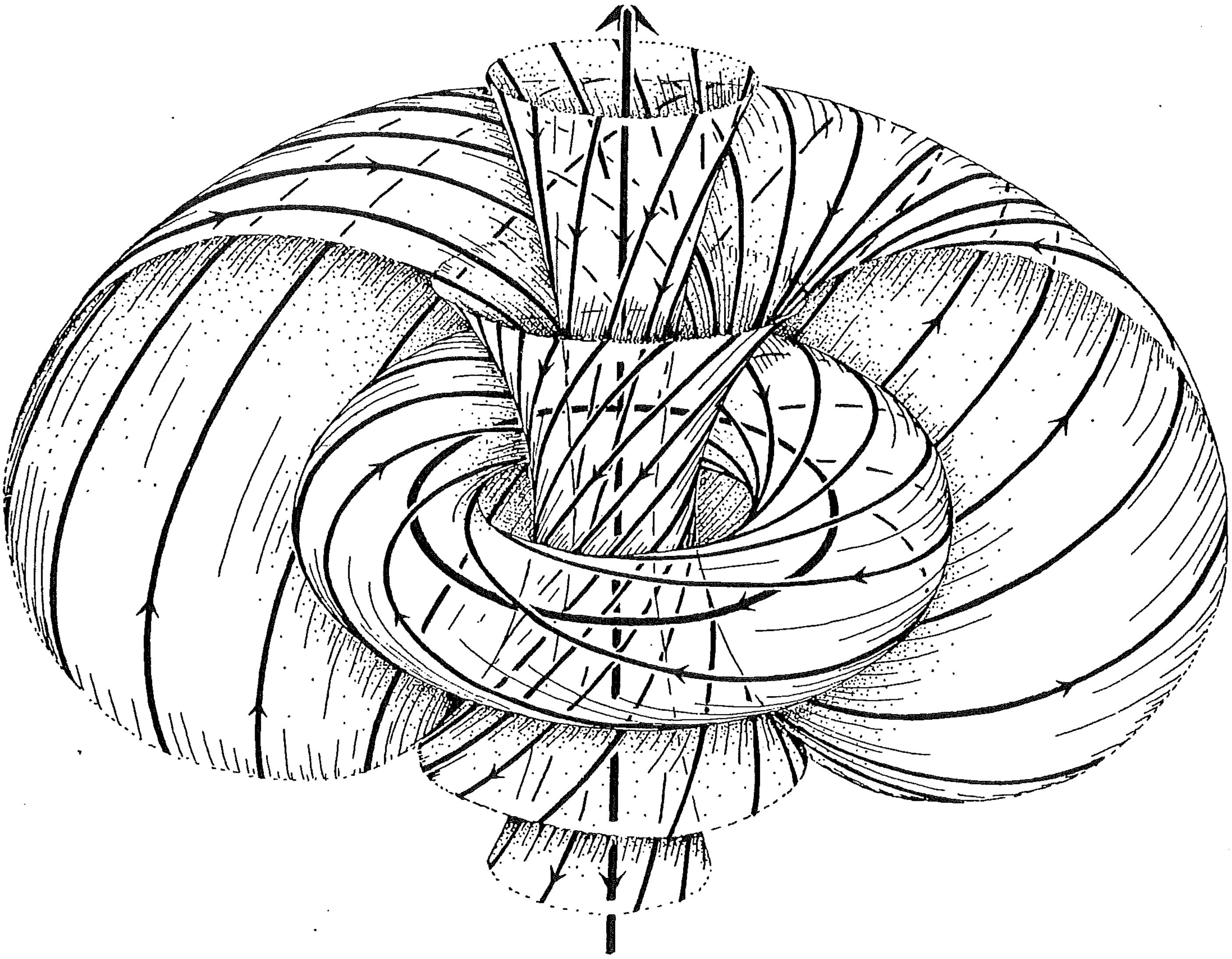
Twistor Newsletter (nº 25: 7, Sept., 1987)



Mathematical Institute, Oxford, England

We find that the expressions (A), (B), (C), (D), (E), (F) provide homogeneous functions with numbers of homogeneity given by, respectively, $(\omega, \xi, \zeta, \eta)$, $(\omega, \eta, \xi, \zeta)$, $(\xi, \omega, \eta, \zeta)$, $(\omega, \eta, \zeta, \xi)$, $(\xi, \eta, \omega, \zeta)$, $(\xi, \eta, \zeta, \omega)$, from which all permutations can clearly be built up. Qadir had previously explicitly worked out the Casimir operators, as explicit expressions in the symmetric functions of ω, η, ξ, ζ.

These contour integrals may be thought of as integrating representative functions for cohomology, and each contour circle reduces the cohomology by one step. Getting from a twistor function $f(\mathbf{Z})$ to a massless field by the standard (generalized Bateman) contour integral arises as a special case:

$$\pi_{A'}\cdots\pi_{L'}\,\phi^{A'\cdots L'}(x) = \oint f(\mathbf{Y}+\lambda\mathbf{Z})\,d\lambda$$

(This is the component in the "π-direction — which suffices, if we regard $\pi_{A'}$ as varying)

In accordance with twistor diagram procedures, etc. one should be able to <u>reverse</u> the reduction in cohomology by using <u>open</u> contours (i.e. contours with boundary). It would be worth while to explore this further, especially in order to relate to the following:

(I) <u>The inverse twistor function</u> (see R.P. in Quantum Gravity I; B.D.B., R.P. & G.A.J.S.). Here we have the standard procedure: twistor function $\rightarrow$ field, as an instance of (A) in a special case. $\mathbf{F}$ is independent of $\mathbf{W}$, $\mathbf{X}$ and $\mathbf{Y}$, so we have $\partial_W F = \partial_X F = \partial_Y F = 0$. Applying (A) to F we get G, which now satisfies $\partial_W G = \partial_X G = 0$, $\partial_Z \partial_Y G = 0$. The inverse of this: field $\rightarrow$ twistor function, should be an <u>open</u> contour version of (A), i.e. (A), which looks awfully like the B.D.B. — R.P. — G.A.J.S. expression in Q.G.I.

(II) <u>The twistor transform</u>

This should be an open (or open & closed, i.e. a cylinder) version of (F) $\therefore$ homogeneity degrees are $1-n, -1, -1, -1$ (if F has degrees $0, 0, 0, -n-2$, for helicity $\frac{1}{2}n$), i.e. $G = \oint\ldots$ is a function of $\lceil \mathbf{XYZ} \div \Delta$ (= "T"), of homogeneity degree $n-2$, multiplied by Δ^{-1} (as it should be).

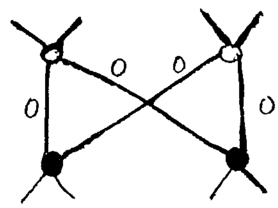
Thanks due to A.P.H. and R.J.B.

Roger Penrose

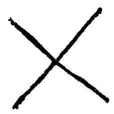
References: R. Penrose (1974) Relativistic Symmetry Groups, in <u>Group Theory in Non-Linear Problems</u>, ed A.O. Barut (Reidel)
A. Qadir, Ph.D. thesis, Birkbeck College, London

A twistor diagram for second-order $\emptyset^4$ scattering

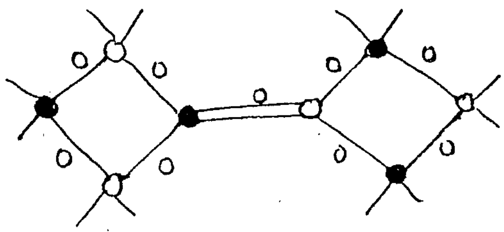
It's well-known that the twistor diagram



can be thought of as corresponding to the first-order $\emptyset^4$ scattering integral represented by the Feynman diagram

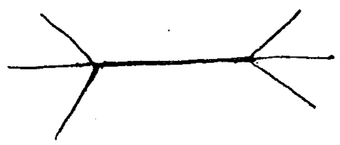


R.P. long ago raised a tentative suggestion (in Quantum Gravity I, pp 366-7) that a twistor diagram of form



(1)

might correspond to a second-order $\emptyset^4$ Feynman diagram of form

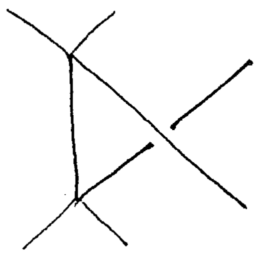


(2)

(No choices of channel are implied in (1) or (2)).

As R.P. pointed out, such a correspondence would require the double line in the twistor diagram to play the role of the "off-shell" Feynman propagator in the Feynman diagram. For some reason no-one has ever examined this suggestion in any detail, perhaps because $\emptyset^4$ theory has not been taken very seriously as a physical theory. However, massless $\emptyset^4$ theory does have interest in its own right as a component of the "standard model" of contemporary field theory, and it is perhaps the simplest theory (being without gauge fields) in which to study higher-order diagrams. It turns out that study of this integral does open the door to a much wider class of Feynman diagrams than has hitherto been considered, and later articles in this Newsletter will go into some of these implications.

In this piece, however, I shall study this second-order amplitude simply as a matter of concrete calculation. To do this I shall consider a specific channel (i.e. allocation of in- and out-states), namely



(3)

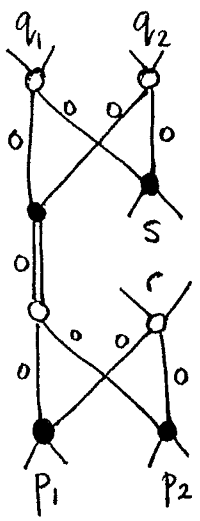
Choosing elementary in- and out- states, this Feynman diagram is simply a notation for the following integral

$$\iint \frac{d^4x\, d^4x'\, \Delta_F(x-x')}{(x-p_1)^2(x-p_2)^2(x-r)^2(x'-q_1)^2(x'-q_2)^2(x'-s)^2} \tag{4}$$

where p_1^a, p_2^a, s^a are in the past tube and q_1^a, q_2^a, r^a are in the future tube. In Fourier-analysed form this is

$$\begin{aligned}\int\cdots\int\, & d^4k_1\cdots d^4k_6\, d^4k\; \delta^+(k_1^2)\,\delta^+(k_2^2)\,\delta^-(k_3^2)\,\delta^-(k_4^2)\,\delta^-(k_5^2)\,\delta^+(k_6^2) \\ & \delta(k_1+k_2+k_3-k)\,\delta(k_4+k_5+k_6+k)\,(k^2)^{-1} \\ & e^{-ik_1\cdot p_1}e^{-ik_2\cdot p_2}e^{-ik_3\cdot r}e^{-ik_4\cdot q_1}e^{-ik_5\cdot q_2}e^{-ik_6\cdot s}\end{aligned} \tag{5}$$

The idea is that the scattering amplitude is now coded as this function of the six points $p_1^a \ldots\ldots s^a$, which we have to show can be represented as the result of integrating a twistor diagram. Following R.P.'s suggestion, we might try for size the twistor diagram



(6)

(using a notation by which external H's are labelled by the corresponding points in $\mathbb{C}M$).

Can this diagram possibly yield the correct function of the points as given by (4), (5)? *No it cannot.* To see this, consider operating on both (5) and (6) with :

$$\left(\frac{\partial}{\partial p_1{}^a} + \frac{\partial}{\partial p_2{}^a} + \frac{\partial}{\partial r^a}\right)^2$$

In (5) this is equivalent to introducing a factor $(k_1^a + k_2^a + k_3^a)^2 = k^2$

into the integral, cancelling the inverse derivative operator $(k^2)^{-1}$

Equivalently we could apply the same operator to (4), noting that

$$\left(\frac{\partial}{\partial p_1{}^a} + \frac{\partial}{\partial p_2{}^a} + \frac{\partial}{\partial r^a}\right)^2 \left\{\frac{1}{(x-p_1)^2(x-p_2)^2(x-r)^2}\right\} = \Box_x \left\{\frac{1}{(x-p_1)^2(x-p_2)^2(x-r)^2}\right\}$$

and then using integration by parts to turn $\Box_x$ on to $\Delta_F(x-x')$

producing (by definition) $\delta(x-x')$. Either way, we obtain the finite,

non-zero " ϕ^6 " integral

$$\int \frac{d^4x}{(x-p_1)^2(x-p_2)^2(x-r)^2(x-q_1)^2(x-q_2)^2(x-s)^2} \tag{7}$$

On the other hand, applying the same operator to (6), we note (again using integration by parts) that

applying $\partial_{p_1} . \partial_{p_2}$ to the integral is equivalent to applying

$$\underbrace{Z\,X}\ \underbrace{\partial_Z\,\partial_X}$$

to its interior, and similarly for $\partial_{p_2}.\partial_r,\ \partial_{p_1}.\partial_r$. Since of course $\partial_{p_1}.\partial_{p_1} = 0$ etc.

the whole operator is equivalent to applying

$$Z\,X\,\partial_Z\,\partial_X - Z\,\partial_W\,\partial_Z\,W - X\,\partial_W\,\partial_X\,W$$

It is easy to check that this operator yields *zero* identically when applied to (6), i.e. without requiring any conditions on the contours employed. So (6) cannot possibly be equivalent to (7), and it is no use hunting for new contours for (6) in an attempt to get it to work.

Thuis R.P.'s suggestion isn't right as it stands. In fact what we have seen is that the double line in (6) must represent an "on-shell" zero-mass propagator. It corresponds not to the Feynman propagator but to the positive frequency propagator

$$\Delta^+(x-x')$$

satisfying $\Box_x \Delta^+(x-x') = 0$

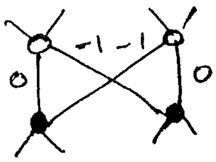
(This all really amounts to no more than the simple idea that a on-twistor function can only represent a zero-mass field). However, these observations point directly to a correct interpretation of (6), namely as the

integral obtained by replacing $\Delta_F(x-x')$ by $\Delta^+(x-x')$ in (4), or

equivalently, by replacing $(k^2)^{-1}$ by $\delta^+(k^2)$ in (5).

One can easily verify this identification by an exact computation of the twistor and space-time integrals.

Before going further it is important to note that the Feynman integral (4), (5) is actually *divergent;* it can be described as divergent at $k^2 = 0$. There is a strong analogy with the likwise divergent Møller scattering amplitude. Recall that the twistor diagram for Møller scattering,



makes finite sense as a genuine contour integral when the internal lines are made *inhomogeneous,* denoting factors like

$$(w.z - k)^{-1}, \qquad \gamma + \log(w.z - k)$$

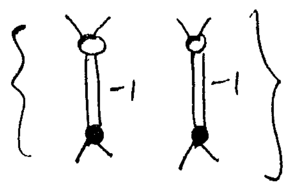
where k is some number, as yet unspecified.
[see my papers in Proc. Roy. Soc. A 397 pp 341-396 (1985)].

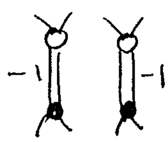
The result of the integration is then of form:

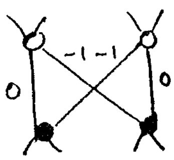
finite amplitude independent of k

plus a k-dependent term which corresponds to the divergence in the space-time integral.

Oversimplifying slightly, to make the essential point, this k-dependent term looks like

$\log k \times$ 

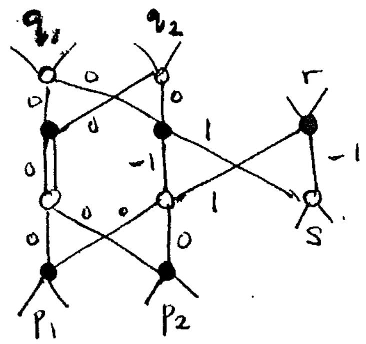
We can thus think of 

as the *period* of 

It will soon become important, I believe, to make these ideas precise.

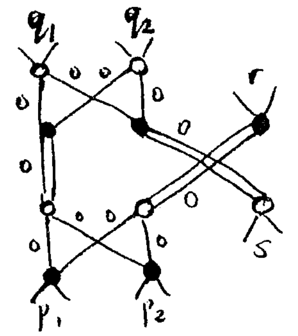
So following the analogy, we are led to look for a twistor diagram whose *period* is equivalent to (6). By "twistor diagram" I now mean an *inhomogeneous* twistor diagram. ALL the diagrams written down from here onwards will be implicitly inhomogeneous.

As I know of no systematic way of deriving the sought-after diagram I shall simply state the thing I guessed and show how it can be verified. This is:



(8)

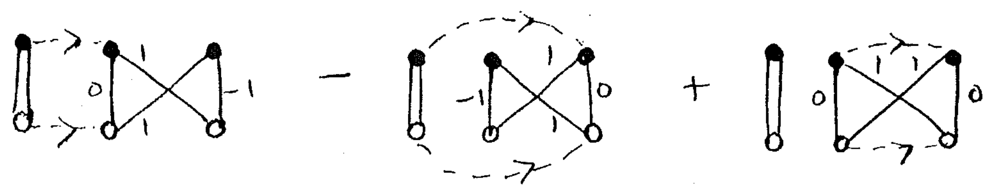
which clearly has a period equivalent to (6), namely



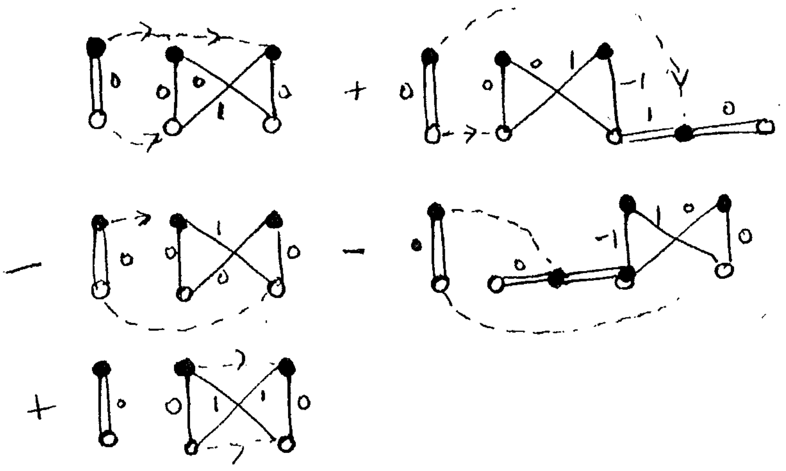
To verify the identity of (8) with (4) and (5) we must operate with

$$\left(\frac{\partial}{\partial p_1}+\frac{\partial}{\partial p_2}+\frac{\partial}{\partial r}\right)^2$$

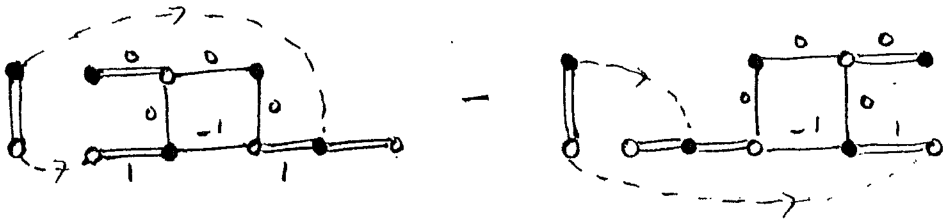
and actually do some calculation! The result of applying the operator is, to employ an obvious abbreviation:



Using integration by parts, this sum can be rewritten as:

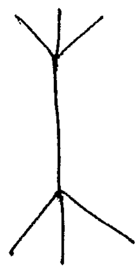


By a diagram identity (see my paper on Compton scattering, Proc. Roy. Soc. A 386, at page 189), the second and fourth terms can be rewritten as



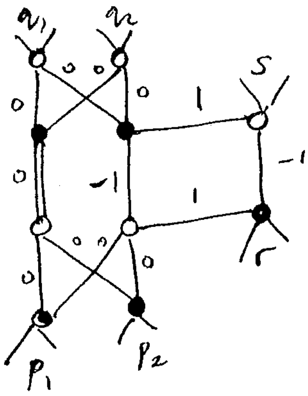
and they cancel, being dual formulations of the same scalar function of the six points. We are left with three terms which - hey presto - agree with the formula for the ϕ^6 integral that I worked out long ago but in which I couldn't detect any recognisable shape. (See Advances in Twistor Theory page 254, and then my article in TN 12 for further comment on how the formula was obtained. The calculation behind this formula should be revamped now that at last it falls into a recognisable pattern.)

We may likwise study the channel represented by the Feynman diagram



(9)

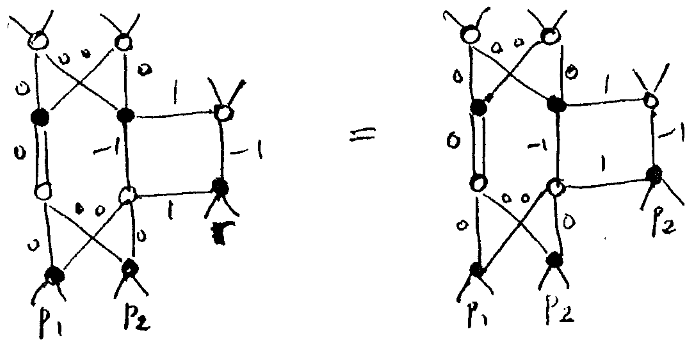
and verify that it corresponds to:



(10)

In this channel there is no divergence and no period. Study of this channel makes it particularly striking how peculiar the result is, by virtue of its asymmetric representation of a function symmetric in $p_1, p_2, r\ ;\ q_1, q_2, s$

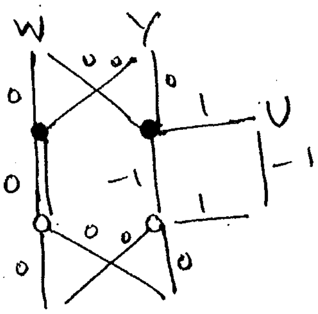
It is very far from obvious that



etc. etc.

(11)

although this must be true. I would expect this to be connected with properties of the 3-twistor symmetry group. One can certainly make the claim (11) look more plausible by noticing a connection with the 2-twistor symmetry group. Consider the operator $U.\partial_Y$ acting on



Much diagram-chasing and copious use of integration by parts massages this into a sum of terms in each of which a numerator just cancels a pole. Thus, if the poles are all essential to the contour, the sum is zero. Actually this is not quite true because the poles are now implicitly *inhomogeneous* like $(U.X - k)^{-1}$ and further argument is required to convince one that in this case inhomogeneity doesn't make any difference.

The reason we are interested in this operator is that it shows that the interior of diagram must project out an eigenfunction with eigenvalue 0 of the operator $Y.\partial_U\, U.\partial_Y$, which here is equivalent to the spin operator for scalar fields. Thus the diagram can only "see" the fields attached at Y and U through the spin-0 part of their product. This is equivalent to saying that it can only yield a functional of the one-point function

$$\{(x-q_2)^2(x-s)^2\}^{-1}$$

and not (as *a priori* one would expect) of the two-point function

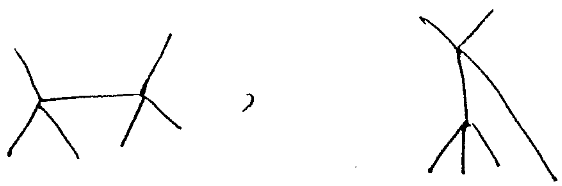
$$\{(x-q_2)^2(y-s)^2\}^{-1}$$

Obviously this applies similarly to U and W, and it's also immediate forW and Y; this makes it reasonable that the diagram actually "sees" only the one-point function

$$\{(x-q_1)^2(x-q_2)^2(x-s)^2\}^{-1}$$

This argument could probably be developed into something rigorous and capable of interesting generalisation.

There are of course other non-zero channels, namely those of form



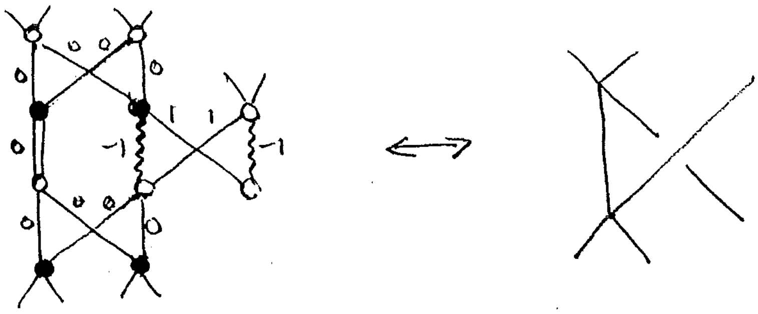
in which the number of particles going in is not the same as that of those emerging. These amplitudes should *also* be calculable from the same twistor diagram. At present I have no results regarding this question; my feeling is that it is not possible without taking on board the re-think of diagram structure and crossing symmetry which emerges in a later paper in this Newsletter, and that it even then will require/inspire some fresh and very interesting geometric ideas.

Andrew Hodges

Some twistor diagrams for Closed-loop Feynman diagrams

This article leads on from the result obtained for second-order ϕ^4 scattering integrals to deduce some third- and fourth-order twistor diagrams. We start with Feynman diagrams which are tree diagrams and then move on to consider some which have closed loops. The discussion is one of gradually diminishing rigour and is not intended to be definitive, although I believe it to be basically correct. It gives a first glimpse of what "ultra-violet-divergences" should look like in twistor diagram terms.

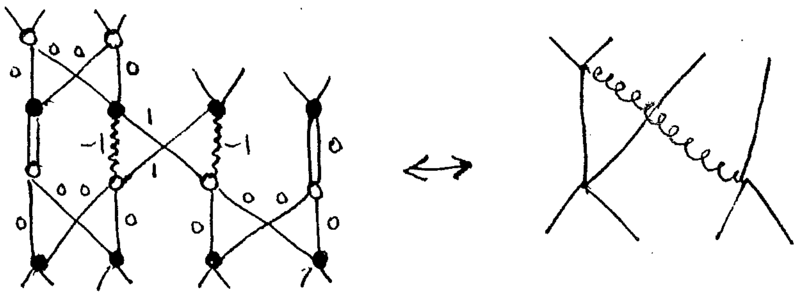
The starting-point is the correspondence:



(1)

(here and later a wavy line  for (-1)-lines is sometimes used - purely to aid the eye in seeing the pattern of the diagram).

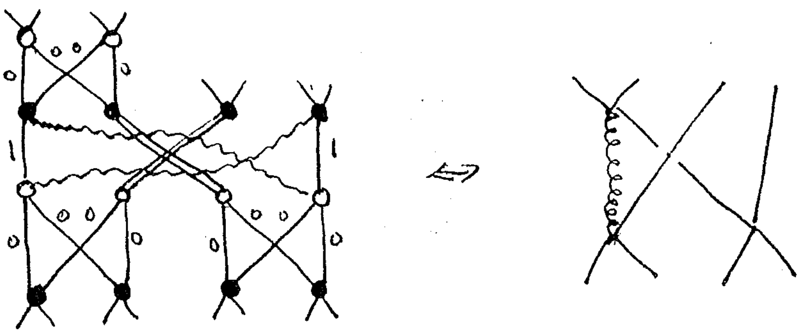
Now we may deduce at once the correspondence



(2)

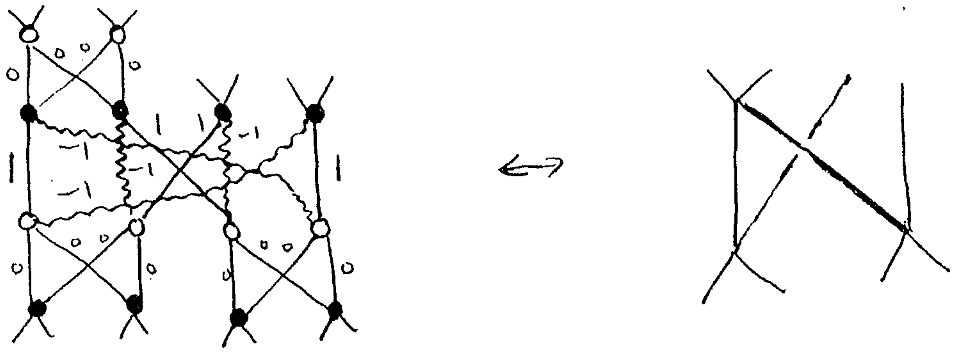
where the springy line in the Feynman diagram prescribes an on-shell propagator Δ^+ instead of the Feynman propagator Δ_F.

By symmetry, we have also



(3)

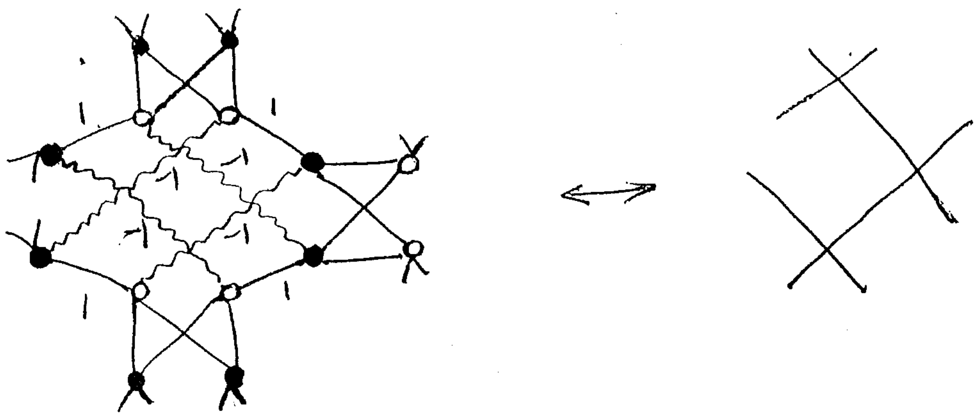
This immediately suggests:



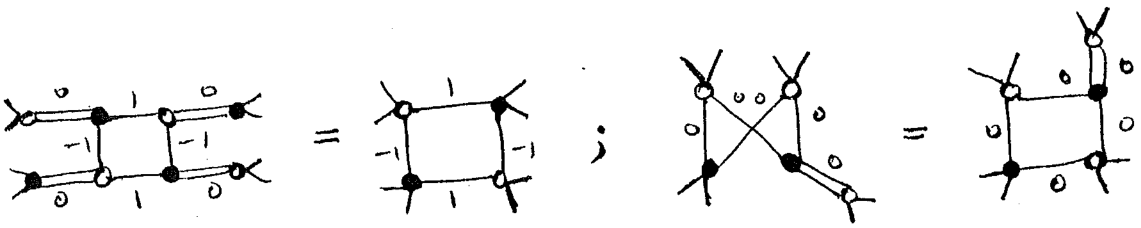
(4)

The point is that (2) and (3) give two necessary conditions to be satisfied by a twistor diagram corresponding to the third-order Feynman diagram in (4), namely that two different periods are correct. We could go on and test (4) by the process of differentiation along the propagator lines, but I haven't yet done this (it would be nicer, of course, to find a more general theorem about how to extend tree diagrams...).

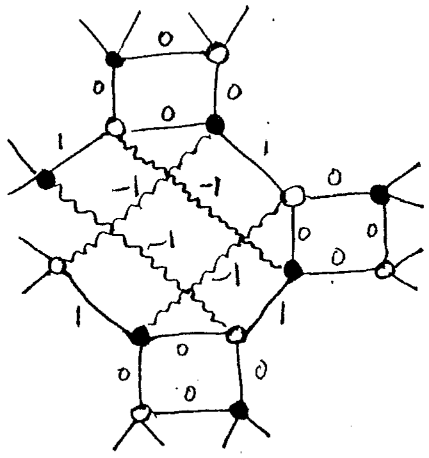
We can now make a further chain of propositions. Take (4) in another channel, thus supposing:



We can put this in a more symmetrical form by using the diagram identities

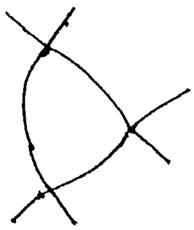


to suggest:



(5)

At this point we can make a bold jump towards representing the simplest possible " finite" Feynman loop diagram, namely



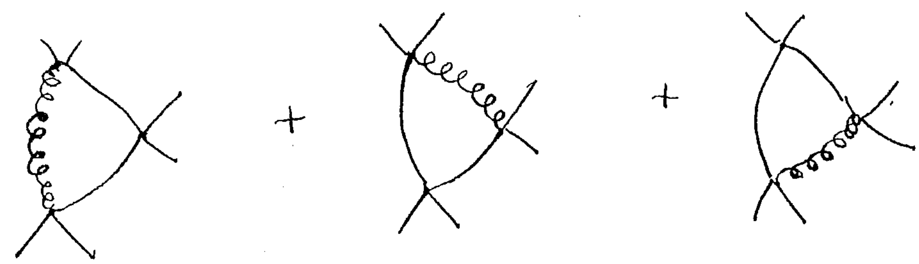
How do we make this bridge from trees to loops? The essential point is that the loop integral in momentum space looks like:

$$\int \frac{d^4k}{((k-u)^2+i\varepsilon)((k-v)^2+i\varepsilon)((k-w)^2+i\varepsilon)}$$

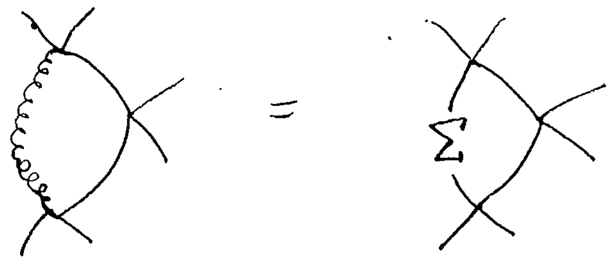
But following the Feynman prescription for how to do the integral, we note from simple observations in one-dimensional complex analysis that this can be reformulated as

$$\int \frac{d^4k \;\; \delta^+(k-u)^2}{((k-v)^2+i\varepsilon)((k-w)^2+i\varepsilon)} \quad + \quad 2 \text{ similar terms}$$

i.e. as

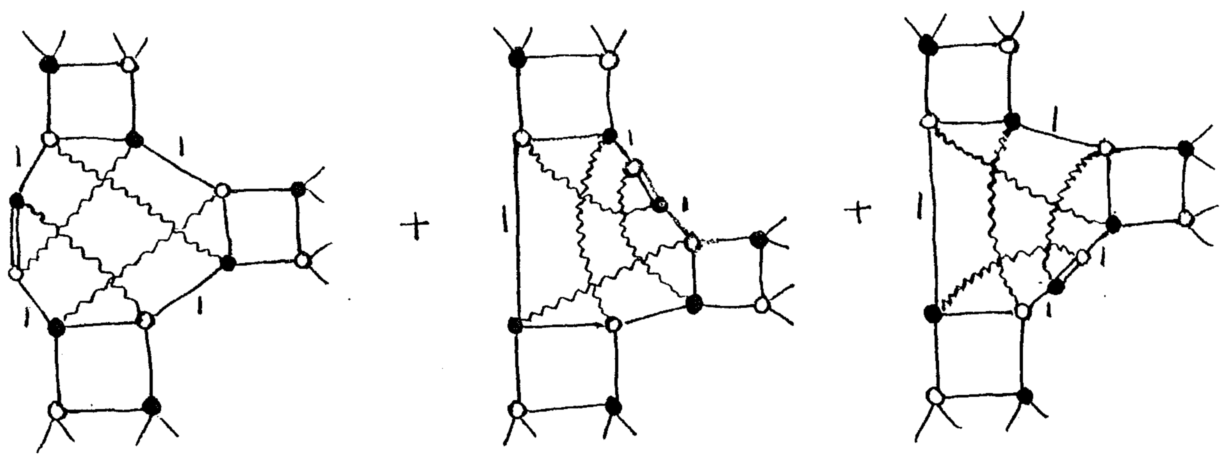


although in fact this is only true formally; each term is divergent and only the sum makes sense. Note that this is the first time that we have actually made use of the *Feynman* propagator as opposed to to other solutions of the inhomogeneous wave equation. Thus, it is only at this point that we really begin to claim a connection between twistor diagrams and the content of *quantum field theory*. Now the positive frequency propagator can be replaced by the idea of summing over a complete set of states, i.e.

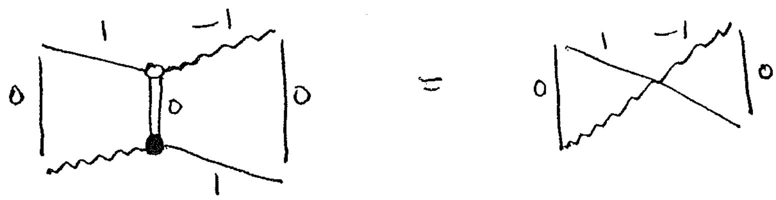


and this relates the loop to a sum over tree diagrams of the type we have just written down. To do the summation over a complete set of states in a twistor diagram we note that

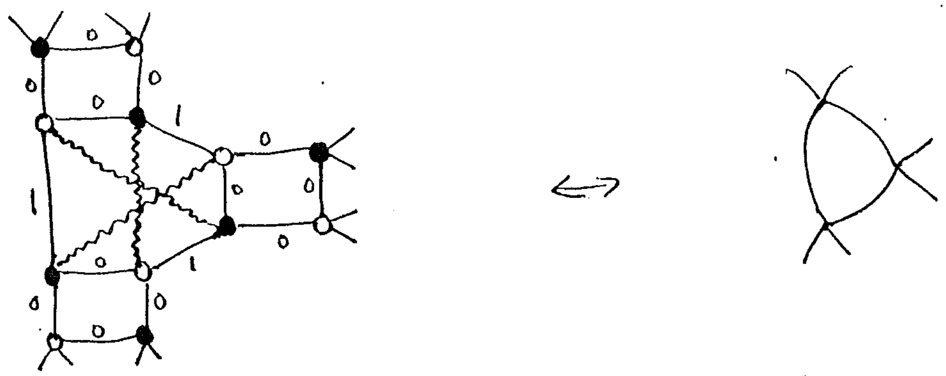
Thus formally, the Feynman loop diagram is translated to:



Now using the diagram identity



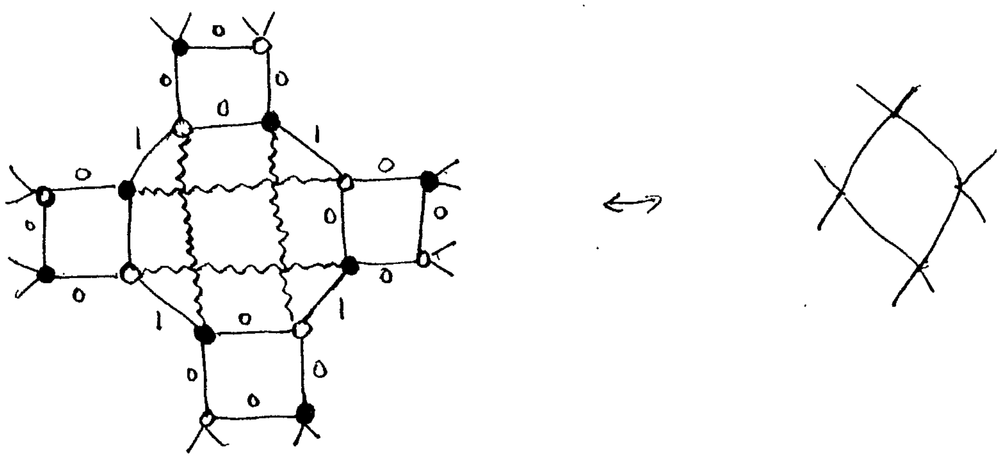
I put forward my guess that the sum can be represented thus:



(6)

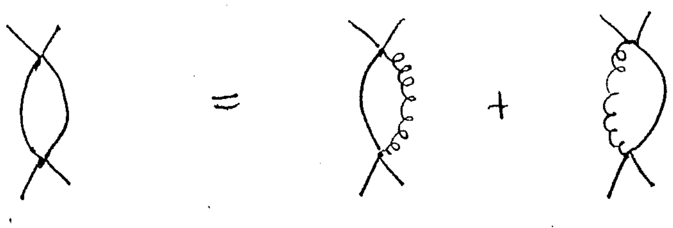
The "hexagon" diagram in the interior should be seen as a new construct, not simply a concatenation of "boxes". Integration of the Feynman diagram shows that calculating the amplitude involves calculating the area of a triangle, and I think something analgous is involved in (6). [I also hope it will make sense of the "hexagon" diagrams that I noticed *could* be used to give another version of the ϕ^6 integral, in TN 12].

We can likewise write down the reasoned guess

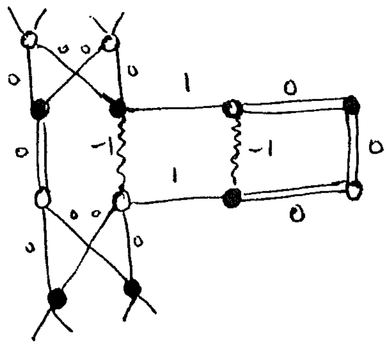


(7)

Finally we can apply these ideas, at least formally, to "ultra-violet divergent" Feynman diagrams. The simplest is perhaps

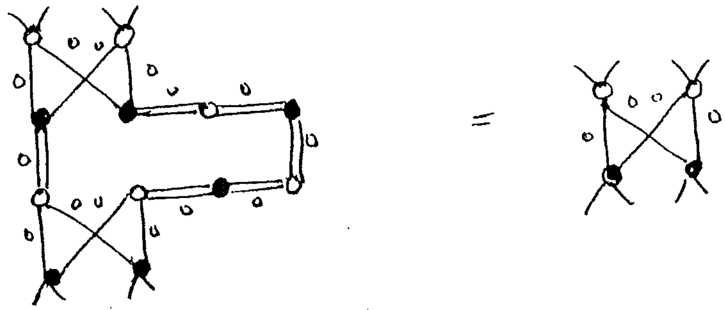


Employing the same ideas about relating to tree diagrams, we go back to (1) and so write down



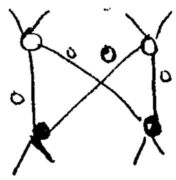
(8)

What can this mean but something whose *period* is



Thus it can only be:

$\log k$



so I expect my "k" to appear as the regulariser of an "ultra-violet" divergence. Is this reasonable? I think so: note that the idea of ultra-violet divergence is usually defined by a power-counting argument which is equivalent counting the order of the pole at infinity in momentum space. Now in a conformal theory I cannot see why such a divergence shoudl be thought different from a pole at zero in momentum space. Explicitly, the divergent integral

$$\int \frac{d^4k}{(k-u)^2(k-v)^2}$$

that we are studying here, is conformally equivalent to

$$\int \frac{d^4k'}{(k'^2)^2(k'-u')^2(k'-v')^2} \tag{9}$$

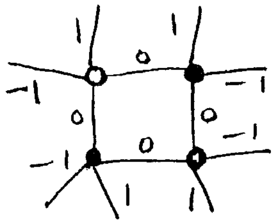
- an integral which would normally be regarded as infra-red divergent, and not "seriously" divergent at all. A particular point of interest is that such poles at zero momentum do arise when external fields of zero energy are attached at vertices, and this is exactly what is done in the "standard model" picture of generating massive fields through interaction with the constant Higgs scalar field (i.e. the elementary field based at I). It is my view that infra-red divergences, ultra-violet divergences, and the conformal breaking which leads to massive states, should all be related.

The field opened up by these observations seems very rich indeed. There seems no reasons why these approaches to higher order diagrams and loop diagrams should not be applied analogously in the other conformal field theories which are components of the standard model.

Andrew Hodges

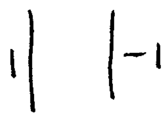
Inhomogeneity and Crossing Symmetry

The picture clearly emerging from the preceding work is that the ϕ^4 vertex inside a Feynman diagram corresponds to the configuration



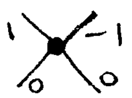
(1)

inside a twistor diagram, and that a virtual ϕ^4 propagator line is reflected in a pair of twistor diagram lines

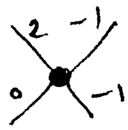


which can be thought of as collapsing together to give the positive frequency field propagator

The fact that there are four (-1) lines in (1) can be thought of as corresponding to the presence of four "off-shell" Feynman propagator lines in the Feynman ϕ^4 vertex. This all fits very well with the general programme for diagram theory outlined in TN 23, with the feature that in this field theory the twistor vertices look like



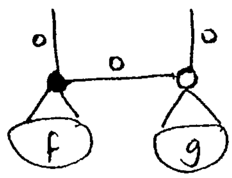
rather than



I now wish to go back to the "box diagram" for the first-order ϕ^4 scattering amplitude from which all this started, and to discuss the issue which has been glossed over in the preceding articles. This is the now venerable question of the so-called "hard channels" - the fact that the box diagram doesn't really possess crossing symmetry.

Recent work by M.A.S. and S.A.H. (q.v., TN 23 and the article in this TN), has improved upon my naive "philosophy 0" approach of doing contour integrals in which the external fields appear as representative functions. They have investigated the question of how the diagram evaluation procedure can be re-interpreted more satisfactorily in terms of genuine mappings of the various external H^1's into the complex numbers. It was vaguely hoped that by finding a procedure which looked properly at the H^1's, rather than at their representative functions, procedures corresponding to the "hard channels" would emerge. But they don't.

The basic, crude, difficulty with the original (projective) box diagram is this: given the piece of diagram



suppose we wish to allow f and g both to represent in-states. Take these to be coincident elementary fields $(x-p)^{-2}$ say. Now if the contour integration is to respect the H^1's, it must have the effect of reducing the twistors W, Z to spinors based on the line corresponding to p^a. But we *can't* do this because then W.Z vanishes. Any proposed rescue of the procedure must find a way round this problem in one way or another. M.A.S. and S.A.H. have shown that looking more seriously at the external fields as H^1's doesn't find such a way round it.

The so-called "hard contour" found by me gets round it by doing a contour integral which *doesn't* see all the external functions as H^1's. The answer turns out to be a functional of fields, although it is a mystery as to why this should be so. Using this procedure has been very useful as a stopgap, not least because it generalises from the simple box to all the other diagrams we are interested in, but it is obviously unsatisfactory.

Given this basic difficulty, we can ask whether the definition of the box diagram should be modified in some way. *One* idea that has popped up repeatedly ever since the question of crossing symmetry first arose in the early 1970s, is that the space in which the integration is done should be *blown up* in some way. Another idea arose when I saw the significance of inhomogeneous propagators for getting massive fields and for regularising divergences. Maybe *inhomogeneity* could solve the hard contour too! After all, if the diagram line becomes $(W.Z - k)^{-1}$ then we can allow $W.Z = 0$. Unfortunately, it is easy to see that just putting k's into the diagram as it stands doesn't make any difference.

Actually I think these *are* relevant and helpful ideas, but to show why I want to approach the question from a fresh and more *physical* angle. The point to be made - which has not before been made - is that we are interested in this ϕ^4 amplitude *not* just as a calculation of the integral

$$\int \phi_1(x)\,\phi_2(x)\,\phi_3(x)\,\phi_4(x)\,d^4x$$

but as a special case of a calculation in a *quantum field theory* - a field theory with all the structure of particle and antiparticle creation and annihiliation which is encoded so beautifully into the Feynman propagator formalism.

The view I take now is that we will only get a twistor transcription of this amplitude by seeing it as a very special case of the general ϕ^4 vertex - very special because all the Feynman propagators are reduced to positive frequency propagators. This will make the theory *look* more difficult, but it is actually a very positive indication for diagram theory. It will suggest that twistor diagrams are naturally associated with the content of quantum field theory, and only make sense when understood in the context of a field theory. A shadow or reflection of the field theory must be retained in the twistor formalism even in this first-order level.

Until recently I would have had no idea of how to write down a twistorial object corresponding to the general ϕ^4 vertex, but now it seems obvious that we should write down (1). Thus I should expect to see the first-order amplitude emerge as some sort of *reduced* version of (1).

We can get an idea of what sort of "reduction" may be involved by looking at the field theory more analytically. The Feynman vertex is

$$V(p,q,r,s)=\int d^4x\ \Delta_F(x-p)\,\Delta_F(x-q)\,\Delta_F(x-r)\,\Delta_F(x-s)$$

which must satisfy the equation

$$\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p}\,V(p,q,r,s) = \Delta_F(p-q)\,\Delta_F(p-r)\,\Delta_F(p-s)$$

What is V? It is connected with the function

$$D(p,q,r,s) = \int_0^\infty\int_0^\infty \frac{d\alpha\, d\beta}{(1+\alpha+\beta)\left((p-q)^2(r-s)^2\alpha + (p-r)^2(q-s)^2\beta + (p-s)^2(q-r)^2\alpha\beta\right)}$$

which one can evaluate using the *dilogarithm* function, although this hardly helps very much. D(p,q,r,s) is the function of four points I called a "super-amplitude" in an earlier stab I made at understanding the ϕ^4 amplitude (see *Advances in Twistor Theory*, page 256). It satisfies the equation

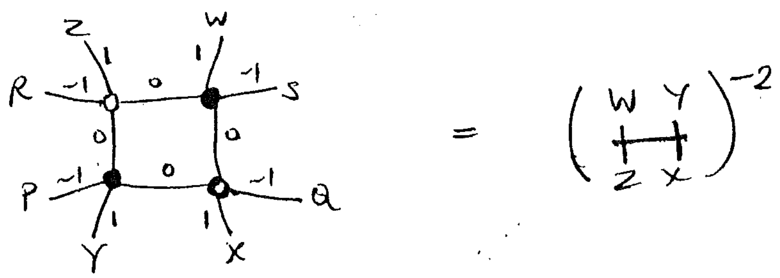
$$\frac{\partial}{\partial p}\cdot\frac{\partial}{\partial p}\, D(p,q,r,s) = \left\{(p-q)^2(p-r)^2(p-s)^2\right\}^{-1}$$

I use the vague phrase "connected with" because V is a function only in the sense that the delta-function is; we can't make any real sense of this without a correct treatment of these things as singular functions (using relative cohomology?).

[*Digression and Confession:* The basic idea of that earlier work in A.T.T. was, I believe, correct; but the calculation given there, which claimed to get the dilogarithmic function D from a twistor diagram, was certainly NOT correct. It relied on an unjustifiable change in the order of integrations. It can't be right because anything resulting from the diagram given there would have to satisfy the zrm equation in each of the p, q, r, s - and D doe sn't.]

However, this connection suggests the following idea: the function D has three *periods* which agree with the three different channel amplitudes. Thus we may consider looking at *periods* of the twistor diagram (1) and see what we get.

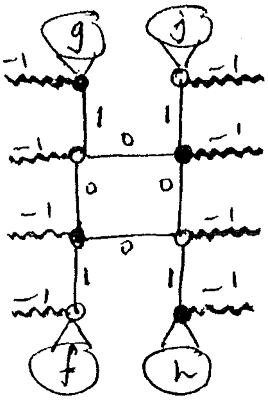
My claim is that we can indeed get the three required contour integrals by doing this. Specifically, there exists a "period" contour such that



independently of P, Q, R, S. Now it is already well-known that

$$\oint \frac{f(Y)\,g(Z)\,h(X)\,j(W)}{\left(\begin{matrix} W & Y \\ Z & X \end{matrix}\right)^2}$$

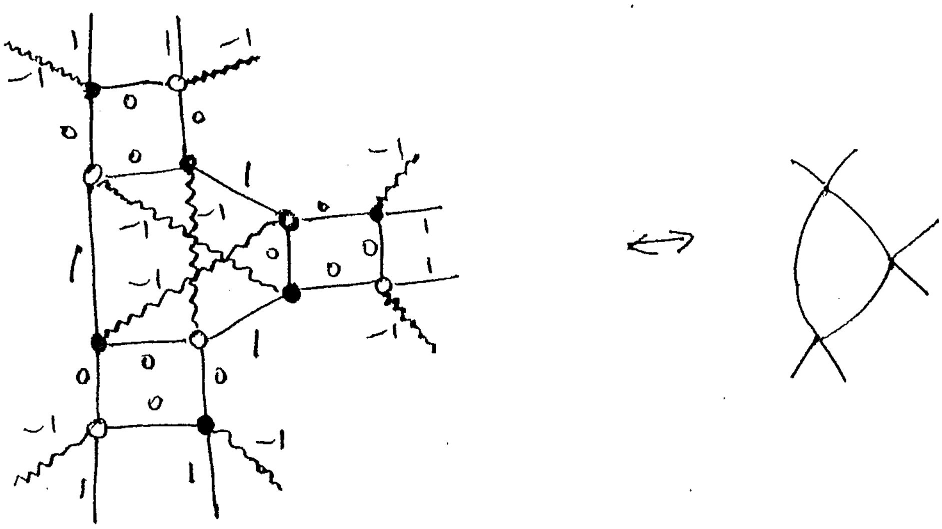
possesses three contours, one for each channel. We can write the whole prescription as:



This, I claim, is how we should think of the first-order ϕ^4 amplitude.

The rationale of these further (-1) lines is given in the Appendix where I have put an outline of the detailed calculation.

Naturally the same idea must be applied to the larger diagrams that have been built up out of the box diagram; e.g. we should write



This may look complicated at first but I believe the complexity is due to the fact that we are necessarily doing *field theory* and that the inhomogeneity and logarithmic (-1) lines, although they do not appear explicitly in the final amplitude calculation, are essential to the predictive theory. The proposed formalism enjoys the following features which I would regard as *criteria* for a satisfactory twistor diagram theory:

(1) There is a single entity (a differential form on a product of twistor spaces) which can be integrated in different ways to yield amplitudes corresponding to the various possible allocations of in- and out-states.

(2) That entity is composed of factors corresponding to twistor diagram lines

(3) The twistor diagram lines are all on the same footing; i.e. consistently represent the same mathematical objects.

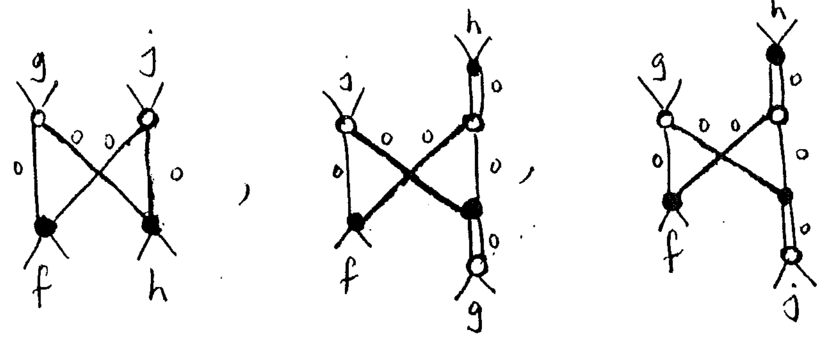
(4) The contour manifestly treats the external free fields as H^1s

(5) The formalism does not depend upon the discrete permutation symmetry peculiar to the ϕ^4 vertex, so that it has a chance of generalising to other field theories.

(6) The formalism naturally extends to higher-order amplitude calculations.

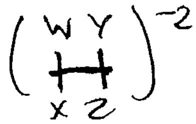
Notes on these features:

(1) This condition is required if the idea of crossing symmetry is to have any content. If it fails, the various channels become completely independent. As an example of where it is NOT met, take the theory which tells you to write down the three diagrams



for the three channels. (M. L. Ginsberg's investigations have used this idea in a more sophisticated form.) This not only loses the idea of crossing symmetry but also turns out to violate condition (5). It only works because of the discrete symmetry.

(2) This condition comes from pure faith that the *predictive power* of the theory will eventually reside in a *generating rule* that uses twistor diagram lines in analogy to Feynman diagram lines. If we scrapped it we could of course just write down



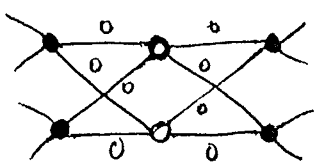
and say that this gives the twistor version of the first-order amplitude. We could do this in any way we liked for other amplitudes in other theories and so put a long list of twistor-encoded amplitudes into a big book. But this I believe would be stamp-collecting, not physics.

(3) This condition is again forced by the idea that the theory can be generated from some rule in analogy to the specification of Feynman diagrams". Each twistor line must be treated alike.

When we come to the integration of the new proposed ϕ^4 vertex diagram (see Appendix) we shall see how each line is treated as a "restricting pole" (relative H^1?).

(4) This rules out my "hard contours".

(5) This rules out the cheating ideas already discussed under point (1); it also rules out a theory which claims the totally symmetric diagram



as the fundamental object. It is peculiar to the ϕ^4 vertex, and has no analogy for massless QED for instance.

Lastly, there is one criterion for a satisfactory theory which I don't believe *is* yet met by the proposed new diagram structure, namely that we should understand this business of taking periods when free fields are attached on the outside. In fact I have no idea how this should be formulated. However, I quite like the fact that it is rather mysterious and apparently more complicated than Feynman diagram evaluation. The mystery may have something to do with the question of what a free particle (observed at infinity) actually *is*. It might even have something to do with wave-function reduction.

Appendix

The contour integrals needed consist of a mountain and a molehill. Let's get the molehill out of the way first: we need to show that

$$\left(\begin{matrix} W \quad Y \\ \vdash\!\!\dashv \\ X \quad Z \end{matrix}\right)^{-2}$$

is a kernel which gives three contours. Taking elementary states, reduce W, X, Y, Z immediately to spinors on the corresponding lines. Now use the formula

$$(2\pi i)^{-3}\oint \frac{d^2\pi \wedge d^2\sigma}{(Q^A{}_B\pi^A\sigma_B)(R^A{}_B\pi^A\sigma_B)} = \int_0^\infty \frac{2\,du}{Q.Q + 2u\,Q.R + u^2 R.R}$$

Contours can be specified precisely for this spinor integral, and thereby induced for our twistor integral (cf. S.A.H. and M.A.S., this TN, who pay particular attention to this question.) In fact we apply the formula twice over, finding three different results according to which pairs of spinors are linked together. The results of the integration are immediate from the formula. By elementary twistor algebra and a change of variable, the results are conveniently put in the forms

$$\int_0^\infty \frac{du}{T(u)}, \quad \int_{-1}^{0} \frac{du}{T(u)}, \quad \int_{\infty}^{-1} \frac{du}{T(u)}$$

together with a period contour which gives result $\oint \frac{du}{T(u)}$

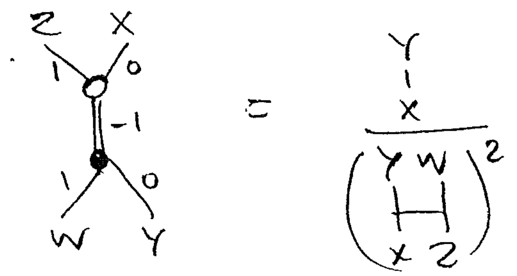
Here T(u) is a standard quadratic in u, its coefficients being formed from the conformal invariants associated with the four points in CM.

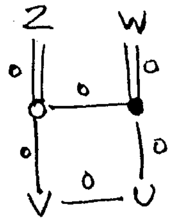
One may (and must) check that given an assignment of in and out states (i.e. points belonging appropriately to past and future tubes), these functions of the points agree with the amplitudes for first-order ϕ^4 scattering, and are unambiguously defined. (One can easily see which contour is the right one for which channel by looking at the special cases where points coincide.)

[SAH and MAS consider the homology of the integral when the points are not allocated to a specific channel but are in general position. In this case the idea of "amplitude" is not well-defined since there is no unique analytic continuation from the "physical" parameter regions where amplitudes are defined, to a general parameter position. However, the u-integral formulation of the results, given above, makes it very easy to see that in general position there is a linear dependence between the three "channel" contours (as analytically continued) and the period contour; there are just three independent contours. These primitive methods cannot show there are no more than three, but MAS and SAH mention that they have a proof of this.]

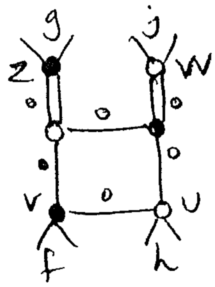
Now, the mountain. I will try to indicate the idea of the large integral without getting into the quite substantial and sophisticated calculations.

We start with the projective twistor diagram formula



Hence it is possible to write:  $= \left(\begin{smallmatrix} W & U \\ & \vdash\!\!| & \\ Z & V \end{smallmatrix} \right)^{-2}$

with the Y.X pole being "cancelled". Hence formally

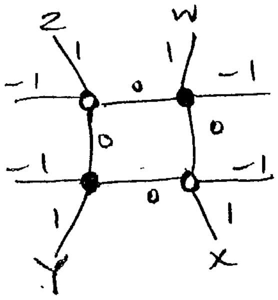


$$= \oint \frac{f(V)g(Z)h(U)j(W)}{\left(\begin{smallmatrix} W & U \\ & \vdash\!\!| & \\ Z, & V \end{smallmatrix} \right)^{2}} \; DZVUW$$

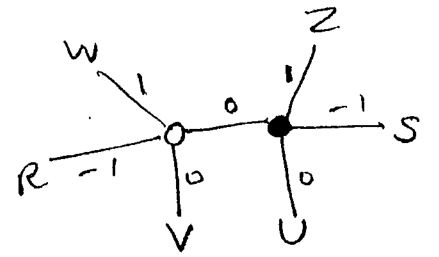
apparently thus allowing three contours to exist. But they don't exist in the original space, because they have to include points where Y.X = 0. They might exist on a space *blown up* at Y.X = 0. We certainly can't regard the pole Y.X as a "restricting" pole. Thus, we should have to make the formalism highly asymmetric, breaking condition (3).

Making the lines inhomogeneous does not in itself solve the problem. However, it turns out if we make the diagram inhomogeneous *and* put in extra (-1) lines *and* make the whole thing symmetrical, we *do* get the effect of this "cancelling of a pole" while still treating every pole in the diagram as a "restricting pole". (I suspect that there is some *connection* with blowing up involved in this.)

Thus, we are led to



as a way of making proper sense of the naive 'cancelling' idea. Now let us study it explicity. The first step is to consider



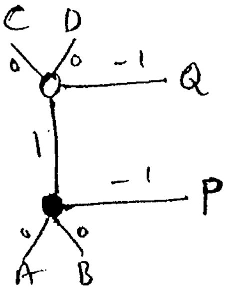
(all lines now inhomogeneous)

and by taking the period contour recover
$$\frac{\begin{smallmatrix}U\\ |\\ V\end{smallmatrix}}{\left(\begin{smallmatrix}W & & U\\ & \mathrm{H} & \\ Z & & V\end{smallmatrix}\right)^2}$$

just as before. (I do not claim this is obvious.) Note that the (-1) lines play the role of "magic". Now we are left with

$$\oint \frac{D^4U\wedge D^4V \;\; \begin{smallmatrix}U\\ \downarrow\\ V\end{smallmatrix}\left\{\gamma+\log\left(\begin{smallmatrix}Y\\ |\\ Q\end{smallmatrix}-k\right)\right\}\left\{\gamma+\log\left(\begin{smallmatrix}P\\ |\\ V\end{smallmatrix}-k\right)\right\}}{\left(\begin{smallmatrix}W & & U\\ & \mathrm{H} & \\ Z & & V\end{smallmatrix}\right)^2\left(\begin{smallmatrix}Y\\ |\\ V\end{smallmatrix}-k\right)^2\left(\begin{smallmatrix}Y\\ |\\ X\end{smallmatrix}-k\right)^2\left(\begin{smallmatrix}U\\ |\\ V\end{smallmatrix}-k\right)}$$

The essential results all come from my earlier work but some extensions are necessary. The integral has basically the same shape as



$$= \oint \frac{D^4U\, D^4V \;\left\{\gamma+\log\left(\begin{smallmatrix}Y\\ |\\ Q\end{smallmatrix}-k\right)\right\}\left\{\gamma+\log\left(\begin{smallmatrix}P\\ |\\ V\end{smallmatrix}-k\right)\right\}}{\left(\begin{smallmatrix}Y\\ |\\ V\end{smallmatrix}-k\right)^2 \begin{smallmatrix}Y\\ |\\ C\end{smallmatrix}\;\begin{smallmatrix}U\\ |\\ D\end{smallmatrix}\;\begin{smallmatrix}A\\ \downarrow\\ V\end{smallmatrix}\;\begin{smallmatrix}B\\ \downarrow\\ V\end{smallmatrix}}$$

which I studied explicitly in connection with Møller scattering, but there are differences because more of the legs are inhomogeneous; the powers are different, and we have the singularity $\left(\begin{smallmatrix}W & & U\\ & \mathrm{H} & \\ Z & & V\end{smallmatrix}\right)^{-2}$ rather than $\left(\begin{smallmatrix}Y & W\\ | & |\\ Z & V\end{smallmatrix}\right)^{-2}$. The methods developed earlier are basically adequate to coping with this generalisation, though it's not easy.

As a guiding light in the calculations, note that

$$\frac{\overset{U}{\underset{V}{\downarrow}}}{\overset{U}{\underset{V}{\downarrow}} - k} \cong \frac{k}{\overset{U}{\underset{V}{\downarrow}} - k}$$

since the pole "enforces" $\overset{U}{\underset{V}{\downarrow}} = k$

and this can be written as $k\dfrac{\partial}{\partial k} \log\left(\overset{U}{\underset{V}{\downarrow}} - k\right)$

Now $k\dfrac{\partial}{\partial k}$ has the effect of extracting the period, i.e. the coefficient of log k. This means that we expect the combination

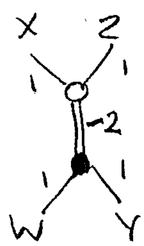
$$\frac{\overset{Y}{\underset{V}{\downarrow}}\left(\gamma + \log\left(\overset{Y}{\underset{a}{\downarrow}} - k\right)\right)\left(\gamma + \log\left(\overset{P}{\underset{V}{\downarrow}} - k\right)\right)}{\overset{U}{\underset{V}{\downarrow}} - k}$$

to have the same effect as putting in $\log\left(\overset{Y}{\underset{V}{\downarrow}} - k\right)$

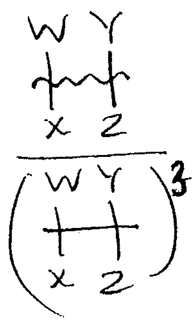
and then treating this as a "double line". But this can be reduced to a projective twistor integral. Explicitly, we expect

$$\oint D^4U \wedge D^4V \, \frac{\overset{U}{\underset{V}{\downarrow}}\left(\gamma + \log\left(\overset{U}{\underset{a}{\downarrow}} - k\right)\right)\left(\gamma + \log\left(\overset{P}{\underset{V}{\downarrow}} - k\right)\right)}{\left(\overset{U}{\underset{Z}{\downarrow}}\right)^2\left(\overset{W}{\underset{V}{\downarrow}}\right)^2\left(\overset{Y}{\underset{V}{\downarrow}} - k\right)\left(\overset{U}{\underset{X}{\downarrow}} - k\right)^2\left(\overset{Y}{\underset{V}{\downarrow}} - k\right)^2}$$

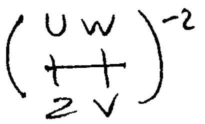
to have the same value as the projective integral

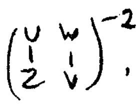


, namely

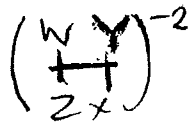


This is the beautiful phenomenon which makes sense of the "cancelled pole" idea, while still keeping all the poles as restricting poles. It is enough to tell us what the result must be, but it be justified properly. To do this I use a CP^4 formalism to go back to my original contour for the "Møller" result, so as to generalise it to the case here, where there are more inhomogeneous poles. We also have to adapt it to the case here where there are *double* poles on the outside and a *single* pole inside, but one advantage of the CP^4 formalism is that this can be effected by differentiation with respect to CP^4 parameters. These methods suffice to justify the result we expected above; now finally, to cope with the appearance of the singularity

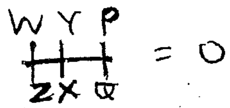


rather than in  , note that these coincide in the case when $W.Z = 0$

and so expand in a power series in $W.Z$. (One could however avoid the use of power series if desired.)

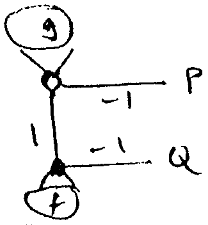
The result anyway is 

independently of P and Q, although there is a subtlety: the contour is not defined if

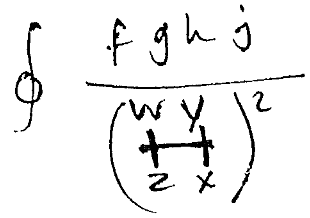


so when we do the W,Y,X,Z integral we must, strictly speaking, avoid this space as well. This can be done, but it reinforces my comment earlier that I don't understand the rules for "taking a period".

Finally, note that this idea of "taking a period" can be used to specify the correct "magic" for integrating the inner product integral: the inner product can be considered as



by a "period" contour, with a result independent of P and Q. Now in doing the final integral



we are effectively doing two copies of the inner product integral, so it's consistent to stick four more (-1) lines on to indicate the two dollops of "magic" required to do this final integral.

Abstract:

Andrew Hodges

Constraint-Free Analysis of Relativistic Strings

L.P. Hughston

Lincoln College, Oxford OX1 3DR, Great Britain

and

W.T. Shaw*

Department of Mathematics†,

Massachusetts Institute of Technology,

Cambridge, Massachusetts 02139, U.S.A.

We present a general solution of the equations of motion for a classical relativistic string in Minkowski space. For an 'open' string the solution is determined by a single real null curve which satisfies a quasi-periodicity condition. A formula is deduced which generates all such curves in terms of freely-specifiable functions. For a 'closed' string two such curves are required.

Our theory establishes a rigorous basis for the numerical investigation of 'cosmic strings', and also forms a starting point for a constraint-free Lorentz covariant analysis of the quantum relativistic theory.

Cohomology and Projective Twistor Diagrams

Introduction:

We demonstrate in sections 1 and 2 the limitations of philosophies 1 and 2 when applied to the box diagram. Then in section 3 we describe a more careful approach to the cohomology of twistor diagrams. Section 4 successfully applies this new method to the reduced version of the *new* "box" diagram [2].

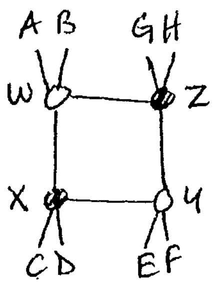
§1 Philosophy 1

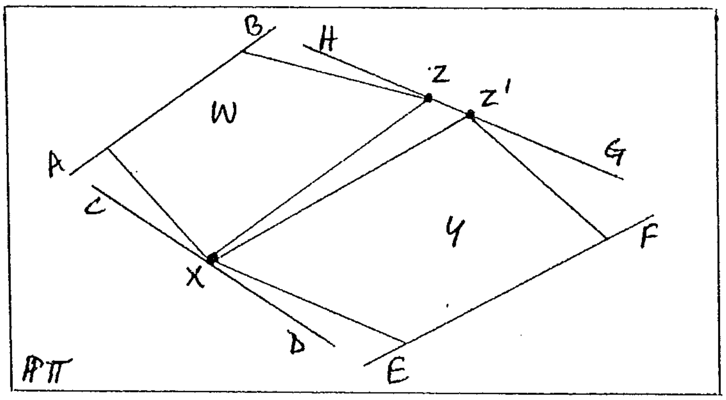
As we mentioned in [1], philosophy 1 applied to the box diagram yields an element of

$$H^{11}(\Pi - Y;\ \Omega^{12})$$

where $\Pi = \mathbb{P}^*_W \times \mathbb{P}_X \times \mathbb{P}^*_Y \times \mathbb{P}_Z$

and Y consists of the two points where $Z = Z'$:





Now the relative cohomology sequence is

$$\begin{array}{ccccccccc}
H^{11}(\Pi;\Omega^{12}) & \to & H^{11}(\Pi - Y;\Omega^{12}) & \to & H^{12}_Y(\Pi;\Omega^{12}) & \to & H^{12}(\Pi;\Omega^{12}) & \to & 0 \\
\| & & & & \|\ \text{Thom} & & \| & & \\
0 & & & & H^0(Y;\mathbb{C}) & & \mathbb{C} & & \\
 & & & & \| & & & & \\
 & & & & \mathbb{C}\oplus\mathbb{C} & & & &
\end{array}$$

so $H^{11}(\Pi - Y;\Omega^{12}) = \mathbb{C}$,

and we have evaluated the box diagram. *But*, if we allow two fields to coincide then $Y = \mathbb{P}^1$ and the sequence becomes

$$
\begin{array}{c}
0 \to H^{11}(\Pi - Y; \Omega^{12}) \to H^{12}_{Y}(\Pi; \Omega^{12}) \to \mathbb{C} \to 0 \\
\qquad\qquad\qquad\qquad\qquad\quad \| \\
\qquad\qquad\qquad\qquad\qquad\quad H^{2}(Y; \mathbb{C}) \\
\qquad\qquad\qquad\qquad\qquad\quad \| \\
\qquad\qquad\qquad\qquad\qquad\quad \mathbb{C}
\end{array}
$$

and our evaluation *fails.* So it must correspond to the use of the *tight* contour, and in order to describe the *physical* contours we would need some "magic" elements. (thanks to RJB for much of this argument.)

§2 Philosophy 2

This time *cup* the four scalar elementary fields together, and simply multiply by

$$
\frac{\mathbb{D}WXYZ}{\begin{matrix} w & w & y & y \\ | & | & | & | \\ x & z & x & z \end{matrix}} .
$$

Obtain an element of $H^{4}(\Box; \Omega^{12})$ and by ignoring its bidegree

map it into $H^{16}(\Box; \mathbb{C})$.

Poincaré duality tells us to look for functionals in $H_{16}(\Box; \mathbb{C})$.

A long algebraic topology calculation shows that

$$
H_{16}(\Box; \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}
$$

and that one of these generators is the tight contour while the other one is the Sparling contour. So this method of identifying functionals misses the two "hard" contours. Worse, it is *not* possible to get new functionals by *dotting* some of the elements and then forgetting the bidegree: recall that the dot product is given by

$$
\alpha \cdot \beta = \delta^{*}(\alpha \cup \beta)
$$

where δ^{*} is the Mayer-Vietoris map, and consider

$$
\begin{array}{ccc}
H^{p}(A \cap B) & \xrightarrow{\delta^{*}} & H^{p+1}(A \cup B) \\
\times & & \times \\
H_{p}(A \cap B) & \xleftarrow{\delta_{*}} & H_{p+1}(A \cup B) \\
\downarrow & & \downarrow \\
\mathbb{C} & & \mathbb{C}
\end{array}
$$

Suppose $\gamma \in H_{p+1}(A \cup B)$ *were* such a new functional. Then $\delta_* \gamma = 0$, otherwise we would already have identified it. But

$$\int_\gamma \delta^* \omega = \int_{\delta_* \gamma} \omega = 0$$

so γ is no good as a functional on $\delta^* \omega$, which is precisely what we want it for.

§3 Serre Duality

This section approaches afresh the question of finding functionals on fields corresponding to (projective) twistor diagrams. We try to avoid eliminating possible functionals until the last possible moment.

The least we can do is to cup the 4 H^1 s together, getting an element of

$$H^4((\mathbb{P}^* - K) \times (\mathbb{P} - L) \times (\mathbb{P}^* - M) \times (\mathbb{P} - N); \mathcal{O}(-2,-2,-2,-2)).$$

In fact, however, the 3 Mayer-Vietoris maps converting these cup products into dot products are all isomorphisms. The following argument is just for one of them.

$$\begin{array}{c} H^2(\mathbb{P}^* \times (\mathbb{P} - L); \mathcal{O}(-2,-2)) \oplus H^2((\mathbb{P}^* - K) \times \mathbb{P}; \mathcal{O}(-2,-2)) \\ \downarrow \\ H^2(\ \cap\ ; \mathcal{O}(-2,-2)) \\ \downarrow \delta^* \\ H^3(\ \cup\ , \mathcal{O}(-2,-2)) \\ \downarrow \\ H^3(\quad \text{ditto} \quad) \oplus H^3(\quad \text{ditto} \quad) \end{array}$$

The groups at either end can be evaluated using the Künneth formula and are all zero for this homogeneity.

So functionals on 4 zero rest mass scalar fields are to be found in the space dual to

$$H^7(\mathbb{P}^* \times \mathbb{P} \times \mathbb{P}^* \times \mathbb{P} - K \times L \times M \times N; \mathcal{O}(-2,-2,-2,-2))$$

which by Serre duality is

$$H^5_c(\Pi - \Lambda\ ; \mathcal{O}(-2,-2,-2,-2))$$

(where Π and Λ are as above). We can go further, though.

If $\omega \in H^7(\Pi - \Lambda)$ and $\eta \in H^5_c(\Pi - \Lambda)$ the duality is given by

$\int_{\Pi-\Lambda} \omega \wedge \eta$. Now consider the sequence

$$\to H^{q-1}(\Pi) \to H^{q-1}(\Lambda) \xrightarrow{\wedge\bar\partial\rho} H^q_c(\Pi - \Lambda) \xrightarrow[\text{by } 0]{\text{extend}} H^q(\Pi) \to$$

where ρ is a bump function equal to 1 on Λ

and 0 on $\Pi - \{$ a neighbourhood of $\Lambda\}$.

Π being what it is, when q = 5 the map $\wedge\bar\partial\rho$ is an isomorphism. Therefore

$$\eta = \zeta \wedge \bar\partial\rho \qquad \text{for some } \zeta \in H^4(\Lambda; \mathcal{O}_\Pi(-2,-2,-2,-2))$$

and $$\int_{\Pi-\Lambda} \omega \wedge \eta = \int_{\Pi-\Lambda} \omega \wedge \zeta \wedge \bar\partial\rho = \int_{\Pi} \omega . \zeta$$

Note that this last integral can be thought of as taking place throughout Π, because although $\zeta\wedge\bar\partial\rho$ would be cohomologous to zero if extended to Π, $\omega\wedge\zeta\wedge\bar\partial\rho$ is not. Indeed, from the usual definition of the dot product we can see that $\omega.\zeta$ is automatically defined on Π . (The fact that $\omega.\zeta$ is the the extension to Π of $\omega\wedge\zeta\wedge\bar\partial\rho$ is discussed in [1], §3.1).

All linear functionals, then, on 4 elementary zrm scalar fields can be obtained by dotting with an element of $H^4(\Lambda; \mathcal{O}_\Pi(-2,-2,-2,-2))$ and integrating. It is important to note that it is only at *this* stage that we make any choices about what our linear functionals are to look like.

Aside: This general procedure for identifying the space of functionals is not specific to the box diagram. It works well for the scalar product:

$$\mu \cdot \frac{1}{\binom{w}{z}^2} \in H^2(K \times L \,;\, \mathcal{O}_{\mathbb{P}^* \times \mathbb{P}}(-2,-2))$$

- where μ is the magic H^1 - is the required functional.

§4 The Moving of the Goalposts

Until recently we thought that the twistor kernel for the massless scalar ϕ^4 integral was of the form

$$\frac{\gamma}{\begin{smallmatrix} W & W & Y & Y \\ | & | & | & | \\ X & Z & X & Z \end{smallmatrix}}$$

and the object was to find a γ corresponding to each of the 3 channels for the process. It seemed like a difficult problem (not surprisingly, given the long history of the "hard contour" question). In [2] however, APH argues that the ("reduced") kernel should look like

$$\frac{\gamma}{\left(\begin{smallmatrix} WY \\ H \\ XZ \end{smallmatrix}\right)^2}.$$

So we need 3 elements of $H^4_c\left(\Lambda - \left\{ \begin{smallmatrix} WY \\ H \\ XZ \end{smallmatrix} = 0 \right\}; \mathcal{O}_{\Pi}(0)\right)$.

We start by constructing some contours in $\Lambda - H \left(= \Lambda - \left\{ \begin{smallmatrix} WY \\ H \\ XZ \end{smallmatrix} = 0 \right\} \right)$ using APH's favourite method. Suppose

$$L_W \leftrightarrow (\lambda_A, -ia^{AA'}\lambda_A), \qquad L_X \leftrightarrow (ib^{AA'}\pi_{A'}, \pi_{A'}),$$

$$L_Y \leftrightarrow (\mu_A, -ic^{AA'}\mu_A), \qquad L_Z \leftrightarrow (id^{AA'}\rho_{A'}, \rho_{A'});$$

then

$$\begin{smallmatrix} W & Y \\ | & | \\ Z & X \end{smallmatrix} = (a-d)^{AB'}(b-c)^{BA'}\lambda_A \pi_{A'} \mu_B \rho_{B'}$$

and

$$\begin{smallmatrix} W & Y \\ | & | \\ X & Z \end{smallmatrix} = -(a-b)^{AA'}(c-d)^{BB'}\lambda_A \pi_{A'} \mu_B \rho_{B'}$$

so the restriction of H to Λ is given by

$$Q^{AA'BB'}\lambda_A \pi_{A'} \mu_B \rho_{B'}$$

where $Q^{AA'BB'} = (a-b)^{AA'}(c-d)^{BB'} + (a-d)^{AB'}(b-c)^{BA'}$.

Consider the degenerate cases which pick out the three channels.

(i) a = b, c = d. The expression for Q is now

$$\{(a-c)^{AB'}\lambda_A\rho_{B'}\}\{(a-c)^{BA'}\mu_B\pi_{A'}\}$$

where a - c must be nonsingular. Let $u_A{}^B$ be a non-singular matrix such that $(a-c)^{AB'}u_A{}^B$ is positive definite. Then setting

$$\lambda_A = u_A{}^B\bar{\rho}_B\ , \quad \mu_B = u_B{}^A\bar{\pi}_A$$

we define a contour $\mathring{\gamma}_1$ on which Q is positive and on which

$\Delta\lambda\wedge\Delta\mu\wedge\Delta\pi\wedge\Delta\rho$ becomes a multiple of $\Delta\rho\wedge\Delta\bar{\rho}\wedge\Delta\pi\wedge\Delta\bar{\pi}$.

Then
$$\int_{\mathring{\gamma}_1} Q^{-2}\,\Delta\lambda\wedge\Delta\mu\wedge\Delta\pi\wedge\Delta\rho \neq 0 .$$

(ii) a = d, b = c. In this case, Q becomes

$$-\{(a-b)^{AA'}\lambda_A\pi_{A'}\}\{(a-b)^{BB'}\mu_B\rho_{B'}\}$$

and a - b is non-singular. Choose a non-singular matrix $V_A{}^B$ so that

$(a-b)^{A'B}V_B{}^A$ is positive definite. Then, as before, the contour

$\mathring{\gamma}_2$ which is defined by $\lambda_A = V_A{}^B\bar{\pi}_B\ , \quad \mu_A = V_A{}^B\bar{\rho}_B$

is one for which the integral of $Q^{-2}\Delta\lambda\pi\rho\mu$ is non-zero.

(iii) a = c, b = d. In this case

$$\begin{aligned} Q^{AA'BB'} &= 2(a-b)^{A[A'}(a-b)^{B']B} \\ &= \tfrac{1}{2}\epsilon^{AB}\epsilon^{A'B'}(a-b)_{CC'}(a-b)^{CC'} \end{aligned}$$

because $(a-b)_{C'}{}^{(A}(a-b)^{B)C'} = 0$.

Let $t^{AA'}$ be a positive-definite matrix. Then on the contour $\overset{\circ}{\gamma}_3$ defined by $\lambda_A = t_A{}^{A'} \bar{\mu}_{A'}$, $\pi_{B'} = t_{B'}{}^{B} \bar{\rho}_B$

we have

$$Q^{AA'BB'}\lambda_A\pi_{A'}\mu_B\rho_{B'} = \pm\tfrac{1}{2}(a-b)^2(t^{AA'}\mu_A\bar{\mu}_{A'})(t^{BB'}\bar{\rho}_B\rho_{B'})$$

which is of one sign for all μ, ρ . Thus $\int_{\overset{\circ}{\gamma}_3} Q^{-2}\,\Delta\lambda\pi\rho\mu \neq 0$.

Moreover these constructions depend on the *non* singularity of such matrices as

a - c in case (i); a - b in case (ii), a - d in case (iii)

and since *non* singularity and positive-definiteness are preserved under small perturbations of Q , we conclude that these contours persist as representatives for non-zero classes in $\Lambda - H$ in the generic cases.

In fact the Leray exact sequence in homology can be used on the space $\Lambda - H$ to show that $H_4(\Lambda - H; \mathbb{C})$ has at *most* 3 generators.

We now have the existence of 3 non-zero classes

$$\overset{\circ}{\xi}_1, \overset{\circ}{\xi}_2, \text{ and } \overset{\circ}{\xi}_3 \in H^4_c(\Lambda, \mathcal{O}_\Lambda(-2,-2,-2,-2))$$

where $\overset{\circ}{\xi}_j = \dfrac{\overset{\circ}{\gamma}_j}{Q^2}$ (j = 1,2,3)

(in the sense of currents) and Q and the $\overset{\circ}{\gamma}_j$ were constructed above. We need to show that there exist extensions ξ_j of $\overset{\circ}{\xi}_j$ for each j where

$$\xi_j \in H^4_c(\Lambda; \mathcal{O}_\pi(-2,-2,-2,-2))$$

and its image under the natural map

$$H^4(-, \mathcal{O}_\pi) \to H^4(-, \mathcal{O}_\Lambda)$$

is $\overset{\circ}{\xi}_j$.

Such ξs are constructed easily enough by replacing Q by $\overset{WY}{\underset{XZ}{H}}$ and $\mathring{\gamma}_j$ by γ_j where γ_j is obtained from $\mathring{\gamma}_j$ by replacing each point P on $\mathring{\gamma}_j$ by a disc of real dimension 16 which is transverse to Λ in Π (Λ is of real codimension 16 in Π).

What is, perhaps, not quite obvious is whether the proposed functionals

$$(\phi_1, \dots, \phi_4) \longmapsto \omega \cdot \xi_j [\Pi] \qquad (j = 1,2,3) \text{ are}$$

not identically zero. It seems to us that the best chance of proving this is to show that the integrals

$$\omega \cdot \xi_j [\Pi] = \int_{\gamma_j} \omega \wedge \bar\partial \beta \wedge \frac{D\,WXYZ}{\left(\overset{WY}{\underset{XZ}{H}}\right)^2}$$

(where γ_j is our 20-dimensional contour and the type of the integrand is (12, 8), β being a C^∞ function which is 1 on Λ and 0 except within about a millimetre of Λ) is equivalent to taking the residue of each field at its line and then integrating the result, multiplied by Q^{-2}, over $\mathring{\gamma}_j$.

(A likely method of proof would be to iterate the formula - which can be established from the generalised Cauchy Integral Formula

$$f(a) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(z)\, \bar\partial\beta \wedge dz}{z - a} .)$$

Showing the equivalence of these integrals would have the further virtue of relating our evaluation procedure to philosophies 0 and 2.

Many thanks to APH and RP for all the useful chatter.

Stephen Huggett Michael Singer

References:

[1] S. A. Huggett and M. A. Singer in TN 23
[2] A. P. Hodges "Inhomogeneity and Crossing Symmetry" in this TN.

Twistor Functions for "Infrared" Maxwell Fields.

Introduction: On Minkowski space, there exists a class of vacuum solutions of the classical Maxwell equations known as the "infrared" solutions. Ashtekars work [1] on the asymptotic quantization of the radiative modes of the Maxwell field led to a geometrical constraint on the fields if the classical solutions are to represent finite-normed one photon wave functions. [That is, in the gauge $\Phi_a n^a \hat{=} 0$, where $\Phi_a(x)$ is any vector potential of the Maxwell field that tends to zero at spatial infinity; n^a is the null normal to $\mathscr{I}$ ($\mathscr{I}$ stands for $\mathscr{I}^+$ or $\mathscr{I}^-$), and "$\hat{=}$" means "equals to" at points of $\mathscr{I}$; then denoting by $\Phi_a^+(x)$ the positive frequency part of Φ_a, the fourier transform $\tilde{\Phi}_a(\omega,\zeta,\bar{\zeta})$ associates a complex vector in momentum space with each position represented by ω, with $\Phi_a^+(u,\zeta,\bar{\zeta}) = \int_0^\infty \tilde{\Phi}_a(\omega,\zeta,\bar{\zeta})e^{-i\omega u}\,d\omega$]. The relationship between elements of a complex vector space and the real solutions of Maxwells equations became apparent as a consequence of working on the null surface $\mathscr{I}$.

Ludvigsen [1, 2] showed that these solutions arise classically as the radiation fields produced from classical scattering of charged particles when the final velocities of the scattered particles differ from the initial ones.

In this note we utilize T.N.B's [3] work on sourced fields to find twistor functions which (via the contour integrals) produce appropriate solutions of the zero-rest-mass equations such that in the terminology of quantum electrodynamics wave functionals over these fields yield the infrared quantum states.

We may conceive a variety of reasons for studying such functions, in particular one may note that,

i) these states arise from <u>interactions</u>, namely a classical external current with a quantized field, and,

ii) they provide "test functions" for twistor regularization schemes (A.P.H. [4]), although it is important to note that the infrared problems arising here are observed as second order effects in QED.

<u>Procedure:</u> Considering the right-handed part of the electromagnetic field, we recall that T.N.B's starting point for finding twistor functions of homogeneity -4 for Liénard-Wiechert solutions was Conways integral,

$$\Phi^{AA'}(x) = \frac{1}{i\pi}\oint \frac{\dot{y}^{AA'}(s)\, ds}{(x-y(s))^2} \qquad {}^{\prime\cdot\prime} = d/ds$$

where s is any monotonic parameter along the worldline of the source $y^a(s)$. The appropriate twistor functions are given by,

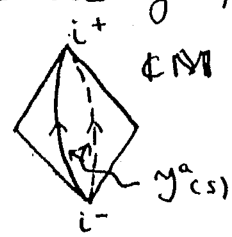
$$f_{-4}\left(\tfrac{1}{Z}\right) = \frac{-2\rho_{A'}}{\rho^{B'}Z_{B'}} \frac{\partial}{\partial Z^A}\left[\frac{1}{i\pi}\oint \frac{\dot{y}^{AA'}(s)\; \alpha.\beta\, ds}{\alpha.\xi(s)\quad \beta.\xi(s)}\right] \qquad (1).$$

The funtion is independent of the choice of the constant spinors $\rho^{A'}$, α^A and β^A. The spinor function of s being defined, $\xi^A(s) := Z^A - i y^{AA'} Z_{A'}$.

We now choose a worldline $y^a(s)$ to meet Ludvigsens criterion that the initial and final velocities of the particle should differ. An appropriate choice would be the rational parameterization,

$$y^a(s) = \left(\left(\frac{s^2-1}{2s}\right), 0, 0, n\left(\frac{s^2+1}{2s}\right)\right) \qquad (2).$$

This yields hyperbolae which, for $0<n<1$ have asymptotes that are timelike. Thus for a choice of n, the field produced by one branch is required.



We note that for some point r^a in $\mathbb{CM}$, the null directions from this point will intersect the hyperbola at 4 points. In $\mathbb{PT}$ this appears as a $\mathbb{CP}^1$ intersecting a ruled surface $\mathcal{L}_s$ (c.f T.N.B [3]) at 4 points. From the theory of primals we know that an arbitrary line in $\mathbb{CP}^3$ meets a k^{th}-order primal in exactly k points (if the points are counted with the correct multiplicity), hence the ruled surface in $\mathbb{PT}$ will be a quartic surface.

We may also deduce that for the range $|s| \leq \infty$, this surface will pass <u>twice</u> through the line I in $\mathbb{PN}$ and through two other lines meeting the line I. These two lines represent the plane in $\mathbb{C}$M$\!$II containing the branches of the hyperbola. To see this recall that a point in $\mathbb{C}$MII with spacelike imaginary part has an S^1's worth of real null lines incident with it (in $\mathbb{PT}$ this is given by the intersection of $\mathbb{PN}$ with the $\mathbb{CP}^1$). Consider such a point and its complex conjugate - no two lines of the same system intersect but every null line in one system will meet every null line in the other. When the $\mathbb{CP}^1$'s meet the line I in $\mathbb{PN}$ then the two families of null lines in $\mathbb{C}$MII sweep over a timelike 2-surface (the plane containing the hyperbola in our case).

Evaluating (1), we follow T.N.B and represent the choice of α^A and β^A as the choice of two points on the line I, $(\alpha^A, 0)$ and $(\beta^A, 0)$. The integral is in the complex s-plane around either the $\alpha\cdot\xi = 0$ or $\beta\cdot\xi = 0$ poles. We note that for our choice of $y^a(s)$, $\alpha\cdot\xi$ and $\beta\cdot\xi$ are quadratic in s, the vanishing of the functions corresponding to intersections of a line through α (or β) and $\overset{1}{z}$ with the ruled surface $\mathcal{L}s$. In order to ensure that we have a contribution due to just one worldine we choose a contour in the s-plane to surround just one of the $\alpha\cdot\xi$ or $\beta\cdot\xi$ poles. [One may think of the point α on the line I as an alpha plane in $\mathbb{C}$MII. Recall that points of $\mathbb{PN}$ correspond to alpha planes which meet their conjugate beta planes. These alpha planes contain real points of Minkowski space. Therefore the intersection of the line in $\mathbb{PT}$ through α (or β) and the ruled surface can correspond to the null separation of three real points in Minkowski space].

Having calculated the residue of (1) and performing the differentiation we are left with a very inelegant expression for the twistor function. It is well known that explicit expressions for fields calculated from the Liénard-Wiechert potentials are either extremely cumbersome or simply not readily obtainable (Conways integral would involve solving a quartic in s), but fortunately there are several simplifications we can make. Introduce a spin frame o^A, ι^A with the standard normalization $o_A \iota^A = 1$, and let $\alpha^A = o^A$, $\beta^A = \iota^A$. Further recall that the global Maxwell field can be determined by the characteristic data on $\mathscr{I}$ via the Kirchoff integrals, hence we could look for a twistor function that would give the asymptotic components of the field $\phi^{\circ}_r(u, \zeta, \bar{\zeta})$, $r = 0, 1, 2$.

Choose a point x^a on $\mathscr{I}$, then let U and V be two null directions through x^a, V pointing along the generators of $\mathscr{I}$. Twistors U and V can be assigned to U and V where $V = X I$ for some X and set $Z = \lambda U + V$. The direction of Z being given by λ in the appropriate manner. These steps would then be consistent with calculating the components of the Maxwell spinor as follows:

$$\phi^0_r(u,\zeta,\bar\zeta) = \frac{\overline{U}V}{i\pi}\oint \frac{Z^C Z^D}{U Z\; V Z} f_{-4}(Z)\, Z\, dZ \wedge dZ \wedge dZ \qquad (3).$$

and integrating around the $\overline{U}_\alpha Z^\alpha$ and $\overline{V}_\alpha Z^\alpha$ poles initially to restrict Z to be of the form $\lambda U + V$. The choice of C and D determines the components are evaluated. [The factor $\overline{U}V$ being inserted as we are working on $\mathscr{I}$].

With this procedure in mind (and some residual freedom left in the choice of $\rho^{A'}$) the final expression for $f_{-4}(Z)$ is,

$$\frac{(1-n^2)+(1+n)^2}{(1-n^2)\rho^{A'}Z_{A'}\,\iota^{B'}Z_{B'}\,(\iota_C Z^C)^2} - \frac{i\sqrt{2}\; o_A Z^A\left[(1-n)^2-(1+n)^2\right]}{(1-n^2)\,\iota^{A'}Z_{A'}\,\rho^{B'}Z_{B'}\,(\iota_C Z^C)^2\sqrt{(1-n^2)(\iota^{A'}Z_{A'})^2-2(o_B Z^B)^2}}$$

$$\cdots (4)$$

We note that if we added the contribution from the other $\alpha\cdot\zeta$ pole then the terms involving the square root would cancel. Similarly surrounding both $\alpha\cdot\zeta$ poles but with a "figure of 8" contour produces a function involving just "square root" terms. For the $f_{-4}(Z)$ given, the limit $n=0$ cancels the "square root" term (as the worldlines become straight-timelike).

"<u>Infrared</u>" Maxwell Fields: Ashtekars condition that wave functionals over the classical solutions should be finite-normed in that,

$$\int_{-\infty}^{\infty} \phi^0_2(u,\zeta,\bar\zeta)\, du = 0 \quad \text{for all values of } \zeta, \bar\zeta. \qquad (5).$$

We may investigate this explicitly. To calculate ϕ^0_2

from (3) using (4) we choose $\overset{c}{|}$ such that $\overset{c}{\underset{v}{|}} = 0$ and integrate,

$$\phi^{0}_{2} = (\overline{u\,v}) \oint \frac{\left(\overset{c}{\underset{z}{|}}\right)^{2}}{\underset{z}{\overset{u}{|}}\ \underset{z}{\overset{v}{|}}}\, f_{-4}(\overset{|}{z})\ \overline{z\ dz \wedge dz \wedge dz}$$

$$= \left(\overset{c}{\underset{u}{|}}\right)^{2} \oint \lambda^{2}\, d\lambda\, f_{-4}(\lambda \overset{|}{u} + \overset{|}{v})$$

$$= \frac{(o_A u^A)(\iota_B u^B)}{\left[(1-n^2)(\iota^{A'}u_{A'})^2(\iota_C v^C)^2 - 2\left(o_D v^D \iota_E u^E - o_F u^F \iota_G v^G\right)^2\right]^{1/2}}$$

$$+ \frac{(1-n^2)(\iota_A v^A\, \iota_B u^B\, \iota^{C'} u_{C'})\left(o_D v^D \iota_E u^E - o_F u^F \iota_G v^G\right)}{\left[(1-n^2)(\iota^{A'}u_{A'})^2(\iota_C v^C)^2 - 2\left(o_D v^D \iota_E u^E - o_F u^F \iota_G v^G\right)^2\right]^{3/2}}$$

Inserting suitable coordinates (labelling $\overset{|}{u}$ and $\overset{|}{v}$) we find

$$\phi^{0}_{2}(u,\zeta,\bar{\zeta}) = \frac{\zeta(1-n^2)(u-\zeta\bar{\zeta})}{\left[(n^2-1)-2(u-\zeta\bar{\zeta})^2\right]^{3/2}} - \frac{u\zeta}{\left[(n^2-1)-2(u-\zeta\bar{\zeta})^2\right]^{1/2}}$$

... (5)

If we now consider (5) and let u take complex values we observe that the result will be non-zero (and hence the field is in the "infrared" sector) as integrating along the real u-axis takes you onto the other sheet of the two-valued function (branch points at $u = -\zeta\bar{\zeta} \pm \frac{i}{2}\sqrt{2(1-n^2)}$). Had we inserted into (5) a field calculated from (3) but with $f_{-4}(\overset{|}{z})$ being an elementary state with the singularities in $\mathbb{PT}^{+}$ such that the field is positive frequency then the result is zero. By looking at the singularities in the complex u-plane one observes that they all lie to one side or the other of the real u-axis so that a contour running along the real axis could be closed in the upper or lower half plane to give zero.

Thanks to A.P.H. and R.P. for useful discussions.

Ian Roulstone.

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Conformal Gravity, The Einstein Equations and Spaces of Complex Null Geodesics.

R.J Baston and L.J. Mason,

The Mathematical institute,
24 - 29 St Giles,
Oxford OX1 3LB.
U.K.

New College,
Oxford OX1 3BN.
U.K.

Abstract

The aim of this work is to give a twistorial characterization of the field equations of conformal gravity and of Einstein space-times. We provide strong evidence for a particularly concise characterization of these equations in terms of 'formal neighbourhoods' of the space of complex null geodesics.

We consider second order perturbations of the metric of complexified Minkowski space. These correspond to certain infinitesimal deformations of its space of complex null geodesics, $\mathbb{PN}$.

$\mathbb{PN}$ has a natural codimension one embedding into a larger space (the product of twistor space and its dual). We show that deformations extend automatically to the fourth order embedding (that is, the fourth formal neighbourhood). They extend to the fifth formal neighbourhood if and only if the corresponding perturbation in the metric has vanishing Bach tensor (these are the equations of conformal gravity). Finally, deformations which extend to the sixth formal neighbourhood correspond to perturbations in the metric that are conformally related to ones satisfying the Einstein equations, at least when the Weyl curvature is sufficiently algebraically general.

One can attempt to construct such formal neighbourhoods in the fully curved case. We present arguments which suggest that our results will also hold when space-time is fully curved.

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Further Remarks on Conserved Vectorial Quantities associated with the Kerr Solution

Here I would like to summarize some of the technical details which lead to the conclusions drawn in my previous article (TN 24, pp. 32-35).

Lemma 1. Suppose $\mathcal{X}_{AB}$ satisfies $\nabla_{A'(A}\mathcal{X}_{BC)} = 0$ and thus $\nabla_{A'A}\mathcal{X}_{BC} = \xi_{A'(B}\varepsilon_{C)A}$ for some $\xi_{A'A}$. Then the divergence of ξ_a vanishes. Moreover if $R_{ab} - \frac{1}{4}Rg_{ab} = 0$ then ξ_a is a Killing vector.

More generally, if R_{ab} is determined by the stress-energy of an electromagnetic field for which the principal spinors are aligned with those of $\mathcal{X}_{AB}$ (as in the Kerr-Newman solution) then ξ_a is a Killing vector.

Proof. See Hughston & Sommers (1973) or Penrose & Rindler (1986, proposition 6.7.17, p. 108). □

In the case of Kerr (K) and Kerr-Newman (K-N) the Killing vector thus constructed can be chosen (by suitable adjustment of the disposable constant factor) to be real, and in the region where it is time-like to be future-pointing with unit norm. (The constant factor in $\mathcal{X}_{AB}$ can be fixed by these conventions)

Lemma 2. Let ξ_a be a real Killing vector constructed as above and let $f_{ab} = \nabla_{[a}\xi_{b]}$. If $f_{ab} = \phi_{AB}\varepsilon_{A'B'} + \phi_{A'B'}\varepsilon_{AB}$ where

$$\phi_{AB} = \frac{1}{2}\nabla_{A'(A}\xi^{A'}_{B)}$$

then the spinor ϕ_{AB} is given by

$$\phi_{AB} = \Psi_{ABCD}\mathcal{X}^{CD}$$

where Ψ_{ABCD} is the Weyl spinor.

Proof. Apply $\nabla^{A'}_{D}$ to $\nabla_{A'A}\mathcal{X}_{BC} = \xi_{A'(B}\varepsilon_{C)A}$ and symmetrize over A and D, then use a Ricci identity. □

Note that the 'charge integral' associated with f_{ab} gives the Komar mass. In the case of the Kerr solution the Komar mass agrees with the charge integral of $\Psi_{ABCD}\mathcal{X}^{CD}$, the spin 1 field obtained by lowering the spin of Ψ_{ABCD} by use of $\mathcal{X}^{AB}$. Thus the 'charge' associated with ϕ_{AB} as defined above is M. An alternative formula for ϕ_{AB} is given by $\phi_{AB} = \Box\mathcal{X}_{AB}$.

Now let us write $L_{ab} = i\mathcal{X}_{AB}\varepsilon_{A'B'} - \bar{\mathcal{X}}_{A'B'}\varepsilon_{AB}$ for the Killing-Yano tensor, so $\nabla_{(a}L_{b)c} = 0$ as a consequence of the twistor equation and the reality of ξ_a. We are interested in the motion of a particle of charge ε under the influence of a field F_{ab} of charge q defined by $F_{ab} = qM^{-1} f_{ab}$ with f_{ab} as defined earlier. The equation of motion is $m U^a \nabla_a U^b = \varepsilon F^{bc} U_b$.

Lemma 3. The vector $G_a = L_{ab} U^b + M_a$ is a constant of the motion (i.e. is parallelly propagated) if and only if there exists a scalar Φ such that $M_a = \nabla_a \Phi$ and

$$\varepsilon L_{bc} F^c{}_a + m \nabla_a \nabla_b \Phi = 0$$

where m is the mass of the particle.

Proof. $U^a \nabla_a G_b = U^a \nabla_a (L_{bc} U^c + M_b) = U^a U^c \nabla_a L_{bc} + L_{bc} U^a \nabla_a U^c + U^a \nabla_a M_b = m^{-1} \varepsilon L_{bc} F^c{}_a U^a + U^a \nabla_a M_b$, which vanishes for all U^a if and only if $\varepsilon L_{bc} F^c{}_a + m \nabla_a M_b$ vanishes. However $L_{bc} F^c{}_a$ is automatically symmetric (on account of the proportionality of ϕ_{AB} and $\mathcal{X}_{AB}$) and thus $\nabla_{[a} M_{b]} = 0$. □

Lemma 4. Suppose $\nabla_{A'A} \mathcal{X}_{BC} = \xi_{A'(B} \varepsilon_{C)A}$. Then the vector N_a defined by $N_{A'A} = \xi^B_{A'} \mathcal{X}_{AB}$ is curl-free.

Proof. We have $\mathcal{X}^{BC} [\nabla_{A'A} \mathcal{X}_{BC} - \xi_{A'(B} \varepsilon_{C)A}] = 0$, whence therefore

$$\tfrac{1}{2} \nabla_{A'A} \mathcal{X}^2 = \xi_{A'B} \mathcal{X}^B_A,$$

where

$$\mathcal{X}^2 = \mathcal{X}_{AB} \mathcal{X}^{AB}. \quad \square$$

Theorem. There exists a scalar Φ such that $\varepsilon L_{bc} F^c{}_a + m \nabla_a \nabla_b \Phi = 0$.

Proof. Let L_{ab} be the Killing-Yano tensor and ξ^a the associated Killing vector, which for the Kerr solution is real. Then

$$\begin{aligned} L_{ab} \xi^a &= i \mathcal{X}_{AB} \varepsilon_{A'B'} \xi^{A'A} - i \bar{\mathcal{X}}_{A'B'} \varepsilon_{AB} \xi^{A'A} \\ &= i \mathcal{X}_{AB} \xi^A_{B'} - i \bar{\mathcal{X}}_{A'B'} \xi^{A'}_B \\ &= -\tfrac{1}{2} i \nabla_b (\mathcal{X}^2 - \bar{\mathcal{X}}^2) \end{aligned}$$

by Lemma 4. Thus by differentiation we obtain

$$\nabla_a (L_{cb} \xi^c) = -\tfrac{1}{2} i \nabla_a \nabla_b (\mathcal{X}^2 - \bar{\mathcal{X}}^2)$$

and therefore

$$(\nabla_a L_{cb}) \xi^c + L_{cb} f_a{}^c = -\tfrac{1}{2} i \nabla_a \nabla_b (\mathcal{X}^2 - \bar{\mathcal{X}}^2)$$

where $\nabla_a \xi_b = f_{ab}$. But $\xi^c \nabla_a L_{cb} = -\xi^c \nabla_c L_{ab}$ and $\xi^c \nabla_c L_{ab} = 0$ since $\xi^c \nabla_c \mathcal{X}_{AB} = 0$, which in turn follows from the relation $\nabla_{A'A} \mathcal{X}_{BC} = \xi_{A'(B} \varepsilon_{C)A}$. Therefore $L_{bc} F^c{}_a = -\tfrac{1}{2} i q M^{-1} \nabla_a \nabla_b (\mathcal{X}^2 - \bar{\mathcal{X}}^2)$ and the scalar Φ is given by

$$\Phi = -\tfrac{1}{2} i \frac{\varepsilon q}{m M} (\mathcal{X}^2 - \bar{\mathcal{X}}^2),$$

as given in TN 24.

L.P. Hughston

Remarks on Sommers' Theorem

L.P. Hughston

Lincoln College, Oxford OX1 3DR, England

Abstract

According to Sommers a spinor field ξ^A which satisfies $\xi^A\xi^B\nabla_{A'A}\xi_B = 0$ and is a p-fold principal spinor ($p \geqslant 1$) of a massless field of spin $\frac{1}{2}(p+q)$ is also a repeated principal spinor of the gravitational field (unless $p = 3q+2$). A new and much simplified proof of this result is established here.

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Twistor Cauchy Riemann Manifolds Associated to Algebraically Special Space-Times.

L.J.Mason

Introduction

It has been known for some time that a residue of the Kerr theorem holds for geodesic shear free null congruences in curved space-times. (See, e.g. Penrose & Rindler, Spinors & Space-time vol II §7, and references therein to P.Sommers. See also recent papers of Robinson & Trautman). This amounts to the fact that the (3-dimensional) quotient space, $\mathbb{P}\mathcal{N}$, of the space-time, $\mathcal{M}$, by the congruence has a naturally defined C.R. structure. From the Goldberg-Sachs theorem we know that, in vacuum, algebraic speciality is equivalent to the existence of a geodesic shear free null congruence. One might therefore hope that these residual twistor C.R. manifolds may be of use in solving the reduced vacuum equations for algebraically special space-times, and also for understanding the global structure of such solutions. Here I wish to report on some preliminary results.

One would like to articulate the vacuum equations on an algebraically special space-time, $\mathcal{M}$, in terms of 'twistorial' quantities on the quotient space, $\mathbb{P}\mathcal{N}$, in the hope of some simplification in the field equations emerging. One may, for instance, in the type N case hope for some generalization of the description of solutions in linearized theory in terms of two free holomorphic functions of 2 variables. I have not yet been able to do this in a satisfactory fashion. (The field equations in the case of a twisting congruence have so far proved very difficult to solve; it is this case, however, which has previously provided the most hope of globally regular solutions. The twisting case is also the most interesting from the point of view of C.R. manifold theory, since only then is the C.R. structure nondegenerate.)

Before one can approach this problem, a preliminary question arises as to what further geometry is required on $\mathbb{P}\mathcal{N}$ over and above the C.R. structure in order to be able to reconstruct the space-time. The answer to this preliminary question turns out to be of interest also because it has interesting implications for the asymptotic structure of algebraically special space-times. In particular, it seems likely that no asymptotically simple vacuum algebraically special space-times exist.

The residual C.R. Manifold

Let $(\mathcal{M},ds^2)$ be a Lorentzian space-time, and let $l^a=o^Ao^{A'}$ be a vector field aligned along a geodesic and shear free null congruence in $\mathcal{M}$. Let $\mathbb{P}\mathcal{N}$ be the quotient space of $\mathcal{M}$ by the congruence. Then a natural C.R. structure on $\mathbb{P}\mathcal{N}$ can be defined as follows. Choose a spinor, $\iota^{A'}$ such that $\iota^{A'}o_{A'}=1$. The vector field $m^a=o^A\iota^{A'}$, by virtue of the geodesic and shear free condition on $o^{A'}$, satisfies $\mathcal{L}_l m^a=al^a+bm^a$ for some complex functions a and b. So the (complex) vector field $m^a\partial_a$ descends to a vector field, δ, on $\mathbb{P}\mathcal{N}$ up to proportionality. In three-dimensions, such a single complex vector field is sufficient to define a C.R. structure on $\mathbb{P}\mathcal{N}$.

[Recall that a C.R. structure on a $(2n+1)$-dimensional manifold models the structure that the manifold would inherit from a codimension one embedding in a complex manifold. The C.R. structure can be specified by an integrable complex n-$\mathbb{C}$dimensional sub-bundle, H, of the complexified tangent bundle. This models the (1,0)-tangent vectors on the ambient complex manifold which are tangent to the embedded submanifold. In this case $n=1$ and so D consists of a vector field up to proportionality.]

The space $\mathbb{P}\mathcal{N}$ can be thought of as a residual twistor space. For a geodesic shear free congruence in Minkowski space, $\mathbb{P}\mathcal{N}$ is the intersection of the Kerr surface defining the congruence in $\mathbb{PT}$ with $\mathbb{PN}$.

The structures required to rebuild the space-time

I will consider only nondegenerate C.R. manifolds, i.e. those for which $[\delta,\bar{\delta}]$ is not a linear combination of δ and $\bar{\delta}$. (Degenerate C.R. manifolds are fairly dull creatures and the vacuum equations and asymptotic structure for that case are relatively easy to handle). This requires that the congruence be twisting.

Since general algebraically special metrics depend on functions of 4 variables, we must first impose some conditions on $\mathcal{M}$. The first conditions include all vacuum algebraically special space-times. These are:

$$\Lambda=0=\Phi_{A'B'AB}o^Ao^B=\Psi_{A'B'C'D'}o^{C'}o^{D'}.$$

The final, slightly awkward condition is that: $(\rho-\bar{\rho})\Psi_2-\Phi_{11}=0$ in the frame given below. This eliminates (2,1,1) vacuum space-times, but includes all type (3,1) and (4) vacuum space-times.

We also need some technicalities concerning C.R. manifolds. On $\mathbb{P}\mathcal{N}$ there is a naturally defined C.R. complex line bundle, $\mathcal{N}\to\mathbb{P}\mathcal{N}$. This is defined to be the $\frac{1}{3}$ power of the canonical bundle $\mathcal{K}$, $\mathcal{N}=\mathcal{K}^{1/3}$. (The canonical bundle on $\mathbb{P}\mathcal{N}$ is the bundle of complex 2-forms orthogonal to $\bar{\delta}$.) The reason for this definition of $\mathcal{N}$ is that when $\mathbb{P}\mathcal{N}$ is realized as a hypersurface in $\mathbb{CP}^2$ then $\mathcal{N}$ is the restriction of $\mathcal{O}(-1)$. (See my article on Chern-Moser connections in $\mathbb{TN}$21.) We can now state the result concerning the structures required on $\mathbb{P}\mathcal{N}$ required to rebuild $\mathcal{M}$.

> **Lemma:** *Space-times in the class defined above are in 1-1 correspondence with pairs consisting of a 3-dimensional C.R. manifold, $\mathbb{P}\mathcal{N}$, together with a (1,0)-form, ι, with values in $\mathcal{N}^2$ whose associated 1-form on $\mathcal{N}$ (also denoted ι) satisfies $\iota\wedge d\iota=0$.*

(Note that by a (1,0)-form, α, I mean that $\alpha(\bar{\delta})=0$, with the complex conjugate definition for a (0,1)-form.)

Proof: I will proceed by writing everything out in coordinates.

The condition $\iota\wedge d\iota=0$ implies that there exists a C.R. function, ζ ($\bar{\delta}\zeta=0$), on $\mathbb{P}\mathcal{N}$, and a trivialization of $\mathcal{N}$ over $\mathbb{P}\mathcal{N}$ such that $\iota=d\zeta$.

A trivialization of $\mathcal{N}$ provides also a fixed scaling for a 1-form, l, on $\mathbb{P}\mathcal{N}$ satisfying $l(\delta)=l(\bar{\delta})=0$. This can be seen as follows. Let λ be a coordinate up the fibre of $\mathcal{N}$ and let $\theta=\lambda\bar{\lambda}l$ for some choice of l. Then $d\theta$ defines all but one component of a Hermitean form on $\mathbb{C}T\mathcal{N}/T^{(0,1)}\mathcal{N}$. On $\mathcal{N}$ we have the pullbacks, ν, from $\mathcal{K}$ of the holomorphic 'volume' forms. The scaling on l can now be fixed by requiring that the determinant of the Hermitean form on $\mathbb{C}T\mathcal{N}/T^{(0,1)}\mathcal{N}$ equals that defined by $d\nu\otimes d\bar{\nu}$. (This procedure is well defined, and is not as ad hoc as it may seem; it is intimately related to those required to define the Chern-Moser connection for the C.R. manifold. See my article in $\mathbb{TN}$21.) Finally, to agree with the reduction of the space-time metric below, we may choose a coordinate u (defined up to the addition of a real function of ζ and $\bar{\zeta}$) such that $l=du+zd\bar{\zeta}+\bar{z}d\zeta$.

We now compare this to a reduction of algebraically special metrics. Proofs of the following can be found in the exact solutions book by M.MacCallum et. al.

Coordinates for $\mathcal{M}$ can be chosen as follows. The curvature conditions $\Lambda=0=\Phi_{A'B'AB}o^A o^B=\Psi_{A'B'C'D'}o^{C'}o^{D'}$ imply that $o^{A'}$ can be scaled so that the 1-form $\iota=o^A\nabla_b o_{A'}dx^b$ is closed and therefore equal to the gradient of a complex function ζ,

$\iota = d\varsigma$. A coordinate r is chosen to be an affine parameter up each geodesic of the congruence with $r=0$ at the hypersurface of minimum expansion, $\rho = -\bar{\rho}$. Finally a coordinate u is chosen so that $l = l_a dx^a = du + z d\bar{\varsigma} + \bar{z} d\varsigma$. The coordinates $(u,\varsigma,\bar{\varsigma})$ are constant along the congruence, and therefore provide coordinates on $\mathbb{P}\mathcal{N}$. With the above curvature conditions and choice of coordinates, the general metric (with diverging rays) is:

$$ds^2 = 2l\cdot n - 2m\cdot\bar{m}$$

$$l = du + \bar{z}d\varsigma + zd\bar{\varsigma} \qquad n = dr + Hdu + \bar{w}d\varsigma + wd\bar{\varsigma}$$

$$\bar{m} = -\rho^{-1}d\varsigma \qquad m = -\bar{\rho}^{-1}d\bar{\varsigma}$$

where $(u,r,\varsigma,\bar{\varsigma})$ are the coordinates and z,w and ρ are complex functions, and H is real. It is convenient to define the vector fields:

$$\delta = z\frac{\partial}{\partial u} - \frac{\partial}{\partial\bar{\varsigma}} \qquad \Delta = \frac{\partial}{\partial u}.$$

The curvature conditions imply:

$$\rho = \frac{-1}{r + i\rho_0} \qquad \rho_0 = \frac{i}{2}(\bar{\delta}z - \delta\bar{z})$$

$$w = \{\frac{\Delta z}{\rho} - i\delta\rho_0\} \qquad H = \tfrac{1}{2}\{\delta\Delta\bar{z} + \bar{\delta}\Delta z + \rho\psi_2^0 + \bar{\rho}\bar{\psi}_2^0\}$$

$$z \equiv z(u,\varsigma,\bar{\varsigma}) \qquad \psi_2^0 \equiv \psi_2^0(u,\varsigma,\bar{\varsigma})$$

The condition $(\rho - \bar{\rho})\Psi_2 + \Phi_{11} = 0$ restricts us to the case in which $\psi_2^0 = 0$. It can now be seen that, with this condition, the knowledge of z as a function of $(u,\varsigma,\bar{\varsigma})$ is sufficient to reconstruct the space-time. This can be found from the coordinate expressions for ι and l from the C.R. manifold point of view.

The combined remaining coordinate and frame freedom is:

$$o^A \rightarrow \lambda(\varsigma)o^A, \qquad u \rightarrow \lambda\bar{\lambda}u, \qquad \varsigma \rightarrow \int\lambda^2(\varsigma)d\varsigma, \qquad r \rightarrow \lambda^{-1}\,\bar{\lambda}^{-1}\,r$$

and $u \rightarrow u + f(\varsigma,\bar{\varsigma})$

This is precisely the freedom associated with deriving the coordinate forms of ι and l from the line bundle valued form ι on $\mathbb{P}\mathcal{N}$, so we see that the correspondence is well defined.□

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