down the correct diagrams for any assignment of isospins, for the rule implies that a diagram has a zero coefficient unless an arrow can be drawn on each (-1) line so as to connect the specified in-states and out-states consistently, the direction of the arrow being conserved at each vertex. (In particular "arrow conservation" automatically incorporates iso-spin conservation.)

We can continue to test this conjectured rule by carrying out the more complicated calculation for GG→GG scattering. A typical process, with its Feynman diagrams, is given by:



Again we can formulate the amplitude for all isospin assignments, and it turns out that can indeed be put in a form consistent with the proposed rules: the amplitude is a sum over the twistor diagrams



with SU(2) coefficients given by the "arrow rule".

Thus, we find a place for the 'missing' diagram (2) above. Again, the separate twistor diagrams have no correspondence whatsoever with the separate Feynman diagrams. (A particular example of this is the case of $G^0 G^0 \to G^0 G^0$; where we find that all three twistor diagrams are non-zero but sum to zero; the Feynman diagrams for this case are however all zero.)

We may also check that the LL → LL scatterings are correctly described by the same rules.

Although these are isolated identities, not following from any general result, they do corroborate the idea of a twistor diagram/Feynman diagram analogy. The U(1) theory is too limited to betray a pattern; the results might tend to suggest that an amplitude should be represented by *one* twistor diagram. By studying the SU(2) theory we find ourselves naturally forced into summation over more than one. Furthermore, we retain and strengthen the idea of a standard vertex analogous to the Feynman vertex.

It's also good for morale to compare the quite *complicated* Feynman rules for SU(2) theory vertices, which involve the isospin indices in a far-from-transparent way, with the very *simple* twistor diagram vertex in which the gauge-field indices are always carried on the (-1) lines.

Of course, we have only justified this comparison for the *first-order* theory, whereas the whole potency of the Feynman rules lies in their prediction of amplitudes to all orders. And it must be remembered that we don't yet have a rule for generating the twistor diagrams for an amplitude, only an indication that the diagrams aren't totally inconsistent with the idea of a combinatorial vertex-based rule! In particular, there is an obvious question as to what the "order" of a twistor diagram is supposed to be: the combinations written down above mix up diagrams with four and six vertices. However, at least we now have a non-trivial pattern in the first-order twistor diagram theory which it seems worth trying to define consistently and so extend to all orders.

It is noteworthy that the "electroweak" theory seems to offer so helpful a line of advance, and I would suggest that as well as taking twistor diagrams more seriously, twistorians might also take the "standard model" of quantum field theory more seriously. It is a striking fact that data from experimental physics has driven theory to the picture of massless leptons and quarks, and a massless Higgs scalar, which provides a sort of foundation within which internal-symmetry-breaking and conformal-symmetry-breaking are supposed to occur. After all, such a picture is remarkably close to the twistor-theoretic belief that conformal structure, massless fields, and left-right asymmetry, are fundamental. My hope is that twistor diagrams provide a proper [manifestly finite, conformally invariant, gauge-invariant] calculus for this *theory with unbroken symmetry*. (Actually, I would prefer to take a stronger line and hope that the standard model will appear as an effort to force into the space-time picture, a theory that really only makes sense in the twistor picture!) Certainly the $\bar{\psi}_L \phi \psi_R$ Higgs-fermion and the ϕ^4 Higgs self- interactions, and hence all the interactions appearing in the standard model with unbroken symmetry, can fit into the twistor diagram formalism, though not with the (-1,-1,0,2) vertices. (Indeed, it appears that these interactions supply an application for a number of diagrams hitherto not considered of direct relevance to physical theory). There seems a chance too of applying twistor diagrams (though perhaps in a perhaps less *fundamental* way) in the theory with *broken* symmetry. We already have the twistor--diagrammatic way of getting mass eigenstates as an infinite sum of terms, and perhaps this can be understood as the counterpart to the "standard model" picture of how mass is imparted to the electron by means of the $\bar{\psi}_L \phi \psi_R$ interaction term in the symmetry-broken theory.

The results so far obtained suggest many more questions for investigation, in particular the study of the other gauge symmetries, including SU(3), of the standard model. I also believe they are robust enough to supply some useful and stimulating guidelines for the *mathematical* programme. These are some of the suggestions:

- There will be a class of 'physical' twistor diagrams defined by some rule about vertices and the attachment of external states. For diagrams in this class there should be a uniquely defined rule for finite evaluation, *including overall sign*. Vacuum diagrams can be expected to fall in this class.
- The existing formalism of evaluation by contour integration must, of course, be superseded by a consistent scheme in which the external states are H^1's of the appropriate spaces. To express crossing symmetry, such a scheme must allow for evaluation in every channel, i.e. with every possible allocation of external states as in- or out-states.
- The "lines" should have a context-independent meaning, and should be interpretable as commutators of twistor-field-theoretic operators. Composing two (or more) copies of a line, to get a multiple line, should be a well-defined operation. It may be that this involves some form of "regularisation", to appear even in the supposedly 'simple' scalar product of free field theory which causes us such trouble.
- The interpretation of the (-1)-lines is intimately concerned with the nature of gauge fields. They must be understood as canonical inverse derivatives somehow giving a gauge-invariant description of gauge fields.
- The diagram calculus should be seen as the perturbative order-by-order calculation of quantities which could in principle be expressed in a non-perturbative way. That is, there should be some analogy to the Lagrangian which supplies the principle for Feynman diagram expansion. We would expect this principle to create a link between twistor diagram theory and other parts of twistor theory, notably those expressing fields as deformations of twistor spaces. **Andrew Hodges**

TWO PHILOSOPHIES FOR TWISTOR DIAGRAMS

1. Duality on Manifolds

Let V be a (C^∞, paracompact oriented) manifold of real dimension n. Then there is a number of ways of investigating the topology of V from the "homology" rather than the "homotopy" point of view. One of the most natural things to do is to define the cohomology of V in terms of the de Rham complex of C^∞ differential forms:

$$\Omega^0(V) \xrightarrow{d} \Omega^1(V) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(V). \tag{1.1}$$

Since $d^2 = 0$, the definition of the p-th cohomology group of V by means of the formula

$$H^p(V) = \ker(d:\Omega^p(V) \to \Omega^{p+1}(V))/d\Omega^{p-1}(V) \tag{1.2}$$

makes good sense.

Another popular invariant of V is singular homology: this is the familiar homology theory of finite chains, cycles and boundaries, which is defined from the complex

$$C_n(V) \xrightarrow{b} C_{n-1}(V) \xrightarrow{b} \ldots \xrightarrow{b} C_1(V) \xrightarrow{b} C_0(V) \to 0$$

where we have written b for the boundary operator - ∂ is also in common use - and $C_p(V)$ for the vector space of finite smooth singular p-chains in V. Since $b^2 = 0$, we set

$$H_p(V) = \ker(b:C_p(V) \to C_{p-1}(V))/bC_{p+1}(V).$$

By the "fundamental theorem of exterior calculus" the pairing $\langle\omega,c\rangle$ of a p-form ω with a p-chain c defined by the formula

$$\langle\omega,c\rangle = \int_c \omega$$

satisfies the relation

$$\langle d\omega,c\rangle = \langle\omega,bc\rangle$$

and it follows that $<,>$ also induces a pairing between $H^p(V)$ and $H_p(V)$. This pairing is always non-degenerate, (Poincaré duality), and, if $H^p(V)$ and $H_p(V)$ are finite dimensional vector spaces, it identifies each with the dual of the other. In particular, the space of linear functionals on $H^p(V)$ is $H_p(V)$.

There is, however, an alternative route to the definition of $H_p(V)$ which proceeds via the complex of C^∞ differential forms on V *with compact support* :

$$\Omega_c^0(V) \xrightarrow{d} \Omega_c^1(V) \xrightarrow{d} \ldots \xrightarrow{d} \Omega_c^n(V).$$

This makes sense since if ω has compact support in V, then so does $d\omega$; d^2 still equals 0 so we set

$$H_c^p(V) = \ker(d:\Omega_c^p(V) \to \Omega_c^{p+1}(V))/d\Omega_c^{p-1}(V)$$

(c.f. (1.1) and (1.2)).

If $\omega \in \Omega^p$ and $\eta \in \Omega_c^{n-p}$ then we may define

$$<\omega, \eta> = \int_V \omega \wedge \eta$$

Once again, the fundamental theorem of exterior calculus guarantees that $<,>$ induces a pairing between $H^p(V)$ and $H_c^{n-p}(V)$: this pairing (usually) identifies each of these spaces with the dual of the other.

Thus any linear functional on $H^p(V)$ that can be defined by a homology class *or* by a compact cohomology class can in fact be defined by *both* a homology and a compact cohomology class. This observation elucidates (at least in abstract) the relationship between the two philosophies for twistor diagrams as described in the next section.

A rather satisfactory framework for the discussion of the apparently different ways of viewing homology and cohomology involves the use of currents and was pioneered by de Rham. Recall that a current is an appropriately continuous linear functional on the vector space $\Omega_c(V)$ of all C^∞ forms with compact support in V. Thus the space of currents is large enough to contain all forms *and* all chains on V. The notion of continuity mentioned above permits one to extend d and b to currents (in fact $d = \pm b$ on (homogeneous) currents); and hence to define the cohomology of V in terms of the complex of all currents on V. Similarly, we define the homology or compact theory of V in terms of the complex of currents with compact support in V.

These theories are Poincaré dual to each other and it can be proved that every cohomology (homology) class can be represented by both a C^∞ (compactly supported) differential form and by a (finite) cycle on V. The proof of the existence of C^∞ representatives rests on the fact that currents can be "smoothed" while representatives that are cycles are constructed from an arbitrary fixed polyhedral subdivision of V.

Much of what has been described above has an analogue for analytic cohomology: if V is now a complex manifold of complex dimension n, the Dolbeault group $H^{p,q}(V) \equiv H^q(V,\Omega^p)$ is defined as the q-th cohomology group of the complex

$$\Omega^{p,q}(V) \xrightarrow{\bar\partial} \Omega^{p,1}(V) \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \Omega^{p,n}(V) \tag{1.3}$$

where, as usual, $\Omega^{p,q}(V)$ is the space of complex-valued C^∞ forms of degree p in the holomorphic directions and of degree q in the anti-holomorphic directions. If (1.3) is twisted by tensoring with a holomorphic vector bundle E, then this definition gives the Dolbeault cohomology groups $H^{p,q}(V,E) \equiv H^q(V,\Omega^p(E))$ of V with coefficients in E.

As above, we may consider also the cohomology of Dolbeault forms with compact support in V: then using a subscript c to denote "compact support" as before we note that if $\omega \in \Omega^{p,q}(V;E)$ and $\eta \in \Omega_c^{n-p,n-q}(V;E^*)$ (where E* is the dual bundle of E) then $\omega \wedge \eta \in \Omega_c^{n,n}(V)$ and so there is a pairing

$$\langle\omega,\eta\rangle = \int_V \omega \wedge \eta.$$

This induces a pairing between $H^{p,q}(V;E)$ and $H_c^{n-p,n-q}(V,E^*)$; the Serre duality theorem then states that this (usually) identifies each of these spaces as the dual of the other.

We wish to conclude with one further observation about the (de Rham or Dolbeault) compact cohomology of an open subset W in a compact V. Let F = V-W and define the space of forms $\Omega(F)$ of C^∞ forms on F as the direct limit

$$\Omega(F) = \varinjlim_{U\supset F} \Omega(U) \qquad \text{(U open in V)}$$

so that a form on F is represented as a certain equivalence class of forms defined on neighbourhoods of F. Note that if F is a submanifold of V so that it admits an intrinsic definition of forms then this one will *not* in general coincide with it. An alternative definition of $\Omega(F)$ is in terms of the exactness of the following sequence

$$0 \to \Omega_c(W) \xrightarrow{\text{"extension by 0"}} \Omega(V) \xrightarrow{\text{"quotient"}} \Omega(F) \to 0$$

which gives rise, in cohomology, to the sequence

$$\ldots \to H^p_c(W) \to H^p(V) \to H^p(F) \to H^{p+1}_c(W) \to \ldots \qquad (1.4)$$

which can be understood to hold for both de Rham or Dolbeault cohomology. This sequence identifies, in effect, $H^p_c(W)$ with the relative cohomology group

$$H^p_W(V) \equiv H^p(V, V-W),$$

a fact that we shall use later on.

2.1 Philosophy 1[1],[2]

Regard all the internal lines as functions, i.e. as elements of $H^0(W.Z \neq 0;\ \mathcal{O}(-m))$. The external H^1s (elementary states based on lines, say, although we will see how to relax this) are then *dotted* together with all the internal H^0s. (The *dot product* $\alpha.\beta$ between $\alpha \in H^p(A;\ S)$ and $\beta \in H^q(B;T)$ is given by first cupping to get $\alpha \cup \beta \in H^{p+q}(A \cap B;\ S \otimes T)$ and then using the Mayer-Vietori's connecting map $m^* : H^{p+q}(A \cap B) \to H^{p+q+1}(A \cup B)$, so that $\alpha.\beta = m^*(\alpha \cup \beta)$.) Let e and v be the numbers of lines and vertices, and d(= 3v) the complex dimension, of a twistor diagram. Then in the case of a tree diagram e = d + 1 and (as long as all the parameters are in general position) thisprocedure of dotting everything in sight together is bound to yield an element of $H^d(X, \Omega^d) \cong \mathbb{C}$. (X is the whole space of the diagram; it is the product of v/2 $\mathbb{PT}$ s and v/2 $\mathbb{PT}^*$s.) But meaningful twistor diagrams are not trees. They have double lines (as in the scalar product) or loops (the box), and in either case e = d so the dotting method (if it works) only yields an element of H^{d-1}. In fact it is worse than that (as pointed out by RJB), because you cannot treat the two lines in a double line separately, as you get zero if you dot a thing with itself. So in the

scalar product case we merely obtain an element of $H^4(X\text{-}Y;\Omega^6)$, where Y, for any diagram, is the intersection of all the subspaces defined when all the lines in the diagram are made springy. (Here Y is the $\mathbb{CP}^1$ given by A.Z = B.Z = W.Z =W.C = W.D = 0). We would need an element $\mu \in H^1(U;\mathcal{O})$ (where U is a neighbourhood of Y) in order to complete the evaluation of the scalar product in this way. We will see in section 3 that it is possible to reinterpret μ as a piece of topology $\tau \in H^2_{X-Y}(X)$, which, by the remarks in section 1, is isomorphic to $H^2_c(X\text{-}Y)$.
The box diagram would yield an element of $H^{11}(X\text{-}Y;\Omega^{12})$. However, RJB has suggested [3] an interesting inerpretation of the dot product. Let

$$r : H^{p-1}(X\text{-}Y;S) \to H^p_Y(X;S)$$

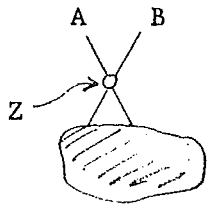
be the connecting homomorphism in the relative cohomology exact sequence. Then in some case, at least, the dot product is

$$\alpha.\beta = r^{-1}(r(\alpha) \cup r(\beta)).$$

Applying this idea to the box diagram (i.e. cupping the relative versions of all the lines together), but omitting to use the final r^{-1} map, yields an element of $H^{12}_Y(X;\Omega^{12})$ which is well worth investigation.

2.2 Philosophy 2

People who *actually* evaluate twistor diagrams do so by contour integrating a holomorphic function. This function is the *cup* product of all the lines in the diagram, thought of as functions (H^0s - even the external lines). It is therefore an element of $H^0(X\text{-}S;\Omega^d)$ (by the time you have included the traditional differential form). Here S is the *union* of all the subspaces defined by making the diagram's lines springy. The contour, an element of $H_d(X\text{-}S;\mathbb{C})$, has to be constructed by hand, although there have been cases where homology theory has helped! Actually this is philosophy 0, since it treats the external lines wrongly. An idea of RP's[4] was resurrected by MAS recently, and led to philosophy 2. We still *cup* everything in sight together, but now the external lines are treated properly, as H's. We get an element of $H^f(X\text{-}T;\Omega^d)$ (where 2f is the number of external lines), which can be regarded as an element of $H^{f+d}(X\text{-}T;\mathbb{C})$ (see [4], or use Dolbeault cohomology, simply forgetting the bidegree d,f and remembering only the total degree d+f). Now our contour (supercontour!) is an element of $H_{f+d}(X\text{-}T;\mathbb{C})$, where T is the union of all the subspaces defined by making internal lines springy, together with the $\mathbb{CP}^1$ subspaces determined by making each pair of external lines springy. This homology group is in general less complicated and more relevant than $H_d(X\text{-}S;\mathbb{C})$. Furthermore, there is an important relationship between the two, best explained by considering one external vertex of an arbitrary diagram:



$$S = \{A.Z=0\} \cup \{B.Z=0\} \cup R$$
$$T = (\{A.Z=0\} \cap \{B.Z=0\}) \cup R$$

There is a Mayer-Vietoris sequence (in homology now) for *each* such vertex:

$$\begin{tikzcd}
H_*(X\text{-}(\{A.Z=0\} \cup R)) \oplus H_*(X\text{-}(\{B.Z=0\} \cup R)) \arrow[r, dash] \arrow[dr, dash] & H_*(X\text{-}T) \arrow[d, "m"] \\
& H_*(X\text{-}S)
\end{tikzcd}$$

and in the cases so far studied (see section 3) themap m is either an isomorphism or an injection at each vertex, and m^f(supercontour) = usual contour. This is not too surprising, given that we applied the Mayer-Vietoris map in *cohomology* once at each external vertex in order to reinterpret $(A.Z\ B.Z)^{-1}$ as an H^1. (In other words, instead of cupping $(A.Z)^{-1}$ and $(B.Z)^{-1}$ together, we dotted them.) This suggests going further, and dotting everything in sight together (as in philosophy 1). In the scalar product case this yields an element of $H^4(X\text{-}Y;\Omega^6)$ which we can think of as in $H^{10}(X\text{-}Y;\mathbb{C})$. Now we need a superduper contour (?) $\kappa \in H_{10}(X\text{-}Y;\mathbb{C})$. Again Mayer-Vietoris maps take us back and forth between these various types of contour (see section 3). The advantage of using this last procedure is that it enables us to compare the two philosophies.

2.3 The relationship between Φ1 and Φ2

Φ1 (philosophy 1) needs a piece of topology $\tau \in H^2_c(X\text{-}Y)$, while Φ2 needs a superduper contour $\kappa \in H_{10}(X\text{-}Y)$. These groups are isomorphic, both being dual to $H^{10}(X\text{-}Y)$, but the details of this relationship between

τ and κ are not yet understood explicitly (In terms of currrents, τ and κ are simply the C^∞ representative and the representation by a cycle of the same class in the "compact theory" of X-Y). The corresponding relationship for the box diagram is even more obscure.

3. The scalar product diagram

3.1 Magic

Let

$$\varphi \in H^{0,1}(\mathbb{P} - L, \mathcal{O}(-2)) = H^1(\mathbb{P} - L, \mathcal{O}(-2))$$

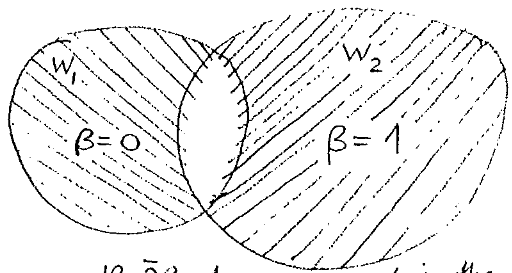
$$\varphi' \in H^{0,1}(\mathbb{P}^* - L', \mathcal{O}(-2)) = H^1(\mathbb{P}^* - L', \mathcal{O}(-2))$$

where L and L' are skew lines in twistor space. Throughout this section we shall confuse appropriate differential forms with the cohomology classes that they represent. Then the dot product of φ with φ'

$$\Phi = \varphi.\varphi' \in H^{0,3}(X - L \times L'; \mathcal{O}(-2,-2)) \tag{3.1}$$

where we have written $X = \mathbb{P} \times \mathbb{P}^*$. In the previous section the dot product between cohomology classes was defined - at the level ofDolbeault forms, it works as follows.

We use $\beta, \beta_1, \beta_2, \ldots$ etc. to denotebump functions on our spaces, that is C^∞ functions with values in [0,1] equal to 1 on some closed set and equal to 0 on some other (disjoint!) closed set. Let W_1 and W_2 be open subsets with $W_1 \cap W_2 \neq \phi$; let β be a bump function on $W_1 \cup W_2$ such that the support of β is contained in W_2 and such that $\beta = 1$ in a neighbourhood of $W_2 - W_1 \cap W_2$: then the support of $d\beta$ and of $\bar\partial\beta$ is contained in $W_1 \cap W_2$. Then if ω_i is a closed form on W_i ($i = 1,2$) the form $\omega_1 \wedge d\beta \wedge \omega_2$ is defined on all of $W_1 \cup W_2$ and represents $\omega_1.\omega_2$. Similarly, if the ω's are $\bar\partial$-closed Dolbeault forms, $\omega_1.\omega_2$ is represented on $W_1 \cup W_2$ by the $\bar\partial$-closed form $\omega_1 \wedge \bar\partial\beta \wedge \omega_2$.



$d\beta$, $\bar\partial\beta$ have support in the unshaded part of $W_1 \cap W_2$.

Thus Φ in (3.1) is represented by $\varphi \wedge \bar\partial\beta \wedge \varphi'$ where β is as defined in the above paragraph for

$$W_1 = (\mathbb{P} - L) \times \mathbb{P}^*, \quad W_2 = \mathbb{P} \times (\mathbb{P}^* - L')$$

and we have confused φ and φ' with their pull-backs to $X-L \times \mathbb{P}^* - \mathbb{P} \times L'$ (this is where RJB writes $\varphi \times \varphi'$). Nowby Serre duality, functionals on the spaceof the RHS of (3.1) appear as elements of

$$H^3_c(X-L\times L';\ \mathcal{O}(-2,-2)) \qquad (3.2)$$

where we have tacitly identified $\mathcal{O}(-4,-4)$ with Ω^6_X. To construct an appropriate Λ in (3.2) we demand that

$$\Phi \wedge \Lambda = \Phi.\lambda$$

for some (0,2)-form λ. In view of the above remarks this amounts to the demand that there be a factorization

$$\Lambda = \bar{\partial}\beta_1 \wedge \lambda \qquad (3.3)$$

where β_1 is a bump function with support in a neighbourhood N of $L \times L'$ and equal to 1 in a smaller neighbourhood of $L \times L'$. This factorization has an interesting interpretation in terms of the sequence (1.4), which in this situation, contains the following segment:-

$$\to H^2(X, \mathcal{O}(-2,-2)) \to H^2(L\times L', \mathcal{O}_X(-2,-2)) \to H^3_c(X-L\times L') \to H^3(X) \to \ldots$$

In this case $H^3(X) = 0 = H^2(X)$ so that

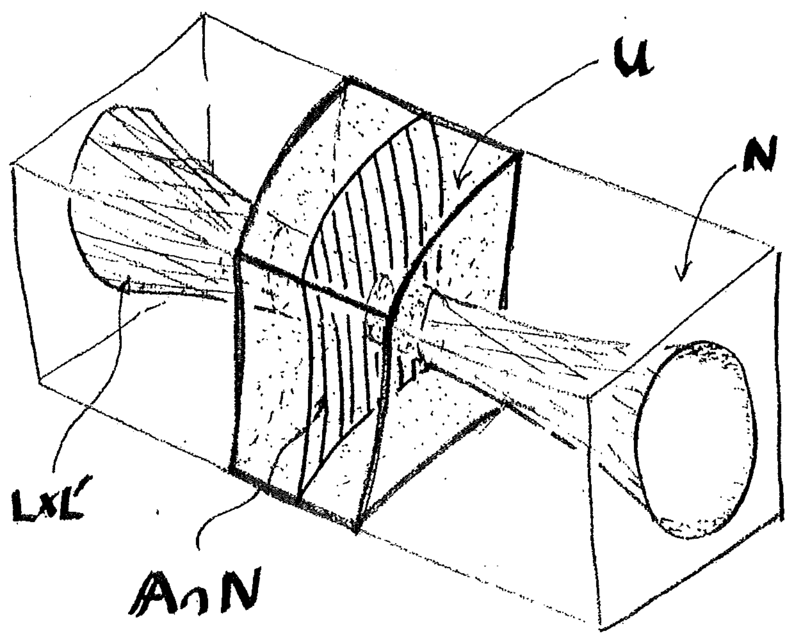
$$H^2(L\times L', \mathcal{O}_X(-2,-2)) \xrightarrow{\cong} H^3_c(X-L\times L', \mathcal{O}(-2,-2))$$

where this map is given precisely by $\lambda \to \Lambda = \bar{\partial}\beta_1 \wedge \lambda$ (c.f. (3.3)).

In order to obtain agreement with the original scalar product contour integral formula (see also [2]) we must be able to factorize λ in the following way:-

$$\lambda = (Z^\alpha W_\alpha)^{-2}.\mu$$

where $\mu \in H^{0,1}(U)$ and U is a neighbourhood in N of $\mathbb{A} = \{(Z,W) : Z^\alpha W_\alpha = 0\}$. Thus U is a neighbourhood of the intersection Y of $\mathbb{A}$ with $L \times L'$.



Ginsberg's construction of the element μ rests on diagram-chasing in the following grid of exact sequences. Each vertical sequence is a relative exact sequence while the rows are induced from the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$. Given an element $\tau \in H^2(N,U;\mathbb{Z})$ we can find

$$\mu = \log(\rho\delta^{-1}i(\tau)) \in H^1(U;\mathcal{O})$$

thus identifying μ as the log of a line bundle (see below).

$$\begin{tikzcd}
 & & & H^1(U;\mathcal{O}) \arrow[d, "r"] \\
 & & H^2(N,U;\mathbb{Z}) \arrow[r, "j"] \arrow[d, "i"] & H^2(N,U;\mathcal{O}) \\
 & H^1(N;\mathcal{O}^*) \arrow[r, "\delta"] \arrow[d, "\rho"] & H^2(N;\mathbb{Z}) \arrow[d] & \\
H^1(U;\mathcal{O}) \arrow[r, "\exp"] \arrow[d, "r"] & H^1(U;\mathcal{O}^*) \arrow[r] & H^2(U;\mathbb{Z}) & \\
H^2(N,U;\mathcal{O}) & & &
\end{tikzcd}$$

Strictly speaking, however, the diagram-chase leads from τ to $r(\mu)$ in $H^2(N,U;\mathcal{O})$ which Ginsberg explained by noting that μ was to be used in a dot product: but we are content to regard the first stage in that product as the map $r : H^1(U,\mathcal{O}) \to H^2(N,U;\mathcal{O})$. Of course, it ismuch easier to use the map $j : H^2(N,U;\mathbb{Z}) \to H^2(N,U;\mathcal{O})$ (having observed that the grid is periodic in a northeasterly direction (!)) but then the interpretation of μ as the log of a line bundle is more obscure.

The perceptive reader may have noticed that in § 2.3 we discussed the use of a contour $\kappa \in H_{10}(X\text{-}Y)$ or of a class $\tau \in H^2_c(X\text{-}Y)$ to evaluate the scalar product but that here we have $\tau \in H^2(N,U;\mathbb{Z})$. In fact, this is not serious, for any class in $H^2_c(X\text{-}Y)$ becomes, under restriction to N, a class in $H^2(N,U;\mathbb{C})$ and conversely any class in this group can be represented by a pair of forms[5] one of which is closed and in $\Omega^2_c(X\text{-}Y)$.

If we start from the element $\mathcal{O}(1,-1)$ of the group $H^1(N,\mathcal{O}^*)$ in the grid, we have that $\delta\rho\,\mathcal{O}(1,-1) = 0$ so that $\mu = \log\rho\,\mathcal{O}(1,-1)$. The spinor versionof this observation (in which the topological triviality of the restriction to a neighbourhood U of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ of the bundle $\mathcal{O}(1,-1)$ is used to guarantee the existence of a log of this bundle on U) is a useful guide to the situation in the twistor case.

It is also not hard to see that N does not have to be a particularly "small" neighbourhood of $L \times L'$ for the diagram-chasing to work and the whole construction can be carried through with $N = \mathbb{P}^- \times \mathbb{P}^{*-}$, for example: this enables one to lift the restriction that φ and φ' be elementary states.

An explicit Čech representative for μ has only been found, however, for the case that U is a "small" neighbourhood of Y. We shall describe how such a thing can be constructed, leaving a study of the relevant geometry to the reader.

Let $\underset{|}{P}$, $\underset{|}{Q}$, $\overset{|}{R}$ and $\overset{|}{S}$ be twistors such that

$$\underset{++}{PQ} = L', \quad \overset{++}{RS} = L, \quad \begin{smallmatrix}Q\\|\\S\end{smallmatrix} = \begin{smallmatrix}P\\|\\R\end{smallmatrix} = 0, \quad \begin{smallmatrix}Q\\|\\R\end{smallmatrix} \Big/ \begin{smallmatrix}P\\|\\S\end{smallmatrix} = -1.$$

Then the sets

$$W_1 = \{(Z,W):\ \begin{smallmatrix}P\\|\\Z\end{smallmatrix} \neq 0,\ \begin{smallmatrix}W\\|\\R\end{smallmatrix} \neq 0\} \text{ and } W_2 = \{(Z,W):\ \begin{smallmatrix}Q\\|\\Z\end{smallmatrix} \neq 0,\ \begin{smallmatrix}W\\|\\S\end{smallmatrix} \neq 0\}$$

cover a neighbourhood of Y and, moreover, for $(Z,W) \in Y$,

$$\begin{smallmatrix}P\\|\\Z\end{smallmatrix}\begin{smallmatrix}W\\|\\S\end{smallmatrix} = \begin{smallmatrix}Q\\|\\Z\end{smallmatrix}\begin{smallmatrix}W\\|\\R\end{smallmatrix} .$$

Thus for a small neighbourhood U of Y, $W_1 \cap U$ and $W_2 \cap U$ cover U and

$$\log \frac{\begin{smallmatrix}P\\|\\Z\end{smallmatrix}\begin{smallmatrix}W\\|\\S\end{smallmatrix}}{\begin{smallmatrix}Q\\|\\Z\end{smallmatrix}\begin{smallmatrix}W\\|\\R\end{smallmatrix}}$$ is defined (its argument is near unity) on $(W_1 \cap U) \cap (W_2 \cap U)$.

We claim that this log represents μ.

3.2

The superduper contour $\kappa \in H_{10}(X-Y)$ can be related to the ordinary tight (physical) contour for the scalar product $\gamma \in H_6(X-S)$ via four applications of Mayer-Vietoris map in homology m. In fact $m^2(\kappa)$ is the super contour in $H_8(X-T)$, while $m^4(\kappa) = \gamma$. The first three m's are isomorphisms, and the last is the injection to γ.
($H_6(X-S) = \mathbb{C}\gamma \oplus \mathbb{C}\lambda$, where λ is the loose, unphysical contour.)

Many thanks to RJB, APH and RP forall the useful discussions and ideas.

Stephen Huggett Michael Singer

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Twistor diagram "Magic" and Cohomology Algebra

Recently, two philosophies have emerged in the attempt to give a cohomological interpretation of Twistor Diagrams. (See [1], say). The point of this note is to relate these for two-vertex diagrams using the algebra structure on the cohomology of projective spaces.

To begin, consider the scalar product . A detailed cohomological interpretation of the corresponding contour integral was first given in [2] by Matt Ginsburg. It may be summarized as follows: each pair of external ears represents an H^1 based on a line, say

$$\alpha \in H^1(\mathbb{PT}-\mathbb{L}, \mathcal{O}(-2+k)) \qquad \beta \in H^1(\mathbb{PT}^*-\mathbb{L}', \mathcal{O}(-2+k))$$

a crossed product in relative cohomology yields $\alpha\times\beta$ in $H^3(\mathbb{PT}\times\mathbb{PT}^*-\mathbb{L}\times\mathbb{L}')$ to which is applied $(Z\cdot W)^{-k-2} \in H^1_A(\mathbb{P}\times\mathbb{P}^*, \mathcal{O}(-2-k,-2-k))$ which gives

$$\theta = (Z\cdot W)^{-k-2}\cdot(\alpha\times\beta) \in H^4(\mathbb{PT}\times\mathbb{PT}^*-Y, \mathcal{O}(-4,-4))$$

where $Y = \mathbb{L}\times\mathbb{L}'\cap A$ and $\mathcal{O}(-4,-4)$ is the canonical bundle, Ω^6, of $\mathbb{PT}\times\mathbb{PT}^*$. A magic element

$$\mu \in H^1(\text{nhd of } Y, \mathcal{O})$$

is sought & found to be $\log \mathcal{O}(1,-1)|_{\text{nhd of } Y}$ (which is well defined), which has the property that

$$\mu\cdot\theta \in H^6(\mathbb{P}\times\mathbb{P}^*, \Omega^6) \cong \mathbb{C}$$

is the scalar product (on spacetime) of the fields represented by α & β. (cf [3], also). This construction is loosely referred to as following "philosophy one".

The second philosophy, initially suggested by RP. & recalled by M.A.S., runs along these lines: construct θ as before, but now pass to the category of $\mathbb{C}$-sheaves (instead of coherent $\mathcal{O}$-Modules) by cupping with the element in $\mathrm{Ext}^6(\Omega^6,\mathbb{C})$ given by the deRham resolution of $\mathbb{C}$. (Equivalently, split the resolution into short exact sequences and repeatedly apply the connecting homomorphism). This gives

$$\begin{array}{ccccc} H^4(\mathbb{PT}\times\mathbb{PT}^*-Y, \Omega^6) & \times & \mathrm{Ext}^6(\Omega^6,\mathbb{C}) & \longmapsto & H^{10}(\mathbb{PT}\times\mathbb{PT}^*-Y; \mathbb{C}) \\ \psi & & \psi & & \\ \theta & \times & dr & \longmapsto & \tilde{\theta} \quad (\text{say}). \end{array}$$

But now it is easy to identify this last group. For, one has the relative cohomology sequence

$$0 \to H^9(X-Y,\mathbb{C}) \to H^{10}_Y(X,\mathbb{C}) \to H^{10}(X,\mathbb{C}) \to H^{10}(X-Y,\mathbb{C}) \to 0 \qquad (1)$$

($X = \mathbb{P}\mathbb{T} \times \mathbb{P}\mathbb{T}^*$) and the Thom isomorphism ([47])

$$\tau : H^i(Y,\mathbb{C}) \xrightarrow{\cong} H^{i+d}_Y(X,\mathbb{C})$$

(for $d = \mathrm{codim}_{\mathbb{R}} Y = 10$, in this case). By the Künneth formula, $H^{10}(X,\mathbb{C}) \cong \mathbb{C}^2$ and ofcourse $H^0(Y) \cong \mathbb{C}$ so (1) becomes

$$0 \to H^9(X-Y,\mathbb{C}) \to \mathbb{C} \xrightarrow{\tau} \mathbb{C}^2 \to H^{10}(X-Y,\mathbb{C}) \to 0$$

Now we shall see that $\tau \neq 0$ (a very general fact, this) so that $H^{10}(X-Y,\mathbb{C}) \cong \mathbb{C}$ and thus $\tilde{\Theta}$ gives, again, a complex number. To see that this is the same as before requires a commutative diagram, given below.

To unite these philosophies we must study the cohomology $H^*(\mathbb{P}\mathbb{T} \times \mathbb{P}\mathbb{T}^*, \mathbb{C})$ as an algebra, and also give the Thom isomorphism in more detail. Recall that the cup (wedge, for some) product endows this cohomology with an algebra structure. Indeed, one has (for z, w in degree 2)

$$H^*(\mathbb{P}\mathbb{T},\mathbb{C}) \cong \mathbb{C}[z]/z^4 \quad \& \quad H^*(\mathbb{P}\mathbb{T}^*,\mathbb{C}) \cong \mathbb{C}[w]/w^4$$

whence, by the Kunneth formula $H^*(\mathbb{P}\mathbb{T}\times\mathbb{P}\mathbb{T}^*,\mathbb{C}) \cong \mathbb{C}[z]/z^4 \otimes \mathbb{C}[w]/w^4$. As said, z and w are of degree two & so generate $H^2(\mathbb{P}\mathbb{T}\times\mathbb{P}\mathbb{T}^*,\mathbb{C}) \cong \mathbb{C}^2$. One may choose to identify these with the (first) chern classes of suitable line bundles:

$$z = c_1(\mathcal{O}(1,0)) \quad . \quad w = c_1(\mathcal{O}(0,1)).$$

Quite generally, if Y is a closed subvariety of X, then the Thom isomorphism is (*)

$$\tau : H^i(Y,\mathbb{C}) \xrightarrow{\cong} H^{i+d}_Y(X,\mathbb{C})$$

where d = real co-dimension of Y in X. If Y is smooth (which is true for twistor diagrams) then the composition of τ with restriction to $H^{i+d}(Y,\mathbb{C})$ is the Euler class of the normal bundle of Y cupped with the original element in $H^i(Y,\mathbb{C})$. Now it often happens (always in Twistor diagrams) that the normal bundle of Y is the restriction to Y of a globally defined bundle on all of X. Similarly, the generators of $H^i(Y,\mathbb{C})$ are often the restriction to Y of generators of $H^i(X,\mathbb{C})$ (for i small enough). Then the Thom mapping τ is realized on $H^*(X,\mathbb{C})$ as cupping with the Euler class of extended normal bundle. When this happens, the middle mapping in sequences such as (1) is easy to identify.

(*) Remark: under the Thom isomorphism, (1) becomes the Leray long exact sequence [5]

<u>Example:</u> Consider $A = \{z \cdot w = 0\} \subset \mathbb{P}\mathbb{T} \times \mathbb{P}\mathbb{T}^*$. The normal bundle of A is the restriction to A of $\mathcal{O}(1,1)$. This is a line bundle, so its Euler class = top chern class = $c_1(\mathcal{O}(1,1)) = z + w$. The generators of $H^2(A, \mathbb{C})$ are $c_1(\mathcal{O}_A(1,0))$ and $c_1(\mathcal{O}_A(0,1))$ which are clearly the restriction to A of z and w. But now the image of $H^2(A, \mathbb{C})$ in $H^4(\mathbb{P} \times \mathbb{P}^*, \mathbb{C})$ is computed as

$$\begin{pmatrix} z \\ w \end{pmatrix} \xmapsto{\times (z+w)} \begin{pmatrix} z^2 + wz \\ wz + w^2 \end{pmatrix}$$

on the other hand, $H^4(\mathbb{P} \times \mathbb{P}^*, \mathbb{C})$ is spanned by z^2, wz, w^2 so that

$$H^3(\mathbb{P} \times \mathbb{P}^* - A, \mathbb{C}) = 0 \qquad H^4(\mathbb{P} \times \mathbb{P}^* - A, \mathbb{C}) \cong \mathbb{C} \qquad \square$$

This completes philosophy two for the scalar product. For, in that, $Y \cong \mathbb{F}_1$. The normal bundle of Y is easily calculated to be the restriction to Y of the bundle

$$N = \{\mathcal{O}(1,0) \oplus \mathcal{O}(1,0)\} \oplus \{\mathcal{O}(0,1) \oplus \mathcal{O}(0,1)\} \oplus \mathcal{O}(1,1)$$

(coming from the requirements $\{P \cdot z = 0 \oplus Q \cdot z = 0\} \oplus \{p \cdot w = 0 \oplus q \cdot w = 0\} \oplus \{z \cdot w = 0\}$). The Euler class of this is calculated by taking the top order part of its Chern character, which is a product of characters of the individual line bundles:

$$c(N) = (1+z)^2 (1+w)^2 (1+z+w)$$

so $e(N) = z^3 w^2 + z^2 w^3$ and spans the image of $H^{10}_Y(X, \mathbb{C})$ in $H^{10}(X, \mathbb{C})$. So $i \neq 0$, as claimed.

Now we can make contact with philosophy one. If one examines [2] one finds that $\mu \in H^1(Y_{\text{nhd}}, \mathcal{O})$ is the image of

$$z - w = c_1(\mathcal{O}(1,-1)) \in \ker\{H^2(\mathbb{P}\mathbb{T} \times \mathbb{P}\mathbb{T}^*, \mathbb{C}) \to H^2(Y, \mathbb{C})\}$$

under the chasing of the following diagram, as pointed out by S.H.:

$$\begin{tikzcd}
0 \arrow[d] & H^1(\mathcal{U}, \mathcal{O}) \ni \mu \arrow[d] \\
z - w \in H^2_{(X-Y)}(X, \mathbb{C}) \arrow[r] \arrow[d] & H^2_{(X-Y)}(X, \mathcal{O}) \arrow[d] \\
H^2(X, \mathbb{C}) \arrow[d] & 0 \\
H^2(\mathcal{U}, \mathbb{C}) \arrow[d] & \\
0 &
\end{tikzcd}$$

($X = \mathbb{P}\mathbb{T} \times \mathbb{P}\mathbb{T}^*$, again)

$\mathcal{U}$ is a tubular neighbourhood of Y; more strictly, a direct limit is taken over all such.

Then applying $\mu \cdot$ is exactly applying $(z-w)^\smile$. But that makes the following diagram commute:

$$\begin{array}{ccc} H^4(\mathbb{PT}\times\mathbb{PT}^*-Y,\Omega^6) & \xrightarrow{\smile dr} & H^{10}(\mathbb{P}\times\mathbb{P}^*-Y,\mathbb{C}) \\ \downarrow \cdot\mu & & \cong\big\downarrow \smile(z-w) \\ H^6(\mathbb{PT}\times\mathbb{PT}^*,\Omega^6) & \xrightarrow{\smile dr} & H^{12}(\mathbb{P}\times\mathbb{P}^*,\mathbb{C})\cong\mathbb{C} \end{array}$$

(notice that $(z^3w^2+z^2w^3)(z-w)=0$ so that $\smile(z-w)$ descends to $H^{10}(\mathbb{P}\times\mathbb{P}^*-Y,\mathbb{C})$).

In other words, both philosophies produce the same result.

The nature of the "magic" in philosophy 1 is now clear: it precisely projects out $H^{10}(\mathbb{P}\times\mathbb{P}^*-Y,\mathbb{C})$ from $H^{10}(\mathbb{P}\times\mathbb{P}^*,\mathbb{C})$, mapping this group to $H^{12}(\mathbb{P}\times\mathbb{P}^*,\mathbb{C})\cong\mathbb{C}$. In a sense, this is superfluous, and philosophy two is the simplest description, unless one wants a spacetime propagator. It seems reasonable, therefore, to use the machinery developed above to compute

$$H^*(X-Y,\mathbb{C})$$

(where $X=\underbrace{\mathbb{P}\times\mathbb{P}\times\dots\times\mathbb{P}}_{r}\times\underbrace{\mathbb{P}^*\times\dots\mathbb{P}^*}_{s}$ and Y is described by a twistor diagram). Appropriate chern classes may then be employed to project this cohomology from $H^*(X,\mathbb{C})$ a so construct propagators "post priori".

By way of example, consider . Let Y be the variety defined by this diagram: then $Y=\mathbb{P}_2\times\mathbb{P}_2'\cap A$. It is not hard to identify this as the complex homogeneous variety $X\times X$, whose cohomology is easy to compute. If one takes

$$\alpha\in H^0(\mathbb{P}_3-\mathbb{P}_2,\mathcal{O}(-1-k))$$
$$\beta\in H^0(\mathbb{P}_3^*-\mathbb{P}_2',\mathcal{O}(-1-k))$$

cross products and applies $(z\cdot w)^{-3-k}$, one obtains

$$\theta=(z\cdot w)^{-3-k}\cdot(\alpha\times\beta)\in H^2(\mathbb{PT}\times\mathbb{PT}^*-Y,\Omega^6)$$

so that $dr\smile\theta=\tilde\theta\in H^8(\mathbb{PT}\times\mathbb{PT}^*-Y,\mathbb{C})$.

The normal bundle of Y is the restriction of $\mathcal{O}(1,0)\oplus\mathcal{O}(0,1)\oplus\mathcal{O}(1,1)$ and one computes that the euler class of this is $zw(z+w)=z^2w+zw^2$. Then the Thom isomorphism is

$$\begin{array}{ccc} H^2(Y,\mathbb{C}) & \xrightarrow{\cong} & H^8_Y(X,\mathbb{C}) \\ \cong\!\!\mid\mid & & \\ \mathbb{C}^2 & & \end{array}$$

and corresponds, on $H^*(\mathbb{P}\times\mathbb{P}^*, \mathbb{C})$ to the map

$$\underset{H^2(\mathbb{P}\times\mathbb{P}^*,\mathbb{C})}{\underset{\cap}{\times(z^2\omega + z\omega^2) : \begin{pmatrix} z \\ \omega \end{pmatrix}}} \longmapsto \underset{H^8(\mathbb{P}\times\mathbb{P}^*,\mathbb{C}) \cong \operatorname{span}(z^3\omega,\ z^2\omega^2,\ z\omega^3).}{\underset{\cap}{\begin{pmatrix} z^3\omega + z^2\omega^2 \\ z^2\omega^2 + z\omega^3 \end{pmatrix}}} \qquad — ②$$

From this and the long exact sequence, one again gets

$$H^8(\mathbb{P}\times\mathbb{P}^* - Y, \mathbb{C}) \cong \mathbb{C}$$

whence an evaluation of the diagram. In this case, the image of $H^8_Y(\mathbb{C})$ in $H^8(\mathbb{P}\times\mathbb{P}^*, \mathbb{C})$ under the Thom map (given in ②) is annihilated by cupping with $\omega^2 - \omega z + z^2$ (check!) which lives in $H^4(\mathbb{P}\times\mathbb{P}^*, \mathbb{C})$ and restricts to zero on $H^4(Y, \mathbb{C})$. A diagram chase, as on the previous page, yields a "magic"

$$\mu \in H^3(\text{nbd of } Y,\ \mathcal{O})$$

with which to pursue philosophy one.

Rob Baston □

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Local Twistors and The Penrose transform for Homogeneous Bundles.

L.J. Mason

This note is intended to explain how to Penrose transform the cohomology of homogeneous bundles on twistor space into local twistor fields on space-time using the minimum of technology. The Penrose transform for homogeneous bundles has already been treated in much greater generality by M.G. Eastwood, [1]. However, I hope it will emerge that the treatment provided here has various advantages, especially for explicit computations. This method works for curved twistor spaces and automatically provides the Ricci curvature correction terms in conformally flat space-time. It also seems likely that it will generalize to twistor spaces of different dimension.

I will first give a brief account of the Penrose transform for tensors $g^{\alpha\ldots\beta}_{\gamma\ldots\delta}$ of homogeneity degree n, and then give various methods for deducing the cohomology of the irreducible bundles on twistor space from this.

Reducible homogeneous bundles

The basic homogeneous bundle that I will work with is $\mathbb{T}^\alpha \to P\mathbb{T}$. Local sections of this bundle can be defined as homogeneity degree -1 vector fields, $A^\alpha\partial/\partial Z^\alpha$, $[Z^\alpha\partial/\partial Z^\alpha, A^\beta\partial/\partial Z^\beta] = -A^\alpha\partial/\partial Z^\alpha$, on the appropriate region in $\mathbb{T}$ over $P\mathbb{T}$. This bundle is of rank four and is trivial; choose a basis in $\mathbb{T}$, $\delta_{\underline{\alpha}}{}^\alpha$, $\underline{\alpha} = 0,\ldots,3$, then $\delta_{\underline{\alpha}}{}^\alpha\partial/\partial Z^\alpha$ is a global holomorphic frame.

This is also a good definition when twistor space is curved although $\mathbb{T}^\alpha$ is then not trivial. The important feature is that for both flat and curved twistor spaces $\mathbb{T}^\alpha$ is the Ward transform of the local twistor bundle on space-time. (see p. 275 of [5], or [2] which uses the alternate definition, $(J^1\mathcal{O}(-1))^*$, of $\mathbb{T}^\alpha$; see [4] for basic facts about local twistors).

In the flat case we can evaluate a tensor $g^{\alpha..}_{\beta..}$ with homogeneity degree (n−2) as a field on $\mathbb{M}$ by first expressing it in terms of components with respect to the global frame, $\delta_{\underline{\alpha}}{}^{\alpha}$, and then evaluating each component as a zero rest mass on $\mathbb{M}$:

$$n < 0 \qquad \Gamma^{\underline{\alpha}..}_{\underline{\beta}..}{}_{C'..E'} = \oint \pi_{C'}..\pi_{E'}\, g^{\underline{\alpha}..}_{\underline{\beta}..}\, \pi^{A'} d\pi_{A'},$$

$$\nabla^{CC'}\, \Gamma^{\underline{\alpha}..}_{\underline{\beta}..}{}_{C'..E'} = 0.$$

$$n = 0 \qquad \Gamma^{\underline{\alpha}..}_{\underline{\beta}..}, \text{ such that } \Box\, \Gamma^{\underline{\alpha}..}_{\underline{\beta}..} = 0,$$

$$n > 0 \qquad \Gamma^{\underline{\alpha}..}_{\underline{\beta}..}{}_{C..E} \text{ such that } \nabla^{CC'}\, \Gamma^{\underline{\alpha}..}_{\underline{\beta}..}{}_{C..E} = 0.$$

(For n > 0 we also have the potential modulo gauge descriptions:

$$\Gamma^{\underline{\alpha}..}_{\underline{\beta}..CC'}{}^{D'..E'} \text{ such that } \nabla^{C}_{(F'}\Gamma^{\underline{\alpha}..}_{|\underline{\beta}..|C')C}{}^{D'..E'} = 0$$

modulo the gauge freedom $\nabla_{CC'}\phi^{\underline{\alpha}..D'..E'}_{\underline{\beta}..}$.)

With an abuse of notation we can denote the local twistor bundle on $\mathbb{M}$ by $\mathbb{T}^\alpha$ and denote by $\delta_{\underline{\alpha}}{}^{\alpha}$ the covariantly constant frame of $\mathbb{T}^\alpha$ on $\mathbb{M}$ corresponding to the global holomorphic frame $\delta_{\underline{\alpha}}{}^{\alpha}$ of $\mathbb{T}^\alpha$ on $P\mathbb{T}$.

Using $\delta_{\underline{\alpha}}{}^{\alpha}$ we can translate the concrete twistor indices on $\Gamma^{..}_{..}$ into local twistor indices. This implies that $\nabla_{AA'}$ acts on twistor indexed quantities via the local twistor connection.

[For these results we dont need the constant frame, $\delta_{\underline{\alpha}}{}^{\alpha}$, we only need the fact that $\mathbb{T}^\alpha$ on $P\mathbb{T}$ is the Ward transform of the local twistor bundle on $\mathbb{M}$. This will therefore work for a curved twistor space so long as we only transform the positive homogeneity cases to potentials modulo gauge on M.]

The Penrose transform for irreducible homogeneous bundles

The bundle $\mathbb{T}^{\alpha}$ is reducible; there is a canonical section, Z^{α}, of $\mathbb{T}^{\alpha}(1)$ defined by the Euler vector field, $Z^{\alpha}\partial/\partial Z^{\alpha}$ on $\mathbb{T}$. (As usual (n) after the symbol for a bundle denotes the tensor product of the bundle with $\mathcal{O}(n)$.) This enables us to reduce $\mathbb{T}_{\alpha}$ and $\mathbb{T}^{\alpha}$ as follows:

$$0 \to \mathcal{O}(-1) \xrightarrow{Z^{\alpha}} \mathbb{T}^{\alpha} \to \Theta(-1) \to 0$$

$$0 \to \Omega^1(1) \to \mathbb{T}_{\alpha} \xrightarrow{Z^{\alpha}} \mathcal{O}(1) \to 0 \qquad (*)$$

where Θ and Ω^1 are the tangent and cotangent bundles of $P\mathbb{T}$ respectively. It is the bundles of homogeneous tensors over Θ or Ω^1 with Young tableau symmetries which are irreducible.

In [1], M.G. Eastwood identifies the irreducible homogeneous bundles on $P\mathbb{T}$ with symbols, $(n|a,b,c)$, a,b,c integers such that $a \leqslant b \leqslant c$. If one allows the use of $\varepsilon_{\alpha\beta\gamma\delta}$, one finds that $(n|a,b,c) \cong (n-r|a-r,b-r,c-r)$. This gives rise to various possible ways of representing the bundles $(n|a,b,c)$ in terms of tensors over $\mathbb{T}^{\alpha}(n)$. A particularly simple case is when $c = 0$. Sections of $(-n|a,b,0)$ correspond to tensors $g_{(\alpha_1..\alpha_a)(\beta_1..\beta_b)}$ that are orthogonal to Z^{α} on all indices and have homogeneity degree n and have the following tableau symmetry:


–a blocks
–b blocks

In general one has a tensor $g^{\alpha..}_{\gamma..}$ that is trace free and has some Young tableau symmetry that is orthoganal to Z^{γ} on its lower indices, and is defined modulo the addition of tensors of the form $Z^{(\alpha}f^{\beta..)(..)}_{\gamma..}$.

The translation from $(n|a,b,c,)$ to $(n-r|a-r,b-r,c-r)$ is implemented by lowering indices using $\varepsilon_{\alpha\beta\gamma\delta}Z^{\delta}$ etc. Each of these different representations gives rise to different fields on $\mathbb{M}$.

(However, the spaces of fields corresponding to the different representations of the same bundle are isomorphic and the ideas explained below can be used to compute the maps explicitly).

Since we now have the Penrose transform (say $\Gamma^{\alpha..}_{\beta..}$ and $\Phi^{\alpha..}_{\beta..}$ resp.) of tensors $g^{\alpha..}_{\beta..}$ and $f^{\alpha..}_{\beta..}$ on $\mathbb{PT}$ we need only compute the Penrose transform of $Z^{\alpha}g^{..}_{..}$ and $Z^{\alpha}f^{..}_{..}$ in terms of $\Gamma^{..}_{..}$ and $\Phi^{..}_{..}$ respectively. We can then impose the Penrose transform of $Z^{\gamma}g^{\alpha..}_{\gamma..} = 0$ as a further condition on Γ and regard the Penrose transform of $Z^{(\alpha}f^{\beta..)(..)}_{\gamma..}$ as gauge freedom.

We can compute the action of Z^{α} component by component. Each component of $Z^{\underline{\alpha}}$ acts on each component of g by helicity raising. The relevant formulae can be computed easily or copied out from [3,4 or 5]. We obtain:

For $n < 0$ we have $Z^{\gamma}g^{\alpha..}_{\beta..} \leftrightarrow \Psi^{\gamma\alpha..}_{\beta..B'..E'}$ where:

$$\Psi^{\gamma\alpha..}_{\beta..D'..E'} = \begin{bmatrix} 0 \\ \Gamma^{\alpha..}_{\beta..C'D'..E'} \end{bmatrix}$$

where the expression on the right is just the primary/secondary part decomposition of the γ index on Ψ in terms of the transform of g, Γ. [This can be made transparent by means of the following "manifestly conformally invariant" method of writing the standard contour integral formula:

$$X^{\mu\nu}\,\Gamma^{\gamma..\epsilon\ \alpha..}_{\beta..} = \oint_{Z^{\rho}X_{\rho\sigma} = 0} Z^{\gamma}..Z^{\epsilon}g^{\alpha..}_{\beta..}\ Z^{[\mu}dZ^{\nu]}.$$

where $X^{\alpha\beta}$, the position twistor, is the canonical section of $\mathbb{T}^{\alpha\beta}$[1] on $\mathbb{M}$ whose value at $x \in \mathbb{M}$ is the skew twistor corresponding to x. Note that the primary part of the spinor decomposition of each of the indices $\gamma..\epsilon$ vanishes; this corresponds to $X_{\gamma\mu}\Gamma^{\gamma..\epsilon\alpha..}_{\beta..} = 0$.]

For $n > 0$ the most elegant expression for $Z^{\gamma}g^{\alpha..}_{\beta..}$ on space-time is in terms of its potential modulo gauge description:

$$Z^{\gamma} g^{\alpha..}_{\beta..} \leftrightarrow \Psi^{\gamma\alpha..}{}_{\beta..DD'}{}^{E'..F'} \quad \text{where:}$$

$$\Psi^{\gamma\alpha..}{}_{\beta..DD'}{}^{E'..G'} = \begin{bmatrix} 0 \\ \varepsilon_{C'}{}^{(E'}\Gamma^{|\alpha..|}_{\beta..}{}_{DD'}{}^{F'..G')} \end{bmatrix}$$

where again the right hand side is the primary/secondary part decomposition on the γ index on Ψ.

For $n \geqslant 0$ the expression for $Z^{\gamma} g^{..}_{..}$ in terms of the field description is: $Z^{\gamma} g^{\alpha..}_{\beta..} \leftrightarrow \Psi^{\gamma\alpha..}{}_{\beta..D..F}$ where

$$\Psi^{\gamma\alpha..}{}_{\beta..\,D..F} = \begin{bmatrix} (n+1)\varepsilon_{(D}{}^{C}\,\Gamma^{\alpha..}_{|\beta..|E..F)} \\ -i\nabla_{C'D}\Gamma^{\alpha..}_{\beta..\,E..F} \end{bmatrix}$$

<u>Remarks</u>

It can be seen that for $n \neq 0$ (using the potential modulo gauge description for $n > 0$) the conditions $Z^{\beta} g^{\alpha..}_{\beta..} = 0$ and the operation of quotienting out by $Z^{(\alpha} f^{..)..}_{\beta..}$ correspond to requiring that the spinor parts of Φ are symmetric over their primed spinor indices. Alternatively if one uses the $c = 0$ representation of homogeneous bundles and evaluate the $n \geqslant 0$ cases as fields, one obtains a description in terms of fields with no gauge.

Example: $H^1(\Omega^1)$. Let $A_\alpha dZ^\alpha \in H^1(\Omega^1)$. So A_α is homogeneous of degree -1, and satisfies $Z^\alpha A_\alpha = 0$. Let $\Gamma_{\alpha B} = (\psi_{AB}, \Phi^{A'}{}_{B})$ be the Penrose transform of A_α. The equations $\nabla^{BB'}\Gamma_{\alpha B} = 0$ become:

$\nabla^{BB'}\psi_{AB} = 0$, and $\nabla^{BB'}\Phi^{A'}{}_{B} - i\varepsilon^{B'A'}\psi^{B}{}_{B} = 0$.

The Penrose transform of $Z^\alpha A_\alpha$ is: $\nabla_{CC'}\Phi^{C'}{}_{B} - \varepsilon_{BC}\psi^{A}{}_{A} = 0$. These imply $\nabla_{[a}\Phi_{b]} = 0$ so $\Phi_a = \nabla_a \phi$ for some ϕ defined up to a constant. So elements of $H^1(\Omega^1)$ are evaluated as fields ψ_{AB} satisfying $\nabla^{AA'}\psi_{AB} = 0$, $\psi^{A}{}_{A} = \Box\phi$. (Note that the machine in [1] gives only a composition series for $H^1(\Omega^1)$, it produces:

$0 \to \{\text{A.S.D. Maxwell}\} \to H^1(\Omega^1) \to \{\phi \mid \Box^2\phi = 0\} \to 0.)$

An irritation is that there are possible finite dimensional errors in the computation of the cohomology of irreducible bundles - the sequences (*) have a nontrivial extension class (the obvious element of $H^1(\mathbb{PT},\Omega^1) \cong \mathbb{C}$) so that the corresponding long exact sequence of (*) will in general have nontrivial connecting homomorphism. This is also, of course, an irritation for those who wish to compute the cohomology of a bundle from that of the irreducibles in its composition series; especially if it emerges that the bundle is trivial.

It is possible to use the helicity lowering formula to transform the action of $\partial/\partial Z^\alpha$ on tensors on $\mathbb{PT}$ to operations on the corresponding fields on space-time. This leads, in the potential modulo gauge case, to an elegant description of Herz potentials in terms of local twistors. An example of this is given in an accompanying article in this TN. (See also [5].)

The formulae for multiplication by Z^α go through in the curved case. However the action of $\partial/\partial Z^\alpha$ is not in general defined.

Many thanks to M.A. Singer for useful discussions.

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The Relationship Between Spin-2 fields, Linearized Gravity and Linearized conformal Gravity.

This note explains the relationship between spin-2 fields, linearized gravity and linearized conformal gravity in terms of local twistors.

A spin 2 field is a spinor field Ψ_{ABCD} with conformal weight -1, satisfying: $\nabla^{AA'}\Psi_{ABCD} = 0$.

Both linearized gravity and conformal gravity can be described by a metric perurbation, h_{ab}, with conformal weight 2. Abuse notation by writing $\Psi_{ABCD} = \nabla^{A'}_{(C}\nabla^{B'}_{D}h_{AB)A'B'}$, then the equations of linearized gravity are the same as for spin-2, although the conformal weight of Ψ_{ABCD} is now zero. The equations for linearized conformal gravity are: $\nabla^{C}_{A'}\nabla^{D}_{B'}\Psi_{ABCD} = 0$.

[We need only concern ourselves with perturbations in the metric such that $\tilde{\Psi}_{A'B'C'D'} = \nabla^{A}_{(A'}\nabla^{B}_{B'}h_{C'D')AB} = 0$ since in linearized theory, metric perturbations for both gravity and conformal gravity split into the sum of self dual and anti self dual parts [1].]

The relationship between these fields is clear on twistor space: Spin-2 fields correspond to elements $g(Z) \in H^1(\mathcal{O}(2))$.

Linearized conformal gravity corresponds to a general linearized deformation of a region U in $\mathbb{PT}$, ie $g^{\alpha}\partial/\partial Z^{\alpha} \in H^1(U,\Theta)$, where Θ is the tangent bundle of $\mathbb{PT}$, g^{α} has homogeneity degree one and is defined modulo multiples of Z^{α}. It is convenient to fix the freedom in g^{α} by setting $\partial g^{\alpha}/\partial Z^{\alpha} = 0$; this implies that the deformation preserves the holomorphic volume form, $\Delta Z = \varepsilon_{\alpha\beta\gamma\delta}Z^{\alpha}dZ^{\beta}\wedge dZ^{\gamma}\wedge dZ^{\delta}$, on $\mathbb{PT}$.

Linearised gravity corresponds to elements of $H^1(U,\Theta)$ which preserve various structures on twistor space; this amounts to saying that they are of the form $(I^{\alpha\beta}\partial g/\partial Z^{\alpha})\partial/\partial Z^{\beta}$ for some skew simple infinity twistor $I^{\alpha\beta}$ and twistor function $g \in H^1(U,\mathcal{O}(2))$.

It can be seen that a spin-2 field determines a linear gravity field once one has made the choice of an infinity twistor, and linear gravity is a subset of linear conformal gravity.

Infinitesimal deformations, g^{α}, of U determine infinitesimal deformations of the bundle $\mathbb{T}^{\alpha}$ on U (see previous article for definition of $\mathbb{T}^{\alpha}$). This determines a variation in the local twistor connection as this is the Ward transform of $\mathbb{T}^{\alpha}$. The linearized deformation is given by $\partial g^{\alpha}/\partial Z^{\beta}$ in $H^1(U,\mathbb{T}^{\alpha}_{\beta})$. The variation in the connection is given by Penrose transforming $\partial g^{\alpha}/\partial Z^{\beta}$ as a homogeneity degree zero matrix valued twistor function to a matrix valued 1-form with A.S.D. (covariant) exterior derivative (which is the variation in the curvature of the local twistor connection).

For linearized gravity the variation in the local twistor connection is therefore the Penrose transform of $I^{\alpha\gamma}\partial^2 g/\partial Z^{\gamma}\partial Z^{\beta}$.

Penrose transforming all these objects according to the scheme presented in my preceding article yields the following interrelationships: Let $g \in H^1(\mathcal{O}(2)) \longleftrightarrow \gamma_{C(A'B'C')}$ modulo $\nabla_{C(C'}{}^{X}{}_{A'B')}$ where $\nabla^{C}_{(D'}\gamma_{A'B'C')C} = 0$. Then $\partial^2 g/\partial Z^{\alpha}\partial Z^{\beta} \longleftrightarrow G_{\alpha\beta} \in \mathbb{T}_{(\alpha\beta)}\otimes\Omega^1(\mathbb{M})$ where:

$$G_{\alpha\beta} = \begin{bmatrix} \Gamma_{cAB}dx^{c} & ih_{cA}{}^{B'}dx^{c} \\ ih_{cB}{}^{A'}dx^{c} & \gamma_{c}{}^{A'B'}dx^{c} \end{bmatrix} .$$

and
$$DG_{\alpha\beta} = \begin{bmatrix} \Psi_{ABCD}d^2x^{CD} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{where } d^2x^{AB} = \varepsilon_{A'B'}dx^{AA'}{}_{\wedge}dx^{BB'}$$

The spinor parts of $G_{\alpha\beta}$ constitute the potential chain for Ψ_{ABCD}:
$$\Psi_{ABCD} = \nabla^{C'}_{(D}\Gamma_{CAB)C'} = \nabla^{C'}_{(D}\nabla_{A|A'|}h_{B}{}^{A'}{}_{C)C'}$$
$$= \nabla^{C'}_{(D}\nabla_{A|A'|}\nabla_{B|B'|}\gamma_{C)C'}{}^{A'B'}.$$

To obtain the variation in the local twistor connection due to a linear gravity field corresponding to g, choose a skew simple twistor $I^{\alpha\beta}$ such that $DI^{\alpha\beta} = 0$, then $I^{\alpha\gamma}G_{\beta\gamma}$ is the corresponding variation in the connection and curvature:

$$I^{\alpha\gamma}G_{\beta\gamma} = \begin{bmatrix} \Gamma_C{}^A{}_B dx^C & h_C{}^{AB'}dx^C \\ 0 & 0 \end{bmatrix}, \quad D(I^{\alpha\gamma}G_{\gamma\beta}) = \begin{bmatrix} \Psi^A{}_{BCD}d^2x^{CD} & 0 \\ 0 & 0 \end{bmatrix}$$

Note that the decomposition of $I^{\alpha\gamma}G_{\gamma\beta}$ into spinor parts is done with respect to the choice of $I^{\alpha\beta}$ as infinity twistor.

Finally we can obtain the relevant quantities for linearized conformal gravity. Let $g^\alpha \in H^1(U,\mathbb{T}^\alpha(1))$ be evaluated as potential modulo gauge: $g^\alpha \longleftrightarrow C^\alpha{}_C{}^{D'} = (ih_C{}^{AD'}, \tilde{\Gamma}_{CA'}{}^{D'})$ satisfying:

$$\nabla^C_{(B'}h_{C'D')C}{}^A + \Gamma^A{}_{(B'C'D')} = 0, \text{ and } \nabla^{B(B'}\tilde{\Gamma}_B{}^{C'}{}_{A'}{}^{D')} = 0.$$

Then $\partial g^\alpha/\partial Z^\beta$ is given by $\Gamma^\alpha{}_\beta$ where:

$$\Gamma^\alpha{}_\beta = \begin{bmatrix} \Gamma_C{}^A{}_B dx^C & ih_C{}^{AB'}dx^C \\ iP_{CA'B}dx^C & \tilde{\Gamma}_{CA'}{}^{B'}dx^C \end{bmatrix}$$

where $\Gamma^\alpha{}_\alpha = 0$ follows from $\partial g^\alpha/\partial Z^\alpha = 0$, and $\Gamma_{CB}{}^A$ and $P_{CA'B}$ are given by the linearized limit of the standard curved space expressions for these quantities in terms of the variation of the metric, h_{ab}.

Of interest for "googly" considerations is that one can obtain these expressions from $f \in H^1(U,\mathcal{O}(-6))$, since then $I_{\gamma\beta}Z^\alpha Z^\gamma f\Delta Z$ can be evaluated directly onto space-time (see [2] for details) to give a 2-form $F^\alpha{}_\beta$ which is the self dual version of $D(G_{\alpha\gamma}I^{\gamma\beta})$ given above. To obtain potentials observe that $d(I_{\gamma\beta}Z^\alpha Z^\gamma f\Delta Z) = 0$ and so $Z^\alpha Z^\beta f\Delta Z = d\Omega^{\alpha\beta}$. We can push $\Omega^{\alpha\beta}$ down to space-time to give a 1-form $G^{\alpha\beta}{}_C dx^C$, which is the self dual version of $G_{\alpha\beta}$, so that $I_{\alpha\gamma}G^{\alpha\beta}$ is the corresponding variation in the local twistor connection. There is here the slight subtlety that the gauge freedom in $I_{\alpha\gamma}G_C{}^{\beta\gamma}$ is constrained by the equation $\Box\nabla^C G_C{}^{\alpha\gamma} = 0$ (as first noticed in a similar context by Eastwood & Singer).

LJ Mason

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Twistor quantization of Open Strings in Three Dimensions.

In this note I will outline the quantization of certain loops in *real* 4-dimensional twistor space. Such loops represent quasi-periodic null curves in 3-dimensional Minkowski space-time $\mathcal{M}_{1,2}$. These null curves themselves represent open strings (in the "conformal gauge") propagating as timelike minimal surfaces in $\mathcal{M}_{1,2}$. The correspondence between null curves and minimal surfaces is valid in any dimension. The focus here is on 3 dimensions because the twistorial picture is particularly straightforward. Recall that in 4 dimensions a null geodesic corresponds to a null twistor Z^{α} modulo a phase. For null *curves* this phase degeneracy grows to be a large gauge symmetry group and one has to contend with a highly degenerate 2-form in attempting to write down the symplectic structure. These problems evaporate in $\mathcal{M}_{1,2}$, where one may define both real spinors and real twistors. The phase degeneracy may be eliminated completely.

The 3-dimensional case is interesting for other reasons. In the light-cone gauge (LCG) quantization of a relativistic string (Goddard et al 1973) a special coordinate system is chosen in order to solve the constraints of the theory before quantization. This breaks Lorentz covariance classically. One checks after quantization that the theory is covariant. One finds that the theory in dimension d is covariant if either

a) $d = 26$ and the ground state is tachyonic

b) $d = 2, 3$, with no other restrictions.

Since a string in 2 dimensions leads a rather dull existence one is led to consider dimension 3. The remarks made here ignore questions regarding second quantization and interactions. Within this rather narrow field of view the 3-dimensional case is much preferred to the 26-dimensional case since there need not be tachyons within the theory! (The LCG first-quantization of a *superstring* is covariant in 3 and 10 dimensions.) The other traditional space-time approach to string quantization is covariant, with the constraints being imposed after quantization as conditions on the "wave-function". This form of the free theory is much less fussy about dimension at the first quantized level. The space of physical states (i.e. solutions of the quantized constraints) has a positive semi-definite inner product provided only that $d \leq 26$ and the ground state $(\text{mass})^2$ is bigger than a certain negative number. Evidently $d = 4$ has a rather ambiguous status within the framework of conventional first-quantization techniques as applied to the free theory. The same can be said of the twistor quantization in that dimension, because of difficulties caused by the phase degeneracy described above. There is very limited evidence that a theory of quantum strings cannot be formulated consistently within a 4-dimensional framework. It is perhaps more appropriate to suggest that the quantization procedures used so far are inadequate to deal with the string problem, except in certain critical dimensions. At the very least one should investigate alternatives. The approach described here builds on the twistor description of strings at the classical level. A fairly comprehensive bibliography of such ideas is given by Hughston and Shaw (1987).

Let $x^a = (t, x, z)$ be coordinates for Minkowskian 3-space $\mathcal{M}_{1,2}$, with metric

$$ds^2 = dx^a dx_a = dt^2 - dx^2 - dz^2 \quad .$$

The points of $\mathcal{M}_{1,2}$ correspond to symmetric 2-index spinors x^{AB} via

$$x^{AB} = \frac{1}{\sqrt{2}} \begin{bmatrix} t - z & x \\ x & t + z \end{bmatrix}$$

and

$$x^a x_a = 2 \det x^{AB} = x^{AB} x^{CD} \varepsilon_{AC} \varepsilon_{BD} \quad .$$

Complex conjugation on tensors is extended to spinors as an *involutory* operation $\dagger$. The inner product on spin space defined by

$$< \alpha, \alpha >= i\alpha^A \alpha^{\dagger B} \varepsilon_{AB}$$

defines the group $SU(1,1)$. The twistor equation is the condition

$$\nabla_{(CD}\omega_{A)}(x) = 0$$

on a spinor field $\omega^A(x)$, where $\nabla_{CD} = \nabla_{(CD)}$ is the spinor covariant derivative. Equivalently one may write

$$\nabla_{CD}\omega_A(x) = \varepsilon_{A(C}\pi_{D)} \tag{1}$$

for some spinor field π_A. Application of the Ricci identity implies that π_A is constant and that consequently the solution to (1) is $\omega^A(x) = \omega^A + x^{AB}\pi_B$, where ω^A is constant. The *twistor correspondence* is defined by the zeroes of $\omega^A(x)$:

$$\omega^A + x^{AB}\pi_B = 0 \quad . \tag{2}$$

Twistor space may be considered as the set of pairs $Z^\alpha = \{\omega^A, \pi_A\}$. It comes equipped with an antisymmetric metric $\varepsilon_{\alpha\beta} = \varepsilon_{[\alpha\beta]}$, defined by

$$Z_1^\alpha Z_2^\beta \varepsilon_{\beta\alpha} = \omega^A p_A - \Omega^A \pi_A \quad ,$$

where $Z_1^\alpha = \{\omega^A, \pi_A\}$; $Z_2^\alpha = \{\Omega^A, p_A\}$. Raising and lowering of indices is defined in the usual way: $Z_\alpha = Z^\beta \varepsilon_{\beta\alpha} = \{\pi_A, -\omega^A\}$ and $Z^\alpha Z_\alpha = 0$.

The incidence relation (2) defines the twistor correspondence between space-time and twistor space. Given a space-time point x^{AB} the set of all twistors incident with x is an isotropic 2-plane: If Z_1^α and Z_2^α are any two points incident with x then

$$Z_1^\alpha Z_{2\alpha} = 0 \quad . \tag{3}$$

Projectively one obtains a null line in projective twistor space. Any two twistors satisfying (3) define a unique point in space-time. Conversely, given a pair $\{\omega^A, \pi_A\}$ the corresponding space-time structure is a γ-plane (in general odd dimension), which in this dimension is a null line:

$$x^{AB} = x_0^{AB} + t\pi^A \pi^B \quad .$$

Suppose that a twistor Z^α is incident with a *real* space-time point: $x^{\dagger AB} = x^{AB}$. Applying $\dagger$ to (2) gives $\omega^{\dagger A} + x^{AB}\pi_B^\dagger = 0$, and so

$$Z^\alpha Z_\alpha^\dagger \equiv \omega^A \pi_A^\dagger - \omega^{\dagger A}\pi_A = 0 \quad .$$

This condition may be solved by requiring that Z^α is itself real: $Z^{\dagger\alpha} = Z^\alpha$ which, with an appropriate choice of spinor basis, means that all the components of Z^α are real. This is assumed from here on.

An open string in $\mathcal{M}_{1,2}$ corresponds to a quasi-periodic null curve in $\mathcal{M}_{1,2}$. If the world sheet of the string is given as $X^a(\tau,\sigma)$ and one works in the conformal gauge, where the induced metric of the world-sheet is conformal to the flat metric $ds^2 = d\tau^2 - d\sigma^2$, then one can write $X^a(\tau,\sigma) = \phi^a(\tau - \sigma) + \phi^a(\tau + \sigma)$, where $\dot\phi^a(s)$ is both null and periodic:

$$\dot\phi^a(s)\dot\phi_a(s) = 0 \quad , \quad \dot\phi^a(s + 2\pi) = \dot\phi^a(s) \quad .$$

Here and throughout the operation $\dot{}$ denotes $\frac{d}{ds}$. The periodicity condition is equivalent to $\phi^a(s + 2\pi) = \phi^a(s) + P^a$, where P^a is indeed the total momentum of the string.

Since $\dot{\phi}^a(s)$ is real and null there is a real spinor field $\pi_A(s)$ such that

$$\dot{\phi}^{AB}(s)\pi_B(s) = 0 \ .$$

One normalizes π^A so that $\dot{\phi}^{AB} = \pi^A \pi^B$. It follows that

$$P^{AB} = \int_{-\pi}^{\pi} ds \ \pi^A \pi^B \ . \tag{4}$$

A curve $Z^\alpha(s)$ in twistor space may be defined by imposing the incidence relations at every point on the curve:

$$\omega^A(s) + \phi^{AB}(s)\pi_B(s) = 0 \ .$$

The space-time curve is also incident with $\dot{Z}^\alpha$:

$$\dot{\omega}^A(s) + \phi^{AB}(s)\dot{\pi}_B = 0 \ .$$

It follows that $Z^\alpha(s)$ must satisfy:

$$Z^\alpha(s)\dot{Z}_\alpha(s) = 0 \ . \tag{5}$$

Given our assumptions about reality (5) is the only constraint on the twistor curve and replaces the nullity assumption on the space-time curve.

The curve $\omega^A(s)$ is not periodic. Indeed,

$$\omega^A(s + 2\pi) = \omega^A(s) - P^{AB}\pi_B(s) \ .$$

Another difficulty is that the translations do not act properly on the twistor fields as given. Under a space-time translation of the string: $X^a \to X^a + V^a$ the null curve ϕ^a transforms according to

$$\phi^a \to \phi^a + \frac{1}{2}V^a \ ,$$

and so

$$\omega^A(s) \to \omega^A(s) - \frac{1}{2}V^{AB}\pi_B(s) \ .$$

This is *half* the displacement in twistor space that one expects. Both these problems can be solved as follows. Define

$$\Pi^{AB} = \frac{1}{2\pi}P^{AB} \ , \quad \gamma^a = \frac{1}{2\pi}\int_{-\pi}^{\pi} ds \ \phi^a(s) \ ,$$

the latter quantity being the average $< \phi^a >_s$ with respect to the given parameterization. Now define $\Omega^A(s)$ by

$$\omega^A(s) = \{\gamma^{AB} - s\Pi^{AB}\}\pi_B(s) + \Omega^A(s) \quad .$$

The new curve $P^\alpha(s) = \{\Omega^A(s),\ \pi_A(s)\}$ has several useful properties. Note first that $\Omega^A(s)$ is periodic, so that $P^\alpha(s)$ defines a *loop* on twistor space. Also, under translations Ω^A transforms correctly:

$$\Omega^A(s) \to \Omega^A(s) - V^{AB}\pi_B(s) \quad .$$

The constraint is modified when expressed in terms of $P^\alpha(s)$. Indeed, we have

$$P^\alpha \dot{P}_\alpha + \Pi^{AB}\pi_A\pi_B = 0 \quad . \tag{6}$$

The loop variables are important for they can be used to cast both the symplectic structure and the angular momentum structure into particularly simple forms. Consider first the symplectic structure. From any Lagrangian field theory one can construct an associated (not necessarily non-degenerate) symplectic 2-form. Applying these ideas to string theory, suppose one has a string $X^a(\tau,\sigma)$ and two nearby strings $X^a + V_1^a$, $X^a + V_2^a$. Working in the conformal gauge the symplectic form is described by an integral:

$$2\omega(V_1, V_2) = \int_0^\pi d\sigma \{V_2^a V_{1a,\tau} - V_1^a V_{2a,\tau}\} \quad .$$

The integral can be taken over the $\tau = 0$ cross-section, for the equations of motion and boundary condtions ensure that ω is independent of τ. Now let V_i^a correspond to variations $\delta_i P^\alpha$ in P^α. Then some calculation leads to:

$$\omega(V_1, V_2) = \int_{-\pi}^{\pi} ds\ \delta_1 P^\alpha \delta_2 P_\alpha \quad . \tag{7}$$

Thus if the open strings in space-time are represented by loops in twistor space then the symplectic structure on the space of open strings is just the integral around the loop of the standard twistor symplectic structure. This formula is significantly more straightforward than its space-time counterpart, although it does not have the *manifest* reparameterization invariance of the latter. (The action of reparameterizations in twistor space is not just to replace the parameter by it's new form.)

Both the momentum and symplectic form are elementary integrals around the loop in twistor space so one can write out the whole theory in terms of Fourier modes. One expands $P^\alpha(s)$ as

$$P^\alpha(s) = \frac{1}{2\sqrt{\pi}} \sum_{n=-\infty}^{\infty} Z_n^\alpha \exp(-ins) \quad ,$$

where the components satisfy $Z_n^\alpha = Z_{-n}^{\dagger\alpha}$. The 2-spinor form of each mode is $Z_n^\alpha = \{\omega_n^A, p_{n,A}\}$. Then the symplectic structure is given by

$$\omega = d\omega_0^A \wedge dp_{0,A} - \sum_{n=1}^{\infty} dZ_{n,\alpha}^\dagger \wedge dZ_n^\alpha \tag{8}$$

and the constraints (6) are given by the vanishing, for $0 \leq r < \infty$, of the quantities

$$\mathcal{L}_r = \sum_{m=1}^{\infty} i(m+\frac{r}{2})Z_{m+r}^\alpha Z_{m,\alpha}^\dagger + \frac{1}{2\pi}P^{AB}p_{m+r,A}p_{m,B}^\dagger$$

$$+ \sum_{m=0}^{r} \frac{im}{2} Z_m^\alpha Z_{r-m,\alpha} + \frac{1}{4\pi}P^{AB}p_{r-m,A}p_{m,B}$$

and their complex conjugates. The total momentum P^{AB} is given by

$$P^{AB} = \frac{1}{2}p_0^A p_0^B + \sum_{n=1}^{\infty} p_n^{(A}p_n^{\dagger B)} \quad .$$

One may write down a corresponding expression for the angular momentum tensor M^{ab} as a sum over modes. This is given in the space-time by

$$M^{ab} = \int_0^\pi d\sigma \, \{X^a X^b_{,\tau} - X^b X^a_{,\tau}\} \quad ,$$

the integral being taken over any constant-τ cross-section. This tensor may be expressed in spinor form as

$$M^{ab} = M^{(AK)(BL)} = \mu^{AB}\varepsilon^{KL} + \mu^{KL}\varepsilon^{AB} \quad ,$$

where μ^{AB} is both symmetric and real. In terms of the twistor loop coordinates one finds that

$$\mu^{AB} = \int_{-\pi}^{\pi} ds \, \Omega^{(A}\pi^{B)} \quad ,$$

and hence that

$$\mu^{AB} = \frac{1}{2}\omega_0^{(A}p_0^{B)} + \sum_{n=1}^{\infty}\frac{1}{2}\{\omega_n^{(A}p_n^{\dagger B)} + \omega_n^{\dagger(A}p_n^{B)}\} \ .$$

The Poisson bracket algebra of the constraints may be computed, using the canonical variables indicated by (8). One finds that for all integers m and n,

$$\{\mathcal{L}_m, \mathcal{L}_n\} = i(m-n)\mathcal{L}_{m+n} \ .$$

One recovers the Lie algebra of $Diff(S^1)$ from the twistor Poisson bracket algebra. The two sets of constraints with $n \geq 0$ and $n \leq 0$ each generate a closed sub-algebra under Poisson brackets. Also, the constraint $\mathcal{L}_0$ defines the mass m of the system, via

$$\mathcal{L}_0 = \frac{1}{2\pi}m^2 + \sum_{n=1}^{\infty} 2inP_n^{\alpha}P_{n,\alpha}^{\dagger} = 0 \ .$$

The path to twistor quantization of this system is now clear. The quantum states are "functions" $f(p_0^A, Z_n^{\alpha})$ of a spinor and a sequence of twistors which are holomorphic in their twistor arguments and one makes the replacements

$$\omega_0^A \rightarrow i\frac{\partial}{\partial p_{0,A}} \ , \quad Z_{n,\alpha}^{\dagger} \rightarrow -i\frac{\partial}{\partial Z_n^{\alpha}} \ .$$

One may make the following hypothesis to deal with normal-ordering ambiguity in $\hat{\mathcal{L}}_0$: **The functions $f(p_0^A)$ are quantum string states with mass zero.** This fixes the quantized $r = 0$ operator as

$$\hat{\mathcal{L}}_0 = \frac{1}{2\pi}\hat{m}^2 + \sum_{n=1}^{\infty} n\hat{E}_n \ ,$$

where $\hat{E}_n = Z_n^{\alpha}\frac{\partial}{\partial Z_n^{\alpha}}$ is the nth homogeneity operator. The hypothesis also demands that one only admits the conjugates of the *non-negative* r constraints when one chooses the holomorphic polarization. These are

$$\hat{\mathcal{L}}_r^{\dagger} = \sum_{m=1}^{\infty}(m+\frac{r}{2})Z_m^{\alpha}i\hat{Z}_{m+r,\alpha}^{\dagger} + \frac{1}{2\pi}P^{AB}p_{m,B}\hat{p}_{m+r,A}^{\dagger}$$

$$+\sum_{m=0}^{r}\frac{-im}{2}\varepsilon^{\alpha\beta}\hat{Z}_{m,\beta}^{\dagger}\hat{Z}_{r-m,\alpha}^{\dagger} + \frac{1}{4\pi}P^{AB}\hat{p}_{r-m,A}^{\dagger}\hat{p}_{m,B}^{\dagger}$$

where $\hat{Z}^{\dagger}_{m,\beta} = -i\frac{\partial}{\partial Z^{\beta}_{m}}$ if $m > 0$ and $\{p_{0,B}, -i\frac{\partial}{\partial p_{0,B}}\}$ if $m = 0$.

In spite of their apparent complexity, exact solutions of the quantum constraint equations may be found quite easily. The functions $f(p_{0,A})$ are the basic allowed states. The next set of states to consider are functions $f(p_{0,A}, Z^{\alpha}_{1})$, independent of Z^{α}_{n} for $n \geq 2$. The constraints for these functions may be solved as follows. Define η by

$$\eta = \frac{p_{0,A}\omega^{A}_{1}}{p_{0,B}p^{B}_{1}} .$$

The solution space is given by $f = G(\eta, p_{0,A}, p_{1,A})$ and contains both massless and massive states. To get massive states one lets k be non-negative and sets $G = \exp(\pm\sqrt{2}m\eta)H(p_{0,A}, p_{1,A})$, where

$$m^2 = 2\pi k \ , \quad H = (p_{0,A}p^{A}_{1})^{-k}\phi(p_{0,A})$$

and ϕ is homogeneous of degree $k-1$. The massive states of this type correspond to functions *homogeneous* in each of their spinor arguments. The full spectrum of states is not yet known.

The quantized constraints define a twistorial representation of the Virasoro (1970) algebra. To write this in standard form define

$$V_m = \hat{\mathcal{L}}_{-m} = \hat{\mathcal{L}}^{\dagger}_{m}, \quad m \neq 0 \ ,$$

$$V_0 = \hat{\mathcal{L}}_0 + \frac{1}{4} \ .$$

Then the full algebra becomes

$$[V_m, V_n] = (m-n)V_{m+n} + \frac{d}{12}(m^3 - m)\delta_{m,-n}$$

and the states $f(p, Z)$ satisfy

$$[V_0 - h]f(p, Z) = 0 \ , \quad V_m f(p, Z) = 0 \ , \quad m \geq 1$$

where $d = 2$ and $h = \frac{1}{4}$. Note that in the space-time covariant quantization the number d would be the dimension of space-time, here 3. The twistor representation has a different central charge.

Interesting questions:

1) What is the inner product?

2) How does one deal with the phase degeneracy in 4 dimensions, where the whole theory is invariant under the transformations $P^\alpha(s) \rightarrow \exp(i\phi(s))P^\alpha(s)$?

3) How much of a coincidence is it that real spinors and twistors also exist in the "critical" string dimensions 10 & 26?

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A General Construction for Classical Strings in Eight-Dimensional Space

The problem is equivalent to that of determining all smooth null curves in eight dimensions, and provides a good example of the utility of twistor methods in higher dimensional geometry.

Let $i = 1 \ldots 10$ and let X^i denote homogeneous coordinates for CP^9. Then eight dimensional flat space, with the usual conformal structure, can be regarded as a quadric Q^8 in CP^9 given by $X^i X_i = 0$, where indices are raised and lowered by use of Ω^{ij} and Ω_{ij}.

A null curve in Q^8 is then a curve $X^i(s)$ which satisfies $X^i X_i = 0$ and $\dot{X}^i \dot{X}_i = 0$, where the dot denotes differenciation with respect to s. We wish to construct the general $X^i(s)$ which satisfies these conditions.

Let us write $A = 1 \ldots 16$ for a spinor index, of given helicity, associated with the orthogonal group $O(10)$. The condition for a spinor ξ^A to be pure is $\Gamma^i_{AB} \xi^A \xi^B = 0$, where Γ^i_{AB} is naturally symmetric on its spinor indices. Let $\rho^A(s)$ be coordinates for a curve in the pure spin space, so $\Gamma^i_{AB} \rho^A \rho^B = 0$ for $\forall s$. By differentiation we have:

(i) $\Gamma^i_{AB} \rho^A \rho^B = 0$

(ii) $\Gamma^i_{AB} \rho^A \dot{\rho}^B = 0$

(iii) $\Gamma^i_{AB} \dot{\rho}^A \dot{\rho}^B + \Gamma^i_{AB} \rho^A \ddot{\rho}^B = 0$

(iv) $3\Gamma^i_{AB} \dot{\rho}^A \ddot{\rho}^B + \Gamma^i_{AB} \rho^A \dddot{\rho}^B = 0$

We require the following lemma: a vector V^i is null if and only if it can be expressed in the form

$$V^i = \Gamma^i_{AB} \eta^A \eta^B$$

for some spinor η^A (impure). Given the pure curve $\rho^A(s)$ we set

$$X^i = \Gamma^i_{AB} \dot{\rho}^A \dot{\rho}^B .$$

It follows from the lemma that $X^i X_i = 0$, so the curve $X^i(s)$ lies in Q^8.

Now we wish to show that $\dot{X}^i\dot{X}_i = 0$. To do this we need another lemma: a vector V^i is null if and only if it can be expressed in the form

$$V^i = \Gamma^i_{AB}\,\xi^A\beta^B,$$

where ξ^A is pure (β^B not necessarily pure).

Suppose then that the curve in Q^8 is given by $X^i(s) = \Gamma^i_{AB}\dot{\rho}^A\dot{\rho}^B$, where ρ^A is pure. By (i)...(iv) above we have

$$\dot{X}^i = 2\Gamma^i_{AB}\dot{\rho}^A\ddot{\rho}^B = -\tfrac{2}{3}\Gamma^i_{AB}\rho^A\dddot{\rho}^B,$$

whence we see that $\dot{X}^i$ contains the pure spinor ρ^A as a 'cofactor' and is thus null by the second lemma.

Therefore we have shown, as desired, that if $X^i(s) = \Gamma^i_{AB}\dot{\rho}^A\dot{\rho}^B$ where $\rho^A(s)$ is pure, then $X^i(s)$ describes a null curve in Q^8.

Conversely, let $X^i(s)$ satisfy $X^iX_i = 0$ and $\dot{X}^i\dot{X}_i = 0$. We wish to show that there generally exists a pure curve $\rho^A(s)$ such that $X^i = \Gamma^i_{AB}\dot{\rho}^A\dot{\rho}^B$.

Suppose we let $\rho^A = \Gamma^{AB}_i X^i\alpha_B$ where α_B is pure. A straightforward calculation then shows that ρ^A is automatically pure. This follows at once by geometrical consideration; alternatively, by use of the Clifford algebra identity $\Gamma^{(i}_{AB}\Gamma^{j)AC} = \Omega^{ij}\delta^C_B$, for arbitrary X^i and α_B we obtain

$$\Gamma^i_{AB}\rho^A\rho^B = X^i(2\Gamma^{CD}_k X^k\alpha_C\alpha_D) - X^jX_j\Gamma^{iCD}\alpha_C\alpha_D,$$

which vanishes by virtue of the purity of α_C.

For the derivative of ρ^A we have the expression

$$\dot{\rho}^A = \Gamma^{AB}_i\dot{X}^i\alpha_B + \Gamma^{AB}_i X^i\dot{\alpha}_B. \qquad (*)$$

Now we suppose (as will be justified in due course) that α_B can be chosen such that the first term on the right side of (*) vanishes; i.e. we suppose $\Gamma^{AB}_i\dot{X}^i\alpha_B = 0$. Then we have

$$\dot{\rho}^A = \Gamma^{AB}_i X^i\dot{\alpha}_B.$$

Furthermore, in that case, we obtain:

$$\Gamma^i_{AB}\dot\rho^A\dot\rho^B = \Gamma^i_{AB}\Gamma^{AC}_j X^j\dot\alpha_C\Gamma^{BD}_k X^k\dot\alpha_D$$

$$= \Gamma^i_{AB}\Gamma^{j\,AC}X_j\dot\alpha_C\Gamma^{BD}_k X^k\dot\alpha_D$$

$$= 2\Gamma^{(i}_{AB}\Gamma^{j)AC}X_j\dot\alpha_C\Gamma^{BD}_k X^k\dot\alpha_D$$

$$- \Gamma^j_{AB}\Gamma^{i\,AC}X_j\dot\alpha_C\Gamma^{BD}_k X^k\dot\alpha_D$$

$$= 2g^{ij}\delta^C_B X_j\dot\alpha_C\Gamma^{BD}_k X^k\dot\alpha_D$$

$$- \Gamma^{i\,AC}\dot\alpha_C\left[\Gamma^j_{AB}\Gamma^{BD\,k}X_jX_k\dot\alpha_D\right]$$

$$= X^i\left(2\Gamma^{CD}_k X^k\dot\alpha_C\dot\alpha_D\right) - X^2\Gamma^{i\,CD}\dot\alpha_C\dot\alpha_D .$$

And since $X^2=0$ it follows therefore that $\Gamma^i_{AB}\dot\rho^A\dot\rho^B$ is proportional to X^i. With a suitable change in the scale of ρ^A, which does not alter its purity, we obtain the desired result

$$X^i = \Gamma^i_{AB}\dot\rho^A\dot\rho^B ,$$

where

$$\rho^A = \Gamma^{AB}_i X^i\alpha_B \Big/ \left(2\Gamma^{CD}_k X^k\dot\alpha_C\dot\alpha_D\right)^{1/2} .$$

What remains to be shown, as was promised above, is that in the generic case α_A can indeed be chosen so that the term $\Gamma^{AB}_i\dot X^i\alpha_B = 0$ in (*).

This is established as follows. Let α_A be given by the formula

$$\alpha_A = \Gamma_{iAB}\dot X^i\beta^B ,$$

where $\beta^A(s)$ is an arbitrary pure curve. It devolves at once, by geometrical argument, that α_A is pure. Furthermore, since $\dot X^i$ is null, we have:

$$\Gamma^{AB}_{i}\dot{X}^{i}\alpha_B = \Gamma^{AB}_{i}\dot{X}^{i}(\Gamma_{BCj}\dot{X}^{j}\beta^{C})$$
$$= \Gamma^{AB}_{i}\Gamma_{jBC}\dot{X}^{i}\dot{X}^{j}\beta^{C}$$
$$= \dot{X}^{2}\beta^{A}$$
$$= 0\,,$$

by which it is shown that α_A satisfies the desired condition.

Therefore we have shown that if $X^i(s)$ describes a null curve in Q^8 then at generic points along the curve there exists a pure spinor field $\rho^A(s)$ such that $\dot{X}^i(s) = \Gamma^i_{AB}\dot{\rho}^A\dot{\rho}^B$.

We hope to discuss these results in greater detail elsewhere. For more information on spinors in higher dimension the reader is advised to consult the appendix in volume 2 of the book by Penrose and Rindler.

Throughout the discussion we have made implicit use of the following identity which holds for spinors in 10 dimensions:

$$\Gamma^{i}_{(AB}\Gamma_{C)D\,i} = 0\,. \qquad (**)$$

From this identity it follows, for example, that a vector of the form $V^i = \Gamma^i_{AB}\zeta^A\zeta^B$ is necessarily null, and that a vector of the form $V^i = \Gamma^i_{AB}\xi^A\beta^B$ is necessarily null if ξ^A is pure, i.e. if $\Gamma^i_{AB}\xi^A\xi^B$. Many of the remarkable features of spinors in ten dimensions (twistors for eight dimensions) can be seen to follow as a consequence of the identity $(**)$.

L. P. Hughston & W. T. Shaw

A Note on Real Null Curves in Minkowski Space

An outstanding problem in classical string theory has been the provision of a general formula for real null curves in flat Minkowskian four-space. By use of a twistor method we have recently been able to produce such a formula, which will be given below.

Straightforward geometrical considerations show that smooth real null curves in Minkowski space correspond in twistor space to smooth curves $Z^\alpha(s)$ in CP^3 (s real) which satisfy $Z^\alpha \bar{Z}_\alpha = 0$, $\dot{Z}^\alpha \bar{Z}_\alpha = 0$, $Z^\alpha \dot{\bar{Z}}_\alpha = 0$, and $\dot{Z}^\alpha \dot{\bar{Z}}_\alpha = 0$. If $Z^\alpha(s)$ satisfies these conditions then $X^{\alpha\beta}(s) = Z^{[\alpha}\dot{Z}^{\beta]}$ describes a real null curve in Minkowski space. The points of the curve in space-time correspond to the tangent lines of the curve in CP^3. We require that the curve in CP^3 should have the property that its tangent lines all lie within the hypersurface PN given by $Z^\alpha \bar{Z}_\alpha = 0$, thus corresponding to real points in space-time. Our main results are as follows:

<u>THEOREM 1</u>. Let $X^\alpha(s)$ be an arbitrary curve, with continuous third derivatives, lying in PN. Define

$$Y^\alpha = X^\alpha + \phi\, I^{\alpha\beta} \bar{X}_\beta$$

where

$$\phi(s) = -i\, \dot{X}^\alpha \bar{X}_\alpha / (2\, I^{\alpha\beta} \dot{\bar{X}}_\alpha \bar{X}_\beta).$$

Then Y^α satisfies $Y^\alpha \bar{Y}_\alpha = 0$, $\dot{Y}^\alpha \bar{Y}_\alpha = 0$, and $Y^\alpha \dot{\bar{Y}}_\alpha = 0$; and thus generates a null curve in (finite) Minkowski space whose displacement into the imaginary is always <u>null</u>. Conversely, any smooth null curve (with non-vanishing torsion) with this property can thus be represented.

This theorem is essentially a stepping stone to our primary result, which is:

THEOREM 2. Let Y^α satisfy $Y^\alpha \bar{Y}_\alpha = 0$, $\dot{Y}^\alpha \bar{Y}_\alpha = 0$, and $Y^\alpha \dot{\bar{Y}}_\alpha = 0$. Define P^α by the relation

$$P^\alpha = Y^\alpha / (I_{\rho\sigma} \dot{Y}^\rho Y^\sigma)^{1/2} ,$$

so we have

$$I_{\alpha\beta} \dot{P}^\alpha P^\beta = 1 .$$

Then let Z^α be defined by

$$Z^\alpha = P^\alpha + \xi I^{\alpha\beta} \bar{P}_\beta + \eta I^{\alpha\beta} \bar{P}_\beta ,$$

where ξ and η are given by

$$\xi = -\tfrac{i}{2} \dot{R} , \quad \eta = R$$

with

$$R = \frac{-i \dot{P}^\alpha \dot{\bar{P}}_\alpha}{(I_{\alpha\beta} \dot{P}^\alpha \ddot{P}^\beta - I^{\alpha\beta} \dot{\bar{P}}_\alpha \ddot{\bar{P}}_\beta)} .$$

Then Z^α satisfies

$$Z^\alpha \bar{Z}_\alpha = 0 , \quad \dot{Z}^\alpha \bar{Z}_\alpha = 0 , \quad Z^\alpha \dot{\bar{Z}}_\alpha = 0 , \quad \dot{Z}^\alpha \dot{\bar{Z}}_\alpha = 0 ;$$

whence

$$X^{\alpha\beta}(s) = Z^{[\alpha} \dot{Z}^{\beta]}$$

is a real null curve in Minkowski space. Conversely, any real null curve, apart from a null geodesic, can be thus represented.

The proofs of these results will be given elsewhere, along with further details and applications to string theory.

References

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Quasi-local mass for small surfaces

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Abstract. We calculate Penrose's quasi-local mass for a small sphere by expanding in powers of the affine parameter along the generators of a light cone. We then check the result and extend it to small surfaces of more general shape by techniques involving more direct use of spinor calculus. In a non-vacuum spacetime, the mass inside a small sphere is obtained from a 4-momentum P_a which is the product of the volume enclosed by the surface and the 4-vector $T_{ab}t^b$, where t^a is a unit timelike vector orthogonal to a spanning 3-surface. In the vacuum case, the mass vanishes at the fifth order in the diameter of the surface, both for small spheres and for small surfaces of more general shape.

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More on Penrose's quasi-local mass

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Abstract. We define the 'data' of a 2-surface S in spacetime in terms of its first and second fundamental forms and show how Penrose's kinematic twistor for S is a functional of these data. We find necessary and sufficient conditions in terms of the data for S to be 'contorted', i.e. for its data not to be the data of a 2-surface in a conformally flat space, and we show that S is non-contorted iff the usual (local) twistor definition of norm is in fact constant on S. We find a large class of non-contorted 2-surfaces in the Schwarzschild solution and show that the Penrose mass M_P at one of this class is zero or M_S, the Schwarzschild mass parameter, according as the 2-surface does not or does go round the central hole. Finally, we calculate the 2-surface twistor space for a stationary black hole and prove the 'isoperimetric inequality' for the Penrose mass of a static black hole.

1. Introduction

This paper continues the investigation of Penrose's quasi-local mass construction (Penrose 1982a) begun in an earlier paper (Tod 1983). In the earlier paper, a number of examples of the construction were given and the mass was calculated for a variety of 2-surfaces in a variety of spacetimes. What enabled the calculations to be carried through was that the usual definition of twistor inner product when applied to a pair of 2-surface twistors gave a quantity constant on the 2-surface (Tod 1983). Consequently, this usual definition defined an inner product on the 2-surface twistor space and the mass could be calculated in terms of the norm of the kinematic twistor with respect to this inner product.

It was remarked in Tod (1983) that a 2-surface S will have this property, that the usual inner product gives a constant on S, if S together with its first and second fundamental form could be embedded in a conformally flat 4-space. Penrose (1984) has proposed that a 2-surface with this latter property be called 'non-contorted', and conversely be called 'contorted' if it cannot be so embedded. Thus a 2-surface S is contorted if the presence of conformal curvature can be deduced from the first and second fundamental forms of S, briefly, from the 'data' of S. Further, if S is non-contorted then the usual inner product is a constant on S.

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