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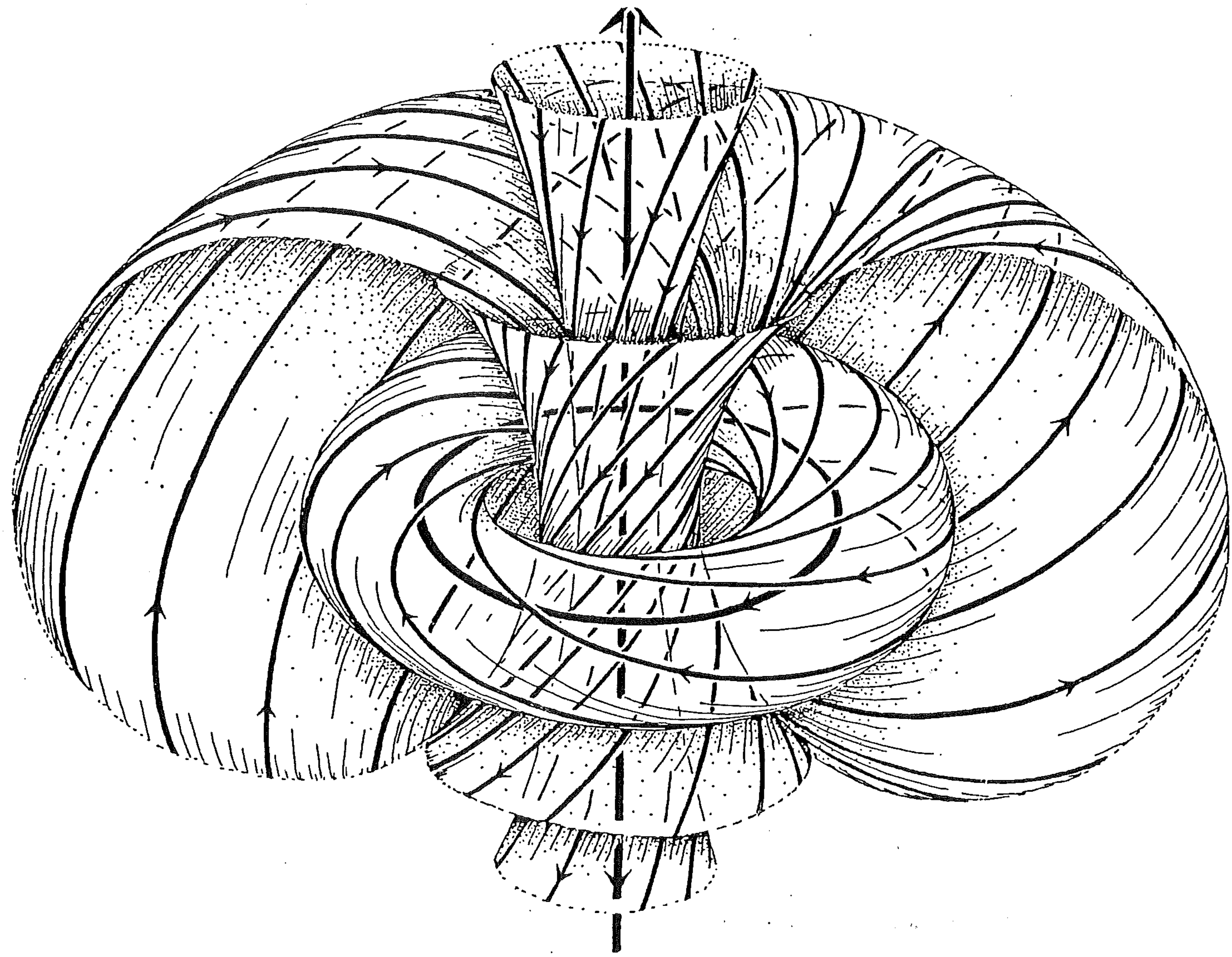

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An Approach to a Coordinate-Free Calculus at II

In attempting to obtain an effective googly graviton construction, it appears that it would be helpful to have a way of calculating at (and near) II (the non-projective blow-up of I in blown-up twistor space III). III is defined by "coordinates" $(\omega^A\pi_{B'}, \pi_{A'}\pi_{B'})$ – i.e. by $\frac{1}{2}\Sigma$ – and II arises in the limit $\lambda \to 0$ of $(\lambda^{-1}\omega^A, \lambda\pi_{A'})$. Various calculations have been carried out earlier (by R.P., T.N.B and M.A.S.) but only using particular coordinate systems. What we need is an invariant way of seeing when a tensor quantity in III is finite at II, etc.

A set of local coordinates for III is provided by

$$w = (\pi_{0'})^2, \quad x = \omega^0\pi_{0'}, \quad y = \omega^1\pi_{0'}, \quad v = -\omega^1\pi_{1'}.$$

(Other similar sets of coordinates would be needed to cover other parts of III.) Here $w=0$ gives II, so we are interested in how quantities behave as $w \to 0$. Taking a twistor basis $-\bullet\!\!\rfloor$, $\circ\!\!\rfloor$, $\circ$, $\bullet$ and dual basis $\circ$, $\bullet$, $\ulcorner\!\bullet$, $-\ulcorner\!\circ$, where $\circ\bullet\!\!\rfloor = 1 = \circ\bullet$, $\circ\!\!\circ = \bullet\!\!\circ = \circ\!\!\bullet = \bullet\!\!\bullet = 0$, we have

$w = \Sigma\bullet\Sigma\bullet$, $x = \circ\Sigma\Sigma\bullet$, $y = \bullet\Sigma\Sigma$, $z = \bullet\Sigma\Sigma\circ$, whence

$$d\Sigma\bullet = \frac{1}{2}\frac{dw}{\Sigma\bullet}, \quad d\Sigma\circ = \frac{dv}{\bullet} - \frac{\Sigma\circ}{\bullet}\left(\frac{dy}{\Sigma\bullet} - \frac{\bullet\Sigma\, dw}{2(\Sigma\bullet)^2}\right), \quad d\bullet\Sigma = \frac{dy}{\Sigma\bullet} - \frac{\bullet\Sigma\, dw}{2(\Sigma\bullet)^2}, \quad d\circ\Sigma = \frac{dx}{\Sigma\bullet} - \frac{\circ\Sigma\, dw}{2(\Sigma\bullet)^2}$$

These can be substituted into a differential form of interest, and then $\circ, \bullet, \circ, \bullet$ eliminated by allowing them to vary. For example, one important differential form is

$$\Sigma dz = \pi_{0'}d\pi_{1'} - \pi_{1'}d\pi_{0'}.$$

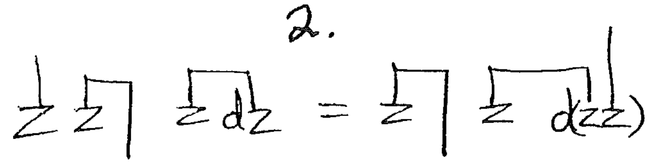
We know from previous work that Σdz vanishes at II. Can we see this in a coordinate-free way? First, let us use our coordinates: direct calculation from the above "blob" notation gives

$$\bullet\Sigma\,\Sigma dz = d(\bullet\Sigma\Sigma)\,\Sigma\circ\bullet$$

which, by $\ulcorner\!\!\urcorner\!\!\!\dashv = -\ulcorner\!\!\urcorner\ulcorner\!\!\urcorner$ gives

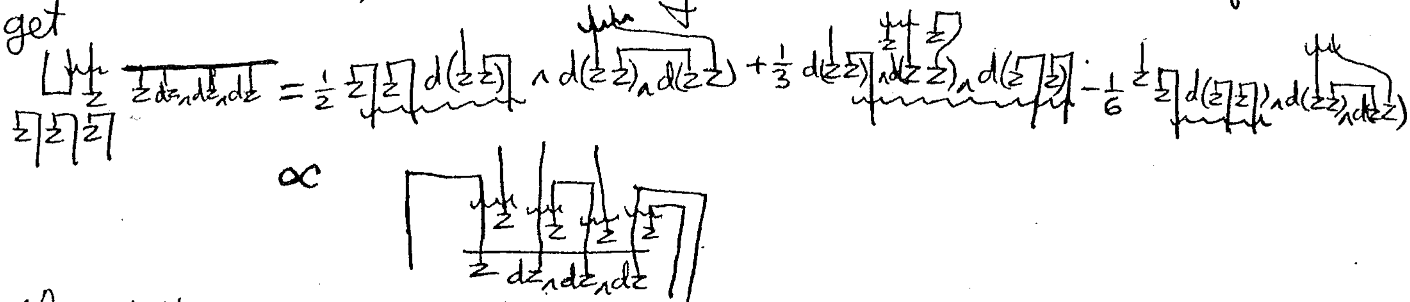
$$\bullet\Sigma\,\Sigma dz = -d(\bullet\Sigma\Sigma)\,\Sigma .$$

We can now remove $\bullet$, since this basis vector can be chosen arbitrarily, so multiplying by Σ we get the identity



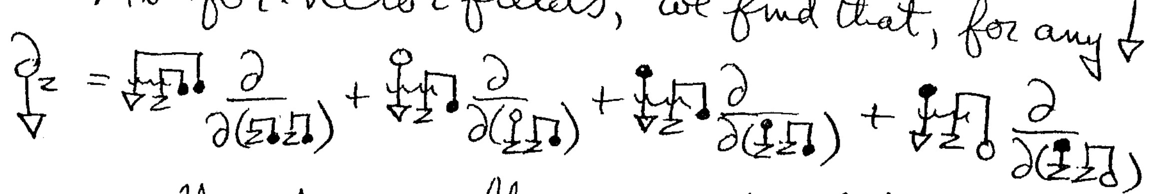
(which we can now see directly — but we might not have guessed this expression without first using ♀, ♀, …). This identity indeed tells us that $Z\,dZ$ is $O(\omega)$, i.e. $O(Z\,Z)$. The ⊓ of $Z\,d(ZZ)$ does double-duty — i.e. making both the $\frac{1}{Z}$ and the $d(\frac{1}{Z})$ hooked (since it can be rewritten ⊓ = −⊓⊓) — telling us that the R.H.S. is $O(Z\,Z)$, being of the form $Z\,Z \times d(ZZ)$, while the L.H.S. = non-singular non-zero$(\frac{1}{ZZ}) \times$ required form $Z\,dZ$. ∴ $Z\,dZ = O(Z\,Z)$.

Now consider the 3-form $Z\,dZ \wedge dZ \wedge dZ$. Calculating in terms of the above coordinates, and eliminating the basis elements, as before, we get



the L.H.S. being $O(Z\,Z) \times Z\,dZ \wedge dZ \wedge dZ$ and the R.H.S. being $O(1)$ (since $\frac{1}{ZZ}$ and $d(\frac{1}{ZZ}) = d\frac{1}{ZZ}$ are both $O(1)$). Thus $Z\,dZ \wedge dZ \wedge dZ$ has a simple pole at I, so $Z\,Z\,Z\,dZ \wedge dZ \wedge dZ$ and $Z\,dZ \otimes Z\,dZ \wedge dZ \wedge dZ$ are both finite at I — as is required for various googly ideas (R.P. in TN 16, 17, 23)

As for vector fields, we find that, for any ↓,



The coefficients are all components of $\frac{1}{Z}$, so we see that ↓ is well-behaved at I if $\boxed{\frac{1}{Z}\text{ is non-singular}}$.

In particular, we can check that that the vector fields representing "googly maps" are appropriately well behaved, using this procedure.

By combining this with the above procedure for forms, we should be able to deal with any tensor field on I. Work is in progress.

~ Roger Penrose & Vanessa Thomas

The Einstein Bundle of a Non-linear Graviton

Suppose M is a complex 4-manifold with holomorphic conformal structure. If M is also self-dual and sufficiently local (e.g. civilized in the terminology of [3]), then Roger Penrose shewed in [5] that M gives rise to a twistor space P, a 3-dimensional complex manifold in which the points $x \in M$ are represented by lines $L_x \subset P$. He shewed, moreover, that this correspondence is 1-1 and that a Ricci flat holomorphic metric in the given conformal class corresponds to additional structure on P roughly in the form of a holomorphic submersion to the Riemann sphere with symplectic fibres. A slightly tricky point in this construction is in shewing that a natural connection which arises on M is torsion free (it then follows that this is the metric connection). This tricky point also arises in the generalization to metrics with cosmological constant due to Ward [6] and, from another point of view, LeBrun [2]. A considerable simplification in the proof of this correspondence was achieved by LeBrun through his Einstein bundle on ambitwistor space [4]. The object of this note is to present a direct simplification on P by means of an Einstein bundle on P. This also shews just how natural LeBrun's Einstein bundle is as the correct generalization of the non-linear graviton construction to the non-self-dual case. Moreover, this proof should lend itself to the hyperKähler case where an argument via ambitwistors is unavailable.

Let F denote the projective primed spin bundle over M with local coordinates $(x^a, \pi_{A'})$ as usual. For any choice of representative metric on M one can lift the Levi-Civita connection horizontally to F. It acts on spinor fields on F but changes under conformal rescaling of the metric. For example,

$$\hat{\nabla}_{AA'}\phi_B = \nabla_{AA'}\phi_B - \Upsilon_{BA'}\phi_A + w\Upsilon_{AA'}\phi_B + \pi_{A'}\Upsilon_{AB'}\frac{\partial \phi_B}{\partial \pi_{B'}}$$

if ϕ_B has conformal weight w. Let ∇_A denote the operator $\pi^{A'}\nabla_{AA'}$ and let $\Upsilon_A \equiv \pi^{A'}\Upsilon_{AA'}$. Then we obtain

$$\underline{\hat{\nabla}_A\phi_B = \nabla_A\phi_B - \Upsilon_B\phi_A + w\Upsilon_A\phi_B .}$$

Notice that if $w = 1$ then $\nabla_{(A}\phi_{B)}$ is conformally invariant whilst if $w = -1$ then $\nabla^A\phi_A$ is conformally invariant. These are two of a whole series of conformally invariant differential operators on F the simplest of which is ∇_A acting on ordinary functions. As such ∇_A gives rise to a distribution on F which is integrable precisely when Ψ_{ABCD} vanishes i.e. when M is self-dual. In this case the twistor space of M is defined to be the space of leaves of this distribution: we have the usual diagram and a resolution of $\mu^{-1}\mathcal{O}$ on F:

$$P \xleftarrow{\mu} F \xrightarrow{\nu} M$$

$$0 \to \mu^{-1}\mathcal{O} \to \mathcal{O} \xrightarrow{\nabla_A} \mathcal{O}_A(1)\{-1\} \xrightarrow{\nabla^A} \mathcal{O}(2)\{-3\} \to 0$$

where round brackets denote homogeneity in π and curly brackets conformal weight. Similar resolutions may be used to <u>define</u> bundles on P (the ambitwistor analogue is discussed in detail in [1]). The most general such resolution is

$$0 \to \mu^{-1}\mathcal{O}(\overset{p}{\times}\!-\!\overset{q}{\bullet}\!-\!\overset{r}{\bullet}) \to \mathcal{O}(\overset{p}{\times}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet}) \to \mathcal{O}(\overset{p+q+1}{\times}\!-\!\underset{-q-2}{\times}\!-\!\overset{q+r+1}{\bullet}) \to \mathcal{O}(\overset{p+q+r+2}{\times}\!-\!\underset{-q-r-3}{\times}\!-\!\overset{q}{\bullet}) \to 0$$

defining the bundle $\overset{p}{\times}\!-\!\overset{q}{\bullet}\!-\!\overset{r}{\bullet}$ on P for integers p,q,r with $q,r \geq 0$. The bundle $\overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}$ on F is defined for integers a,b,c with $c \geq 0$ by $\mathcal{O}(\overset{a}{\times}\!-\!\overset{b}{\times}\!-\!\overset{c}{\bullet}) = \mathcal{O}_{\overbrace{(AB\ldots C)}^{c}}(a)\{b+c\}$ and the perculiar notation is designed to reflect the flat case where all these bundles are homogeneous and thus determined by appropriate weights. This resolution may be used to construct a Penrose transform. For this note the only case of relevance is the transform of

$\Gamma(P, \mathcal{O}(\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}))$ for the <u>Einstein bundle</u> $\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$.

In this case, the first part of the resolution is

$$0 \to \mu^{-1}\mathcal{O}(\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}) \to \mathcal{O}\{1\} \xrightarrow{\nabla_{(A}\nabla_{B)} + \Phi_{AB}} \mathcal{O}_{(AB)}(2)\{-1\}$$

where $\Phi_{AB} = \pi^{A'}\pi^{B'}\Phi_{ABA'B'}$ so $\phi \in \Gamma(P, \mathcal{O}(\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}))$ corresponds to a solution of $\nabla^{(A}_{(A'}\nabla^{B)}_{B')} f + \Phi^{AB}_{A'B'} f = 0$ so, if f is nowhere vanishing, it provides a rescaling such that $\Phi_{ab} = 0$, as required. This is the LeBrun-Ward conclusion since $\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} = \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet} \otimes \overset{2}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet} = \Omega^1 \otimes \kappa^{-1/2}$. It is this intrinsic description of the Einstein bundle which is lacking for ambitwistors.

<u>References</u>

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6. RSW Self-dual space-time with cosmological constant. Comm. Math. Phys. <u>78</u> (1980) 1-17.

Many thanks to Claude LeBrun for useful chat. Michael Eastwood, Adelaide.

5

Quantization of Strings in Four Dimensions

1. Preliminaries

In TN23 it was shown how one can define a novel quantization scheme for open strings in three dimensions based on the use of real twistor coordinates for the underlying null curve. This is described in greater detail in [1]. Our purpose here is to present a tentative approach to the corresponding problem in four dimensions for real Minkowski space **M**. The fundamental space-time object is a curve $\dot{\phi}^a(s)$ which is both null and periodic:

$$\dot{\phi}^a(s)\dot{\phi}_a(s) = 0 \ , \quad \dot{\phi}^a(s+2\pi) = \dot{\phi}^a(s) \ .$$

The periodicity condition is equivalent to

$$\phi^a(s+2\pi) = \phi^a(s) + P^a \ ,$$

where P^a, a future-pointing vector, is the total momentum of the corresponding string when the string tension T is set at unity. The world-sheet of an open string is then given by $X^a(\tau,\sigma) = \phi^a(\tau-\sigma) + \phi^a(\tau+\sigma)$. Closed strings are obtained by combining pairs of such null curves with equal total momentum. For further discussion see [2-9].

Since $\dot{\phi}^a(s)$ is null there is a spinor field $\pi_{A'}(s)$ such that

$$\dot{\phi}^{AA'}(s)\pi_{A'}(s) = 0 \ .$$

One normalizes $\pi^{A'}$ so that $\dot{\phi}^{AA'} = \bar{\pi}^A\pi^{A'}$. It follows that

$$P^{AA'} = \int_{-\pi}^{\pi} ds\bar{\pi}^A\pi^{A'} \ .$$

A curve $Z^\alpha(s)$ in twistor space may be defined, up to a phase, by imposing the following incidence relations at every point on the curve:

$$\omega^A(s) = i\phi^{AA'}(s)\pi_{A'}(s) \ .$$

The space-time curve is also incident with $\dot{Z}^\alpha$:

$$\dot{\omega}^A(s) = i\phi^{AA'}(s)\dot{\pi}_{A'}(s) \ .$$

The reality of ϕ^a implies that various inner products of the twistor fields vanish. A minimal set of these conditions, sufficient to imply the rest, is

$$Z^\alpha\bar{Z}_\alpha = 0 \ , \quad \dot{Z}^\alpha\bar{Z}_\alpha = 0 \ , \quad \dot{Z}^\alpha\dot{\bar{Z}}_\alpha = 0 \ .$$

In three dimensions the constraints corresponding to the first and third of these conditions can be solved [1] by demanding that the twistor is real since the inner product is symplectic rather than unitary. In four dimensions, however, these two constraints must be retained.

2. Loop Variables

Now the curve $\omega^A(s)$ is not periodic. Indeed,

$$\omega^A(s+2\pi) = \omega^A(s) + iP^{AA'}\pi_{A'}(s) \ .$$

Also, the translations do not act properly on the twistor fields as given. Under a space-time translation of the string: $X^a \to X^a + V^a$ the null curve ϕ^a transforms according to

$$\phi^a \to \phi^a + \frac{1}{2}V^a \ ,$$

and so

$$\omega^A(s) \to \omega^A(s) + \frac{i}{2} V^{AA'} \pi_{A'}(s) \quad .$$

This is *half* the displacement in twistor space that one expects. Howver, one may define a new twistor curve which is periodic and transforms suitably under translations as follows. Define

$$\gamma_s^a = \frac{1}{2\pi} \int_{-\pi}^{\pi} ds \ \phi^a(s) \ ,$$

this quantity being the average $< \phi^a >_s$ with respect to the given parametrization. Now define $\Omega^A(s)$ by

$$\omega^A(s) = i \left[\frac{s}{2\pi} P^{AA'} - \gamma_s^{AA'} \right] \pi_{A'}(s) + \Omega^A(s) \ .$$

The new curve $P^\alpha(s) = \{\Omega^A(s),\ \pi_{A'}(s)\}$ has several useful properties. Since $\Omega^A(s)$ is periodic, $P^\alpha(s)$ defines a *loop* on twistor space. Also, under translations Ω^A transforms correctly:

$$\Omega^A(s) \to \Omega^A(s) + i V^{AA'} \pi_{A'}(s) \quad .$$

The constraints are modified when expressed in terms of $P^\alpha(s)$. An independent set can be taken to be

$$P^\alpha \bar{P}_\alpha = 0 \quad ,$$

$$\dot{P}^\alpha \bar{P}_\alpha + \frac{i}{2\pi} P^{AA'} \bar{\pi}_A \pi_{A'} = 0 \quad ,$$

$$\dot{P}^\alpha \dot{\bar{P}}_\alpha + \frac{i}{2\pi} P^{AA'} [\dot{\bar{\pi}}_A \pi_{A'} - \bar{\pi}_A \dot{\pi}_{A'}] = 0 \quad ,$$

where the reality of γ^a has been used.

3. Symplectic Structure

The loop variables are important for they can be used to cast both the symplectic structure and the angular momentum structure into particularly simple forms. Consider first the symplectic structure. From any Lagrangian field theory one can construct an associated 2-form. Applying these ideas to string theory, suppose one has a string $X^a(\tau, \sigma)$ and two nearby strings $X^a + V_1^a$, $X^a + V_2^a$. Working in the conformal gauge the symplectic form is described by an integral:

$$2\omega(V_1, V_2) = \int_0^\pi d\sigma \{ V_2^a V_{1a,\tau} - V_1^a V_{2a,\tau} \} \quad .$$

The integral can be taken over the $\tau = 0$ cross-section, for the equations of motion and boundary condtions ensure that ω is independent of τ. This can be written in terms of variations in the corresponding null curve as

$$2\omega(V_1, V_2) = \int_{-\pi}^{\pi} ds [\delta_2 \phi^a \delta_1 \dot{\phi}_a - \delta_2 \dot{\phi}_a \delta_1 \phi^a] + \delta_1 \phi^a(\pi) \delta_2 \phi_a(-\pi) - \delta_2 \phi^a(\pi) \delta_1 \phi_a(-\pi)$$

Now let V_i^a correspond to variations $\delta_i P^\alpha$ in P^α. Then some calculation leads to:

$$\omega(V_1, V_2) = \frac{i}{2} \int_{-\pi}^{\pi} ds \ \{ \delta_1 P^\alpha \delta_2 \bar{P}_\alpha - \delta_2 P^\alpha \delta_1 \bar{P}_\alpha \} \quad .$$

Thus if the open strings in space-time are represented by loops in twistor space then the symplectic structure on the space of open strings is just the integral around the loop of the standard twistor symplectic structure.

The twistor expression for the energy-momentum of the string has already been given in terms of the loop variables. One may write down a corresponding expression for the angular momentum tensor M^{ab}. This is given in the space-time by

$$M^{ab} = \int_0^\pi d\sigma \ \{ X^a X_{,\tau}^b - X^b X_{,\tau}^a \} \quad ,$$

the integral being taken over any constant-τ cross-section. One may express this in terms of the null curve as

$$M^{ab} = \int_{-\pi}^{\pi} ds\{\phi^a\dot{\phi}^b - \phi^b\dot{\phi}^a\} + \phi^a(-\pi)\phi^b(\pi) - \phi^a(\pi)\phi^b(-\pi) \ .$$

This tensor may be expressed in spinor form as $M^{ab} = \mu^{AB}\varepsilon^{A'B'} + \bar{\mu}^{A'B'}\varepsilon^{AB}$, where μ^{AB} is symmetric. In terms of the twistor loop coordinates one finds that

$$\mu^{AB} = \int_{-\pi}^{\pi} ds\ i\Omega^{(A}\bar{\pi}^{B)} \ .$$

These expressions may be combined into a simple expression for the angular momentum twistor as an integral over the twistor loop variables:

$$A^{\alpha\beta} = \int_{-\pi}^{\pi} ds P^{(\alpha}I^{\beta)\gamma}\bar{P}_\gamma \ .$$

Note that both the 2-form defining the symplectic structure and the angular momentum twistor have a particularly simple form when expressed in terms of the twistor *loop* variables. This is not the case if one uses the primary twistor curve $Z^\alpha(s)$. Then one obtains integrals of similar type together with boundary terms like those which appear in the space-time expressions involving the null curve $\phi^a(s)$. When expressed in terms of the loop variables the structures describing the string are just the integrals around a loop in twistor space of the structures appropriate to the space of null geodesics.

4. Symmetries and Constraints

There are two infinite-dimensional symmetry groups acting on the twistor representation of a null curve. The first is the phase symmetry. The twistor curve and the loop curve $P^\alpha(s)$ are only defined up to phase rotations:

$$P^\alpha(s) \longrightarrow e^{i\phi(s)}P^\alpha(s) \ .$$

Infinitesimally we have

$$\delta P^\alpha = i\phi(s)P^\alpha \ ,$$

where $\phi(s)$ is a real-valued function. This symmetry is associated with a degeneracy in the 2-form ω. If we set $\delta_1 P^\alpha = i\phi P^\alpha$ we obtain

$$\omega(phase, V_2) = -\frac{1}{2}\int_{-\pi}^{\pi} ds\phi\delta_2(P^\alpha\bar{P}_\alpha)$$

which vanishes on the constraint surface where $P^\alpha\bar{P}_\alpha = 0$.

The corresponding computation for reparametrizations or diffeomorphisms of the parameter set is rather more involved. If, infinitesimally, we set $s = \sigma + f(\sigma)$ and for simplicity set $f = 0$ at $\sigma = \pm\pi$ then some integration by parts leads to

$$\omega(diffeo, V_2) = \frac{i}{2}\int_{-\pi}^{\pi} d\rho f\delta_2\left[\frac{1}{2}(\dot{P}^\alpha\bar{P}_\alpha - P^\alpha\dot{\bar{P}}_\alpha) + \frac{i}{2\pi}P^{AA'}\bar{\pi}_A\pi_{A'}\right] \ .$$

On the constraint surface, where $\dot{P}^\alpha\bar{P}_\alpha + \frac{i}{2\pi}P^{AA'}\bar{\pi}_A\pi_{A'} = 0$ and $P^\alpha\bar{P}_\alpha = 0$, one obtains zero, as expected. One may also verify that the angular momentum twistor satisfies

$$\delta A^{\alpha\beta} = 0$$

under changes in the twistor loop corresponding to phase rotations or reparametrizations The first invariance is obvious; the second requires some integration by parts.

Thus far the last constraint has not appeared. The only symmetry apparent in the space-time description is the set of reparametrizations. We have injected a local phase symmetry by the use of twistor coordinates. Both these symmetries correspond to a degeneracy of the 2-form ω. The last constraint plays a rather curious role in that it corresponds to a degeneracy of ω in a direction which points *off* the constraint surface, whereas the vector fields corresponding to phase changes and reparametrizations are tangential to the constraint surface. Specifically, it preserves the first two constraints but not the third one. One way of interpreting this is to note that the vector field takes us from real null curves to those which are *semi-complex*, i.e., null curves whose imaginary part is a null vector. The simplest way of describing this vector field is in terms of the primary twistor curve $Z^\alpha(s)$. The vector field is given by

$$\delta Z^\alpha = \frac{d}{ds}[ig\dot{Z}^\alpha] \ ,$$

where g is a real-valued function.

5. The Constraint Algebra

Further insight into the structure of the constraints can be obtained by considering their Poisson bracket algebra. At this point we must decide on a suitable bracket. We will use the brackets suggested by the 2-form ω. As usual, we may work with the loop variables or their Fourier coefficients. Choosing the latter approach we set

$$P^\alpha = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} P_n^\alpha e^{ins}$$

so that

$$A^{\alpha\beta} = \sum_{n=-\infty}^{\infty} 2P_n^{(\alpha} I^{\beta)\gamma} \bar{P}_{n\gamma} \ ,$$

$$\omega = \sum_{n=-\infty}^{\infty} i dP_n^\alpha \wedge d\bar{P}_{n\alpha} \ ,$$

with brackets

$$\{f,g\} = i \sum_{r=-\infty}^{\infty} \left\{ \frac{\partial f}{\partial P_r^\alpha} \frac{\partial g}{\partial \bar{P}_{r\alpha}} - \frac{\partial g}{\partial P_r^\alpha} \frac{\partial f}{\partial \bar{P}_{r\alpha}} \right\} .$$

Now let

$$U_{n,N} = P_n^\alpha \bar{P}_{n-N\alpha} \ , \quad V_{n,N} = \frac{1}{2\pi} P^{AA'} \pi_{nA'} \bar{\pi}_{n-NA} \ .$$

The Fourier modes of the constraints may be written as A_N, B_N, C_N respectively, where

$$A_N = \sum_{n=-\infty}^{\infty} U_{n,N}$$

$$B_N = \sum_{n=-\infty}^{\infty} \{nU_{n,N} + V_{n,N}\}$$

$$C_N = \sum_{n=-\infty}^{\infty} \{n^2 U_{n,N} + 2nV_{n,N}\} \ .$$

The algebra of the U, V quantities is

$$\{V_{n,N}, V_{m,M}\} = 0 \ ,$$

$$\{U_{n,N}, V_{m,M}\} = i[V_{m,M+N}\delta_{n,m-M} - V_{n,M+N}\delta_{m,n-N}],$$

$$\{U_{n,N}, U_{m,M}\} = i[U_{m,M+N}\delta_{n,m-M} - U_{n,M+N}\delta_{m,n-N}],$$

from which we deduce the following constraint algebra:

$$\{A_N, A_M\} = 0$$

$$\{B_N, B_M\} = i(N-M)B_{M+N}$$

$$\{A_N, B_M\} = iNA_{N+M} \ ,$$

$$\{A_N, C_M\} = i[2NB_{M+N} - N^2 A_{M+N}]$$

$$\{B_N, C_M\} = i[(2N-M)C_{M+N} - N^2 B_{M+N}] \ ,$$

$$\{C_N, C_M\} = i[(M^2 - N^2)C_{M+N} + 2(N - M)D_{M+N}] \ .$$

The quantity $D_N = \sum_n \{n^3 U_{n,N} + 3n^2 V_{n,N}\}$. We note that a simple way of appreciating the structure of this algebra is to construct the operators

$$\tilde{A}_N = ie^{-iNs} \ , \quad \tilde{B}_N = e^{-iNs}\frac{d}{ds} \ , \quad \tilde{C}_N = -ie^{-iNs}\frac{d^2}{ds^2} \ .$$

The algebra of these quantities under commutation is then isomorphic to that given above, with $\tilde{D}_N = -e^{-iNs}\frac{d^3}{ds^3}$.

6. Quantization

The form of the constraint algebra poses some interesting questions when we consider quantization. The form of ω suggests that we consider a holomorphic polarization with the quantum states as functions $|\Psi(P^\alpha_n) >$ of all the modes. We make the replacements

$$\bar{P}_{n\alpha} \longrightarrow -\frac{\partial}{\partial P^\alpha_n} \ .$$

The structure of the angular momentum twistor indicates that this scheme leads to a manifestly Poincaré invariant set of commutation relations for all the relevant operators. However, the constraint algebra fails to close: the constraints are *second class* in the sense of Dirac [10,11]. In general the imposition of such constraints as quantum conditions can lead to inconsistencies and one is led to the construction of Dirac brackets and the elimination of the second class class constraints before quantization. However, in the present context it appears that with some care inconsistencies can be avoided, while the implementation of the Dirac procedure presents formidable technical problems. We remark that our procedure is to be regarded as a "covariant quantization" scheme. There are also schemes analogous to "light-cone gauge" (LCG) quantization. However, in four dimensions it turns out that the LCG scheme also has second class constraints, so that this form of gauge fixing offers little advantage over the manifestly covariant scheme.

Now we define operators

$$\hat{U}_{n,N} = -P^\alpha_n \frac{\partial}{\partial P^\alpha_{n-N}} \ , \quad \hat{V}_{n,N} = -\frac{1}{2\pi}\hat{P}^{AA'}\pi_{nA'}\frac{\partial}{\partial \omega^A_{n-N}} \ .$$

Here $\hat{P}^a$ is the quantum momentum operator. In these expressions only $\hat{U}_{n,0}$ has potential factor-ordering problems. First we assume that these expressions are ordered with all derivative terms on the right. Then some simple calculation gives

$$[\hat{V}_{n,N}, \hat{V}_{m,M}] = 0 \ ,$$

$$[\hat{U}_{n,N}, \hat{V}_{m,M}] = [\hat{V}_{m,M+N}\delta_{n,m-M} - \hat{V}_{n,M+N}\delta_{m,n-N}],$$

$$[\hat{U}_{n,N}, \hat{U}_{m,M}] = [\hat{U}_{m,M+N}\delta_{n,m-M} - \hat{U}_{n,M+N}\delta_{m,n-N}].$$

We might also consider operators with a factor-ordering contribution:

$$\hat{U}_{n,N} \longrightarrow \hat{U}_{n,N} + \lambda_n\delta_{N,0} \ .$$

The same algebra is obtained provided that $\lambda_n = c$, a constant independent of n. When we define the quantum constraints in the obvious way, e.g. by

$$\hat{A}_N = \sum_{n=-\infty}^{\infty} \hat{U}_{n,N}$$

and so on, inspection of the $\hat{A}$, $\hat{B}$, $\hat{C}$ constraints shows that any non-zero c will lead to infinities in the operators $\hat{A}_0$ and $\hat{C}_0$, while the operator $\hat{B}_0$ is unchanged. So we may consider $\hat{A}_0$ and $\hat{C}_0$ as only defined up to a constant. If we speculate on regularization the infinity in $\hat{C}_0$ may be regarded in a natural way as $2c\sum_n n^2 \equiv 0$, by ζ-function techniques, but there seems to be no good reason for assigning any particular value to the constant in $\hat{A}_0$. Apart from this factor-ordering term, the quantum constraint algebra is precisely equivalent to the classical constraint algebra with $\{\ ,\ \} \to i[\ ,\ ]$. So, for example, we have

$$[\hat{B}_N, \hat{B}_M] = (N-M)\hat{B}_{M+N} \ ,$$

so that the twistor quantization scheme has *vanishing conformal anomaly*. The price we pay for this is the introduction of unphysical degrees of freedom and extra constraints to go with them. [This is similar in spirit to the BRST quantization procedure where the introduction of ghosts removes the anomaly but extra constraints appear to fix the ghost degrees of freedom.]

Now we consider how the constraints are to be imposed. The absence of a conformal anomaly means that one might initially consider imposing the constraints for all N. This appears to generate far too many conditions. For example, we would like the quantum analogue of the rigid rotator to be part of the scheme. Classically this is a massive two-twistor state which does not satisfy the condition $D_0 = 0$. It turns out that the quantum condition $\hat{D}_0|\Psi>=0$ eliminates the massive solutions, so we do not wish to impose it. Now classically the reality constraints satisfy identities such as $A_{-N} = \bar{A}_N$, so we really only need to impose the constraints for $N \geq 0$. The algebra for $N \geq 0$ closes, provided we adjoin suitable $\hat{D}_N, \hat{E}_N, \ldots$ constraints. The first $\hat{D}$ condition which appears is just $\hat{D}_1$ from $[\hat{C}_0, \hat{C}_1]$. Further commutation with $\hat{C}_0$ generates $\hat{E}_1$ and so on. So we have a new set of constraints consisting of a block starting at the zero Fourier mode and a block starting at the $N = 1$ mode. The entire set is generated by a finite subset. There are several choices; a minimal one is $\hat{A}_0$, $\hat{A}_1$, $\hat{B}_0$, $\hat{C}_0$. We remark that at this stage, since $\hat{A}_0$ cannot be generated by commutation of any of the other $N \geq 0$ constraints, we may assign the c-number in it as we wish.

There are two ways in which constraints can be imposed quantum mechanically. Unfortunately the strong form

$$C|\Psi>=0$$

leads to difficulties. One cannot find any solutions which are functions of finitely many twistors! The weak form, and all that is needed ultimately, is that the matrix element of any constraint between physical states should be zero:

$$<\Phi|C|\Psi>=0 \ . \tag{*}$$

However, we do not know in detail what inner product is to be used. Nevertheless, we imagine that it is related to the standard twistor inner products in some way. In particular, if Ψ and Φ are each functions of k twistors, this expression should reduce to an integral over a k-fold product of twistor spaces (and their duals), the relevant variables being those of which Ψ and Φ are functions. One might imagine then that the operation of working out the inner product involves setting all but k twistor variables to zero and integrating over the remainder. Suppose we denote by K the subspace where all twistors but the arguments of $|\Psi>$ vanish. If our speculations about the inner product are correct, then one way of arranging that the matrix element $*$ vanishes is to work out $C|\Psi>$ and demand that its restriction to K is zero.

A few remarks are pertinent at this point. First, if we do not impose the constraints in their strong form, the problems arising from the failure of the algebra to close are less significant, since further strong constraints cannot be generated by commutation. So in fact several schemes are possible: we may apply just the $\hat{A}, \hat{B}, \hat{C}$ constraints, or add the $\hat{D}, \hat{E}, \hat{F}\ldots$ conditions. The examples worked out are not able to distinguish between these possibilities, though the restriction to $N \geq 0$ for $\hat{A}, \hat{B}, \hat{C}$ and to $N \geq 1$ for the others appears to be necessary for an interesting theory.

Second, it should not be surprising that we cannot find solutions to the *strong* constraints in terms of finitely many twistors. Any state which solves all the constraints must be a state which is quantum mechanically invariant under phase changes and reparametrizations. Consequently it would have to depend in a highly symmetrical way on all the twistor modes. For example, suppose we consider a classical null geodesic, which corresponds to a single twistor up to a phase. This could be represented by any one P_n^α, or by a collection of proportional P_n^α. The corresponding quantum state would have to depend in a totally symmetrical way on all the P_n^α to be phase invariant. [This can be seen in practice by looking at the action of the $\hat{A}$ constraints.] Since the investigation of such totally symmetric states is rather impractical, we consider invoking some form of "gauge fixing". We can regard the restriction procedure described above as a rather brutal form of this. The operation "set all but finitely many twistors to zero" can be regarded as a possible gauge condition.

First let us suppose that all but P_k^α are zero. All the $\hat{V}$ operators vanish, and the only contributing constraints are the $N=0$ conditions. Let E_k denote the homogeneity operator $-\hat{U}_{k,0}$. From $\hat{A}_0$ we find

$$\{E_k + \lambda\}|\Psi(P_k^\alpha)>= 0 \ ,$$

where λ is the undefined constant in $\hat{A}_0$. Then $\hat{C}_0$ is just k times $\hat{B}_0$, and this gives us

$$kE_k|\Psi(P_k^\alpha)>= 0 \ .$$

So if $k=0$, we can consider single twistor states of homogeneity λ, but if $k \neq 0$ we have states of homogeneity zero. If we set $\lambda = 2$, the $k=0$ states are just massless scalars according to the standard scheme. However, the most natural choice in this context is to set $\lambda = 0$, which puts all values of k on an equal footing, and suggests that the single twistor states are massless with helicity -1, i.e. negative helicity photons.

7. Regge Trajectories

Now suppose that all twistors except P_r^α and P_s^α are set to zero, where $s > r$. Again the restricted constraints reduce to a few simple conditions. Let $\hat{\mathcal{M}}^2$ denote the quantum mass-squared operator $\hat{P}^a\hat{P}_a$. After some preliminary manipulations we find that the independent conditions are as follows. From $\hat{A}_0$ we find

$$\{E_r + E_s + \lambda\}|\Psi>= 0 \ .$$

From $\hat{B}_0$ we obtain

$$\Big\{\frac{1}{2\pi}\hat{\mathcal{M}}^2 - rE_r - sEs\Big\}|\Psi>= 0 \ ,$$

while $\hat{C}_0$ gives

$$\Big\{\frac{r+s}{2\pi}\hat{\mathcal{M}}^2 - r^2E_r - s^2Es\Big\}|\Psi>= 0 \ .$$

Finally $\hat{A}_{s-r}$ gives

$$P_s^\alpha\frac{\partial}{\partial P_r^\alpha}|\Psi>= 0 \ .$$

Comparing the middle two constraints we deduce that

$$rs[E_r + E_s]|\Psi>= 0 \ .$$

For consistency we must have $\lambda = 0$ unless either r or s is zero, in which case it can be assigned freely.

A detailed analysis of two-twistor states was made in [12]. There is an underlying $SU(2)$ symmetry group with ladder operators

$$\mathcal{I}_+ = P_s^\alpha\frac{\partial}{\partial P_r^\alpha} \ , \quad \mathcal{I}_- = P_r^\alpha\frac{\partial}{\partial P_s^\alpha}$$

satisfying $[\mathcal{I}_+, \mathcal{I}_-] = 2\mathcal{I}_3$, where

$$\mathcal{I}_3 = \tfrac{1}{2}\{E_s - E_r\} \ .$$

For a mass eigenstate (here any state homogeneous in each of its arguments) with non-zero mass, the spin operator is just the Casimir

$$\hat{S}^2 = \mathcal{I}_-\mathcal{I}_+ + \mathcal{I}_3(\mathcal{I}_3 + 1) \ .$$

Comparing these relations with the constraints, we see that the states are annihilated by $\mathcal{I}_+$ and everything about the spectrum is determined by a knowledge of λ (subject to the requirements above) and the homogeneity of one of the twistors. We can write

$$\mathcal{I}_3 = E_s + \frac{\lambda}{2}\,,$$

$$\hat{\mathcal{M}}^2 = 2\pi[(s-r)E_s - r\lambda]$$

acting on $|\Psi>$. Thus we obtain a linear relation between the mass-squared and the spin $\mathcal{I}_3$:

$$\hat{\mathcal{M}}^2 = 2\pi[(s-r)\mathcal{I}_3 - (r+s)\lambda/2]\,,$$

yielding the desired Regge trajectory behaviour for the quantum rigid rotator.

Recall that we set $s > r$, so that the coefficient of $\mathcal{I}_3$ in this relation is positive. Furthermore, since the states are annihilated by $\mathcal{I}_+$ we must arrange that $\mathcal{I}_3 \geq 0$. Thus all the states with $\lambda = 0$ have non-negative mass-squared and integral spin. If either r or s is zero a non-zero λ is possible, and it is possible (e.g. with $\lambda = 1$) to get half-integral spin states. This example justifies one of our assumptions used in applying the constraints; that only the $N \geq 0$ constraints be imposed. If we had allowed the negative N conditions to be imposed also, we would have had to set $\mathcal{I}_-|\Psi>= 0$ also, which kills off all but a few trivial states and destroys the Regge trajectories.

The constraint equations restricted to three or more twistors are rather more involved and so far a definitive picture has not emerged. We hope to discuss these and related issues (such as the Dirac bracket and LCG schemes) elsewhere.

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Fattening Complex Manifolds

by Claude LeBrun

Let Y be a complex manifold, and let $X \subset Y$ be a closed complex submanifold. The m-th order infinitesimal neighborhood of X in Y is [G] the ringed space $X^{(m)} = (X, \mathcal{O}_{(m)})$, where

$$\mathcal{O}_{(m)} = (\mathcal{O}_Y/I^{m+1})|_X$$

where $\mathcal{O}_Y$ is the sheaf of holomorphic functions on Y and where $I \subset \mathcal{O}_Y$ is the ideal sheaf of functions vanishing on X; these objects arise as the natural setting for formal power-series solutions of extension problems in which analytic objects on X are to be extended to a neighborhood of X in Y. We may use the above example as our guide in defining a <u>fattening</u> of a complex manifold $(X, \mathcal{O})$ of order m and codimension k to be a ringed space $X^{(m)} = (X, \mathcal{O}_{(m)})$, where $\mathcal{O}_{(m)}$ is locally isomorphic to

$$\mathcal{O}_{m,k} := \mathcal{O}[\zeta^1, \dots, \zeta^k]/(\zeta^1, \dots, \zeta^k)^{m+1}$$

and is equipped with an augmentation homomorphism $\alpha\colon \mathcal{O}_{(m)} \longrightarrow \mathcal{O}$ which, with respect to the above local isomorphisms, just "forgets" $\zeta^1, \dots, \zeta^k$.

When $k = 1$, a fattening is the same as a thickening in the terminology of [EL]. The purpose of the present article is to provide a solution to the following problem: given a fattening $X^{(m)}$ of X, when does there exist a fattening $X^{(m+p)}$ of higher order which extends it? How many such extensions are there? The results given here will strengthen and generalize the result for thickenings given in [EL], and will be applied to a twistor correspondence for conformal gravity in a forthcoming paper [L].

Let $X^{(m)}$ be a fattening of X, and observe that there is a natural collection $\{X^{(\ell)} \mid \ell = 0, 1, \dots, m\}$ of associated fattenings obtained by setting

$$\mathcal{O}_{(\ell)} := \mathcal{O}_{(m)}/I^{\ell+1}_{(m)},$$

where $I_{(m)} \subset \mathcal{O}_{(m)}$ is the ideal of nilpotents; we say that $X^{(m)}$ extends $X^{(\ell)}$. These lower order fattenings come equipped with a useful family of $\mathcal{O}_{(\ell)}$-modules, namely $I^q_{(p)}$ for $p \leq \ell + q$. In particular, provided $m > 0$, we may define the conormal bundle $N^* \longrightarrow X$ by

$$\mathcal{O}(N^*) = I_{(1)}$$

because the right-hand sheaf is locally free over $\mathcal{O} = \mathcal{O}_{(0)}$, and the normal bundle $N \longrightarrow X$ is defined to be its dual; these definitions are dictated by the archetypal examples of the infinitesimal neighborhoods of $X \subset Y$. In the special case of a thickening, $I_{(p)}$ turns out to be locally free on $X^{(p-1)}$ for $p \leq m$, so that we may sensibly define a vector-bundle extension of N^* to $X^{(p-1)}$ by

$$\mathcal{O}_{(p-1)}(N^*) := I_{(p)} \text{ if } k = 1,\ p \leq m\ ,$$

and duals or tensor products of N^* will similarly be extended to $X^{(p-1)}$ by dualizing and tensoring this sheaf as an $\mathcal{O}_{(p-1)}$ -module.

In a similar spirit, we may mimic $TY|_X$ by defining the extended tangent bundle $\hat{T} \longrightarrow X$ by

$$\begin{aligned}\mathcal{O}(\hat{T}) &= \mathrm{Der}_{\mathcal{O}_{(1)}}(\mathcal{O}_{(1)}, \mathcal{O}) \\ &:= \{D\colon \mathcal{O}_{(1)} \longrightarrow \mathcal{O}\ \mathbb{C}\text{-linear} \mid D(fg) = fDg+gDF\}\end{aligned}$$

This extended tangent bundle fits into an exact sequence

$$0 \longrightarrow T \longrightarrow \hat{T} \longrightarrow N \longrightarrow 0\ ,$$

where $T := T^{1,0}X$, from whence [EL] $\mathcal{O}_{(1)}$ can be reconstructed, so that the family of all first order fattenings of X with normal bundle N is naturally identified with $H^1(X, \mathcal{O}(T \otimes N^*))$. We can extend this vector bundle to $X^{(p-1)}$, $p \leq m$, by letting

$$\mathcal{O}_{(p-1)}(\hat{T}) := \mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)}, \mathcal{O}_{(p-1)}), \quad p \leq m .$$

If $p \geq 0$, there is a natural restriction homomorphism $\pi\colon \mathrm{Aut}(\mathcal{O}_{(m+p)}) \longrightarrow \mathrm{Aut}(\mathcal{O}_{(m)})$, where Aut denotes ring automorphisms. Set $\mathrm{Aut}_{(m)}(\mathcal{O}_{(m+p)}) = \ker \pi$.

Lemma 1. Suppose that $p \leq m$. Then for any fattening $X^{(m+p)} = (X, \mathcal{O}_{(m+p)})$ there is a natural isomorphism

$$\mathrm{Aut}_{(m)}(\mathcal{O}_{(m+p)}) \cong \mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)}, I^{m+1}_{(m+p)}) .$$

Proof. First we construct an isomorphism

$$\sigma\colon \mathrm{Aut}_{(m)}(\mathcal{O}_{(m+p)}) \longrightarrow \mathrm{Der}_{\mathcal{O}_{(m+p)}}(\mathcal{O}_{(m+p)}, I^{m+1}_{(m+p)})$$

by

$$\sigma(\phi) = \phi - 1 .$$

Let us see that this is well defined. If ϕ is an automorphism of $\mathcal{O}_{(m+p)}$ acting trivially on $\mathcal{O}_{(m)}$, we have $(\phi-1)\,\mathcal{O}_{(m+p)} \subset I^{m+1}_{(m+p)}$ and

$$\begin{aligned}(\phi-1)\,(fg) &= \phi(f)\,\phi(g) - fg \\ &= f(\phi-1)(g) + g(\phi-1)(f) + [(\phi-1)(f)](\phi-1)(g) \\ &= f(\phi-1)(g) + g(\phi-1)(f)\end{aligned}$$

because $p \leq m$ implies $[I^{m+1}_{(m+p)}]^2 = 0$. Thus $\sigma(\phi)$ is indeed a derivation if $\phi \in \mathrm{Aut}_{(m)}(\mathcal{O}_{(m+p)})$. Conversely, if $D \in \mathrm{Der}_{\mathcal{O}_{(m+p)}}(\mathcal{O}_{(m+p)}, I^{m+1}_{(m+p)})$,

$$\begin{aligned}(1+D)(fg) &= fg + fDg + gDf \\ &= fg + fDg + gDf + (Df)Dg \\ &= [(1+D)(f)][(1+D)(g)] ,\end{aligned}$$

so that $1+D$ is an automorphism acting trivially on $\mathcal{O}_{(m)}$, and σ is bijective.

To see that σ is an isomorphism of sheaves of groups, notice that if $D_j \in \mathrm{Der}_{\mathcal{O}_{(m+p)}}(\mathcal{O}_{(m+p)}, I^{m+p}_{(m+p)})$, $j = 1,2$, then

$$D_j(I^{p+1}_{(m+p)}) \subset I^{p}_{(m+p)} D_j I_{(m+p)}$$

$$\subset I^{m+p+1}_{(m+p)} = 0 ,$$

so that $p \leq m$ implies $D_1 D_2 = 0$ and

$$(1+D_1)(1+D_2) = 1 + (D_1+D_2).$$

Thus σ is indeed an isomorphism, as claimed.

Finally, notice that these is a natural isomorphism

$$\rho\colon \mathrm{Der}_{\mathcal{O}_{(m+p)}}(\mathcal{O}_{(m+p)}, I^{m+p}_{(m+p)}) \longrightarrow \mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)}, I^{m+1}_{(m+p)})$$

because elements of $\mathrm{Der}_{\mathcal{O}_{(m+p)}}(\mathcal{O}_{(m+p)}, I^{m+1}_{(m+p)})$ annihilate $I^{p+1}_{(m+p)}$ by the above correspondence. This gives an isomorphism

$$\rho\sigma\colon \mathrm{Aut}_{(m)}(\mathcal{O}_{(m+p)}) \cong \mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)}, I^{m+1}_{(m+p)}).$$

Q.E.D.

The right-hand sheaf would be more useful if it were constructed out of $\mathcal{O}_{(p)}$ alone. The next lemma shows that such a construction is indeed possible.

<u>Lemma 2</u>. For any fattening $X^{(m+p)} = (X, \mathcal{O}_{(m+p)})$,

$$I^{m+1}_{(m+p)} \cong \odot^{m+1}_{\mathcal{O}_{(p-1)}} I_{(p)}$$

as a sheaf of $\mathcal{O}_{(p-1)}$-modules, where $\odot$ denotes the symmetric tensor proudct.

<u>Proof</u>. There is a natural surjective morphism

$$\mu\colon \odot^{m+1}_{\mathcal{O}_{(p-1)}} L_{(p)} \longrightarrow I^{m+1}_{(m+p)}$$

given by $\mu(\sum_J \gamma_J \bigodot_{j=0}^{m} \beta_{Jj}) := \sum_J \tilde{\gamma}_J \prod_{j=0}^{m} \tilde{\beta}_{Jj}$, where $\beta_{Jj} \in I_{(p)}$, $\gamma_J \in \mathcal{O}_{(p-1)}$, and $\tilde{\gamma}_J$, $\tilde{\beta}_{Jj} \in \mathcal{O}_{(m+p)}$ have projections to $\mathcal{O}_{(p-1)}$ (respectively $\mathcal{O}_{(p)}$) equal to γ_J (respectively β_{Jj}) but are otherwise arbitrary.

To show that μ is injective, notice that the question is local, so that we may take $\mathcal{O}_{(m+p)} = \mathcal{O}_{m+p,k}$, which we will treat as an $\mathcal{O}$-module. Each element of $\odot^{m+1}_{\mathcal{O}_{p-1,k}} I_{p,k}$ may be expressed as

$$\sum_{q=0}^{p-1} \sum_{1\leq i_0\leq\cdots i_{m+q}\leq k} f_{i_0\cdots i_{m+q}} \zeta_{i_0}\odot\cdots\odot\zeta_{i_{m-1}} \odot(\zeta_{i_m}\cdots\zeta_{i_{m+q}}) ,$$

where the coefficients $f_{i_0\cdots i_{m+q}}$ are elements of $\mathcal{O}$. The image of this element under μ is

$$\sum_{q=0}^{p-1} \sum_{1\leq i_0\leq\cdots\leq i_{m+q}\leq k} f_{i_0\cdots i_{m+q}} \zeta_{i_0}\cdots\zeta_{i_{m+q}} ,$$

which can only be zero if all the coefficients $f_{i_0\cdots i_{m+q}}$ vanish, since the terms $\zeta_{i_0}\cdots\zeta_{i_{m+q}}$, $q < p$, are independent generators over $\mathcal{O}$.

Q.E.D.

With these tools in hand, we now give the main result of this article.

Theorem. Suppose that $0 < p \leq m$. The obstruction to the existence of a fattening $X^{(m+p)}$ extending a given fattening $X^{(m)} = (X,\mathcal{O}_{(m)})$ is an element of $H^2(X,\mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)},\odot^{m+1}_{\mathcal{O}_{(p-1)}} I_{(p)}))$. If this obstruction vanishes, the family of all such fattenings is acted upon freely and transitively by $H^1(X,\mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)},\odot^{m+1}_{\mathcal{O}_{(p-1)}} I_{(p)}))$.

Proof. The family of all fattenings of X of order m+p and codimension k is the non-abelian sheaf cohomology set $H^1(X,\mathrm{Aut}(\mathcal{O}_{m+p,k}))$. By lemmata 1 and 2 there is a short exact sequence of sheaves of groups

$$\mathrm{Der}_{\mathcal{O}_{p,k}}(\mathcal{O}_{p,k},\odot^{m+1}_{\mathcal{O}_{p-1,k}} I_{p,k}) \overset{j}{\rightarrowtail} \mathrm{Aut}(\mathcal{O}_{m+p,k}) \twoheadrightarrow \mathrm{Aut}(\mathcal{O}_{m,k}) ,$$

where π is the natural restriction map and j is obtained by following $(p\sigma)^{-1}$ with the natural inclusion of $\mathrm{Aut}_{(m)}(\mathcal{O}_{m+p,k})$ into $\mathrm{Aut}(\mathcal{O}_{m+p,k})$. Since

$\mathrm{Der}(\mathcal{O}_{(p)},\Theta^{m+1}_{\mathcal{O}_{(p-1)}}I_{(p)})$ is abelian, we can apply the obstruction theory developed in [EL] to the present situation. Thus, if $t \in H^1(\mathrm{Aut}(\mathcal{O}_{m,k}))$, the obstruction to t being in the image of π is an element of $H^2(X,t\mathrm{Der}_{\mathcal{O}_{p,k}}(\mathcal{O}_{p,k},\Theta^{m+1}_{\mathcal{O}_{p-1,k}}I_{p,k})t^{-1})$. Here the sheaf $t\mathrm{Der}_{\mathcal{O}_{p,k}}(\mathcal{O}_{p,k},\Theta^{m+1}_{\mathcal{O}_{p-1,k}}I_{p,k})t^{-1}$ is obtained from local copies of $\mathrm{Der}_{\mathcal{O}_{p,k}}(\mathcal{O}_{p,k},\Theta^{m+1}_{\mathcal{O}_{p-1,k}}I_{p,k})$ by using a Čech representation of t as the set of transition functions via the action of $\mathrm{Aut}(\mathcal{O}_{m,k})$ on $\mathrm{Aut}_{(m)}(\mathcal{O}_{m+p.k})$ by conjugation. But

$$
\begin{aligned}
&t\mathrm{Der}_{\mathcal{O}_{p,k}}(\mathcal{O}_{p,k},\Theta^{m+1}_{\mathcal{O}_{p-1,k}}I_{p,k})t^{-1}\\
&\qquad = \mathrm{Der}_{t\mathcal{O}_{p,k}}(t\mathcal{O}_{p,k},\Theta^{m+1}_{t\mathcal{O}_{p-1}}tI_{p,k})\\
&\qquad = \mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)},\Theta^{m+1}_{\mathcal{O}_{(p-1)}}I_{(p)})
\end{aligned}
$$

if t is a Čech cohomology element representing $\mathcal{O}_{(m)}$. This yields the desired result.

Q.E.D.

Part of this result, namely the H^1 clause, may be checked more directly, in the spirit of Rothstein's approach to deformations of complex supermaniolds [R]. Namely, given $X^{(m+p)}$, the family of all fattenings of order m+p agreeing with $X^{(m)}$ is $H^1(X,\mathrm{Aut}_{(m)}(\mathcal{O}_{(m+p)}))$. If $p \leq m$, we may use lemmata 1 and 2 to rewrite this as $H^1(X,\mathrm{Der}(\mathcal{O}_{(p)},\Theta^{m+1}_{\mathcal{O}_{(p-1)}}I_{(p)}))$.

A somewhat different way of stating the same result involves noticing that $\mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)},\Theta^{m+1}_{\mathcal{O}_{(p-1)}}I_{(p)})$ may be rewritten as

$\mathrm{Der}_{\mathcal{O}_{(p)}}(\mathcal{O}_{(p)}, \mathcal{O}_{(p-1)}) \otimes_{\mathcal{O}_{(p-1)}} \odot^{m+1} I_{(p)} = \mathcal{O}_{(p-1)}(\hat{T}) \otimes_{\mathcal{O}_{(p-1)}} \odot^{m+1}_{\mathcal{O}_{(p-1)}} I_{(p)}$, since a derivation from $\mathcal{O}_{(p)}$ to an $\mathcal{O}_{(p-1)}$-module is determined precisely by arbitrarily chosen images for $\zeta^1,\ldots,\zeta^k, z^1,\ldots,z^n \in \mathcal{O}_{(p)}$, where $\alpha(z^1),\ldots,\alpha(z^n) \in \mathcal{O}$ form a local coordinate system. This is particularly appealing in the cases when either k or p is one, in which case $\mathcal{O}_{(p-1)}(N^*) := I_{(p)}$ is a vector bundle on $X^{(p-1)}$. When we insert these changes, the following is obtained:

Corollary. Suppose that either p=1 or that k=1, and let $m \geq p$. The obstruction to finding $X^{(m+p)}$ extending a given fattening $X^{(m)}$ is an element of $H^2(X, \mathcal{O}_{(p-1)}(\hat{T} \otimes \odot^{m+1}N^*))$. If this obstruction vanishes, $H^1(X, \mathcal{O}_{(p-1)}(\hat{T} \otimes \odot^{m+1}N^*))$ parameterizes the family of all such fattenings $X^{(m+p)}$.

When both k and p are one, this is the result of [EL].

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On The Weights of Conformally Invariant Operators

Introduction: In [3] the conformally invariant differential operators on $\mathbb{M}$ (compactified complexified Minkowski space) were classified and it was shewn that each has an invariant curved analogue by the addition of suitable curvature correction terms ($\Box + R/6$ is the prototype). In this article, however, there is one vital point which was skipped over in the process of complexification. I am grateful to Robin Graham for drawing my attention to this omission and to Rice University for hospitality earlier this year during which the point was cleared up as follows.

On a general (pseudo-) Riemannian manifold M one can consider rescaling the metric according to $g_{ab} \mapsto \hat{g}_{ab} = \Omega^2 g_{ab}$ for an arbitrary nowhere-vanishing function Ω known as a conformal factor. There are various reasons for rescaling by the square of a conformal factor rather than just insisting on a positive rescaling or rescaling by e^{λ} for some function λ, the main reason being that Ω has units of length rather than length^2. Also, in four dimensions, if M is spin then Ω rescales the spinor epsilons which is a pleasant feature: in general, and also in the complexification, rescaling the metric by Ω^2 is preferred since it corresponds to rescaling a spin structure by Ω. In the complex case, rescaling by e^{λ} is unnecessarily restrictive. Another popular convention is $g_{ab} \mapsto \hat{g}_{ab} = u^{4/n-2} g_{ab}$ for $n = \dim M$ but this is specifically designed so that the Yamabe equation simplifies. Thus, for the purposes of this article, a conformal density of weight w is a "function" f which rescales according to $f \mapsto \hat{f} = \Omega^w f$ when $g_{ab} \mapsto \hat{g}_{ab} = \Omega^2 g_{ab}$. As usual, f may be equivalently regarded as a section of an appropriate line bundle. Similar comments apply to conformally weighted spinor fields. There is now a significant difference between the real and complex cases for on a real manifold w is an unrestricted real number whereas on a complex manifold with complex-valued Ω the weight w must be integral else Ω^w makes no sense. In [3] the argument for classification proceeded on the complexification $\mathbb{M}$ and so the weights were integral. It turns out that this assumption is justified but does require additional argument as follows.

Dimension 4: To deal with the real case one should attempt to classify invariant differential operators on real compactified Minkowski space $M = SO(4,2)/P$. As in [3] the question is equivalent to the classification of homomorphisms between Verma modules induced from finite-dimensional representations of P. Since the Verma modules are purely Lie-algebraic constructs, there is no harm in complexification at this level. Thus, one searches for homomorphisms (see [1] for notation)

$$V(\overset{s}{\bullet}\!\!-\!\!\overset{t}{\times}\!\!-\!\!\overset{u}{\bullet}) \longrightarrow V(\overset{p}{\bullet}\!\!-\!\!\overset{q}{\times}\!\!-\!\!\overset{r}{\bullet})$$

as equivalent to invariant differential operators

$$\overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet} \longrightarrow \overset{s}{\bullet}\!-\!\overset{t}{\times}\!-\!\overset{u}{\bullet}$$

except that it is no longer the case that q and t are required to be integral. More specifically, a section of $\overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet}$ is a spinor field

$$\phi_{\overbrace{AB\cdots D}^{r}\overbrace{E'F'\cdots H'}^{p}} \quad \text{(slight notational change from [1])}$$

of conformal weight $w = p+q+r$ (which is no longer required to be integral). This coincides with asking for a finite-dimensional representation of the Lie algebra (<u>not</u> Lie group)

$$\mathfrak{p} = \bullet\!-\!\times\!-\!\bullet = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

These are classified by $\overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet}$ with $p, r \in \mathbb{Z}_{\geq 0}$ and $q \in \mathbb{C}$ in the usual way ($\overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet}$ gives <u>minus</u> the weight of the <u>lowest</u> weight vector).

The argument for classification is much the same as that for integral weights [3]. Thus, the first requirement in order that there be a non-zero homomorphism between two Verma modules is that they have the same <u>central character</u> which, in this case, reduces one to looking for invariant operators between two of:

$$\overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet} \qquad \overset{p+q+1}{\bullet}\!-\!\underset{-q-2}{\times}\!-\!\overset{q+r+1}{\bullet} \qquad \overset{p+q+r+2}{\bullet}\!-\!\underset{-q-r-3}{\times}\!-\!\overset{q}{\bullet} \qquad \overset{q}{\bullet}\!-\!\underset{-p-q-3}{\times}\!-\!\overset{p+q+r+2}{\bullet} \qquad \overset{q+r+1}{\bullet}\!-\!\underset{-p-q-r-4}{\times}\!-\!\overset{p+q+1}{\bullet} \qquad \overset{r}{\bullet}\!-\!\underset{-p-q-r-4}{\times}\!-\!\overset{p}{\bullet}$$

whence, the only possibility for an invariant differential operator with non-integral conformal weight is

$$\overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet} \longrightarrow \overset{r}{\bullet}\!-\!\overset{-p-q-r-4}{\times}\!-\!\overset{p}{\bullet}.$$

There are two ways of eliminating this possibility:

<u>Method 1.</u> Consider the possible symbols for this differential operator.

$\Omega^1 = \overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet}$, $\odot^2\Omega^1 = \overset{2}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{2}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{0}{\bullet}$, $\odot^3\Omega^1 = \overset{3}{\bullet}\!-\!\overset{-6}{\times}\!-\!\overset{3}{\bullet} \oplus \overset{1}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{1}{\bullet}$,

$\odot^4\Omega^1 = \overset{4}{\bullet}\!-\!\overset{-8}{\times}\!-\!\overset{4}{\bullet} \oplus \overset{2}{\bullet}\!-\!\overset{-6}{\times}\!-\!\overset{2}{\bullet} \oplus \overset{0}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{0}{\bullet}$ etc. so one is lead to ask whether one can have $\overset{k}{\bullet}\!-\!\overset{-2n}{\times}\!-\!\overset{k}{\bullet} \otimes \overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet} = \cdots \oplus \overset{r}{\bullet}\!-\!\overset{-p-q-r-4}{\times}\!-\!\overset{p}{\bullet} \oplus \cdots$.

If so then, amongst other restrictions, $\left.\begin{array}{l} k+p+2(-2n+q)+k+r = r+2(-p-q-r-4)+p \\ \text{and } 2 \text{ divides } k+p+r \end{array}\right\}$

so $2q = 2n-4-(k+p+r)$ and q must be an integer.

<u>Method 2.</u> By means of the <u>translation principle</u>, as in [3], a differential operator $\overset{p}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{r}{\bullet} \to \overset{r}{\bullet}\!-\!\overset{-p-q-r-4}{\times}\!-\!\overset{p}{\bullet}$ gives rise to operators

$$\overset{p}{\bullet}\!-\!\underset{q+1}{\times}\!-\!\overset{r-1}{\bullet} \longrightarrow \overset{r-1}{\bullet}\!-\!\underset{-p-q-r-4}{\times}\!-\!\overset{p}{\bullet} \quad \text{and} \quad \overset{p-1}{\bullet}\!-\!\underset{q}{\times}\!-\!\overset{r}{\bullet} \longrightarrow \overset{r}{\bullet}\!-\!\underset{-p-q-r-3}{\times}\!-\!\overset{p-1}{\bullet}$$

so one can reduce to the case $p = r = 0$ and ask whether there is an operator

$$\overset{0}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{0}{\bullet} \longrightarrow \overset{0}{\bullet}\!-\!\overset{-q-4}{\times}\!-\!\overset{0}{\bullet}$$

and now consider possible symbols. Evidently, one is forced into

$$\overset{0}{\bullet}\!-\!\overset{-2n}{\times}\!-\!\overset{0}{\bullet} \otimes \overset{0}{\bullet}\!-\!\overset{q}{\times}\!-\!\overset{0}{\bullet} = \overset{0}{\bullet}\!-\!\overset{-q-4}{\times}\!-\!\overset{0}{\bullet}$$

so $q = n-2$ is integral (and the operator is $\Box^n$).

Notice that these arguments apply only to flat space. Presumably there is an argument directly applicable to curved space: I anticipate that operators on a curved 4-dimensional spacetime which are invariant under conformal rescaling must act on fields with integral conformal weight. As shewn below, the corresponding result is false in 3-dimensions (and presumably the pattern continues with even versus odd dimensions (cf. Huygens' principle etc.)). Notice that complexification is more satisfactory in dimension 4.

<u>Dimension 3</u>: The same general reasoning applies to the search for invariant operators

$$\overset{q}{\times}\!\Rightarrow\!\overset{r}{\bullet} \longrightarrow \overset{t}{\times}\!\Rightarrow\!\overset{u}{\bullet}$$

where again r is non-negative integral and q is unrestricted. Central character considerations restrict the search to

$$\overset{q}{\times}\!\Rightarrow\!\overset{r}{\bullet} \quad \overset{-q-2}{\times}\!\Rightarrow\!\overset{2q+r+2}{\bullet} \quad \overset{-q-r-3}{\times}\!\Rightarrow\!\overset{2q+r+2}{\bullet} \quad \overset{-q-r-3}{\times}\!\Rightarrow\!\overset{r}{\bullet}.$$

If q is integral one obtains, by translation, the deRham sequence

$$\begin{array}{ccccccc} \overset{0}{\times}\!\Rightarrow\!\overset{0}{\bullet} & \longrightarrow & \overset{-2}{\times}\!\Rightarrow\!\overset{2}{\bullet} & \longrightarrow & \overset{-3}{\times}\!\Rightarrow\!\overset{2}{\bullet} & \longrightarrow & \overset{-3}{\times}\!\Rightarrow\!\overset{0}{\bullet} \\ \| & & \| & & \| & & \| \\ \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \end{array}$$

and its siblings (the Bernstein-Gelfand-Gelfand resolutions). If q is not integral then there is the possibility of an invariant operator

$$\overset{q}{\times}\!\Rightarrow\!\overset{r}{\bullet} \longrightarrow \overset{-q-r-3}{\times}\!\Rightarrow\!\overset{r}{\bullet}.$$

To investigate this case (and also to shew that there are no other operators in the deRham case) one can start with symbol considerations:

$\Omega^1 = \overset{-2}{\times}\!\Rightarrow\!\overset{2}{\bullet}$, $\odot^2\Omega^1 = \overset{-4}{\times}\!\Rightarrow\!\overset{4}{\bullet} \oplus \overset{-2}{\times}\!\Rightarrow\!\overset{0}{\bullet}$, $\odot^3\Omega^1 = \overset{-6}{\times}\!\Rightarrow\!\overset{6}{\bullet} \oplus \overset{-4}{\times}\!\Rightarrow\!\overset{2}{\bullet}$,

$\odot^4\Omega^1 = \overset{-8}{\times}\!\Rightarrow\!\overset{8}{\bullet} \oplus \overset{-6}{\times}\!\Rightarrow\!\overset{4}{\bullet} \oplus \overset{-4}{\times}\!\Rightarrow\!\overset{0}{\bullet}$, etc. so one asks for

$$\overset{-2n}{\times}\!\Rightarrow\!\overset{2k}{\bullet} \otimes \overset{q}{\times}\!\Rightarrow\!\overset{r}{\bullet} = \cdots \oplus \overset{-2-r-3}{\times}\!\Rightarrow\!\overset{r}{\bullet} \oplus \cdots$$

whence $2(-2n+q) + 2k + r = 2(-q-r-3) + r$ so $2q = 2n-k-r-3$. Thus, $2q$ is integral but q itself need not be. By applying the translation functor (tensor with $\overset{0}{\bullet}\!\Rightarrow\!\overset{1}{\bullet} = \overset{0}{\times}\!\Rightarrow\!\overset{1}{\bullet} + \overset{-1}{\times}\!\Rightarrow\!\overset{1}{\bullet}$) one is reduced to deciding on the existence of a differential operator $\overset{-1/2}{\times}\!\Rightarrow\!\overset{0}{\bullet} \longrightarrow \overset{-5/2}{\times}\!\Rightarrow\!\overset{0}{\bullet}$.

The multiplication table for $so(5,\mathbb{C})$ is (entries give $[a,b]$).

$b \backslash a$	H	h	X	x_1	x_2	x_3	Y	y_1	y_2	y_3
H	0	0	$-2X$	$2x_1$	0	$-2x_3$	$2Y$	$-2y_1$	0	$2y_3$
h	0	0	X	$-2x_1$	$-x_2$	0	$-Y$	$2y_1$	y_2	0
X	$2X$	$-X$	0	$-x_2$	$-x_3$	0	$-H$	0	$2y_1$	$2y_2$
x_1	$-2x_1$	$2x_1$	x_2	0	0	0	0	$-h$	$-Y$	0
x_2	0	x_2	x_3	0	0	0	$2x_1$	$-X$	$-H-2h$	$-2Y$
x_3	$2x_3$	0	0	0	0	0	$2x_2$	0	$-2X$	$-2H-2h$
Y	$-2Y$	Y	H	0	$-2x_1$	$-2x_2$	0	y_2	y_3	0
y_1	$2y_1$	$-2y_1$	0	h	X	0	$-y_2$	0	0	0
y_2	0	$-y_2$	$-2y_1$	Y	$H+2h$	$2X$	$-y_3$	0	0	0
y_3	$-2y_3$	0	$-2y_2$	0	$2Y$	$2H+2h$	0	0	0	0

The Lie subalgebra $sl(2,\mathbb{C}) \cong \langle h, x_1, y_1 \rangle$ corresponds to this node

$$\bullet\!\Rightarrow\!\bullet$$

The Lie subalgebra $sl(2,\mathbb{C}) \cong \langle H, X, Y \rangle$ corresponds to this node

$\mathfrak{p} = \times\!\Rightarrow\!\bullet$ is therefore generated by everything <u>except</u> y_1, y_2, y_3 <u>namely</u> $H, h, X, x_1, x_2, x_3, Y$

The possible differential operator $\overset{-1/2}{\times}\!\Rightarrow\!\overset{0}{\bullet} \longrightarrow \overset{-5/2}{\times}\!\Rightarrow\!\overset{0}{\bullet}$ is second order by symbol considerations: $\bigodot^2\Omega^1 = \overset{-4}{\times}\!\Rightarrow\!\overset{4}{\bullet} \oplus \overset{-2}{\times}\!\Rightarrow\!\overset{0}{\bullet}$ and $\overset{-2}{\times}\!\Rightarrow\!\overset{0}{\bullet} \otimes \overset{-1/2}{\times}\!\Rightarrow\!\overset{0}{\bullet} = \overset{-5/2}{\times}\!\Rightarrow\!\overset{0}{\bullet}$.
By Poincaré-Birkhoff-Witt $V(\overset{-1/2}{\times}\!\Rightarrow\!\overset{0}{\bullet}) = \mathbb{C}[y_1,y_2,y_3]\alpha$ where α is <u>highest</u> (annihilated by X and x_1) with weight given by $h\alpha = -\tfrac{1}{2}\alpha$ and $H\alpha = 0$. One easily checks that $(y_2{}^2 - 2y_1y_3)\alpha \equiv \beta$ satisfies $h\beta = -\tfrac{5}{2}\alpha$, $H\beta = 0$, and $X\beta = 0$ so β corresponds to the proposed symbol and lifts to an invariant operator iff $x_1\beta = 0$. But $x_1\beta = x_1(y_2{}^2 - 2y_1y_3)\alpha = (Yy_2 + y_2x_1y_2 - 2hy_3 - 2y_1x_1y_3)\alpha$
$= (-y_3 + y_2Y + y_2Y + y_2{}^2x_1 - 2y_3h - 2y_1y_3x_1)\alpha = -y_3\alpha + y_3\alpha = 0$. The operator in question is, of course, the conformally invariant Laplacian in dimension 3. There is also an invariant first order operator $\overset{-3/2}{\times}\!\Rightarrow\!\overset{1}{\bullet} \longrightarrow \overset{-5/2}{\times}\!\Rightarrow\!\overset{1}{\bullet}$ (if α is the highest weight vector in $V(\overset{-3/2}{\times}\!\Rightarrow\!\overset{1}{\bullet})$ then $(y_2 + 2y_1Y)\alpha$ is also maximal). These operators $\overset{-1/2}{\times}\!\Rightarrow\!\overset{0}{\bullet}$ $\overset{-3/2}{\times}\!\Rightarrow\!\overset{1}{\bullet} \longrightarrow \overset{-5/2}{\times}\!\Rightarrow\!\overset{1}{\bullet}$ $\overset{-5/2}{\times}\!\Rightarrow\!\overset{0}{\bullet}$ and translates together with the Bernstein-Gelfand-Gelfand operators comprise a complete list of invariant operators between bundles of non-singular character. In addition, there is one singular case:

$$\overset{-1}{\times}\!\Rightarrow\!\overset{1}{\bullet} = \overset{-1}{\times}\!\Rightarrow\!\overset{1}{\bullet} \longrightarrow \overset{-3}{\times}\!\Rightarrow\!\overset{1}{\bullet} = \overset{-3}{\times}\!\Rightarrow\!\overset{1}{\bullet}.$$

These operators all have curved analogues either by a curved translation principle [3] or (better) by direct application of the Cartan connection [2]. The half integral weights which occur in this example suggest an analogue for G_2:

<u>Case of $\times\!\Rrightarrow\!\bullet$</u>: Central character considerations restrict attention to

$$\overset{q}{\times}\!\Rrightarrow\!\overset{r}{\bullet} \quad \overset{-q-2}{\times}\!\Rrightarrow\!\overset{3q+r+3}{\bullet} \quad \overset{-q-r-3}{\times}\!\Rrightarrow\!\overset{3q+2r+4}{\bullet} \quad \overset{-2q-r-4}{\times}\!\Rrightarrow\!\overset{3q+2r+4}{\bullet} \quad \overset{-2q-r-4}{\times}\!\Rrightarrow\!\overset{3q+r+3}{\bullet} \quad \overset{-q-r-3}{\times}\!\Rrightarrow\!\overset{r}{\bullet}$$

and symbol considerations force $6q \in \mathbb{Z}$. The multiplication table for G_2 is:

	H	h	X	x_1	x_2	x_3	x_4	x_5	Y	y_1	y_2	y_3	y_4	y_5
H	0	0	$-2X$	$3x_1$	x_2	$-x_3$	$-3x_4$	0	$2Y$	$-3y_1$	$-y_2$	y_3	$3y_4$	0
h		0	X	$-2x_1$	$-x_2$	0	x_4	$-x_5$	$-Y$	$2y_1$	y_2	0	$-y_4$	y_5
X			0	$-x_2$	$-x_3$	$-x_4$	0	0	$-H$	0	$3y_1$	$4y_2$	$3y_3$	0
x_1				0	0	0	x_5	0	0	$-h$	$-Y$	0	0	$-y_4$
x_2					0	$-x_5$	0	0	$3x_1$	$-X$	$-H-3h$	$-4Y$	0	$3y_3$
x_3						0	0	0	$4x_2$	0	$-4X$	$-8H-12h$	$-12Y$	$-12y_2$
x_4							0	0	$3x_3$	0	0	$-12X$	$-36H-36h$	$36y_1$
x_5								0	0	$-x_4$	$3x_3$	$-12x_2$	$36x_1$	$-36H-72h$
Y									0	y_2	y_3	y_4	0	0
y_1										0	0	0	$-y_5$	0
y_2											0	y_5	0	0
y_3												0	0	0
y_4													0	0
y_5														0

[This table can be computed from the root diagram.

It is now elementary to check that, with obvious notation, $(y_2 + 3y_1Y)\alpha$ provides an invariant differential operator $\overset{-4/3}{\times}\!\Rrightarrow\!\overset{1}{\bullet} \longrightarrow \overset{-7/3}{\times}\!\Rrightarrow\!\overset{2}{\bullet}$.
However, although

$$\begin{pmatrix} 9y_1{}^2y_4{}^2 - 18y_1y_2y_3y_4 + 6y_1y_3{}^3 + 8y_2{}^3y_4 \\ -3y_2{}^2y_3{}^2 - 27y_1y_4y_5 + 3y_2y_3y_5 \end{pmatrix}\alpha$$

provides a symbol for a differential operator $\overset{-1/2}{\times}\!\Rrightarrow\!\overset{0}{\bullet} \longrightarrow \overset{-5/2}{\times}\!\Rrightarrow\!\overset{0}{\bullet}$, it does <u>not</u> lift and, by translation, it follows that there are no differential operators with $\tfrac{1}{2}$-integral weights.

<u>Refs</u>: 1. RJB & MGE TN 20. 2. RJB TN 20 and D.Phil. thesis Oxford 1985.
3. MGE & John Rice: Commun. Math. Phys. <u>109</u> (1987), 207-228.
<u>Michael Eastwood (Adelaide)</u>.

Tensor products of Verma modules and conformally invariant tensors.

Let $P = \bullet\!-\!\times\!-\!\bullet$ be the conformal complex Poincaré group in $SL(4,\mathbb{C})$. If $G = SL(4,\mathbb{C})$ then the flag variety $G/P = Gr_2(\mathbb{C}^4) = \mathbb{C}M$. At least locally (and globally, modulo topological obstructions) on any 4-dim$_{\mathbb{C}}$ conformal manifold there is an unique P-principal bundle $\mathcal{G} \to \mathcal{M}$ together with a $\mathfrak{g}$ = Lie algebra $\mathfrak{sl}(4,\mathbb{C})$ — valued one-form on $\mathcal{G}$, ω giving an isomorphism $T_g\mathcal{G} \to \mathfrak{g}$ (so if $\mathcal{M}$ is conformally flat, $\mathcal{G}$ is covered by G and the pull-back of ω is the Maurer-Cartan form). ω has the property that $\omega^{-1} : \mathfrak{g} \to \Gamma(T\mathcal{G})$ restricts to a homomorphism of Lie algebras on $\mathfrak{p}$ = Lie Alg. of P and for $u \in \mathfrak{p}$, $v \in \mathfrak{g}$, $\omega^{-1}[u,v] = [\omega^{-1}u, \omega^{-1}v]$. The failure of ω^{-1} to be a homomorphism of algebras is thus determined on a complement to $\mathfrak{p}$ in $\mathfrak{g}$: set $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}_-$ and take $x, v \in \mathfrak{u}_-$. Then $\Omega(\omega^{-1}x, \omega^{-1}v) = [x,v] - \omega[\omega^{-1}x, \omega^{-1}v]$ is the curvature of ω. All of this is well known in twistor theory as the <u>local twistor transport</u> and its curvature. (See [1]). The same sort of picture holds in dims$_{\mathbb{C}} \geq 3$. (In dim 2, the conformal group is ∞-dim, which is what all the fuss is about). If V is any representation of P, one has a homogeneous sheaf of holomorphic sections of the vector bundle $\mathcal{G} \times_P V$ induced by V. So $\overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet} = \mathcal{O}^{A'}$ etc (see [2]). From the flat case, we know there is a good class of linear differential invariants of the conformal structure of $\mathcal{M}$ (see [3], [4], [5]): the calculation of these depends on ω and the identification of $J^\infty(V)$ (the formal jet bundle of $\mathcal{G}\times_P V$) as a homogeneous bundle, induced by the ($\mathfrak{b}$-finite) dual of a <u>Verma module</u> $\mathcal{M}_{\mathfrak{p}}(V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^*$. The differential invariants we want correspond to the translation invariant differential operators on $\mathcal{G} \times_P V$. These are classified by $\mathfrak{p}$-module homomorphisms $\mathcal{M}_{\mathfrak{p}}(V^*) \leftarrow W^*$ (giving a differential operator on homogeneous sheaves $\mathcal{O}(V^*) \to \mathcal{O}(W^*)$). Frobenius reciprocity says such an operator is equivalent to a $\mathfrak{g}$-module homomorphism of Verma modules $\mathcal{M}_{\mathfrak{p}}(V^*) \leftarrow \mathcal{M}_{\mathfrak{p}}(W^*)$.

<u>Recap</u>: homomorphisms of Verma modules $\Longleftrightarrow$ distinguished differential invariants of conformal structure of $\mathcal{M}$.

<u>Remark</u>: In particular, any conformally invariant operator on M (e.g. $\Box$) gives rise to one on $\mathcal{M}$ (e.g. $\Box + R/6$). One may even calculate the curvature terms explicitly.

The invariant operators on G/P fit into certain Bernstein/Gelfand/Gelfand resolutions (generalized deRham resolutions). On $\mathcal{M}$ these fail to be complexes ($\mathcal{D}^2 \neq 0$) by virtue of curvature (i.e. ω^{-1} is not a homomorphism on all of $\mathfrak{g}$). Their composition gives further differential invariants: thus

$$\overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet} \xrightarrow{d_1} \overset{2}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet} \xrightarrow{d_2} \overset{3}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{0}{\bullet} \qquad \text{by } \pi^{A'} \xrightarrow{d_1} \nabla^A_{(A'}\pi_{B')} \quad \text{and}$$

$$\varphi^A_{A'B'} \longrightarrow \nabla^A_{(A'}\varphi_{B'C')}$$

so that $d_2 \circ d_1 : \pi^{A'} \longrightarrow \nabla^A_{(A'}\nabla_{|A|B'}\pi_{C')} \propto \tilde{\Psi}_{A'B'C'D'}\pi^{D'}$ (see [6])

It has been conjectured that taking all possible combinations of this kind, in various linear combinations and repeatedly yields all differential invariants of the conformal structure (see [6]). To begin to try to prove such a result, one can examine the known results more closely. For instance

$$E_{abc} \triangleq \tilde{\psi}_{A'B'C'D'} \nabla^{AA'} \psi_{ABCD} - \psi_{ABCD} \nabla^{AA'} \tilde{\psi}_{A'B'C'D'} \qquad (1)$$

is invariant (and extremely useful). It is clear that this ought to be thought of as some kind of anti-symmetrization "$\alpha \cdot \beta - \beta \cdot \alpha$". Now both $\tilde{\psi}$ and ψ, as sections of $\underset{\bullet\!-\!\times\!-\!\bullet}{4\;{-4}\;0}$ and $\underset{\bullet\!-\!\times\!-\!\bullet}{0\;{-4}\;4}$, induce sections of the corresponding jet sheaves. Over each point of M this gives elements of dual Verma modules, $M(\underset{\bullet\!-\!\times\!-\!\bullet}{4\;{-4}\;0})^*$ and $M(\underset{\bullet\!-\!\times\!-\!\bullet}{0\;{-4}\;4})^*$. Just as in the finite dimensional case, we may take tensor products of these modules and decompose (hopefully into dual Verma modules) so obtaining fresh spinorial invariants, like (1). Indeed, an explicit calculation readily checks that (1) comes this way.

<u>Recap</u>: Decompose tensor products to compute conformal invariants. (<u>Conjecture</u>: all come this way).

So far, I do not know the general theory of such tensor products (which must be complicated, to recapture, in particular, the finite dimensional theory). But, using the Penrose transform, one can consider a special case:

$P \subset G$, with "transpose" determines a pt at infinity in $\mathbb{M}$ and so a line $\mathbb{L}$ in $\mathbb{PT}$ and its annihilator $\mathbb{L}^*$ in $\mathbb{PT}^*$. Then, as $\mathfrak{g}$-modules

$$\begin{aligned} H^1(\underset{\times\!-\!\bullet\!-\!\bullet}{1\;0\;1}) \;(\text{on } \mathbb{PT}-\mathbb{L}) &\overset{\mathfrak{p}}{\cong} \text{unique irreducible submodule of } M(\underset{\bullet\!-\!\times\!-\!\bullet}{0\;{-4}\;4})^* = L(\underset{\bullet\!-\!\times\!-\!\bullet}{0\;{-4}\;4}) = L^1 \\ H^1(\underset{\bullet\!-\!\bullet\!-\!\times}{1\;0\;1}) \;(\text{on } \mathbb{PT}^*-\mathbb{L}^*) &\cong \text{unique irreducible submodule of } M(\underset{\bullet\!-\!\times\!-\!\bullet}{4\;{-4}\;0})^* = L(\underset{\bullet\!-\!\times\!-\!\bullet}{4\;{-4}\;0}) = L^2 \end{aligned}$$

(sheaf $\underset{\times\!-\!\bullet\!-\!\bullet}{1\;0\;1} \not\cong \Theta$ so this concerns linearized deformations). Under $\mathfrak{p}$ (<u>not</u> under $\mathfrak{g}$) one has a splitting (differential) $M(\underset{\bullet\!-\!\times\!-\!\bullet}{0\;{-4}\;4})^* \longrightarrow L(\underset{\bullet\!-\!\times\!-\!\bullet}{0\;{-4}\;4})$ (etc). (It may be (iff Bach tensor $\equiv 0$) that, already, ψ & $\tilde{\psi}$ lie in the submodules). So we try to decompose $L^1 \otimes L^2$ under $\mathfrak{g}$ (strictly, we should do it under $\mathfrak{p}$ but, since we will encounter only $\mathfrak{g}$-irreducibles, this is irrelevant). Compute:

$$\text{Künneth} \Longrightarrow L^1 \otimes L^2 \cong H^2((\mathbb{P}-\mathbb{L}) \times (\mathbb{P}^*-\mathbb{L}^*), \underset{\times\!-\!\bullet\!-\!\bullet}{1\;0\;1} \otimes \underset{\bullet\!-\!\bullet\!-\!\times}{1\;0\;1}) \cong H^4_{\mathbb{L}\times\mathbb{L}^*}(\mathbb{P}\times\mathbb{P}^*, \underset{\times\!-\!\bullet\!-\!\bullet}{1\;0\;1} \otimes \underset{\bullet\!-\!\bullet\!-\!\times}{1\;0\;1}). \quad (*)$$

(Mayer Vietoris + rel cohom. sequence). The map is just <u>cross product</u>. But then multiplication by $\frac{1}{(z\cdot w)^k}$ gives maps

$$\smile \frac{1}{(z\cdot w)^k} : H^3_{\mathbb{L}\times\mathbb{L}^*}(\mathbb{A}, (\underset{\times\!-\!\bullet\!-\!\bullet}{1\;0\;1} \otimes \underset{\bullet\!-\!\bullet\!-\!\times}{1\;0\;1})|_{\mathbb{A}} \otimes \underset{\times\!-\!\bullet\!-\!\times}{k\;0\;k}) \longrightarrow L^1 \otimes L^2 \quad (2) \qquad (k \geq 1)$$

whose images span $L^1 \otimes L^2$ (standard theory). The groups (2) are easy to calculate, using $(\underset{\times\!-\!\bullet\!-\!\bullet}{1\;0\;1} \otimes \underset{\bullet\!-\!\bullet\!-\!\times}{1\;0\;1})|_{\mathbb{A}} \cong \underset{\times\!-\!\bullet\!-\!\times}{2\;0\;2} + (\underset{\times\!-\!\bullet\!-\!\times}{2\;1\;0} \oplus \underset{\times\!-\!\bullet\!-\!\times}{0\;1\;2}) + (\underset{\times\!-\!\bullet\!-\!\times}{0\;2\;0} \oplus \underset{\times\!-\!\bullet\!-\!\times}{1\;0\;1})$. If λ is one of these wts, tensored by $\underset{\times\!-\!\bullet\!-\!\times}{k\;0\;k}$ $(k \geq 1)$ then one has a contribution $L(\sigma_2\sigma_1\sigma_3 \cdot \lambda)$ in (2) (See [2] for notation, Weyl group etc.)

So one gets invariants as sections of $\underset{\bullet\!-\!\times\!-\!\bullet}{3+k \quad -8-2k \quad 3+k}$

$$\cong \mathcal{O}_{(\underbrace{A'\ldots D'}_{3+k})(\underbrace{A\ldots D}_{k+k})}[-2] \qquad (\text{from } \underset{\times\!-\!\bullet\!-\!\times}{2\;0\;2})$$

and the other constituents give the following (NB: no cancellation in long sequence because all occurs in one degree, viz 3).

(i)	$\underset{\times\!-\!\times}{210} \longmapsto \overset{k+2\quad -2k-8\quad k+4}{\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}$	$\ni$	$\tilde{\psi}_{A'B'C'D'}\nabla^{A'}_{(E}\psi_{ABCD)} - \psi_{(ABCD}\nabla^{A'}_{E)}\tilde{\psi}_{A'B'C'D'}$
(ii)	$\underset{\times\!-\!\times}{012} \longmapsto \overset{k+4\quad -2k-8\quad k+2}{\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}$	$\ni$	$\tilde{\psi}_{(A'B'C'D'}\nabla^{A}_{E')}\psi_{ABCD} - \psi_{ABCD}\nabla^{A}_{(E'}\tilde{\psi}_{A'B'C'D')}$
(iii)	$\underset{\times\!-\!\times}{101} \longmapsto \overset{k+2\quad -2k-6\quad k+2}{\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}$	$\ni$	$E_{abc} = \tilde{\psi}_{A'B'C'D'}\nabla^{AA'}\psi_{ABCD} - \psi_{ABCD}\nabla^{AA'}\tilde{\psi}_{A'B'C'D'}$
(iv)	$\underset{\times\!-\!\times}{020} \longmapsto \overset{k+3\quad -6-2k\quad k+3}{\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}$	$\ni$	$\psi_{ABCD}\tilde{\psi}_{A'B'C'D'}$
	$(k\geq 1)$		<u>case $k=1$ examples</u>

Notice the "generalizations" at (i) & (ii) of E_{abc} (which are easily & explicitly checked). (R.P. points out that E_{abc} occurs in Dighton's thesis). All the tensors obtained in this way are subject to invariant differential operators, constructed as at the start of this note. Iff the Bach tensor vanishes, they are automatically in the kernel of this operator, since then each lies in the irreducible subsheaf of the jet sheaf. (Explicit calculations verify this for E_{abc}!)

<u>Remark:</u> A linear deformation of $\mathbb{A}$, preserving its contact structure lies in $(H^1(\mathbb{A}, \underset{\times\!-\!\times}{101}))$ cf. [7]. The usual obstruct. theory + rep theory implies that this extends to $H^1(\mathbb{P}, \underset{\times\!-}{101}) \otimes H^1(\mathbb{P}^*, \underset{-\!\times}{101}) \subset H(\mathbb{P}\times\mathbb{P}^*, \circledast)$. iff Bach tensor vanishes (use argument at $\circledast$, previous page). Then the commutator (as in Kodaira-Spencer deformation theory) $[v,w]$, $(v\in H^1(\mathbb{P}, \underset{\times\!-}{101})$, $w\in H^1(\mathbb{P}^*, \underset{-\!\times}{101}))$ lies in $H^2(\mathbb{P}\times\mathbb{P}^*, \circledast)$. Restriction to formal neighbourhoods of $\mathbb{A}$ gives one elements of

$$H^2(\mathbb{A}, \underset{\times\!-\!\times}{101} \otimes \underset{\times\!-\!\times}{k'\ 0\ {-k'}}) \qquad k'\geq 5\ ;\ \text{set } k=k'-5$$

$$\cong$$

$$\ker\left(\overset{k+2\quad -2k-6\quad k+2}{\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet} \xrightarrow{\nabla^{AA'}} \overset{k+1\quad -2k-6\quad k+1}{\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet}\right)$$

But this must be a constituent in $L^1\otimes L^2$ & $[v,w] \mapsto$ projection of $[V]\otimes[W]$. So the obstructions are identified as the invariants of (iii). The other invariants admit similar geometric descriptions.

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[2] MGE & RJB: TN 20
[3] MGE & John Rice: Comm. Math Phys, April 1987
[4] RJB: TN 20
[5] RJB: Thesis: The Algebraic construct. of inv diff ops (1985)
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[7] LM + RJB: Class & Quantum grav (to appear).

Rob Baston

An Algebraic form of the Penrose Transform

(or: Dolbeault rules, $\mathcal{O}(k)$)

The Penrose transform of homogeneous vector bundles on complex homogeneous manifolds is now well understood as an exercise involving the Bernstein/Gelfand-Gelfand resolution & the Bott-Borel-Weil theorem. The result may be viewed as an isomorphism of $\mathfrak{g}$-modules ($\mathfrak{g}$-complex semi-simple), often irreducible, realized as a cohomology group or a space of solutions of invariant equations. Usually, the common module is the irreducible quotient of a Verma module, and one gets a construction of such modules & homomorphism between them "for free".

The question arises: can the Penrose transform act on more general $\mathfrak{g}$-modules? The answer is yes; it is then closely related to Zuckerman's functors. (see [1]). Consider the stalk at the identity coset of a homogeneous bundle, $\mathcal{O}_{\mathfrak{g}}(\lambda)$ on G/P (notation, see [2]). This is naturally a $\mathfrak{g}$-module (by action of left translation). If $\mathcal{O}_{\mathfrak{p}}(\lambda)$ has global sections, these must naturally form a finite dimensional sub-$\mathfrak{g}$-module. So the global section functor on sheaves corresponds to the functor $\Gamma_{\mathfrak{g}}$ on $\mathfrak{g}$-modules which picks out the vectors whose orbits under $\mathfrak{g}$ are finite. Not all sheaves have global sections. Not all $\mathfrak{g}$, modules will have nontrivial image under $\Gamma_{\mathfrak{g}}$. One needs derived functors $H^i_{\mathfrak{g}} = R^i\Gamma_{\mathfrak{g}}$, generalizing cohomology. In an appropriate category, these are computed by taking injective resolutions, applying $\Gamma_{\mathfrak{g}}$ and taking cohomology. How does one get such a resolution? The answer, for the kind of $\mathfrak{g}$-module which arises as the stalk at eP of a homogeneous sheaf is stunningly simple:

<u>Lemma</u>; If $0 \to \mathcal{O}_{\mathfrak{p}}(\lambda) \to \mathcal{E}^{0,\cdot}_{\mathfrak{p}}(\lambda)$ is the <u>Dolbeault resolution</u> of $\mathcal{O}_{\mathfrak{p}}(\lambda)$ then the stalk of this resolution at the identity coset is an injective resolution of $\mathcal{O}_{\mathfrak{p}}(\lambda)|_{eP}$

(This is an easy consequence of Schur's lemma). (So <u>Dolbeault</u>-protagonists were right all along!): it follows that one has an isomorphism of $\mathfrak{g}$-modules

$$R^i\Gamma_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{p}}(\lambda)|_{eP}\right) \cong H^i(G/P, \mathcal{O}_{\mathfrak{p}}(\lambda))$$

If $\mathfrak{l} \subset \mathfrak{g}$ is another parabolic, with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ then $R^i\Gamma_{\mathfrak{l}}$ are defined on the category $\mathcal{O}_{\mathfrak{p}}$ of $\mathfrak{g}$-modules (of which $\mathcal{O}_{\mathfrak{p}}(\lambda)|_{eP}$ is an object), and are an algebraic form of the Penrose transform. They are the Zuckerman functors (though not quite employed like this, usually). (One computes $R^i\Gamma_{\mathfrak{l}}$ exactly as one computes the Penrose transform, via a BG^2-resolution and the application of $R^i\Gamma_{\mathfrak{l}}$ to the resolvents)

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[2] RJB & MGE: The Penrose Transform, its interaction with rep theory or TN 19.

Rob Baston 1·IV·87

Classical Strings in Ten Dimensions

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Abstract.

This article analyses the motion of a classical relativistic string in flat complex ten dimensional space-time. A general solution is presented for the equations of motion. The solution is given in terms of essentially freely specifiable functions.

In deriving these results extensive use is made of spinors for ten dimensions, the basic properties of which are described in some detail. In particular, a significant role is played by those spinors in ten dimensions which satisfy the 'purity' property of E. Cartan. These constitute an eleven dimensional algebraic variety V in the sixteen dimensional linear space of reduced (Weyl) spinors for the group $SO(10)$. A general classical relativistic string in ten dimensions can be represented by means of a set of arbitrarily specifiable twice-differentiable curves in V.

As a byproduct of the investigation a general solution is also given for the equations of motion of a classical relativistic string in *eight* dimensional space-time.

Seldom have thinkers become so absorbed in revery, or so far estranged from reality, as to imagine for our space a number of dimensions exceeding the three of the given space of sense...

-Ernst Mach, *Space and Geometry*, 1906

To appear in Proc. Roy. Soc.

The Isoperimetric Inequality for Black Holes.

The Isoperimetric Inequality for black holes is the name which has become attached to the inequality

$$A \leqslant 16\pi M^2 \qquad 1$$

which is conjectured to hold between the area A of a 2-surface S and the mass M contained by or associated with S in some way, when S is a black hole or an outermost trapped surface in an initial data set (Penrose 1973, Gibbons 1984). It forms part of what Penrose (1973, 1982) calls *the establishment picture* of the Cosmic Censorship Hypothesis.

In a variety of situations, (1) is equivalent to an integral inequality on smooth, strictly positive functions on the unit sphere S^2. In this note I shall recall what these situations are and say something about the integral inequality.

The inequality is

$$4\pi \int f^2 \, dS \;\leqslant\; \left(\int (f + f^{-1}|\nabla f|^2) \, dS \right)^2 \qquad 2$$

Since f is positive we can substitute $f = g^2$ to simplify the right-hand-side a little:

$$4\pi \int g^4 \, dS \;\leqslant\; \left(\int g^2 + 4|\nabla g|^2 \, dS \right)^2 \qquad 3$$

I shall also want to consider a slightly different inequality and ask: for what k is it true that

$$4\pi \int g^4 \, dS \;\leqslant\; \left(\int g^2 + k|\nabla g|^2 \, dS \right)^2 \qquad 4$$

The first inequality was found by Gibbons and Penrose (Penrose 1973, Gibbons 1984) during an attempt to violate the Cosmic Censorship Hypothesis. They considered an ingoing shell of null matter, or equivalently a matter-distribution supported on an ingoing null-cone N, in flat space. The space-time to the future of the null-cone is curved, and the matter density can be adjusted so as to give a trapped surface S on N. Now (2) turns out to be equivalent to (1) for S, where f defines the location of S on N. Penrose and Gibbons argued further that the CCH would be violated if (1) was.

The same inequality arose in a study of Robinson-Trautman metrics (Penrose 1973). It is possible to prove that these metrics have a past-trapped surface on each of the special shear-free null hypersurfaces. Inequality (2) implies that the area of this trapped surface and the Bondi mass associated with the corresponding null hypersurface satisfy (1).

Inequality (2) arose for the third time in the work of Ludvigsen and Vickers (1983) where it is equivalent to (1) for the general case that

- A is the area of a marginally future trapped surface S on a null hypersurface N which extends to $\mathscr{I}^-$
- M is the Ludvigsen-Vickers quasi-local mass on N which increases into the past and tends to the Bondi mass at $\mathscr{I}^-$.

Finally, I found (2) as being equivalent to (1) for a static 'black-hole-in-a-box' (Tod 1985, 1986) where

- A is the area of the cross-over 2-surface S on the Killing horizon
- M is Penrose's quasi-local mass at S.

It is rather remarkable that exactly the same inequality turns up in all four places.

Penrose (1982) inquired whether a proof of (2) was known. The first proof that I knew of was mine (Tod 1985) where I deduced it from the Sobolev inequality in $\mathbb{R}^4$ (see eg Aubin 1982). Actually what this proves is inequality (4) but with k=3. Of course this is sufficient, but its rather unaesthetic since it is precisely (2) which keeps turning up. Later I used the Sobolev inequality in $\mathbb{R}^3$ to deduce (4) but with k=8/3 (Tod 1986). Now the question naturally arises of what is the smallest k which will do?

At GR11 Malcolm Ludvigsen told me of a proof of (2) due to R. Osserman and R. Schoen and contained in a letter to him. The proof in this letter (from 1984) goes as follows: start with

$$4\pi \int f^2 \, dS \leqslant \left(\int f \, dS\right)^2 + \left(\int |\nabla f| \, dS\right)^2 \qquad 5$$

which we shall justify later. Now use

$$\left(\int |\nabla f| \, dS\right)^2 \leqslant \left(\int f \, dS\right)\left(\int f^{-1} |\nabla f|^2 \, dS\right)$$

on the right-hand side in (5):

$$4\pi \int f^2 \, dS \leqslant \left(\int f \, dS\right)^2 + \left(\int f \, dS\right)\left(\int f^{-1} |\nabla f|^2 \, dS\right)$$

$$\leqslant \left(\int f + \tfrac{1}{2} f^{-1} |\nabla f|^2 \, dS\right)^2 \qquad 6$$

Here I have introduced a 2 in the denominator of the second term which Osserman didn't do in the letter, I presume because he was aiming at precisely (2). This is now (4) but with k=2. Thus from (5) we obtain a still further sharpening of (4). To justify (5), Osserman remarks that it reduces to the isoperimetric inequality on the unit sphere if f is the characteristic function of a domain D. This is

$$4\pi A \leqslant L^2 + A^2$$

$$\text{or} \qquad A(4\pi - A) \leqslant L^2 \qquad 7$$

where A is the area of D and L is the length of the boundary.

The final step in proving (4) this way is a proof of (5). Göran Bergqvist, a student of Malcolm Ludvigsen, recently sent me a preprint by G. Talenti of the University of Florence (Talenti 1986)which does just that. The paper is to appear in *General Inequalities 5* pub. Birkhauser and the proof uses the isoperimetric inequality (7).

This still leaves open the question of the smallest k in (4). I have two reasons for thinking it might be k=1. The first is that, if we linearise g about one, say $g=1+\epsilon V$ then for k=1 (4) is equivalent to the Poincaré inequality:

$$\int |\nabla V|^2\, dS \geqslant 2\int V^2\, dS \qquad \text{if} \qquad V\int dS = 0 \qquad 8$$

which in turn is essentially the statement that 2 is the smallest non-zero eigenvalue of $-\nabla^2$ on the unit sphere. The second is that, if $g=1+t\cos\theta$, $|t|<1$, which is the most testing case in the linearised inequality, then (4) holds with k=1.

In conclusion, (4) is proved for k=2 although it is k=4 which turns up in the examples, and the proofs of (4) use various isoperimetric inequalities which adds support to calling (1) the isoperimetric inequality for black holes.

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K. P. Tod 1985 Class.Quant.Grav.**2** L65

1986 Class.Quant.Grav.**3** 1169

Paul Tod

Abstract :

Twistor description of the symmetries of Einstein's equations for stationary axisymmetric spacetimes.

by N.M.J. Woodhouse.

It is shown that the "hidden symmetries" of the reduced Einstein equations investigated by Kinnersley and others have a simple description in terms of Ward's twistor construction of stationary axisymmetric vacuum spacetimes.

Published in Classical and Quantum Gravity 4 (1987) 799.

A Note on Conserved Vectorial Quantities associated with the Kerr Solution

1. Introduction. Chandrasekhar (1983) points out that whereas the equations governing gravitational perturbations of the Kerr solution can be decoupled and separated, and hence solved, despite their byzantine complexity, the corresponding equations governing the combined gravitational-electromagnetic perturbations of the Kerr-Newman solution cannot, apparently, be decoupled; — that although a definitive proof to this effect is lacking, nevertheless numerous efforts to separate the equations were unsuccessful.

The separability in the one case and apparent unseparability in the other cannot simply be an accident, since we know in the case of the Kerr solution that the separability of various systems of partial differential equations is intimately linked with the existence of a Killing tensor for that space-time, and hence ultimately to the existence of a solution of the twistor equation $\nabla_{A'(A}\chi_{BC)} = 0$ (Walker & Penrose 1970). In a general way therefore we conclude that there must very likely be some feature of the Kerr geometry in connection with the twistor field χ_{AB} that does not carry through to the Kerr-Newman geometry; — and that this feature, or rather lack thereof, is responsible for the non-separability of the perturbation equations in the Kerr-Newman case. In this note I shall examine an unusual feature of just this sort which arises in connection with the Killing-Yano tensor associated with the Kerr and Kerr-Newman space-times.

2. The Killing-Yano Tensor. Let us begin by fixing a few conventions. In the case of both the Kerr solution (K) and the Kerr-Newman solution (K-N) there exists a solution χ_{AB} of the twistor equation $\nabla_{A'(A}\chi_{BC)} = 0$. The freedom of a constant factor is fixed by requiring that the vector $\xi_{A'A}$, defined by

$$\nabla_{A'A}\chi_{BC} = \xi_{A'(B}\varepsilon_{C)A} \qquad (1)$$

or equivalently by

$$\nabla^{A'A}\chi_{AB} = \tfrac{3}{2}\xi^{A'}{}_{B}\,, \qquad (2)$$

which is necessarily a Killing vector (cf. Hughston & Sommers 1973, Penrose & Rindler 1976, p. 108), should be future-pointing and asymptotically of unit norm. Then the skew tensor L_{ab} defined by

$$L_{ab} = i\chi_{AB}\varepsilon_{A'B'} - i\bar{\chi}_{A'B'}\varepsilon_{AB} \qquad (3)$$

satisfies the Killing-Yano equation $\nabla_{(a}L_{b)c} = 0$: it is the trace-free part of this equation which is equivalent to the twistor equation, whereas the vanishing divergence $\nabla^a L_{ab} = 0$ is implied by the reality of ξ_a.

In both K and K-N an absolutely conserved (i.e. parallelly propagated) vectorial quantity can be constructed for geodesic orbits by use of L_{ab} (Floyd 1973; cf. also Penrose & Rindler 1986 p. 110). Thus if $U^a\nabla_a U^b = 0$ then the vector H_a defined by $H_a = L_{ab}U^b$ is parallelly propagated: $U^a\nabla_a H_b = 0$, which follows as a simple consequence of the Killing-Yano relation.

3. Charged Particle Orbits. Here the situation is rather more complicated, for in the case of K-N we consider a charged particle moving under the influence of the associated electromagnetic field of the K-N solution; — whereas in the case of K we consider the motion of a charged particle in a special 'test' electromagnetic field of the Kerr type with its principal directions aligned with those of the gravitational field.

In each case, as with geodesic orbits, we may construct a Carter-type fourth integral of the motion (cf. Hughston et al 1972) given by $Q = H_a H^a$ with H_a defined as above, where U^a is tangent to the orbits in question.

On the other hand the situation as regards conserved vectorial quantities for charged particle orbits is somewhat more subtle. Floyd shows in his thesis that in K-N although a conserved (i.e. parallelly propagated) vector cannot apparently be constructed for charged particle orbits nevertheless a vectorial quantity can indeed be found which satisfies a natural modified propagation law. Suppose F_{ab} is the electromagnetic field associated with K-N, and let the equations of motion for a charged particle in that background be

$$m\, U^a \nabla_a U^b = \varepsilon F^{bc} U_c \qquad (4)$$

where m is the mass and ε is the charge of the particle. The vector $H_a = L_{ab}U^b$, defined as before, satisfies according to Floyd the following propagation equation:

$$m\, U^a \nabla_a H^b = \varepsilon F^{bc} H_c . \qquad (5)$$

This arises from the fact that the tensor $\theta_{ab} = L_{ac}F^c{}_b$ is automatically symmetric, a circumstance deriving from the fact that the spinor ϕ_{AB} associated with F_{ab} is proportional to χ_{AB} (cf. Hughston et al 1972).

In the case of K the vector H^a is likewise 'Lorentz propagated', i.e. according to equation (5), if here we interpret F^{ab} as the special test field as described above (with a specified charge).

4. Conserved Vectorial Quantities for Charged Particle Orbits.

Now what about the prospects for constructing an absolutely conserved vector in the case of charged particle motion? In this connection it is helpful to recall a lemma (Hughston et al 1972) concerning the energy integral for charged particle motion:

Lemma. Let F_{ab} satisfy the Maxwell equation $\nabla_{[a}F_{bc]} = 0$, and suppose U^a satisfies the Lorentz equation (4). Suppose moreover that ξ^a is a Killing vector and that the Lie derivation of F_{ab} with respect to ξ^a vanishes: $\mathcal{L}_\xi F_{ab} = 0$. Then $F_{ab}\xi^b = \nabla_a \phi$ for some scalar field ϕ, and hence $I = mU^a\xi_a + \varepsilon\phi$ is a constant of motion for the charged particle: $U^a\nabla_a I = 0$.

The important point to note is the existence, under the given assumptions, of a scalar ϕ such that $F_{ab}\xi^b = \nabla_a\phi$, which is essentially the potential energy of the charged particle. By analogy with this situation we are led in the present case to consider a vectorial integral of the form

$$G_a = L_{ab}U^b + M_a . \qquad (6)$$

In order for G_a to be absolutely conserved, i.e. $U^a\nabla_a G_b = 0$, we must by (4) have

$$\varepsilon L_{bc}F^c{}_a + m\nabla_a M_b = 0 \qquad (7)$$

and hence $\nabla_{[a}M_{b]} = 0$ since the other term is symmetric. Thus if G_a is to be conserved there must exist a scalar Φ such that $M_a = \nabla_a\Phi$ and hence that $\varepsilon L_{bc}F^c{}_a + m\nabla_a\nabla_b\Phi = 0$.

Amazingly in the case of the Kerr solution, where F_{ab} is a test field (of charge q), such a scalar Φ does exist, and is given by the following formula:

$$\Phi = -\tfrac{1}{2} i \frac{\varepsilon q}{mM}\left(\chi_{AB}\chi^{AB} - \bar{\chi}_{A'B'}\bar{\chi}^{A'B'}\right). \qquad (8)$$

Thus for a charged particle moving under the influence of a test electromagnetic field F^{ab} in the Kerr background there exists an absolutely conserved vectorial integral H_a — a feature of the Kerr geometry which insofar as I am aware has not hitherto been pointed out. (Here M is the mass parameter of K.)

In the case of the K-N geometry, on the other hand, the construction breaks down, and it is apparently not possible to construct a conserved vectorial quantity for charged particle motion, at least along the lines suggested above.

Interestingly enough the terms which arise in the K-N case which 'obstruct' the existence of a conserved vector quantity are essentially the same expressions as those arising in the combined gravitational-electromagnetic perturbations studied by Chandrasekhar which render the equations non-separable there (cf. p. 583 in his book). By the same token however we might therefore expect the vector H_a to play a role in the Kerr case in the decoupling and separability of Maxwell's equations in the Kerr background, though this is far from being obviously the case.

L. P. Hughston

I hope to give a more extended account of this material elsewhere. Gratitude is expressed to Ian Gatenby for helpful discussions and for checking through details of some of the calculations.

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Real Classical Strings

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Abstract.
A general solution is constructed for the equations of motion of the classical relativistic string. All allowable string configurations are encompassed, apart from a special set of measure zero characterized by the vanishing of a certain invariant for which the motion of the string is restricted to a time-like hyperplane. The solution is given in terms of an essentially freely specifiable curve lying in the hypersurface PN, the space of 'null twistors'.

To appear in Proc Roy Soc.

Notice to readers of Twistor Newsletter:

Articles, abstracts, (quotations from T'catcher) should now be sent to Ian Roulstone (Editor) at The Mathematical Institute, 24-29 St Giles, Oxford. OX1 3LB, England. [owing to Lionel Mason's retirement]. Thank-you.

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Examples of Anti-Self-Dual Metrics

by Claude LeBrun

Recall that an oriented Riemannian 4-manifold (M,g) is called anti-self-dual if its Weyl curvature C satisfies $C = -*C$, where $*$ is the Hodge star operator. An interesting class of such manifolds is provided by the Kähler manifolds of scalar curvature zero ([2], [7]). In this note we produce such manifolds with isometry group $SU(2)$; we pay particular attention to certain special cases which are complete and asymptotically locally flat.

We begin by considering a Kähler potential $\phi(u)$, where $u = |z_1|^2 + |z_2|^2$; thus

$$\omega = i\partial\bar{\partial}\phi$$

is the Kähler form of a metric on a spherical shell centered at the origin in $\mathbb{C}^2$. Letting $V(u)$ be defined by

$$\omega \wedge \omega = -V\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2,$$

the Ricci form of ω is given by

$$\rho = i\partial\bar{\partial} \log V$$

and the scalar curvature R satisfies

$$R\omega\wedge\omega = \rho\wedge\omega.$$

The equation $R = 0$ may therefore be written as

$$(\partial\bar{\partial}\phi) \wedge (\partial\bar{\partial} \log V) = 0.$$

The symmetry of the situation allows us to merely carry out our calculations at the z_1-axis. Letting $\psi = \log V$,

$$\partial\bar{\partial}\phi = (\phi' + u\phi'')dz_1\wedge d\bar{z}_1 + \phi' dz_2 \wedge d\bar{z}_2$$
$$\partial\bar{\partial}\psi = (\psi' + u\psi'')dz_1\wedge d\bar{z}_1 + \psi' dz_2 \wedge d\bar{z}_2$$

at the z_1-axis, so that our equation reads

$$0 = (\phi' + u\phi'')\psi' + (\psi' + u\psi'')\phi'$$

or

$$0 = \frac{1}{u}\frac{d}{du}\,(u^2\phi'\psi').$$

Hence, for some constant A, $u^2\phi'\psi' = A$.

Now let $y = u\phi'$. Then

$$V = \phi'(\phi' + u\phi'') = \frac{yy'}{u}$$

and $\psi' = \frac{d}{du}\log V = \frac{y'}{y} + \frac{y''}{y'} - \frac{1}{u}$. Thus

$$A = u^2\phi'\psi' = uy' + \frac{uyy''}{y'} - y,$$

and

$$Ay' = u(y')^2 - yy' + uyy''$$

$$= \frac{d}{du}\,[uyy' - y^2].$$

It follows that

$$y^2 + Ay + B = uyy'$$

and

$$\log u = \int \frac{y\,dy}{y^2+Ay+B}.$$

Hence, letting $(y^2+Ay+B) = (y+\alpha_1)(y+\alpha_2)$,

(1) $$u = \begin{cases} C(y+\alpha_2)^{\alpha_2/(\alpha_2-\alpha_1)}\,(y+\alpha_1)^{\alpha_1/(\alpha_1-\alpha_2)}, & \alpha_1 \neq \alpha_2 \\ C(y+\alpha)\,e^{\alpha/(y+\alpha)}, & \alpha_1 = \alpha_2 = \alpha. \end{cases}$$

Now

$$\phi = \int \frac{y}{u}\,du = y\log u - \int \log u\,dy$$

$$= (y+\alpha_1)\frac{\alpha_1}{\alpha_1-\alpha_2} + (y+\alpha_2)\frac{\alpha_2}{\alpha_2-\alpha_1}$$

$$- \frac{\alpha_1^{\,2}}{\alpha_1-\alpha_2}\log(y+\alpha_1) - \frac{\alpha_2^{\,2}}{\alpha_2-\alpha_1}\log(y+\alpha_2)$$

$$+ \text{const},$$

so

(2) $$\phi(u) = y - \alpha_1\log u - \alpha_2\log(y+\alpha_2) + \text{const},$$

where $y(u)$ is obtained by inverting equation (1).

In order to give the metric explicity, we now introduce a new coordinate $r = \sqrt{y}$, and let σ_1, σ_2, σ_3 be a left-invariant coframe for S^3 coinciding with dx^2, dy^2, dy^1 at $(z_1,z_2) = (1,0)$. The Kähler form is then given by

$$\frac{i\omega}{2} = (\phi'+u\phi'')\ (d(\sqrt{u}) \wedge \sqrt{u}\sigma_3) + u\phi'(\sigma_1\wedge\sigma_2)$$

$$= \frac{dy}{du}\ (\frac{du^2}{4u} + u\sigma_3{}^2) + r^2(\sigma_1{}^2+\sigma_2{}^2).$$

But since $uyy' = y^2+Ay+B$,

$$\frac{dy}{du}\ \frac{du^2}{4u} = \frac{y\ dr^2}{uy'} = \frac{y^2\ dr^2}{y^2+Ay+B} = \frac{dr^2}{1+\frac{A}{r^2}+\frac{B}{r^4}}.$$

Similarly, $uy'\sigma_3^2 = (1 + \frac{A}{r^2} + \frac{B}{r^4})\ r^2\sigma_3{}^2$, and hence

$$g = \frac{dr^2}{1+\frac{A}{r^2}+\frac{B}{r^4}} + r^2(\sigma_1{}^2+\sigma_2{}^2+(1 + \frac{A}{r^2} + \frac{B}{r^4})\sigma_3{}^2). \tag{3}$$

If $A = 0$, this is the metric of Eguchi-Hanson [3]; it is Ricci-flat, and the potential ϕ is explicitly given by

$$\phi(u) = u^2+b^2 + \frac{b^2}{2}\log(\sqrt{u^2+b^2}-b) - \frac{b^2}{2}\log(\sqrt{u^2+b^2}+b)$$

for $B = -b^2$. (Eguchi and Hanson adopt the opposite orientation convention, and so consider the metric to be self-dual instead of anti-self-dual.)

If $B = 0$, the potential ϕ may again be given explicitly, namely by

$$\phi(u) = u + A\log u.$$

Provided that $A > 0$, this metric is the standard metric [5] on the blow-up of $\mathbb{C}^2$ at the origin; the fact that it has zero scalar curvature was discovered by Burns [1]. Its anti-self-duality is of importance because it is conformally isometric to the Fubini-study metric (in an orientation reversing manner), the map being

$$\Phi : \mathbb{C}^2-\{0\} \longrightarrow \mathbb{C}^2-\{0\}$$

$$\Phi(z_1,z_2) = \left(\frac{z_1}{|z_1|^2+|z_2|^2}, \frac{z_2}{|z_1|^2+|z_2|^2}\right).$$

(We have restricted the Fubini-study metric to $\mathbb{C}^2 \subset \mathbb{CP}_2$; by dilation we take its Kähler potential to be log (1+Au), thereby making Φ the required conformal map.)

Both the Eguchi-Hanson and Burns metrics can be made complete by replacing the origin with $\mathbb{P}_1$ and, in the Eguchi-Hanson case, modding out by Z_2; thus, the metrics naturally live on complex line bundles over $\mathbb{P}_1$. We now generalize procedure .

Suppose that $\alpha_1 = -a^2$, $\alpha_2 = ka^2$, where $a > 0$ and k is a non-negative integer. Restrict attention, for a moment, to fibers of the Hopf map $\mathbb{C}^2-\{0\} \longrightarrow \mathbb{P}_1$. Then g becomes

$$g|_{\text{fiber}} = \frac{dr^2}{(1-\frac{a^2}{r^2})(1+\frac{ka^2}{r^2})} + (1-\frac{a^2}{r^2})(1+\frac{ka^2}{r^2})\, r^2 d\theta^2.$$

Now divide out by the action of Z_{k+1}, introducing a new angular coordinate $\hat{\theta} = (k+1)\theta$, and simultaneously introducing a new radial coordinate $\hat{r} = r^2 - a^2$. The fiber-wise metric thus becomes

$$g|_{\text{fiber}} = \frac{\hat{r}^2 d\hat{r}^2}{\hat{r}^2\left(1+\frac{ka^2}{\hat{r}^2+a^2}\right)} + \left(1+\frac{ka^2}{\hat{r}^2+a^2}\right)\frac{\hat{r}^2}{(1+k)^2}\, d\hat{\theta}^2$$

$$= \frac{1}{1+\frac{ka^2}{\hat{r}^2+a^2}}\left[d\hat{r}^2 + \left(\frac{1+\frac{k}{1+(\hat{r}/a)^2}}{1+k}\right)^2 d\hat{\theta}^2\right].$$

Since $\left(1+\frac{k}{1+(\hat{r}/a)^2}\right)/(1+k)$ differs from 1 by terms of order $\hat{r}^2$, this metric extends smoothly across the origin $\hat{r} = 0$.

Let us now notice that this process is compatible with the complex structure by inspecting equation 1, which in the present circumstances yields

$$|z_1|^2 + |z_2|^2 = (r^2+ka^2)^{k/(k+1)}(r^2-a^2)^{1/k+1}.$$

If we identify the complex ζ-plane with the quotient of a fiber by Z_{k+1} via the map

$$\zeta \longrightarrow \left(\frac{c_1\zeta^{1/(k+1)}}{\sqrt{c_1^2+c_2^2}}, \frac{c_2\zeta^{1/(k+1)}}{\sqrt{c_1^2+c_2^2}} \right)$$

we obtain

$$|\zeta|^2 = (r^2-a^2)(r^2+ka^2)^k,$$

so that the radial coordinate $\hat{r}$ is smoothly related to $|\zeta|$ by

$$|\zeta| = \hat{r}(\hat{r}^2 + (k+1)a^2)^{1/k}.$$

Essentially the same argument shows that the only other complete metric that can be obtained from the above family is the flat metric $A = B = 0$. Thus, we have the following:

<u>Proposition</u>. Let $L \longrightarrow \mathbb{P}_1$ denote the line bundle with chern class +1. For any integer $k \geq 0$, L^{-k} admits a complete zero scalar curvature asymptotically flat Kähler metric. Up to multiplication by an overall constant, there is exactly one such metric which is SU(2) invariant.

(The last observation follows from the fact that the metric (3) is only multiplied by an overall constant if A and B are changed in a fashion preserving A^2/B.)

Adding a point at infinity to any of these metrics gives rise to a compact self-dual orbifold.

It remains to be seen if these metrics are closely related to the metrics of Gibbons-Hawking [4] and of Hitchin [6].

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Abstract.:

Ambiguities in the definition of quasilocal mass.

by N.M.J. Woodhouse.

For a small surface in a vacuum spacetime, some components of the kinematic twistor are non-zero at the fifth order in the diameter of the surface even though, according to Penrose's modified construction, the leading term in the quasilocal mass M_P is sixth order (or higher). This implies that the leading term in M_P cannot be defined independently of higher order terms in the twistor norm.

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