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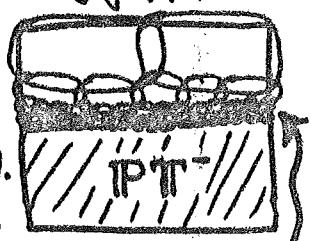
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Contents.

A Finite Covering for $\overline{PTT^+}$ (R. Penrose)	Front page.
Replacing the Snail Contour by a Branched Contour (R. Ward)	Page 0.
Twistors and Multipole Moments (G. Curtis)	Page 1.
H-Space - A New Approach (M. Ko et al.)	Page 5.
Global Properties of Massless Free Fields (D. Lerner)	Page 7.
Evaluation of Strand Networks (J. Mousouris)	Page 9.
The Back-Handed Photon etc. (R. Penrose)	Page 12.
TN12 Errata.	Page 17.
A Generalized Photon Construction (L. Hughston)	Page 18.
2-Twistor Functions for Potentials (N. Bell)	Page 22.

A Finite Covering for $\overline{PTT^+}$

A difficulty arises, in connection with the sheaf cohomology approach to twistor functions (see TN.2), if one tries to find a fixed covering of $\overline{PTT^+}$ which is "sufficiently fine" for all elements of $H^1(\overline{PTT^+}, \mathcal{O}(n))$ (i.e. for all analytic massless wave functions). This arises basically because $\overline{PTT^+}$ is not holomorphically (pseudo-)convex and admits no finite covering by Stein manifolds. Any covering of $\overline{PTT^+}$ by Stein manifolds must necessarily be infinite, with open sets crowding up against the boundary $P\Gamma$. This is reflected in the fact that while for any fixed analytic massless wave function, a twistor function may be found (by way of the inverse twistor function) whose domain is just $U_1 \cap U_2$ where U_1 and U_2 are open sets with $U_1 \cup U_2 = \overline{PTT^+}$, this particular covering can never and for which the corresponding twistor function's domain may need to be arbitrarily narrow. However, if we allow ourselves to use closed sets for our sheaf cohomology (and there seems no reason against this), then the covering of $\overline{PTT^+}$ by the two closed sets given respectively by $|Z^0 + Z^1| > |Z^1 + Z^2|, Z^a Z_a > 0$ and by $|Z^0 + Z^2| > |Z^1 + Z^2|, Z^a Z_a > 0$ will suffice (as the inverse twistor function shows) for all such wave functions.



~ Roger Penrose

Replacing the snail contour by a branched contour. O.

In TN1, R.P. & B.B. describe the "snail contour" method of converting a sphere integral into a closed contour integral of a definite integral. What we'll show here is how one can use a "branched contour" (see R.P.'s article in TN2) instead of a snail contour.

The setup is as follows. We're given a sphere \mathbb{S}^2 $I = \oint_{\Sigma} G(\zeta, \bar{\zeta}) d\zeta_1 d\bar{\zeta}_1 = \oint_{\Sigma} G(\zeta, \eta) d\zeta_1 d\eta_1$, where the contour Σ is given by $\bar{\eta} = \zeta$ and where $G(\zeta, \eta)$ is holomorphic in a neighbourhood of $\bar{\eta} = \zeta$.

Cover the ζ -sphere with patches U_j , with the property that $G(\zeta, \eta)$ is holomorphic for ζ and $\bar{\eta}$ ranging over the same patch U_j (the idea is to make the U_j sufficiently small, so that ζ and $\bar{\eta}$ are forced to lie close to each other). In each patch U_j , pick some point p_j . For $\zeta \in U_j \cap U_k$, define $f_{jk}(\zeta) = \int_{T_{jk}} G(\zeta, \eta) d\eta_1$,

where T_{jk} is some contour in $U_j \cap U_k$ from p_j to p_k . It's not hard to check that (a) $f_{jk}(\zeta)$ is holomorphic on $U_j \cap U_k$; (b) $f_{jk} = -f_{kj}$; (c) $f_{jk} + f_{k\ell} + f_{\ell j} = 0$ on $U_j \cap U_k \cap U_\ell$. So the f_{jk} constitute a 1-cocycle (see TN2) and determine an element of the cohomology group $H^1(CP_1, \theta(-2))$.

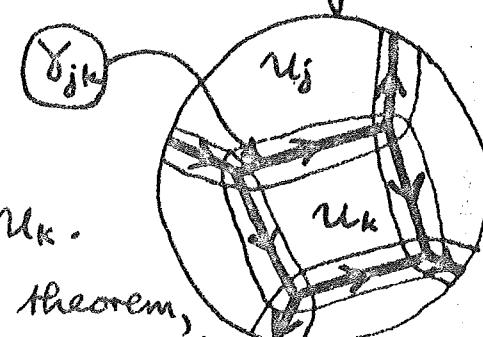
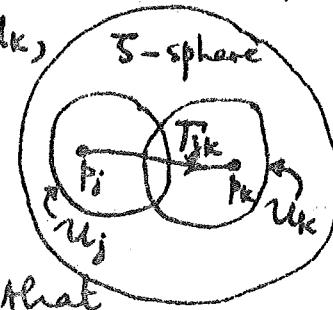
The cocycle $\{f_{jk}\}$ can be evaluated using a branched contour integral:

$$I' = \sum_{j,k} \int_{\gamma_{jk}} f_{jk}(\zeta) d\zeta_1$$

the γ_{jk} are contours lying in $U_j \cap U_k$.

One can now show, using Stokes' theorem, that $I' = I$; in other words, we've succeeded in expressing the sphere integral as a branched contour integral.

Richard Ward.



A BRIEF OUTLINE OF TWISTORS AND MULTIPOLE MOMENTS

It has been known for some time how to construct charge and angular momentum twistor integrals for the sources of massless spin 1 and 2 fields. It is interesting to ask if there is a similar formulation for the multipole moments. In fact, there is one for stationary massless spin s fields on Minkowski space. It may be possible to extend this to non-stationary fields and asymptotically flat space times.

For simplicity we deal only with a spin 2 field ψ_{ABCD} . A field is stationary if there is a timelike, constant vector t^a with $t^a t_a = 1$ and $t^a \psi_{ABCD} = 0$. It follows from the work of Geroch and Bramson [1,3] that multipoles may be described by a set of symmetric spinor fields $A'_1 \dots A'_{2n}$ on Minkowski space which satisfy

$$\nabla_L^{L'} A'_1 \dots A'_{2n} = - \frac{2}{3} n(2n-1) \epsilon^{L' (A'_1 A'_2 \dots A'_{2n-1} A'_{2n})} t^L . \quad (1)$$

The mass m is $\frac{1}{3} i w$. These are related to the (essentially 3-dimensional) trace-free, symmetric multipole tensors by

$$\Omega^{a_1 \dots a_n} = - \frac{1}{2} t^{a_1} \dots t^{a_n} A'_{n+1} \dots A'_{2n} . \quad (2)$$

They also have $t_{a_1} \Omega^{a_1 \dots a_n} = 0$. The $\Omega^{a_1 \dots a_k}$ are complex quantities and contain information about both the electric and magnetic moments.

From (1) $\omega^{A'_1 \dots A'_{2n}}$ satisfies the twistor equation $\nabla_L^{(L' A'_1 \dots A'_{2n})} \omega^{L'} = 0$

and so may be used to construct a $2n$ index symmetric twistor

$\Omega_{a_1 \dots a_{2n}} = \begin{array}{c} 2n \\ \vdots \\ 2n \end{array}$. By virtue of (1) these satisfy the algebraic constraints

$$\begin{array}{c}
 3i(2n-2)! \quad \text{Diagram: } \begin{array}{c} \text{2n} \\ \cdots \end{array} = \begin{array}{c} \text{T} \\ \text{2n-2} \end{array} \\
 \text{Diagram: } \begin{array}{c} \alpha \\ \text{T} \\ \beta \end{array} \leftrightarrow \begin{pmatrix} 0 & t^{AB'} \\ 0 & 0 \end{pmatrix}.
 \end{array} \quad (3)$$

In particular, for $n=1$ we see $\Omega_{\alpha\beta} = -m \langle T \rangle = \frac{1}{2}\eta_{\alpha\beta}$. The momentum π^{α} is mt^{α} .

We shall give a construction for a set of twistors satisfying (3). They are obtained by constructing iteratively a sequence of spin 1 fields and looking at the corresponding charge integrals. It is a generalisation of the method used by Penrose [5] for the angular momentum twistor. We first construct the higher spin fields

$$(n) \psi_{A_1 \dots A_{2n}} = t_{A'A_1} \nabla^{A'}_{A_2} \psi_{A_3 \dots A_{2n}}, n \geq 2 \quad (4)$$

$$(2) \psi_{ABCD} = \psi_{ABCD}$$

Then, for a solution $\alpha_{A_1 \dots A_{2n}}$ of the twistor equation $\nabla_{A'} \alpha_{A_1 \dots A_{2n}} = 0$,

$$(n) \phi_{AB} = -i \left(-\frac{1}{3}\right)^{n-1} \alpha_{A_1 \dots A_{2n}} \quad (n+1) \psi_{AB A_1 \dots A_{2n}}, n \geq 1 \quad (5)$$

is a spin 1 field. We let $f_n(w_\alpha)$ be the twistor function corresponding to $(n) \psi_{A \dots K}$ and $A^\alpha \dots \delta$ be the symmetric twistor corresponding to $\alpha^A \dots D$. The twistor charge integrals for (5) are [6]

$$q(A_{A_1 \dots A_{2n}}) = -\frac{1}{4\pi^2} \left(-\frac{1}{3}\right)^{n-1} \oint_{W_{\alpha_1 \dots \alpha_{2n}}} A_{A_1 \dots A_{2n}} f_{n+1}(w_\alpha) d^4 w_\alpha. \quad (6)$$

The maps q define twistors $Q_{\alpha \dots \delta}$ dual to the $A^\alpha \dots \delta$ by

$$q = Q_{\alpha_1 \dots \alpha_{2n}} A_{A_1 \dots A_{2n}}. \quad (7)$$

There are $\binom{3+2n}{3}$ independent solutions for $\alpha_{A_1 \dots A_{2n}}$, so we will obtain this number of conserved (complex) quantities. However, this reduces to $2(n^2 + 2n + 2)$ real quantities for stationary fields [cf. 6].

It follows from (4) that

$$f_{n+1} = 1 \quad \begin{array}{c} R \\ \diagup \quad \diagdown \\ T \end{array} \quad \begin{array}{c} \partial \\ \diagup \quad \diagdown \\ W \end{array} \quad f_n \quad (8)$$

for an arbitrary twistor R_α . Hence from (6) and (8) the $Q_{\alpha \dots \beta}$ satisfy (3). Hence they are multipole twistors. $Q_{\alpha \beta}$ is just the usual angular momentum twistor [5].

On integrating by parts $(n - 1)$ times

$$\begin{array}{c} 2n \\ \dots \end{array} = \frac{(-1)^{n+1}}{4\pi^2(n+1)!} \left(-\frac{1}{3} \right)^{n-1} \int \begin{array}{c} R \\ \diagup \quad \diagdown \\ T \end{array} \dots \begin{array}{c} R \\ \diagup \quad \diagdown \\ T \end{array} \begin{array}{c} W \\ \diagup \quad \diagdown \\ W \end{array} \dots \begin{array}{c} W \\ \diagup \quad \diagdown \\ W \end{array} \frac{f_2}{(RW)^{n-1}} d^4W. \quad (9)$$

It is possible to put this in a rather nice form. We set

$$Q_{\alpha \beta} \leftrightarrow \begin{pmatrix} O & mt_B' \\ mt_A & 2i\mu^{A'B'} \\ mt_A' & mt_B \end{pmatrix}. \quad (10)$$

and let $S^{\alpha \beta}$ be the simple skew twistor defining the complex centre of mass $p^\alpha = \frac{2}{m} t_B^A \mu^{A'B'}$ with normalisation $\bar{S} = 2$. By making the choice

$$R = \overline{Q}_S W \text{ where } Q_S = 1:$$

$$\begin{array}{c} 2n \\ \dots \end{array} = \frac{(-1)^{n+1}}{4\pi^2(n+1)!} \frac{(2/3)^{n-1}}{m^{n-1}} \int \begin{array}{c} \overset{n-1}{\overbrace{W \dots W}} \\ S \end{array} \begin{array}{c} \overset{n+1}{\overbrace{W \dots W}} \\ S \end{array} \frac{f_2}{(WW)^{n-1}} \frac{d^4W}{Q}. \quad (11)$$

It is possible to rewrite (6) as an integral on cuts of \mathbb{S}^+ for $A'_1 \dots A'_{2n}$. This agrees with the work of Bramson [1] for $n = 1, 2$. The integrands contain $\psi_1^0, \psi_0^0, \psi_0^1, \dots$ successively which are intuitively the pieces containing the relevant multipole information. The w are naturally defined on the space H of shear free cuts of \mathbb{S}^+ for Minkowski space. H is just a copy of Minkowski space and has the usual twistor space T associated with it. The multipole twistors may be considered to be elements of T^n . By working in H it is possible to show (1) directly [2].

It is possible to put a Lie algebra structure on the space \mathcal{J} of symmetric twistors R . This turns out to be the Lie algebra of $SO(5, \mathbb{C})$ and is semi-simple. Hence there is a natural way of raising and lowering the indices of the multipole twistors by using the non-singular Killing

form as metric.

As an example we look at linearised Kerr type solutions [6]. We take $f_2(w_\alpha) = \left(\frac{w_w}{\lambda}\right)^{-3}$ with $\begin{bmatrix} \Gamma \\ \Lambda \end{bmatrix} = \begin{bmatrix} \Gamma \\ \bar{\Lambda} \end{bmatrix}$. Λ is symmetric with inverse B . We set $\Delta = (\det \Lambda)^{1/2}$ and then $\Omega = \frac{\pi}{8i\Delta} B$. On using (6) we find

$$\Omega_{\alpha_1 \dots \alpha_{2n}} = \frac{1}{2} \left(\frac{1}{6m} \right)^{n-1} \frac{(2n)!}{n!} \Omega_{(\alpha_1 \alpha_2 \dots \alpha_{2n-1} \alpha_{2n})} \quad (12)$$

Using this formula the electric and magnetic parts for linearised Kerr agree with the results of Hansen [4].

The very natural algebraic structure of twistor multipoles suggests that these should be regarded as the more basic objects. The expression (9) contains no explicit reference to the stationarity condition and so perhaps should be used in the non-stationary case. However this may lead to contour problems. The spinor version can be directly extended to asymptotically flat stationary space times [1]. The problem of how to extend the definitions to general asymptotically flat space times is still open.

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J. E. Curtis

X-Space - A New Approach.

This note is intended as a sequel to the article "Local Metric Properties of \mathcal{X} -space" by Richard Hansen in TN1 and sets out new expressions for the metric and curvature tensors of \mathcal{X} -space which do not involve the use of integrals.

As in the previous article, we start with the hybrid object $Z_a(\tilde{\gamma}, \tilde{\delta}, \tilde{\zeta})$ in \mathcal{X} -space satisfying the equation $\tilde{\gamma}^2 Z_a = \dot{\sigma} Z_a$. — (1)

For fixed $\tilde{\gamma}$ and $\tilde{\zeta}$, Z_a may be regarded as a vector field in \mathcal{X} -space and we define three other vector fields as $\tilde{\delta}Z_a$, $\tilde{\gamma}Z_a$ and $\tilde{\delta}\tilde{\delta}Z_a$ at the same values of $\tilde{\gamma}$ and $\tilde{\zeta}$. These are found from the integral form of the metric in Hansen's article to have the normalisations $Z^a Z_a = \tilde{\gamma}^a \tilde{\delta}Z_a = \tilde{\gamma}^a \tilde{\gamma}Z_a = \tilde{\gamma}^a \tilde{\delta}\tilde{\delta}Z_a = \tilde{\delta}Z^a \tilde{\delta}Z_a = 0$,

$$\tilde{\gamma}^a \tilde{\delta}\tilde{\delta}Z_a = -\tilde{\delta}Z^a \tilde{\delta}Z_a = 1.$$

Introducing the quantity \mathcal{F} by $\tilde{\delta}Z^a \tilde{\delta}Z_a = -2\mathcal{F}$,

we observe $\tilde{\delta}Z^a \tilde{\delta}\tilde{\delta}Z_a = D = -\tilde{\delta}\mathcal{F}$,

$$\tilde{\delta}\tilde{\delta}Z^a \tilde{\delta}\tilde{\delta}Z_a = -2E = -2(1 - \mathcal{F}\dot{\sigma} + \frac{1}{2}\tilde{\delta}^2\mathcal{F}),$$

so that the following four vectors form a null tetrad with the usual normalisation:

$$L_a = Z_a \quad \tilde{M}_a = -\frac{1}{2}D Z_a - \mathcal{F} \tilde{\delta}Z_a + \tilde{\delta}Z_a$$

$$M_a = \tilde{\delta}Z_a \quad N_a = E Z_a + \frac{1}{2}D \tilde{\delta}Z_a + \tilde{\delta}\tilde{\delta}Z_a.$$

The metric is then simply

$$\begin{aligned} g_{ab} &= L_a N_b + L_b N_a - M_a \tilde{M}_b - \tilde{M}_a M_b \\ &= 2 Z_{(a} \tilde{\delta}\tilde{\delta}Z_{b)} - 2 \tilde{\delta}Z_{(a} \tilde{\delta}Z_{b)} + 2 D Z_{(a} \tilde{\delta}Z_{b)} \\ &\quad + 2 E Z_a Z_b + 2 \mathcal{F} \tilde{\delta}Z_a \tilde{\delta}Z_b. \end{aligned} — (2)$$

The circularity in the definition of \mathcal{F} may be avoided by expanding the vector $\tilde{\delta}^2 Z_a$ as $\tilde{\delta}^2 Z_a = \Lambda_1 Z_a + \Lambda_2 \tilde{\delta}Z_a + \Lambda_3 \tilde{\delta}Z_a + \Lambda_4 \tilde{\delta}\tilde{\delta}Z_a$, so that the Λ_i may be found algebraically, and observing that $\Lambda_4 = 2\mathcal{F}$.

The covariant derivative $\tilde{Z}_{a;b}$ may be found by observing that it satisfies the following equation, obtained from (1),

$$\tilde{\delta}^2 Z_{a;b} = \dot{\sigma} Z_{a;b} + \ddot{\sigma} Z_{a;b} Z_b.$$

We find a particular integral of this,

$$Z_{a;b} = \alpha Z_a Z_b + 2\beta Z_{(a} \tilde{\delta}Z_{b)} + \gamma \tilde{\delta}Z_a \tilde{\delta}Z_b,$$

with

$$\gamma = Z^a \tilde{\delta}Z_a$$

$$\beta = -\frac{1}{3} \tilde{\delta}\gamma$$

$$\alpha = \frac{1}{6} \tilde{\delta}^2 \gamma - \frac{1}{2} \gamma \dot{\sigma}.$$

The general solution is obtained by adding the complementary function $\gamma^c{}_{ab} Z_c$ to this, and the demand that the covariant derivative of the metric vanish leads to the requirement that $\gamma^c{}_{ab}$ be zero.

The Riemann tensor may be calculated from the second derivative of Z_a using the definition $Z_{a;[bc]} = \frac{1}{2} R^d{}_{abc} Z_d$

and $\tilde{\delta}$, $\tilde{\gamma}$ and $\tilde{\delta}\tilde{\delta}$ applied to this.

It is seen to have

It is seen to have vanishing traces and be left flat, and may be expressed in terms of its components, $\tilde{\Psi}_o = R_{abcd} L^a \tilde{M}^b L^c \tilde{M}^d$, etc. as $\tilde{\Psi}_o = Z^a \gamma_a$ from which the other components are determined by

$$\begin{aligned}\delta \tilde{\Psi}_o &= 4 \tilde{\Psi}_1 \\ \delta \tilde{\Psi}_1 &= 3 \tilde{\Psi}_2 + \dot{\sigma} \tilde{\Psi}_0 \\ \delta \tilde{\Psi}_2 &= 2 \tilde{\Psi}_3 + 2\dot{\sigma} \tilde{\Psi}_1 \\ \delta \tilde{\Psi}_3 &= \tilde{\Psi}_4 + 3\dot{\sigma} \tilde{\Psi}_2 \\ \delta \tilde{\Psi}_4 &= 4\dot{\sigma} \tilde{\Psi}_3.\end{aligned}$$

A natural choice of coordinates in \mathcal{X} -space is

$$\begin{aligned}p &= Z \\ q &= -\tilde{\gamma}Z \\ x &= \tilde{\gamma}\tilde{\gamma}Z + Z \\ y &= \tilde{\gamma}Z\end{aligned}$$

for some fixed values of $\tilde{\gamma}$ and $\tilde{\gamma}$. The metric then takes the "already linearised" form $g_{ab} dz^a dz^b = 2dpdx + 2dqdy + 2\tilde{\gamma}dq^2 - 2Ddpdq + 2(\epsilon-1)dp^2$. It is then found that

$$\begin{aligned}2\tilde{\gamma} &= (\tilde{\gamma}^2 Z)_x \\ D &= (\tilde{\gamma}^2 Z)_{,y} = (-\tilde{\gamma}\tilde{\gamma}^2 Z)_x \\ 2(\epsilon-1) &= (\tilde{\gamma}\tilde{\gamma}^2 Z)_{,y},\end{aligned}$$

so that there exists a function $\Theta(p, q, x, y)$ with

$$\Theta_{,x} = -\frac{1}{2}\tilde{\gamma}^2 Z, \quad \Theta_{,y} = \frac{1}{2}\tilde{\gamma}\tilde{\gamma}^2 Z$$

and

$$\begin{aligned}g_{yy} &= -2\Theta_{,xx} \\ g_{pq} &= 2\Theta_{,xy} \\ g_{pp} &= -2\Theta_{,yy}.\end{aligned}$$

The metric is now in Plebanski's second form for the \mathcal{X} -space metric, and the remaining equation which he requires Θ to satisfy is for us an identity:

$$\Theta_{xx} \Theta_{yy} - (\Theta_{xy})^2 + \Theta_{xp} + \Theta_{yp} = 0.$$

The work described here is due to M.Ko, E.T.Newman and K.P.Tod and further details may be found in " \mathcal{X} -space - A New Approach" in the proceedings of SOASST at Cincinnati, June 1976, to be published by Plenum Press.

Some Global Properties of Massless Free Fields.

Several conceptual problems regarding the association of a massless free field $\Psi_{\alpha \dots \beta}(x)$ with a set of $n+1$ twistor functions $\{\psi_r(u^\alpha, v^\beta) : 0 \leq r \leq n\}$ can be resolved by the use of induced representations. In particular, one can understand the origin of the "Gargin phenomenon"¹ and to what extent a pair of twistors passing through x can be regarded as determining a spin-frame at x .

The Bundle of Twistor Dyads:

Let \mathcal{N} be the $O(2,4)$ null cone. Regard M^4 as a plane section of this via the embedding

$$\sigma(x^\mu) = (x^\mu, \frac{1}{2}(1+x_\nu x^\nu), \frac{1}{2}(1-x_\nu x^\nu)). \quad (1)$$

Let $\tilde{\mathcal{M}} = \mathcal{N}/\mathbb{R}^+$ be the quotient space obtained by identifying positive multiples of points on \mathcal{N} . This is a double cover of \mathcal{M} , the conformal compactification. There is a 1:1 correspondence between simple twistor birectas and points of $\mathbb{P}\mathcal{N}$. We fix this uniquely by requiring that if $U^\alpha = (x^{n\alpha}, \xi_n, \bar{\xi}_n)$ and $V^\alpha = (ix^{n\alpha}, \eta_n, \bar{\eta}_n)$ pass through $x^{n\alpha} = x^{n\alpha'}$, then $UV \mapsto (\xi_n \eta_n) \sigma(x^\mu)$. Let $\mathcal{A}^* = \{(u, v) : u \neq \lambda v, \lambda \in \mathbb{C}\}$. Define $\pi : \mathcal{A}^* \rightarrow \mathbb{P}\mathcal{N}$ by $\pi(u, v) = p(UV)$. Then the set $\mathcal{A} = \pi^{-1}(\mathcal{N})$ is called the bundle of twistor dyads over \mathcal{N} . Note that for any $y \in \mathcal{N}$, $\pi^{-1}(y) \cong SL(2, \mathbb{C})$, so that \mathcal{A} can be made into a principal bundle by defining $U_\alpha^\alpha = U^\alpha, U_{\alpha'}^\alpha = V^\alpha$ $(2a)$

$$U_\alpha^1 \circ \alpha = U_{\alpha'}^1 \bar{\alpha}_{\alpha'}^{B_1^1}, \alpha \in SL(2, \mathbb{C}) \quad (2b)$$

Similarly, \mathcal{A} becomes an $\mathbb{R}^+ \otimes SL(2, \mathbb{C})$ bundle over $\tilde{\mathcal{M}}$.

Representations.

The utility of \mathcal{A} lies in the fact that its elements parametrize the coset space $SU(2,2)/O_C$, where

$$O_C = \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} : A = A^T \right\} \quad (3)$$

are the special conformal transformations. Now the stabilizer of $[U(v)] \in \tilde{\mathcal{M}}$ in $SU(2,2)$ is $\mathcal{W} = \{\mathbb{R}^+ \otimes SL(2, \mathbb{C})\} \otimes O_C$. For any representation of $SU(2,2)$ constructed by inducing with a representation T of \mathcal{W} such that $0 \in \ker(T)$, the "Mackey functions"² will be constant on the cosets of O_C . They may therefore be realized as functions on \mathcal{A} , i.e. as vector-valued functions of $U + V$. If the vector space happens to be the symmetric spinors with n primed indices, then any such function will have $n+1$ independent components $\{\psi_r(u, v) : 0 \leq r \leq n\}$, each component transforming as a scalar under $SU(2,2)$.

The massless free fields of helicity $\frac{1}{2}$ are obtained by inducing with the representation

$$T(r, d, A) : \underbrace{\mathbb{S}_{\alpha'_1 \dots \beta'_n}}_n \longrightarrow r^{n+2} \bar{\alpha}_{\alpha'_1}^{c'_1} \dots \bar{\alpha}_{\alpha'_r}^{c'_r} \mathbb{S}_{c'_1 \dots c'_r} \quad (4)$$

A spinor density of conformal weight $-\frac{1}{2} - 1$ on M^4 is obtained from the resulting $\{\Psi_r(U, V)\}$ by defining:

$$\Psi_{r, \dots, \alpha, \dots, \alpha}(x) \equiv \Psi_r(\hat{U}(x), \hat{V}(x)), \text{ where } (5a)$$

$$\hat{U}(x) = (i x^{0'}, i x^{1'}, \dots, i x^{n-r}, 0, 1), \quad \hat{V}(x) = (-i x^{0'}, -i x^{1'}, \dots, -i x^{n-r}, 0, 0) \quad (5b)$$

To obtain a solution to the free field equations, it is necessary to impose an additional condition, the simplest such being $\frac{\partial^2 \Psi_0}{\partial U^{\alpha} \partial V^{\beta}} = 0$, in the analytic case. By construction this will behave properly under $SU(2,2)$.

Spin Frames and the "Grgin Phenomenon"

S may also be regarded as a bundle over M .

In this case, the structure group is the set of all 2×2 complex matrices with real determinant ($\neq 0$) and is disconnected. Since S itself is connected, there are no global cross-sections of $S \rightarrow M$. (Local cross-sections will exist over any simply connected region, cf. (5b).) Thus a spinor-valued function on S (assuming its support contains a closed null geodesic) cannot possibly be thought of as an ordinary spinor field on M , since the bundle of spin-frames (when it exists) is always trivial. Note that this result holds independently of whether or not any field equations are satisfied by the Minkowskian spinor field defined by (5a).

Another way to see this: S is a 4-fold covering of $C_f^+(1,3)/OC =: P_0$. Now $P_0 \rightarrow M$ has structure group $R^+ \otimes L_f^+$ and does admit a global cross-section. (Thinking of M as $U(2)$, it is easy to construct a cross-section for $C_f^+(1,3) \rightarrow M$.) Another, "natural", bundle with the same structure group exists on M as well: the subbundle C' of the frame bundle consisting of tetads orthonormal w.r.t. some metric conformally related to g_M . M has spinor structures $\Rightarrow C'$ is trivial $\Rightarrow C' \cong S$. While there is no canonical isomorphism, there is an obvious one: if $T(x^\mu)$ denotes translation by x^μ in M^4 , and $T(x^\mu)$ is its image in P_0 , identify $T(x^\mu)$ with the standard Minkowskian α -tetrad at x^μ . This is enough to fix the isomorphism. One can now identify any vector-valued function on P_0 (obtained by inducing a rep. of $C_f^+(1,3)$ by $R^+ \otimes L_f^+$) with an appropriate conformally weighted density on M (as opposed to M^4).

Now an element $\{\mathcal{E}_r(x)\} \in C' \cong S$ is covered by precisely 4 elements of S : if (U, V) is one such, the others are $\{(J^i U, J^i V); i=1,2,3\}$, where $J = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

generates the center of $SU(2,2)$. Consider a set of functions $\{\Psi_r(U, V)\}$ obtained according to (4) above. In order that one be able to interpret these as spinors on M it is necessary + sufficient that $\Psi_r(JU, JV) = \Psi_r(U, V)$, $\forall r$. But, by virtue of the homogeneity,

$$\Psi_r(JU, JV) = (i)^{-n-2} \Psi_r(U, V), \quad (6)$$

and a equality holds only for $n \equiv 2 \pmod{4}$. Note that in this case the center acts trivially so that we actually have a rep. of $C_f^+(1,3)$ + didn't need spinor indices in the first place.

David Leinen

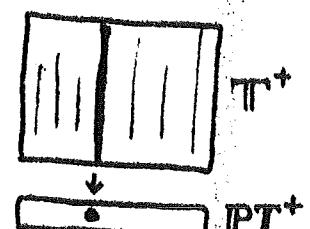
L.E. Grgin, thesis, Syracuse, 1966.

2: See, for example, Niederer + O'Raifeartaigh, Forts. der Phys., 22, 111 (1974).

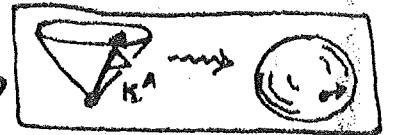
question: how do you know whether $\Psi(x^a, y^a, z^a)$ describes a hadron, or three massless particles, or a lepton and a massless particle? Elements of $H^1(Q, \mathcal{O})$, $H^3(Q, \mathcal{O})$ and $H^2(Q, \mathcal{O})$ are really quite different kinds of animal, after all! But the detailed implementation of this idea has proved elusive — not the least problem being to understand why the spin-statistics relation should hold for many-particle states.

Perhaps the most primitive problem, in trying to push forward with (iii), lies in the fact that a twisted photon is only half a photon, namely the left-handed half. Thus, the Ward-Sparling construction gives a deformation of $Q = T^+$ starting from a twistor function $f(Z^a)$, homogeneous of degree zero ($f \in H^1(T^+, \mathcal{O})$ or $f \in H^1(PT^+, \mathcal{O}(0))$), to give a left-handed photon. Of course a right-handed photon can be produced by using $f(W_a)$, of degree zero (or, conceivably, by using $f(W_a)$). But such would be to defeat the purpose of the economy of the twistor description (i). Simply changing the helicity quantum number (or any other quantum number) should not involve us in changing the space Q over which the twistor function is defined. Thus, maintaining the general programme (i) — (iii) seems to lead us to the view that some form of deformation of T^+ must be possible, which effectively encodes the information provided by an element $f \in H^1(T^+, \mathcal{O})$, when f is homogeneous of degree -4 (i.e., in effect, $f \in H^1(PT^+, \mathcal{O}(-4))$). My suggestion for this is as follows.

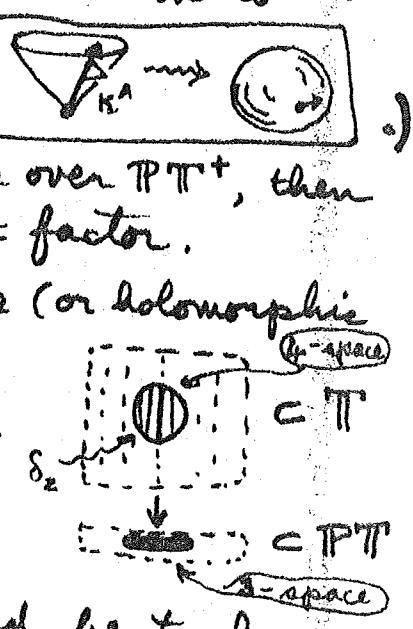
Consider, first, the standard Ward twisted photon. This is obtained by deforming the bundle, where the base space remains the flat PT^+ and the fibre $\mathbb{C} - \{0\}$. Furthermore, the Euler operator $\partial_z = T_z$ remains globally defined. However, with non-trivial twisting, the forms $DZ = \frac{1}{24} dz_1 dz_2 dz_3 dz_4$ and $DZ = \frac{1}{6} \sum dz_i dz_j$ are not well-defined; indeed, globally defined analogues of these forms do not exist (at all) on a non-trivially twisted photon. There is



PT^+

a good reason for this. In the case of flat T , there is a canonical way of representing a twistor Z^α (up to the fourfold ambiguity $\pm Z^\alpha, \pm iZ^\alpha$) in terms of PT , namely as the projective twistor $Z \in PT$, together with a 3-form at Z in PT . This 3-form lifts into T along the fibre over Z ; where it agrees with $\mathcal{D}Z$ defines us the point Z^α (or iZ^α , or $-Z^\alpha$, or $-iZ^\alpha$). Thus, if $\mathcal{D}Z$ is canonically known in the bundle, then the complete bundle structure is determined uniquely in terms of PT , \therefore no twisting in the "photon" can occur. (This construction is the analogue of how one defines a spinor κ^A (up to sign) in terms of the celestial sphere \rightarrow \mathcal{D}Z exists globally in a bundle over PT^+ , then it must be unique up to an overall constant factor.

Suppose, generally, we have a bundle (or holomorphic fibration ... ? ...) just locally, which is just a complex 4-space over a complex 3-space with fibre a complex 1-space. Then all we know locally in the 4-space is a direction field S_z (1-foliation). To know $\mathcal{D}Z$ in the 4-space would be to know rather more structure than S_z . Being a 3-form, $\mathcal{D}Z$ is orthogonal to (i.e. annihilates) precisely one complex direction and so serves to define S_z . (I shall always assume $\mathcal{D}Z$ to be "a $\mathcal{D}Z$ ".) Now $\mathcal{D}Z$ also defines a volume 4-form DZ by $d\mathcal{D}Z = 4 DZ$. Furthermore $\mathcal{D}Z$ and DZ together define the Euler operator γ_z , roughly speaking by " $\gamma_z = \mathcal{D}Z \div DZ$ ", or more precisely by $\mathcal{D}Z = \gamma_z \wedge DZ$. Conversely, this relation shows that $\mathcal{D}Z$ is determined by the pair (γ_z, DZ) . To know one or the other of γ_z, DZ is, by itself, not sufficient to determine $\mathcal{D}Z$, but the two together are equivalent to $\mathcal{D}Z$. Furthermore, assuming that γ_z is to point along S_z (where S_z is given), there is precisely as much information in γ_z locally as there is in DZ . Each provides us with a kind of local



scaling, but it is a "homogeneity degree 4" scaling for DZ and a "homogeneity degree 0" scaling for Υ_z . In a sense, DZ and Υ_z seem to be sorts of duals to one another.

We can regard the Ward photon as arising when we retain only the Υ_z scaling and throw out DZ . Let us try to do the "dual" thing and retain DZ while throwing out Υ_z . I shall proceed in a fairly explicit way, assuming that two "coordinate" patches are given, where a standard twistor description is given in each patch, with X^α on the left and Z^α on the right.

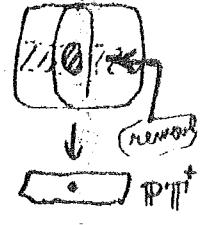
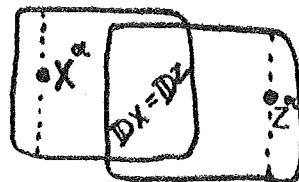
I am assuming there is no monkey-business in the base space, so $X^\alpha \propto Z^\alpha$ may be assumed on the overlap region. (The fibres in each half are given when X^α or Z^α is held constant up to proportionality.) The hypothesis is now $IDX = IDZ$ on the overlap (instead of Ward's $\Upsilon_x = \Upsilon_z$), i.e. $dDx = dDz$, i.e. $d\{Dx - Dz\} = 0$. But $\delta_x = \delta_z$ so $Dx \propto Dz$. Put $Dx = \{1 + f(z)\}Dz$; then $d\{f(z)Dz\} = 0$ (\pm for Z^α) which holds iff $f(z)$ is homogeneous of degree -4. The transition relation is then:

$$x = \{1 + f(z)\}^{-1} z$$

--- A

(This appears to be a particular case of a Sparling-Ward (cf. T.N.I.) construction given earlier for massless fields of any helicity.)

We run into trouble owing to branch points arising from the fourth root unless we exclude an extended region about the origin of each fibre. But things are O.K. near ∞ on the fibres. Thus, I envisage the fibres as being like \mathbb{C} with some bounded (probably connected) region removed (i.e. biholomorphic to $\mathbb{C} - \{z \mid |z| > 1\}$). Note that Υ_z is not preserved by the patching, but related by $\Upsilon_x = (1 + f(z))\Upsilon_z$ (which does strange things to the notion



of a homogeneous function).

The intention is to regard $f(z)$ as a twistor function for a right-handed photon. But, in accordance with (ii), such $f(z)$ is really describing an element of a sheaf cohomology group. Thus we need to check that (A) is appropriately cohomological. Suppose we have a covering of the base space (say $P\mathbb{P}^1$) by a number of open sets U_i . Then on each $U_i \cap U_j$ we need $f_{ij} = -f_{ji}$, such that on each $U_i \cap U_j \cap U_k$, $f_{ik} = f_{ij} + f_{jk}$ (cocycle condition). If we use standard twistor "coordinates" ζ for the entire base space, then the f_{ij} will simply be homogeneous functions of ζ , of degree -4 . Thus, $f_{ij}(\zeta) = -f_{ji}(\zeta)$, $f_{ik}(\zeta) = f_{ij}(\zeta) + f_{jk}(\zeta)$. The cocycle defining the right-handed photon can be taken to be just this collection of homogeneous functions. We piece together our bundle by taking "coordinates" ζ_i for the portion lying above U_i , where on each overlap (say, the portion of the bundle lying above $U_i \cap U_j$) we have $\zeta_i = \{1 + f_{ij}(\zeta)\}^{1/4} \zeta$. We must first check the consistency of this with $f_{ij} = -f_{ji}$. Avoiding messy indices, we have (A): $k = \{1 + f(\zeta)\}^{1/4} \zeta$ and $\zeta = \{1 + g(k)\}^{1/4} k$, and we need to check that $g = -f$. Now $\zeta = \{1 + g(k)\}^{1/4} \zeta = \{1 + f(\zeta)\}^{1/4} \zeta$, and we require $(1 + g(\zeta))(1 + f(\zeta)) = 1$; that is, $1 = (1 + g(\{1 + f(\zeta)\}^{1/4} \zeta))(1 + f(\zeta))$ (because of the -4 homogeneity of g), so $g(\zeta) = -f(\zeta)$ as required. Next we check the compatibility of (A) with $f_{ik} = f_{ij} + f_{jk}$. Again avoiding indices we have, on the triple intersection $U_i \cap U_j \cap U_k$, relations $\zeta = \{1 + p(k)\}^{1/4} k$, $k = \{1 + f(\zeta)\}^{1/4} \zeta$, $\zeta = \{1 + q(\zeta)\}^{1/4} \zeta$. For consistency, we require $\{1 + p(\{1 + f(\zeta)\}^{1/4} \zeta)\} \cdot \{1 + f(\zeta)\} = 1 + q(\zeta)$, i.e. $\{1 + \frac{p(\zeta)}{1 + f(\zeta)}\} \cdot \{1 + f(\zeta)\} = 1 + q(\zeta)$, i.e. $q(\zeta) = p(\zeta) + f(\zeta)$, as required. We need also to show that if the cocycle is a coboundary, i.e. if $f_{ij} = f_j - f_i$ for each i, j , then the bundle is the same as for flat twistor space (f_i being holomorphic throughout U_i). For this we set $\zeta = \{1 + f_i(\zeta_i)\}^{1/4} \zeta_i$ and find that the ζ , so constructed, is defined globally over the whole bundle. The compatibility over each $U_i \cap U_j$ follows by a calculation which is basically the same as the ones given above. Finally we need the fact that if the

bundle is the same as that for flat twistor space (where here and above "the same" must be suitably interpreted in terms of "analytically extendible to" — owing to the gaping holes in the fibres), then the $\{f_i\}$, f -cocycle is a coboundary. To prove this, we simply reverse the above argument. The flatness implies the existence of a global \tilde{z} for the bundle. The local \tilde{z} for each patch (above U_i) must be related to z by a formula $\tilde{z} = \{w_i(z)\}^{\frac{1}{n}}$ from which follows (by the reverse of the above calculation) $f_{ij} = f_j - f_i$ above $U_i \cap U_j$.

The argument just given effectively shows that although (A) appears to be a highly non-linear relation, the system of bundles constructed has nevertheless, a linear structure (given by simply adding the sheaf cohomology group elements). This may be contrasted with Sparling's method of patching together non-linear gravitons: $\star = \exp\{\partial_z f(z) \partial_z\} \tilde{z}$ (where $\partial_z = \partial_{\bar{z} z^*}$, $\star = I^{(n)}$); see TN1 introduction. For this, the cocycle condition fails. This is a manifestation of the very non-linearity of the non-linear graviton. On the other hand, the "4" in the above construction is not essential for linearity and could be replaced by other powers; i.e. f of homogeneity $= n$, with $\star = (1 + f(z))^{1/n} \tilde{z}$. When $n = 1, 0, -1, -2, -3, \dots$ this is implicit in the Sparling-Ward construction (see R.W. in TN1). However, when $n < 0$ the global structure of the fibres would have to be something different (because $(1 + z^{-n})^{1/n} z \rightarrow 1$ as $|z| \rightarrow \infty$ if $n < 0$, whereas $(1 + z^{-n})^{1/n} z \sim z$ as $|z| \rightarrow \infty$ if $n > 0$). In any case, the motivation from "preservation of Dz" exists only when $n = 4$.

A direct construction of the right-handed Maxwell field $\tilde{\Phi}_{\alpha\beta}^+$ (satisfying $\tilde{\Phi}_{\alpha\beta}^+ \tilde{\Phi}_{\gamma\delta}^+ = 0$) from the bundle structure has not yet emerged. But the most serious problem is that of fitting the right-handed and left-handed ways of deforming η^{+-} together into one bundle...? One possibility for doing this is the method suggested by L.P.H. in the following article. However it is unclear, as yet, how to extract left- and right-handed Maxwell fields which do not interact with one another. If this problem can be resolved, then it might (among other things) suggest an analogous approach to an ambidextrous graviton!

Some TN2 ERRATA (not corrected in all editions) p. 8, l. 5 ... β is a cocycle ... ~ Roger Penrose
 p. 9: As Richard Jozsa points out, the "combined covering" ought to have been a refinement $\{\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{U}_4\}$, where $\hat{U}_1 = U_1 \cap \tilde{U}_1$, $\hat{U}_2 = U_1 \cap \tilde{U}_2$, $\hat{U}_3 = U_2 \cap \tilde{U}_1$, $\hat{U}_4 = U_2 \cap \tilde{U}_2$, so that $(\hat{f}_{12}, \hat{f}_{13}, \dots, \hat{f}_{34}) = (f, f, f-f, f+f, f, f)$ (on l. 8) and on l. 11 should be: $(0, f, f, f, f, 0)$.

A Generalized Right-Handed Photon Construction

One of the most intriguing ideas of twistor theory is that it should eventually be possible to see how to incorporate the quantum states of elementary particles "intrinsically" into the geometry of twistor space. Roughly speaking, the idea runs as follows. Initially, a quantum state is thought of as being represented in terms of certain classes of holomorphic functions defined on some (background) twistor geometry. But then these functions are reinterpreted in such a way that they appear in an active guise: they are incorporated into the structure of a set of holomorphic coordinate transition functions which are used to define the complex analytic structure of a new "curved" twistor space. The new curved twistor space now carries built directly into its structure the information of the original quantum state.

While the general methodology for such a scheme remains yet to be defined, nevertheless one of the requirements which has tentatively been adopted in this connection is that the entire construction should be "cohomologically natural": By this what is meant is that the deforming of the twistor space should depend only on the sheaf cohomology class of the twistor function that represents the state of the particle. Thus particle states are, in effect, interpreted as being elements of twistor sheaf cohomology groups⁽¹⁾, and these in turn induce finite deformations of twistor space itself.

The right-handed photon construction⁽²⁾ can be given a natural cohomological description in a setting of considerable generality. The background space on which the photon is defined (and which the photon subsequently "twists up") can be essentially any curved twistor space (or generalized twistor space) on which the Euler homogeneity operator can be globally defined. Suppose the twistor space is covered by coordinate patches U_i , with coordinates Z_i^α , and transition functions H_{ij}^α defined by $Z_i^\alpha = H_{ij}^\alpha(Z_j^\alpha)$ in U_{ij} ⁽³⁾. The condition for the existence of a global Euler operator is that H_{ij}^α should be homogeneous of degree one. To see this note that in U_{ij} one has $\partial/\partial Z_j^\alpha = (\partial H_{ij}^\alpha/\partial Z_j^\alpha)\partial/\partial Z_i^\alpha$; from which it follows, transvecting this expression with Z_j^α , that $Z_j^\alpha \partial/\partial Z_j^\alpha = Z_j^\alpha (\partial H_{ij}^\alpha/\partial Z_j^\alpha) \partial/\partial Z_i^\alpha$. The Euler operator ∇_j in the patch U_j is defined by $Z_j^\alpha \partial/\partial Z_j^\alpha$. Requiring that ∇_j agrees with ∇_i in U_{ij} now gives the desired result $Z_j^\alpha \partial H_{ij}^\alpha / \partial Z_j^\alpha = H_{ij}^\alpha$.

The transition functions $H_{ij}(Z_j)$ are also, of course, required to satisfy the usual compatibility conditions $H_{ij} \circ H_{jk} \circ Z_k = H_{ik} \circ Z_k$ on the triple overlap region U_{ijk} , as well as the inverse

relations $H_{ij} \circ H_{ji} \circ Z_i = Z_i$, on U_{ij} .⁽⁴⁾ Denoting this background space \mathcal{M} now consider an element g in the cohomology group $H^1(\mathcal{M}, \Omega_4)$, where Ω_4 is the sheaf of germs of holomorphic functions homogeneous of degree minus four. The element g defines a photon state relative to the background provided by the space \mathcal{M} . A representative for g will consist of a certain collection of holomorphic functions $g_{ij}(Z_j)$ defined on the intersection regions U_{ij} . These functions must satisfy the cocycle condition. If Z_j is the coordinate of the point p in the patch U_j then the cocycle condition is that $g_{ij}(Z_j) + g_{jk}(Z_k) + g_{ki}(Z_i) = 0$ for each $j \in ijk$.⁽⁵⁾ The functions g_{ij} must also satisfy the skew-symmetry condition, which is $g_{ij}(Z_j) + g_{ji}(Z_i) = 0$. (Note that if $g_{ij}(Z_j)$ is modified by the addition of a term $-g_i(Z_i) + g_j(Z_j)$, where $g_i(Z_i)$ is a holomorphic function definable over the whole of U_i , then both the cocycle condition and the skew-symmetry condition are left invariant. Such a term is a coboundary term. Two cocycles which differ by a coboundary term define the same element in the cohomology group.)

Now a new space $\mathcal{M}(g)$ [\mathcal{M} , twisted by g^*] will be defined by the transition relations $Z_i = [1 + g_{ij}(Z_j)]^{1/4} H_{ij}(Z_j)$. If this patching is denoted by $Z_i = \tilde{H}_{ij}(Z_j)$, then it must be demonstrated that (a) \tilde{H}_{ij} correctly satisfies the compatibility conditions and the inverse relations, and (b) \tilde{H}_{ij} is cohomologically natural, i.e. if g_{ij} is modified by the addition of coboundary terms, the complex analytic structure of $\mathcal{M}(g)$ is leftuntainted (so that the manifold $\mathcal{M}(g)$ depends only on the sheaf cohomology class to which g_{ij} belongs).

Proof of (a). We must prove the compatibility conditions $\tilde{H}_{ij} \circ \tilde{H}_{jk} \circ Z_k = \tilde{H}_{ik} \circ Z_k$, and the inverse relations $\tilde{H}_{ij} \circ \tilde{H}_{ji} \circ Z_i = Z_i$. Note that providing the compatibility conditions hold true the inverse relations are equivalent to $\tilde{H}_{ij}(Z_i) = Z_i$. Since $\tilde{H}_{ii}(Z_i) = Z_i$, and $g_{ii} = 0$ (by the skew condition), it follows that the inverse relations will certainly hold for \tilde{H}_{ij} once the compatibility conditions have been ensured. Now if we are given $Z_i = [1 + g_{ij}(Z_j)]^{1/4} H_{ij}(Z_j)$ together with $Z_j = [1 + g_{jk}(Z_k)]^{1/4} H_{jk}(Z_k)$ we wish to substitute so as to obtain the composition $Z_i = [1 + g_{ik}(Z_k)]^{1/4} H_{ik}(Z_k)$:

$$Z_i = [1 + g_{ij}(Z_j)]^{1/4} H_{ij}(Z_j)$$

$$= [1 + \{1 + g_{jk}(Z_k)\}^{-1} g_{ij} \circ H_{jk} \circ Z_k]^{1/4} [1 + g_{jk}(Z_k)]^{1/4} H_{ij} \circ H_{jk} \circ Z_k$$

$$\begin{aligned}
 &= [1 + g_{jk}(z_k) + g_{ij}(z_j)]^{1/4} H_{ik}(z_k) \\
 &= [1 + g_{ik}(z_k)]^{1/4} H_{ik}(z_k) .
 \end{aligned}$$

Note that the last step of the proof employs the cocycle condition on $g_{ij}(z_j)$. \square

Proof of (b). Suppose, for simplicity, that g_{ij} is cohomologically trivial, i.e. a pure coboundary, of the form $g_{ij} = -g_i(z_i) + g_j(z_j)$. We must show under these circumstances, that $\mathcal{M}(g)$ is equivalent to \mathcal{M} as a complex manifold. Thus there must exist a set of holomorphic functions φ_i on U_i such that $z_i = \varphi_i^{-1} \circ H_{ij} \circ \varphi_j(z_j)$, where $\varphi_i^{-1} \circ \varphi_i = 1$. One obtains:

$$\begin{aligned}
 z_i &= [1 - g_i(z_i) + g_j(z_j)]^{1/4} H_{ij}(z_j) \\
 &= [1 - \{1 + g_j(z_j)\}^{-1} g_i(z_i)]^{1/4} [1 + g_j(z_j)]^{1/4} H_{ij}(z_j) \\
 &= [1 - g_i \circ H_{ij} \circ \varphi_j(z_j)]^{1/4} H_{ij} \circ \varphi_j(z_j) ,
 \end{aligned}$$

where the function φ_j has been defined by $\varphi_j(z_j) = [1 + g_j(z_j)]^{1/4} z_j$. If we define similarly $\varphi_j^{-1} = [1 - g_j(z_j)]^{1/4} z_j$, it can be verified with no difficulty that $\varphi_j^{-1} \circ \varphi_j = 1$. Referring back now to the last step in the calculation above, the desired result $z_i = \varphi_i^{-1} \circ H_{ij} \circ \varphi_j(z_j)$ is obtained. Thus it has been shown that if g_{ij} is cohomologically trivial, then $\mathcal{M}(g) = \mathcal{M}$.

More generally, one can show, using essentially the same sort of argument, that if g_{ij} is replaced with $g_{ij} - g_i + g_j$ then for the new transition functions one obtains the result $z_i = \varphi_i^{-1} \circ H_{ij} \circ \varphi_j(z_j)$, with φ_j defined as before. These transition relations give the same complex manifold as $z_i = H_{ij}(z_j)$, and thus it follows, as desired, that $\mathcal{M}(g)$ is completely independent of the choice of representative cocycle for the cohomology element g . \square

Remarks. The algebra involved in the proofs of (a) and (b) is for all practical purposes the same as that involved in the flat space construction for the right-handed photon, just modified a bit for the accommodation of the more general structure of the background space.

The non-linear graviton⁽⁶⁾, the left-twisted photon⁽⁷⁾, and the combined non-linear graviton-left-twisted photon are all examples

of spaces for which the coordinate transition functions H_{ij} are homogeneous of degree one. These spaces accordingly all qualify as examples of background spaces acting on which the construction described above can be initiated.⁽⁸⁾

- Lane Hughston

Notes

- 1) For the basic ideas of twistor sheaf cohomology, see R.P.'s article in *TIN2*, and R. Jozsa's Oxford MSc. thesis.
- 2) See R.P.'s article in this issue.
- 3) The notation here is: $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, etc.
- 4) Note that from here on twistor indices are suppressed, and only cohomology indices are retained. Note that at no stage is it really necessary to assign values to the cohomology indices; their role is essentially combinatorial. It would be interesting in this connection for someone to try to devise some sort of an "abstract index calculus" for cohomology.
- 5) The notation can be simplified here if we say by convention that the index on the coordinate appearing in the argument of a cochain should always be the same as the last index on the cochain itself (suggested by N.M.J.W.). With this convention g_{ij} stands for $g_{ij}(z_j)$, and similarly the cocycle condition reduces to $g_{ij} + g_{jk} + g_{ki} = 0$, for $p \in U_{ijk}$. [Another useful convention is to introduce a "restriction" map ∂_i which restricts the domain of whatever it's applied to to its intersection with U_i . Thus the cocycle condition can be written $\partial_k g_{ij} + \partial_i g_{jk} + \partial_j g_{ki} = 0$, or simply, $\partial_k g_{ij} = 0$ (suppose the antisymmetrizer carries no combinatorial factor with it). Similarly g_{ij} is a coboundary if \exists a g_i such that $g_{ij} = \partial_i g_j$, and because of $\partial_i \partial_j$ it follows obviously that coboundaries are always cocycles, and so forth.] For other purposes it is useful to refer a cohomological relation entirely to one patch. For such purposes the coordinate transition functions must be used. Referred to the patch U_j , for example, the cocycle condition above is expressed $g_{ij}(z_j) + g_{jk} \circ H_{kj}(z_j) + g_{ki} \circ H_{ij}(z_j) = 0$.
- 6) R.P., in *Gen. Rel. and Grav.*, Vol. 7, No. 1, 31 (1976).
- 7) See R.Ward's article in *TIN1*, and George Sparling's "Twistor Quadrille".
- 8) The construction doesn't actually even require that the background space be a simple Z^d type of twistor space. Various analogs exist, in the same spirit, for product twistor spaces, and also for higher-valence twistor spaces.

In TN 2 it was suggested that the electromagnetic potential could be represented in a 2-twistor formalism via:

$$\hat{\Phi}_{AA'} = \frac{1}{(2\pi i)^4} \oint \left(Z_A' \frac{\partial}{\partial Z^A} - X_A' \frac{\partial}{\partial X^A} \right) F d^2 Z_n d^2 X \quad \text{--- (1)}$$

where $F(\bar{z}, \bar{x})$ is homogeneous of degree $(-2, -2)$. This automatically guarantees the Lorenz gauge condition $\nabla^A \hat{\Phi}_A = 0$. Now, using the above prescription we can generate the electromagnetic potential for the Coulomb field (due to a charge e) by choosing F to be:

$$F(\bar{z}, \bar{x}) = 2ie \log \left(\frac{\bar{\Pi}_X}{\bar{z}\bar{x}} \right), \text{ where the object } \bar{\Pi} \text{ is}$$

$$\text{a kind of "inverse angular momentum twistor" defined by } \bar{\Pi} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ so that } \bar{z}\bar{x} = (Z^0 X^3 - Z^1 X^2 - Z^2 X^1 + Z^3 X^0). \text{ The choice of contour for the integration is an } S' \times S'. \text{ We may further note that if we apply the (mass)² operator } \bar{z}\bar{x} \frac{\partial z}{\partial x} \text{ to the function } F(\bar{z}, \bar{x}) \text{ we get:}$$

$$\bar{z}\bar{x} \frac{\partial z}{\partial x} \left[\log \left(\frac{\bar{\Pi}_X}{\bar{z}\bar{x}} \right) \right] = \frac{1}{(\bar{z}\bar{x})^2} \rightarrow \int_{S' \times S'} \bar{z}\bar{x} \frac{\partial z}{\partial x} \left[\log \left(\frac{\bar{\Pi}_X}{\bar{z}\bar{x}} \right) \right] d^2 Z_n d^2 X = \int_{S' \times S'} \frac{d^2 Z_n d^2 X}{(\bar{z}\bar{x})^2}$$

$= 0$, where we integrate over an $S' \times S'$. Thus $\bar{z}\bar{x} \frac{\partial z}{\partial x} F$ is weakly zero, verifying the fact that here we have a massless field. Although the function F is an eigenstate of the (mass)² operator, it does not appear to be an eigenstate of the spin operator.

We can follow a similar procedure for the linearized Schwarzschild gravitational potential. We start with a slight variant of the formula in TN 2 for $h_{AA'BB'}$, i.e. $h_{AA'BB'} = \frac{1}{(2\pi i)^4} \oint Z(A' X_B') \frac{\partial^2 G}{\partial Z^A \partial X^B} d^2 Z_n d^2 X$, where again $G(\bar{z}, \bar{x})$ is homogeneous of degree $(-2, -2)$. Consider the function

$$G(\bar{z}, \bar{x}) = (-16\sqrt{2}iM) \frac{\bar{\Pi}_X}{\bar{z}\bar{x}} \log \left(\frac{\bar{\Pi}_X}{\bar{z}\bar{x}} \right), \text{ where } \bar{\Pi}_X = (Z^0 X^1 - Z^1 X^2 - Z^2 X^1 + Z^3 X^0)$$

and $M = \text{mass of source}$.

If we perform (2), using this G , where we again use an $S' \times S'$ contour, we obtain for the first component: $h_{00'00'} = 2M \left(\frac{1}{r} - \frac{3\bar{z}}{r^3} \right)$ where $\bar{z} = \frac{(y+i\bar{x})}{\sqrt{2}}$, which differs from the desired first component ($h_{00'00'} = \frac{2M}{r}$) by a gauge transformation of the form $h_{ab} \mapsto h_{ab} - \nabla_a \bar{z}_b - \nabla_b \bar{z}_a$, if we choose $\bar{z}_b = 2M(0, \frac{y}{r}, \frac{y}{r}, \frac{0}{r})$. So, modulo the gauge, we have the right first component, which, from the spherical symmetry of the problem, plus the symmetric nature of our original prescription (2), will lead to the other components being correct.

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