

Twistor Newsletter (no 4 : 1, April, 1977)

Mathematical Institute, Oxford, England

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COHOMOLOGY IS REALLY NEEDED!

The point of this note is to exhibit something that can be done with 1-cocycles that couldn't be done with the old contour integral formalism.

Recall the projective spinor integrals

$$\oint \frac{\pi \cdot d\pi}{(\pi \cdot \alpha)(\pi \cdot \beta)} = \frac{2\pi i}{(\alpha \cdot \beta)}, \quad \oint \frac{\lambda \cdot d\lambda \wedge \mu \cdot d\mu}{(\lambda \cdot \mu)^2} = 2\pi i$$

and consider the expression

$$\oint \frac{\pi \cdot d\pi \wedge \lambda \cdot d\lambda \wedge \mu \cdot d\mu}{(\pi \cdot \lambda)(\pi \cdot \beta)(\lambda \cdot \mu)^2} (\beta \cdot \lambda) \dots \dots \dots \quad (1)$$

Is there a non-trivial contour? At first sight it looks as though we could integrate over π , and then do the (λ, μ) integration. But this is erroneous: For λ has to pass through all possible values in the course of the (λ, μ) integration, including $\lambda = \beta$. But when $\lambda = \beta$, the π integral contour is pinched. There's no sense in which the numerator $(\beta \cdot \lambda)$ can be said to "cancel" this pinching. Numerators cannot affect the existence or non-existence of contours, which depend only on the homology of the integration space with singular regions removed.

2.

Instead, consider the following. Patch π space by $\mathcal{U}_1 = \{\pi \mid \pi \cdot \lambda \neq 0\}$,
 $\mathcal{U}_2 = \{\pi \mid \pi \cdot \beta \neq 0\}$, $\mathcal{U}_3 = \{\pi \mid \pi \cdot \gamma \neq 0\}$
and define

$$f_{12} = (\lambda \beta)(\pi \cdot \lambda)^{-1}(\pi \cdot \beta)^{-1} \text{ on } \mathcal{U}_1 \cap \mathcal{U}_2$$

$$f_{23} = (\beta \gamma)(\pi \cdot \beta)^{-1}(\pi \cdot \gamma)^{-1} \text{ on } \mathcal{U}_2 \cap \mathcal{U}_3$$

$$f_{31} = (\gamma \lambda)(\pi \cdot \gamma)^{-1}(\pi \cdot \lambda)^{-1} \text{ on } \mathcal{U}_3 \cap \mathcal{U}_1.$$

Note $\oint_1 f_{12} + \oint_2 f_{23} + \oint_3 f_{31} = 0$, $\boxed{\begin{matrix} \oint_i \text{ means} \\ \text{"restricted to } \mathcal{U}_i \end{matrix}}$
so that we have defined a 1-cocycle; hence the
branch-contour (see TN2) is well-defined,

as
$$\int_{Y_{12}} f_{12} \pi \cdot d\pi + \int_{Y_{23}} f_{23} \pi \cdot d\pi + \int_{Y_{31}} f_{31} \pi \cdot d\pi.$$

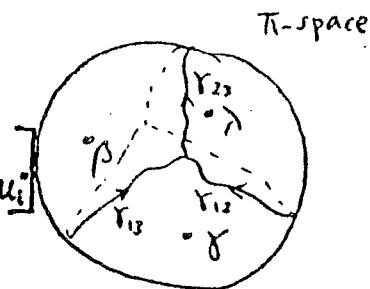
Provided $\lambda \neq \beta$, the branch contour can be transformed

to
$$\oint f_{12} \pi \cdot d\pi = \oint \frac{\beta \cdot ?}{(\pi \cdot \lambda)(\pi \cdot \beta)} \pi \cdot d\pi. \quad \dots \dots (2)$$

If $\lambda = \beta$ we cannot make this transformation - but the branch contour still exists perfectly well - there is no "pinching". So we find the cocycle to extend the expression (2) in a well-defined way to the case $\lambda = \beta$, and hence to give us a way of obtaining a non-trivial result for the integral (1).

The cocycle can only perform this extension because the numerator factor $\lambda \beta$ is what it is. If the β in this numerator is considered as a parameter, and is displaced infinitesimally, the cocycle structure disappears. This is in complete contrast to contour integration of functions in the old way, where we can always argue that small variation of parameters must preserve the existence of a contour.

So we have here the example of something manifestly finite (it's still compact integration) which only exists when a certain parameter condition is met. This is exactly what twistor integral theory needs! For instance, it's now logically possible that there can be a twistorial inner product for massive states which exists only if the two masses are equal and is then manifestly finite. This is impossible in the old scheme. Again, another major stumbling-block in twistor integral theory has been the fact that twistor diagrams involving both a massive in-state and a massive out-state couldn't have contours - contours were always pinched between the singularity at infinity of the in-state and the singularity at infinity of the out-state. But it now appears that when such integrals are in fact finite, there are enough numerator factors (involving the infinity twistor) to allow a reformulation in terms of cocycles in which these numerators "cancel" the pinching. It was in fact a simplified version of this problem that led me to consider the integral (1).



Branch-contour.
The section Y_{13} has
to avoid β and γ
but need not avoid λ
- similarly for the
other two segments.

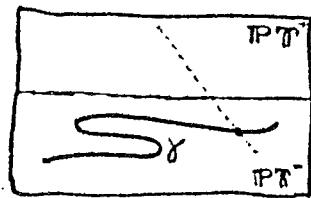
The Twisted Camel: or, how to get through the eye of a singularity.

I make, here, some remarks concerning the following type of situation, for a function of one or more twistor variables, where two pole singularities pinch together, but at the same time the pinching is compensated by the presence of a zero, i.e. we are concerned with integrating an expression of the form $\frac{A}{BC}$ over some contour, where some relevant common zero set of B and C is also a zero set of A .

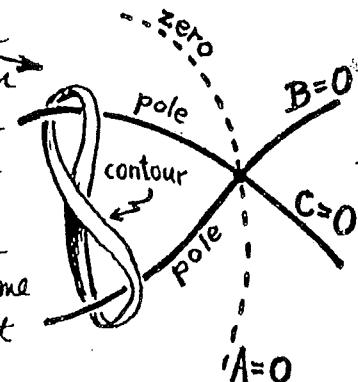
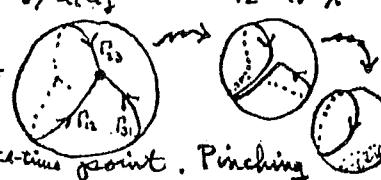
The question of how to handle the contour integration when the contour approaches the point of pinching has some interest in a number of different contexts. In particular, it has relevance to the integrals that A.H. has been considering (cf. T.N. 4, p 1). In a sense, the contour can (at least in certain such situations) be "pushed through" the point of pinching and give a finite result at all stages. I shall indicate two ways of so pushing our camel contour through the needle's eye:

- (i) take the needle away and put it somewhere else; and (ii) blow up the needle!

In method (i), the twistor function is regarded as part of a cochain; we are supposed not to be interested in the actual function itself but in some kind of cohomology group element that it defines. So the function can be replaced by another one in which (hopefully) the pinching now occurs somewhere else. A prototype of this type of behaviour arises when we try to represent certain generalizations of elementary states in terms of twistor functions, such "generalized elementary states" having arisen from a suggestion by Atiyah as to how best to apply R.S.W.'s striking Yang-Mills twistor construction (cf. T.N. 4, p 4). (See A.H. T.N. 4 p.1, for other uses of method (i) which could well be of considerable significance)



Such a generalized elementary state is obtained from some algebraic curve γ lying in PPT^- , the point being that the field is to be singular only for complex space-time points which are described by lines in PPT meeting γ . I shall consider only one example here, namely when γ is a twisted cubic (it had to happen!) and I shall not worry about arranging that γ actually lies in PPT^- (which is easy enough to organize by means of a coordinate change). Choose twistor coordinates W, X, Y, Z and let γ be given by $W:X:Y:Z = 1:\theta:\theta^2:\theta^3$ ($\theta \in \mathbb{C} \cup \{\infty\}$). Set $Q_1 \equiv X^2 - WY$, $Q_2 \equiv WZ - XY$, $Q_3 \equiv Y^2 - XZ$. Then γ is the common intersection of the three quadrics $Q_i = 0$. Set $\mathcal{U}_i = PPT - \{Q_i = 0\}$. Then $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ cover $PPT - \gamma$. Set $f_{ij} = L_{ij}/Q_i Q_j$ where $L_{12} \equiv W-X$, $L_{23} \equiv Y-Z$, $L_{31} \equiv X-Y$. Then the cocycle condition $f_{12} + f_{23} + f_{31} = 0$ is satisfied. Using f_{ij} as a twistor cocycle function, the field (a neutrino field) may be evaluated using a branched contour integral \int_{γ} . But one always finds that the branched contour can be deformed until it is just one loop involving just one f_{ij} , for generic space-time point. Pinching



4.

occurs only for exceptional space-time points — and then we simply switch to a different f . Thus, we have an example of (i) above.

To illustrate method (ii) I consider 2 dimensions only, say \mathbb{C}^2 , where the origin is the point of pinching. Take w, z as coordinates for \mathbb{C}^2 and consider $\oint f dw \wedge dz$ where $f(w, z) = A(w, z)/B(w, z)C(w, z)$ the functions A, B, C having simple zeros on curves a, b, c through $(0,0)$ with distinct tangents there. Consider the small sphere $w\bar{w} + z\bar{z} = \epsilon^2$ ($\epsilon > 0$). Holomorphic curves through $(0,0)$ meet this S^3 in Clifford parallels (Hoff S's) (essentially) two of which are $b \cap S^3$ and $c \cap S^3$. Contour Γ can be a torus in S^3 avoiding $S^3 \cap S^3$ and separating these S^1 's. If we now let $\epsilon \rightarrow 0$, Γ shrinks down to the origin, which is singular: pinching! So we blow up the origin, i.e. replace $(0,0)$ by a whole S^2 (holomorphically!) in which each Clifford-Hoff S' converges down to a different point of S^2 . Γ can be fibred (if chosen suitably) by twisted G-H S's (cf. Robinson congruence pictures) each of which collapses to a point as $\epsilon \rightarrow 0$, but now b and c are both avoided even at $\epsilon = 0$. So the contour slips off completely in the blown-up space! But we need to check that the form $\varphi = f dw \wedge dz$ is non-singular explicitly by mapping (w, z) to the point $(w^2, wz, z^2, w/z)$ of \mathbb{CP}^4 . Arrange words, $q = w/z$ for the blown-up space. Then $\varphi = \frac{A(3q, 3)}{B(3q, 3)C(3q, 3)} dq \wedge dz$ which has a finite value on S^2 (which is defined by $z=0$).

This shows that every f of the above form must vanish, since the contour slips off through the blown-up origin.

~ Roger Penrose

Twistor Construction for Left-Handed Gauge Fields.

This note is concerned with generalizing the twisted photon construction (see TN 1) to non-Abelian gauge theories. First a brief outline of gauge theory (for more details, see Abers & Lee, Phys. Reports 9C, 1 and Cho, J.M.P. 16, 2029). Let G be a Lie group and $\{L_j\}$ a matrix representation of its Lie algebra; suppose that the L_j are $n \times n$ matrices. A gauge potential $\Phi_a = \Phi_a^j L_j$ is an $n \times n$ matrix of 1-forms which is a linear combination of the L_j 's. The corresponding gauge field F_{ab} is defined by

$$F_{ab} = 2 \nabla_{[a} \Phi_{b]} + [\Phi_a, \Phi_b] ,$$

and is said to be left-handed iff $F_{ab}^* \equiv \frac{1}{2} \epsilon_{abcd} F^{cd} = -i F_{ab}$. Such a field automatically satisfies the Yang-Mills equations

$$\nabla^a F_{ab} + [\Phi^a, F_{ab}] = 0.$$

The gauge field may be pictured geometrically as a connection on an n -dimensional complex vector bundle V over complex Minkowski space-time CM . The connection tells one how to propagate a vector $\psi \in V$ along a curve γ in CM : the propagation law is

$$v^a (\nabla_a + \Phi_a) \psi = 0 , \quad (1)$$

where v^a is tangent to γ and where ψ is regarded as a column n -vector, so that Φ_a acts on ψ by matrix multiplication.

If one propagates ψ round a closed path in CM , it does not, in general, return to its original value: in other words, the propagation law (1) is not integrable. But suppose we restrict attention to closed curves which

- (a) can be continuously shrunk to a point without crossing singularities of Φ_a ;
- (b) lie in totally null 2-planes in CM (of the type having tangent vectors of the form $\beta^A \pi^{A'}$, with $\pi^{A'}$ fixed and β^A arbitrary).

Then it is not hard to show that (1) is integrable, provided the gauge field is left-handed. In fact, this integrability characterizes left-handed gauge fields.

Let us assume that Φ_a is holomorphic in the future tube CM^+ . The space of totally null 2-planes in CM^+ is just \mathbb{PT}^+ . Consider the space K of pairs (Z, ψ) , where Z is a totally null 2-plane in CM^+ , and where ψ is a section of V over Z , satisfying (1). If the gauge field is left-handed, then there is an n -complex-dimensional family of such sections. It follows that K has the structure of an n -dimensional vector bundle over \mathbb{PT}^+ . So, starting with a left-handed gauge field in CM^+ , we've built a vector bundle K over \mathbb{PT}^+ (it is possible to think of K as a deformation of the space of n twistors, all of which are proportional to one another). The crucial point is that K contains, in its complex structure, all the information about the gauge field. Given K , one can reconstruct Φ_a ; one way of doing this is as follows.

Let us suppose that \mathbb{PT}^+ is covered by two patches U and \hat{U} and that K is determined by the transition matrix $f(Z)$. So $f(Z)$ is an $n \times n$ matrix of twistor functions homog. of degree zero and holomorphic on $U \cap \hat{U}$. Write $F(x^a, \pi_{A'}) = f(ix^{AA'} \pi_{A'}, \pi_{A'})$. Then for fixed x , $F(x^a, \pi_{A'})$ is holomorphic on $W \cap \hat{W}$, where W and \hat{W} are two patches covering the π -sphere. Now "split" F as follows:

$$F(x^a, \pi_{A'}) = \hat{H}(x^a, \pi_{A'}) H(x^a, \pi_{A'}), \quad (2)$$

where H and \hat{H} are homog. of degree zero in π , and holomorphic in W and \hat{W} respectively. It is always possible to find such a splitting, provided that the bundle K , restricted to the line in \mathbb{PT}^+ corresponding to x^a , is analytically trivial. It is now a simple matter to derive the gauge field: the matrix $H^{-1} \pi^{A'} \nabla_{AA'} H$ turns out to have the form $\pi^{A'} \bar{\Phi}_{AA'}(x^a)$. This defines $\bar{\Phi}_a(x)$ and one can check that it is indeed a left-handed solution of the Yang-Mills equations. The splitting (2) is not unique: the choice of a particular splitting corresponds exactly to a choice of gauge for $\bar{\Phi}_a$.

Remarks. (1) The hope is that this technique will enable us to find new solutions of the Yang-Mills equations, in particular solutions of the pseudoparticle/instanton type (see e.g. Belavin et al, Phys. Lett. 59B, 85), which have been in fashion for the last few years.

(2) The encoding of the gauge field into the vector bundle structure expresses the way in which the gauge field interacts with other fields. For example (roughly speaking) a holomorphic cross-section of K gives rise (via contour integration) to a multiplet of zero-rest-mass fields on CM^+ , which are coupled to the gauge field in the correct way.

Richard Ward.

6.

A Review of Hypersurface Twistors.

We shall consider two types of fields: 1. Electromagnetic (in a flat background)¹ and 2. Gravitational; and two types of hypersurfaces: a. spacelike and b. null.

Case 1a. Given a spacelike hypersurface \mathcal{S} in Minkowski space, let $\mathcal{C}\mathcal{S}$ be its complexification. Projective twistors are represented in $\mathbb{C}\mathcal{M}$ by totally null 2-planes, which intersect $\mathcal{C}\mathcal{S}$ in hypersurface twistor curves (see [1], pp 382-3). The tangent vector to one of these curves has the form $n^{AB'} \pi_B^A, \pi_C^A$, where n^a is normal to \mathcal{S} ; knowing π_A^a exactly (not just its direction) gives one a non-projective hypersurface twistor. Usually the π -spinor is taken to be parallelly propagated along the twistor curve ([1], equation 5.14), but in the presence of an electromagnetic field $F_{ab} = 2\nabla_{[a} \Phi_{b]}$ we replace this condition by

$$n^{AB'} \pi_B^A, \pi_C^A (\nabla_{AA'} - i e \bar{\Phi}_{AA'}) \pi_C^A = 0,$$

where e is some number (the "charge" of the twistor). The non-projective hypersurface twistor space constructed in this way has the structure of a line bundle over the projective space. The projective space remains flat, i.e. a subspace of $\mathbb{C}\mathbb{P}_3$.

Theorem: The information contained in this line bundle is exactly that of the magnetic field $B_a = F_{ab} n^b$. In other words, there is a natural 1-1 correspondence between line bundles over $\mathbb{PTW}(\mathcal{S})$ and vectors \vec{B} in $\mathcal{C}\mathcal{S}$ satisfying $\text{div } \vec{B} = 0$.

Case 2a. The definition of the hypersurface twistor space is given in [1]. An analogous result to the above theorem is not known, although it has been conjectured that the information contained in $\mathbb{PTW}(\mathcal{S})$ is that of $B_{ab} = C_{acde} e_b^c n^e n_f$. Sparling looked at the case of Schwarzschild, with \mathcal{S} being the hypersurface $t=\text{const.}$; he showed that the hypersurface twistor space is flat.

Penrose conjectures in [1], p388 that holomorphic functions on $\mathcal{T}(\mathcal{S})$ can be used to generate fields on $\mathcal{C}\mathcal{S}$ satisfying constraint equations. One example of this is that a function homog. of degree -4 leads (via contour integration) to a field $\phi_{A'B'}$ on $\mathcal{C}\mathcal{S}$ satisfying

$$n_A^{A'} \nabla^{AB'} \phi_{A'B'} = 0,$$

which is equivalent to the constraint equations $\text{div } \vec{E} = \text{div } \vec{B} = 0$.

Case 1b. In the case of a null hypersurface, there are some choices to be made. First, the hypersurface \mathcal{S} may be either topologically trivial or of the "null cone" type (i.e. with topology $\mathbb{R} \times S^2$). Second, the twistors entirely on \mathcal{S} may be either omitted or included (cf. [2], p301-2). Let us consider the case where \mathcal{S} is the null cone of a point and where twistors entirely on \mathcal{S} are omitted. If $F_{ab} = \phi_{AB} \epsilon_{A'B'} + \phi_{A'B'} \epsilon_{AB}$ is a spinor decomposition of the em. field into left- and right-handed parts, then

Theorem: The information contained in the non-projective hypersurface twistor space is that of the null datum $\phi_* = \phi_{AB} o^A o^B$, and hence that of ϕ_{AB}^A . (Here o^A is the spinor pointing up \mathcal{S} .)

¹ It is possible to generalize the electromagnetic construction to non-Abelian gauge fields: cf. the construction for self-dual gauge fields elsewhere in this \mathbb{PTW} .

Case 2b. In this case, the hypersurface twistor space has the additional structure of a scalar product, defined in exactly the same way as for \mathcal{G}^+ ([2], p307; [1], p398). This gives rise to a Kähler structure on $\mathcal{I}(\mathcal{S})$, but very little is known about this structure (except in the special case when $\mathcal{S} = \mathcal{G}^+$). If the null hypersurface \mathcal{S} is shear-free, then $\mathbb{P}\mathcal{I}(\mathcal{S})$ also has the structure of a bundle over $\mathbb{C}\mathbb{P}_1$ (assuming that \mathcal{S} has topology $\mathbb{R} \times S^2$). For example, in the case of \mathcal{G}^+ , $\mathbb{P}\mathcal{I}(\mathcal{G}^+)$ is a bundle over the projective asymptotic spin-space ([2], pp302-3).

Examples. (i) pp-wave $ds^2 = 2 du dv - d\zeta d\bar{\zeta} + 2 h(v, \zeta, \bar{\zeta}) dv^2$, with hypersurface $v = \text{constant}$.

(ii) Reissner-Nordstrom $ds^2 = (1 - 2m/r + e^2/r^2) dv^2 + 2 dv dr - r^2(d\theta^2 + \sin^2\theta d\phi^2)$, with hypersurface $v = \text{constant}$.

In both these cases, the twistor space $\mathcal{I}(\mathcal{S})$ turns out to be flat. It is instructive to consider the propagation of twistors from one hypersurface $v=v_1$ to a neighbouring one $v=v_2$, following a programme similar to that which Penrose uses in the case of an impulsive plane wave ([2], pp270-3). It turns out that the propagation is described by the continuous unfolding of a canonical transformation:

$$\frac{dz^\alpha}{dv} = -i \frac{\partial H}{\partial \bar{Z}^\alpha},$$

where the Hamiltonian H is given by

$$\left\{ \begin{array}{l} \text{in case (i): } H(v, z^*, \bar{z}_*) = z^* \bar{z}^* h\left(v, \frac{i\bar{z}^* - v\bar{z}^3}{\bar{z}^*}, \frac{-i z^* - v z^3}{z^*}\right); \\ \text{in case (ii): } H(v, z^*, \bar{z}_*) = 2im \Lambda^3 U^{-2} + \sqrt{2}e^2 \Lambda^4 U^{-3}, \\ \Lambda \equiv \frac{1}{2}(\bar{z}^* z^2 + \bar{z}^1 z^3 - z^0 \bar{z}^2 - z^1 \bar{z}^3), \quad U \equiv z^0 \bar{z}^0 + z^1 \bar{z}^1. \end{array} \right.$$

In case (i), the vanishing of the Ricci tensor implies that H has the form $H = H^+ + H^-$, where $H^+(v, z, \bar{z}) = \bar{z} \partial_z g(v, z)$ and $H^- = H^+$ (cf. [2], p273). But in case (ii) it is not obvious how to characterize the vanishing of the Ricci tensor.

Finally, let us consider the \mathcal{H} -space $\mathcal{H}(\mathcal{S})$. Assume that \mathcal{S} has the topology $\mathbb{R} \times S^2$. $\mathcal{H}(\mathcal{S})$ may be defined as the space of shear-free sections of $\mathbb{C}\mathcal{S}$, or, equivalently, as the space of compact holomorphic curves in $\mathbb{P}\mathcal{I}(\mathcal{S})$ belonging to a suitable homology class. It follows from a theorem of Kodaira (see [3]) that, provided the curvature of the space-time is not too severe, $\mathcal{H}(\mathcal{S})$ has the structure of a 4-dimensional complex manifold. It also has a natural conformal structure (two points of $\mathcal{H}(\mathcal{S})$ are said to have a null separation iff the corresponding curves in $\mathbb{P}\mathcal{I}(\mathcal{S})$ intersect). This conformal structure is a genuine quadratic one (proof in [3]) and is right-conformally-flat. In general, it is not clear whether there exists a natural metric.

In the special case when \mathcal{S} is shear-free, the bundle structure of $\mathbb{P}\mathcal{I}(\mathcal{S})$ enables one to define a connection on $\mathcal{H}(\mathcal{S})$ which is compatible with its conformal structure (details in [3]). So one gets a Weyl geometry. It seems very likely that there exists a right-flat metric in this case, but this remains conjectural.

References. [1] Penrose in Quantum Gravity.

[2] Phys. Reports 6 241.

[3] G.R.G. 7 31.

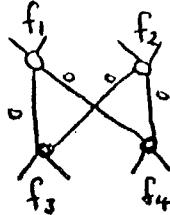
Richard Ward.

8.

The Integrated Product of Six Massless Fields

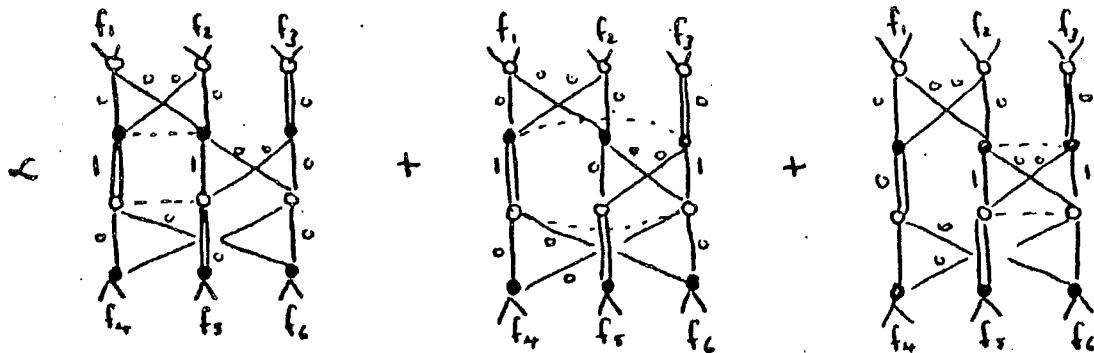
It's well known that for four (scalar) zero-mass fields, two of positive and two of negative frequency, we have

$$\int d^4x \underbrace{\phi_1(x)\phi_2(x)}_{+ve} \underbrace{\phi_3(x)\phi_4(x)}_{-ve} \propto$$



An analogous result for six fields (three in, three out) is:

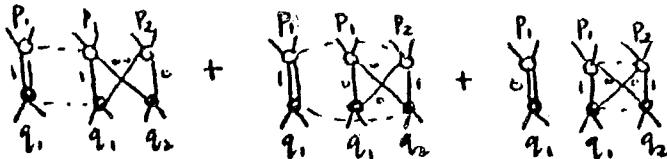
$$\int d^4x \underbrace{\phi_1(x)\phi_2(x)\phi_3(x)}_{+ve} \underbrace{\phi_4(x)\phi_5(x)\phi_6(x)}_{-ve}$$



[Proof (long) is done only for elementary states. Results in my thesis show that it's sufficient to consider]

$$\int d^4x \frac{1}{[(x-p_1)^2(x-p_2)^2(x-q_1)^2(x-q_2)^2]}$$

Computation, using Fourier transforms and the ϕ^4 result, shows that this is



[and the stated result follows from this.]

The interest of this result is that it should be possible to replace (say) $\phi_1\phi_2$ by a massive in-state, $\phi_4\phi_6$ by a massive out-state. We would then have (a scalar version of) the Fermi 4-particle vertex. Recent observations on cohomology make this seem a credible and useful program.

The analogous problem for eight massless fields is much tougher, and a result has not yet been established.

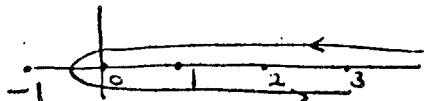
Andrew Hedges

More on the Universal Bracket Factor (see TN1)

We can define "elementary" (scalar) eigenstates of mass by the two-twistor function:

$$F(x, z) = \frac{1}{2\pi i} \int_C \frac{\left(\frac{x}{z}\right)^{1-s}}{(x-z)^{1-s}} \frac{\Gamma(1-s)\Gamma(1-s)}{\left(\frac{A}{x}\right)^{1-s} \left(\frac{B}{z}\right)^{1-s}} \frac{ds}{s^{1-s}},$$

where C is the contour (in the s-plane):



This gives rise to a massive scalar field $\phi(x) \propto \frac{m H_n^{(1)}(m\sqrt{(x-p)^2})}{\sqrt{(x-p)^2} (AB)^2}$,

where $H_n^{(1)}$ is the Hankel function order n, and where $p \leftrightarrow \frac{AB}{x}$.

If $m=0$, this can easily be seen to reduce to

$$F(x, z) = \frac{1}{x-z} \frac{B}{z-AB}, \text{ which is essentially the simplest way}$$

of writing the usual scalar "elementary" field with a two-twistor function.

However, it would be more satisfactory to have a formulation in which the value of the mass plays some role in the singularity structure. Formally,

$$F(x, z) = \frac{\left[\frac{A}{x} + \frac{B}{z}\right]}{ABx^2 - \frac{1}{2}m^2} \text{ enjoys the required properties, where}$$

$[x]$ represents the Universal Bracket Factor, with essential property $\frac{d}{dx}[x] = [x]$.

R.P. mentions in TN1 that the expression $\sum_1^\infty \frac{-T(n)}{(-x)^n}$,

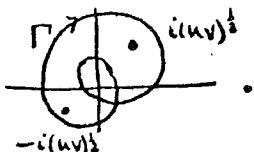
which is involved in the U.B.F., is an asymptotic series for the function

$$E(x) = e^x e^{ix} = e^x \left\{ -\gamma - \log x + x - \frac{x^2}{2.21} \dots \right\} = \int_0^\infty \frac{e^{-w}}{x+w} dw.$$

So it's encouraging that this E function can be related to the Hankel function. In fact,

$$H_0^{(1)}(2i(uv)^\frac{1}{2}) = \frac{1}{2\pi i} \oint_C \frac{E(uz+\frac{v}{z})}{z} dz$$

where C is:



Can anyone make the whole thing work?

Andrew Hodges.

10.

LINEARIZED SCHWARZCHILD IN A 2-TWISTOR FORMALISM

In TN 3 the gravitational potential h_{ab} for linearized Schwarzschild was derived from the twistor prescription:

$$h_{AA'BB'} = \frac{1}{(2\pi i)^4} \oint Z(A'X_B') \frac{\partial^2 G(z, x)}{\partial z(A) \partial X^B} d^2 Z \wedge d^2 X \quad (1).$$

where $G(z, x)$ is homogeneous of degree $(-2, -2)$. The purpose of this note is to show how we can generate linearized Schwarzschild, using the more satisfactory prescription:

$$h_{AA'BB'} = \frac{1}{(2\pi i)^4} \oint (Z_A' \frac{\partial}{\partial Z^B} - X_A' \frac{\partial}{\partial X^B}) (Z_B' \frac{\partial}{\partial Z^A} - X_B' \frac{\partial}{\partial X^A}) G(z, x) d^2 Z \wedge d^2 X. \quad (2).$$

where again $G(z, x)$ is homogeneous of degree $(-2, -2)$. (2) has the following advantages: (a) it automatically guarantees the De Donder gauge condition $\nabla^a (h_{ab} - \frac{1}{2} h^c c \eta_{ab}) = 0$ whereas (1) does not; (b) it does not require the rather artificial symmetrizations of (1); (c) it is in a form directly analogous to the electromagnetic potential prescription $\Phi_{AA'} = \frac{1}{(2\pi i)^4} \oint (Z_A' \frac{\partial}{\partial Z^A} - X_A' \frac{\partial}{\partial X^A}) F d^2 Z \wedge d^2 X$ (see page 22 in TN 3); (d) the actual calculations are more transparent - the integrals giving the components of $h_{AA'BB'}$ being much easier to evaluate than in (1), (they become in fact just the same integrals that are involved in the electromagnetic case).

We use the same twistor function for prescription (2) as was used for (1), namely

$$G(z, x) = (-4i\sqrt{2} M) \frac{\tilde{\Omega}_x}{z^2 x^2} \log\left(\frac{\tilde{\Omega}_x}{z^2 x^2}\right) \quad (3)$$

where $\tilde{\Omega}$ is the inverse of the angular momentum twistor, and $\tilde{\Omega}_x = (z^0 x^3 - z^1 x^2 - z^2 x^1 + z^3 x^0)$. If we write $S_{AB'} = (Z_B' \frac{\partial}{\partial Z^A} - X_B' \frac{\partial}{\partial X^A})$ then

$$S_{00'} G = S_{11'} G = \frac{1}{z^2 x^2} \left\{ \log\left(\frac{z \tilde{\Omega}_x}{z^2 x^2}\right) + 1 \right\} \quad \text{and} \quad S_{01'} G = S_{10'} G = 0 \quad (4)$$

Note: if we took $G = \frac{1}{z^2 x^2} \left\{ \frac{z \tilde{\Omega}_x}{z^2 x^2} \log\left(\frac{z \tilde{\Omega}_x}{z^2 x^2}\right) - \frac{z \tilde{\Omega}_x}{z^2 x^2} \right\}$ (which generates the same space-time field as (3)), then $S_{AB'} G = \frac{1}{z^2 x^2} \log\left(\frac{z \tilde{\Omega}_x}{z^2 x^2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = F.I$ where $F(z, x)$ is the function which in the electromagnetic scheme generates the potential for the Coulomb field (see page 22 in TN 3).

From (4) and (2) we obtain, using an $S^1 \times S^1$ contour for the integration:

$$h_{00'00'} = h_{11'11'} = h_{10'01'} = h_{01'10'} = \frac{2M}{r} \quad \text{with all other components zero. The twistor function (3) thus gives rise to a tensor } h_{ab} = \frac{2M}{r} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5)$$

Now the linearized Schwarzschild metric in cartesian coordinates is,

$$h_{ab} = \frac{2M}{r} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_{ij} \end{pmatrix} \quad \text{where } a_{ij} = \frac{x^i x^j}{r^2}. \quad \text{This does not satisfy De-}$$

Donder. However if we make a gauge transformation $\tilde{h}_{ab} = h_{ab} - \nabla_a \tilde{z}_b - \nabla_b \tilde{z}_a$ where $\tilde{z}_a = M(0, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r})$, then $\tilde{h}_{ab} = \frac{2M}{r} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which now satisfies De-Donder, and is in agreement with (5).

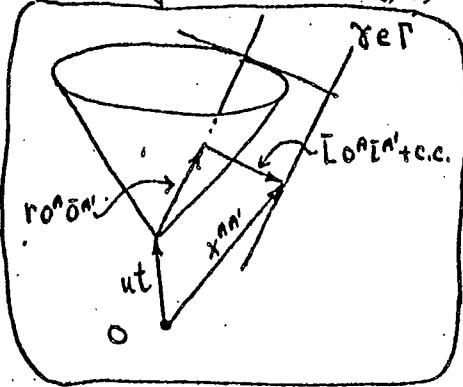
Nicholas P. Jell.

The Non-Analytic Version of Ken's Theorem

11.

An arbitrary C^1 null geodesic congruence Γ in M^4 determines a 3-surface S_Γ in PN ; the congruence is real analytic and shearfree $\Leftrightarrow S_\Gamma$ is locally the intersection of a complex 2-manifold with PN (Ken's theorem). If this is the case, then S_Γ has the following property: at each $z \in S_\Gamma$, $\exists X, Y \in T_z(S_\Gamma)$ such that $JX = Y$, where J is the complex structure on PT . (S_Γ is maximal complex.) In this note, we show that S_Γ is maximal complex $\Leftrightarrow \Gamma$ is a C^1 geodesic shearfree congruence, using a different method from that outlined by R.P. this summer.

Following Newman et.al., introduce coordinates (u, r, \bar{s}, \bar{t}) in M^4 adapted to Γ according to $x^{AA'}(u, r, \bar{s}, \bar{t}) = u t^{AA'} + r o^A \bar{o}^{A'} + \bar{L} o^A \bar{r}^{A'} + L \bar{r}^A \bar{o}^A$, where



$$o^A = \frac{1}{\sqrt{1+j\bar{j}}} (\bar{j}, 1); \quad L^A = \frac{1}{\sqrt{1+j\bar{j}}} (-1, j); \quad t^a = \sqrt{2} \delta^a_0;$$

$t^{AA'} o_A \bar{o}_{A'} = 1$; and $L = L(u, \bar{s}, \bar{t})$ is essentially a function of spin weight $\frac{1}{2}$ on g^+ describing the direction of the line $X \in \Gamma$ which strikes g^+ at (u, \bar{s}, \bar{t}) .

The surface S_Γ is given in homogeneous coords. by $Z^\alpha(u, \bar{s}, \bar{t}) = (i x^{AA'} o_A, \bar{o}_A)$. In the chart $Z^2 \neq 0$,

$$\text{we have the local coords. } w^1 = \bar{Z}^0/Z^2 = i(u - \bar{L}\bar{t}); \quad w^2 = \bar{Z}^1/Z^2 = -i(u\bar{j} + \bar{L}); \quad w^3 = \bar{Z}^3/Z^2 = -\bar{j}.$$

In this chart, PN is given by $\varphi(w, \bar{w}) = w^1 + \bar{w}^1 + w^2 \bar{w}^3 + w^3 \bar{w}^2 = 0$. A convenient set of 3 l.i. vector fields on S_Γ is given by

$$\partial_u = i(\bar{L} + \bar{j}) \frac{\partial}{\partial w^1} - i(j + \bar{L}) \frac{\partial}{\partial w^2} + \text{c.c.} \quad \langle \bar{L} := \frac{\partial \bar{L}}{\partial u} \rangle$$

$$\begin{aligned} \bar{\partial}_0 &= (1 + j\bar{j}) \frac{\partial}{\partial \bar{j}} = -i[\bar{j}j + 1] \left\{ \bar{j} \frac{\partial \bar{L}}{\partial \bar{j}} \frac{\partial}{\partial w^1} + (u + \bar{L}) \frac{\partial}{\partial w^2} - i \frac{\partial}{\partial w^3} \right\} + [i(j - \bar{j}) \frac{\partial}{\partial w^1} - i(L + \bar{j}) \frac{\partial}{\partial w^2}] \\ &\quad + i\chi \left\{ j \frac{\partial}{\partial \bar{w}^1} + \bar{j} \frac{\partial}{\partial \bar{w}^2} \right\}, \quad (\text{where we have made the substitution: } \bar{\partial} L = -L\bar{L} + \chi) \end{aligned}$$

$$\bar{\partial}_0 = \text{c.c. of } \partial_0.$$

Now Γ is shearfree $\Leftrightarrow \chi = 0$ (Hansen+Newman, GRG (1975)). Let $X = a \partial_u + \bar{\chi} \bar{\partial}_0 + \bar{\chi} \bar{\partial}_0$ be an arbitrary tangent vector to S_Γ ; if $X = t^b \frac{\partial}{\partial w^b} + \bar{t}^b \frac{\partial}{\partial \bar{w}^b}$, let $H(X) = t^b \frac{\partial}{\partial w^b}$. A necessary condition for JX to be tangent to S_Γ , and hence to PN , is that $t^b \frac{\partial H}{\partial w^b} = 0$. This happens $\Leftrightarrow a = \bar{\chi}L + \bar{\chi}\bar{L}$. Assuming a to have this form, we find that

$$JX = iH(X) - \bar{i}H(\bar{X}) = i\bar{\chi}\bar{\partial}_0 - i\bar{\chi}\bar{\partial}_0 + i[\bar{\chi}L - \bar{\chi}\bar{L}] \partial_u + 2\chi \bar{\chi} \left\{ j \frac{\partial}{\partial \bar{w}^1} + \bar{j} \frac{\partial}{\partial \bar{w}^2} \right\} + 2\bar{\chi}\bar{\chi} \left\{ \bar{j} \frac{\partial}{\partial w^1} + j \frac{\partial}{\partial w^2} \right\}.$$

The first 3 terms comprise a real tangent to S_Γ . Thus, clearly, $\chi = 0 \Rightarrow S_\Gamma$ is maximal complex. Conversely, any 3-surface S in PN is locally an S_Γ for some Γ and may be given local coordinates in the above fashion. If S ($= S_\Gamma$) is maximal complex, then the entire r.h.s. of the above eqn. must be tangent to S . This happens $\Leftrightarrow \chi = 0 \Leftrightarrow \Gamma$ is shearfree.

David Leiter

12.

The Twistor Cohomology of Local Hertz Potentials

This note examines several of the well-known properties of Hertz-type potentials from a sheaf theoretic viewpoint, using twistor cohomological techniques.

I. An exact sequence for Hertz potentials. A few elementary facts and definitions will be reviewed first. If a field $\phi_R(x)$ on Minkowski space satisfies the zero rest mass [z.r.m.] equation $\nabla^R \phi_R = 0$, then a Hertz potential for ϕ_R is defined to be a solution ψ^{RA} of the wave equation $\square \psi^{RA} = 0$ such that $\phi_R = \nabla_{RA} \psi^{RA}$. A theorem can be proved to the effect that locally a ψ^{RA} can always be found which satisfies the required conditions. There is some gauge freedom in the choice of the field ψ^{RA} , for if the transformation $\psi^{RA} \rightarrow \psi^{RA} + \varrho^A$ is made, then ϕ_R remains unchanged, providing that ϱ^A satisfies the z.r.m. equation $\nabla_{RA} \varrho^A = 0$.

This information can be synthesized neatly in a sheaf theoretic fashion. The following three sheaves are introduced: $\Phi^{R'}$ ~ the sheaf of germs of spinor fields satisfying $\nabla_{RA'} \varrho^{A'} = 0$; $\Psi^{R'}$ ~ the sheaf of germs of spinor fields satisfying $\square \psi^{RA'} = 0$; and Φ_R ~ the sheaf of germs of spinor fields satisfying $\nabla^R \phi_R = 0$. It's a straightforward matter to prove that the relations cited in the previous paragraph amount to the fact that the sequence

$$0 \longrightarrow \Phi^{R'} \xrightarrow{\alpha} \Psi^{R'} \xrightarrow{\beta} \Phi_R \longrightarrow 0 \quad [A]$$

is exact. Here α is simply the inclusion map, whereas β is the sheaf homomorphism which induces the differential mapping $\beta^*: \Psi^{R'} \rightarrow \nabla_{RA'} \psi^{RA'}$ on local sections of $\Psi^{R'}$.

The exactness of the sequence is equivalent to the following set of three conditions:

- (a) the map α is injective.
- (b) the composition $\beta \circ \alpha$ vanishes.
- (c) the map β is surjective.

Each of these conditions is satisfied: (a), since α is simply the inclusion map; (b), since sections of $\Phi^{R'}$ satisfy the z.r.m. equation; and (c), on account of the local existence theorem for Hertz potentials, which was mentioned earlier.⁽²⁾

II. Sequences for higher spins. A similar but somewhat more intricate result applies in the case of higher spin z.r.m. fields. For a spin-2 field ϕ_{AB} satisfying Maxwell's equations $\nabla^{AB} \phi_{AB} = 0$, a Hertz potential is a field ψ^{AB} which satisfies the wave equation and is such that $\phi_{AB} = \nabla_{AB} \psi^{AB}$. In this case there also exists an "intermediate" potential $\psi^{AB}_B := \nabla_B \psi^{AB}$, which satisfies the equa-

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(3) $\nabla^B \gamma^{R'}$ = 0. The gauge freedom in $\gamma^{R'}$ is of two distinct sorts. First there is the transformation $\gamma^{R'} \rightarrow \gamma^{R'} + \varphi^{R'}$, with $\nabla_{R'} \varphi^{R'}$ = 0. Such a transformation preserves both $\gamma^{R'}$ and ϕ_{AB} . Then there is the transformation $\gamma^{R'} \rightarrow \gamma^{R'} + \nabla^R \gamma^B \varphi$, where φ is a scalar subject to $\square \varphi = 0$; this is just the "usual" electromagnetic gauge transformation. Introducing appropriate sheaves for these various fields using a notation patterned in the style of the spin 1/2 case, the following set of exact sequences is obtained:

$$0 \longrightarrow \Phi^{A'B'} \xrightarrow{\Psi^{A'B'}} \Psi^{A'} \xrightarrow{\parallel} \Psi^B \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow \Phi \longrightarrow \Psi^B \longrightarrow \Phi_{RB} \longrightarrow 0 \quad [8]$$

Note that in the second of these two sequences an extra term appears, corresponding to the additional freedom $\varphi \rightarrow \varphi + c$, which leaves γ^B invariant under the transformation $\gamma^B \rightarrow \gamma^B + \nabla^B \varphi$, where c is any complex constant (*).

III. A contour integral formula for Hertz potentials. From a twistor point of view, Hertz potentials can be obtained by means of a contour integral method. In order to produce a z.r.m. field of spin 1/2, a twistor function $f(Z^a)$ which is homogeneous of degree -1 [$\text{hom}(-1)$] is used in the following contour integral formula:

$$\phi_h(x) = \frac{1}{2\pi i} \oint \partial_A f \Delta \pi .$$

where ∂_A is the operator $\partial/\partial w^A$, and $\Delta \pi = \pi_A d\pi^A$. A standard argument shows that ϕ_h satisfies the z.r.m. equations. To form a Hertz potential for ϕ_h , take a "spinor-valued" holomorphic function $f^A(Z^a)$, which is $\text{hom}(-2)$, and form:

$$\gamma^{R'}(x) = \frac{1}{2\pi i} \oint f^R \Delta \pi .$$

Differentiating, it is easy to verify that $\gamma^{R'}$ is a Hertz potential for ϕ_h providing that $i\pi_A f^{R'} = f$. There is some freedom available in the specification of $f^{R'}$, for the invariance of f under the transformation $f^{R'} \rightarrow f^{R'} + \pi^{R'} g$ is readily ascertainable, where $g(Z^a)$ is any holomorphic function which is $\text{hom}(-3)$. This transformation on $f^{R'}$ induces a corresponding transformation on $\gamma^{R'}$, which is given by $\gamma^{R'} \rightarrow \gamma^{R'} + \varphi^{R'}$, where :

$$\varphi^{R'}(x) = \frac{1}{2\pi i} \oint \pi^{R'} g \Delta \pi .$$

This expression for $\varphi^{R'}$ is again a standard formula for a z.r.m. field, hence it follows that the freedom in $f^{R'}$ induces the proper

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gauge freedom available to π^R .

IV. Twistor cohomology groups. In order to obtain a cohomological interpretation for the results mentioned in I, II, and III, it is necessary to introduce several sheaves on twistor space⁽⁵⁾:

$\mathcal{O}(n)$... the sheaf of germs of holomorphic functions which are $\text{hom}(n)$.

$\mathcal{O}^R(n)$... the sheaf of germs of primed spinor-valued holomorphic functions, $\text{hom}(n)$.

$\mathcal{O}_R(n)$... the sheaf of germs of unprimed spinor-valued holomorphic functions, $\text{hom}(n)$.

$\partial_R \mathcal{O}(n)$... the sheaf of germs of unprimed spinor-valued holomorphic functions which are $\text{hom}(n-1)$ and satisfy $\partial^R f_R = 0$. [i.e. f_R must be of the form $\partial_R f$, with $f \in \text{hom}(n)$.] Note that $\partial_R \mathcal{O}(n+1)$ is a subsheaf of $\mathcal{O}_R(n)$.

$\mathcal{O}_R(n)$... the sheaf of germs of holomorphic functions of π_R^R alone (i.e. functions which satisfy $\partial_R f = 0$), which are $\text{hom}(n)$.

Higher and mixed valence analogs of these sheaves can be formed in various ways, and will be denoted using an analogous notation.

Using these sheaves a number of exact sequences can be constructed; the most "primitive" of these sequences is

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\alpha} \mathcal{O}^R(-2) \xrightarrow{\beta} \mathcal{O}(-1) \longrightarrow 0, \quad [C]$$

where α is the injection map consisting of multiplication by π^R , and β is the projection map consisting of contraction with $i\pi_R$; the composition $\beta \circ \alpha = 0$ follows from the trivial spinor identity $\pi^R \pi_R = 0$. The idea now is to form the exact cohomology sequence⁽⁶⁾ associated with the sheaf sequence above, relative to a neighborhood \mathcal{L} of some line in projective twistor space. Since H^n (when $n \neq 1$) vanishes, over \mathcal{L} , for these sheaves, the only non-trivial segment of the cohomology sequence turns out to be:

$$0 \longrightarrow H^1(\mathcal{L}, \mathcal{O}(-3)) \longrightarrow H^1(\mathcal{L}, \mathcal{O}^R(-2)) \longrightarrow H^1(\mathcal{L}, \mathcal{O}(-1)) \longrightarrow 0 \quad [D]$$

The elements of the group $H^1(\mathcal{L}, \mathcal{O}(-3))$ correspond to local sections of the sheaf \mathbb{P}^A over the region of Minkowski space corresponding to \mathcal{L} . Similarly, the elements of $H^1(\mathcal{L}, \mathcal{O}^R(-2))$ and $H^1(\mathcal{L}, \mathcal{O}(-1))$ correspond to local sections of the sheaves \mathbb{P}^A and \mathcal{O}_R , respectively⁽⁷⁾. Letting \mathcal{L} vary, the usual construction of a sheaf can be applied to these sections, and the original Hertz potential sequence [A] is recovered.

In the case of electromagnetism, a pair of algebraic exact sequences of sheaves of twistor functions can be constructed,

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following the pattern suggested by sequence [C] in the spin $\frac{1}{2}$ case:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-4) & \xrightarrow{\xi} & \mathcal{O}^{R'B'}(-2) & \xrightarrow{\eta} & \mathcal{O}^{R'}(-1) \longrightarrow 0 \\ & & & & & & \parallel \\ 0 & \longrightarrow & \mathcal{O}(-2) & \xrightarrow{\alpha} & \mathcal{O}^{R'(-1)} & \xrightarrow{\beta} & \mathcal{O}(0) \longrightarrow 0 \end{array} \quad [E]$$

Here the maps α and β are defined in the same manner as before; the map ξ is multiplication by $\pi^R \pi^{B'}$, and η is contraction with $i \pi_{R'}$. The exact cohomology sequences obtainable from these sheaf sequences are:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathcal{L}, \mathcal{O}(-4)) & \longrightarrow & H^1(\mathcal{L}, \mathcal{O}^{R'B'}(-2)) & \longrightarrow & H^1(\mathcal{L}, \mathcal{O}^{R'}(-1)) \longrightarrow 0 \\ & & & & & & \parallel \\ 0 & \longrightarrow & H^0(\mathcal{L}, \mathcal{O}(0)) & \longrightarrow & H^1(\mathcal{L}, \mathcal{O}(-2)) & \longrightarrow & H^1(\mathcal{L}, \mathcal{O}^{R'(-1)}) \longrightarrow H^1(\mathcal{L}, \mathcal{O}(0)) \longrightarrow 0 \end{array} \quad [F]$$

Now we have a little surprise: notice the way in which the exact cohomology sequence picks up the extra term $H^0(\mathcal{L}, \mathcal{O}(0))$, since this group does not vanish. In fact, $H^0(\mathcal{L}, \mathcal{O}(0)) \cong \mathbb{C}$, and the extra term arising here corresponds precisely to the additional freedom which appears in diagram [B]. The cohomology groups appearing in diagram [F] give rise to the spaces of local sections of the sheaves appearing in diagram [B]. Essentially the same sort of analysis goes through for higher spins.

nanc Hughston

Notes.

- 1) See section 4 ("Zero Rest-Mass Potentials") in Penrose (1965) for a proof of the local existence theorem for Hertz-type potentials.
- 2) In fact, as indicated in Penrose (1965), a sufficient condition for the existence of a Hertz potential is that the region on which the field P_R is defined should have vanishing first and second homotopy groups (i.e. be simply connected and be such that any 2-sphere can be shrunk to a point.) Thus, if \mathcal{M} is any such region then the exact sequence [A] can be modified into the following exact sequence of spaces of sections over \mathcal{M} :

$$0 \longrightarrow \Gamma(\mathcal{M}, \Phi^R) \longrightarrow \Gamma(\mathcal{M}, \Psi^R) \longrightarrow \Gamma(\mathcal{M}, \Phi_R) \longrightarrow 0.$$

- 3) This relation implies that the potential Ψ^R 's is divergence-free (a gauge condition), and is purely 'left-handed', that is, $\nabla^S (\epsilon^R \Psi^S)_S = 0$.
- 4) It is not difficult to see that for each spin s a diagram similar to [B] is obtained. Each such diagram is composed of a set of 2s exact sequences, and shows a "cascade" of 2s various potential fields leading down to the basic r.m. field. Each potential has a certain amount of gauge freedom at its disposal, and

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this freedom is reflected in the structure of its accompanying exact sequence. [Cf. Penrose (1965), footnote on p. 168.]

- 5) Since the sheaves appearing here all involve homogeneous functions, they can be regarded as being defined (a) directly on twistor space, as sheaves of homogeneous functions, or (b) on projective twistor space, as sheaves of germs of holomorphic cross-sections of certain bundles (labeled by the integer n). For an explicit account, see Griffiths and Adams (1974), pp. 42-44. This reference also contains, incidentally, on pp. 51-55 a proof of the useful result that $H^q(\mathbb{P}^r, \mathcal{O}(n))$ vanishes, unless $q=0$ and $n \geq 0$, or $q=r$ and $n \leq -r-1$; and that in these cases $H^0(\mathbb{P}^r, \mathcal{O}(n))$ is isomorphic with the space of polynomials of degree n in the homogeneous coordinates for \mathbb{P}^r , and $H^r(\mathbb{P}^r, \mathcal{O}(n))$ is isomorphic with the dual of $H^0(\mathbb{P}^r, \mathcal{O}(-r-n-1))$.

- 6) If a sequence of sheaves $0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0$ over a space X is exact, then it follows as a well-known theorem that there exists a mapping $\delta^*: H^r(X, C) \rightarrow H^{r+2}(X, A)$ called the "connecting homomorphism", and that the sequence $\cdots \xrightarrow{\delta^*} H^r(X, C) \xrightarrow{\alpha^*} H^{r+2}(X, A) \xrightarrow{\beta^*} H^{r+2}(X, B) \xrightarrow{\delta^*} \cdots$ is exact. This is the "exact cohomology sequence". [For a proof, see Spanier (1966), chapter 4, section 5. The theorem there is given for homology, rather than cohomology, but it's not difficult to make the necessary adjustments. The proof is also given in Gunning (1966), as Theorem 1, an extremely lucid and enjoyable book, which I highly recommend.] Referring back to sequence [A] now, note that, for any region \mathcal{N} of Minkowski space, the associated exact cohomology sequence contains the segment:

$$0 \rightarrow H^0(\mathcal{N}, \Phi^A) \xrightarrow{\alpha^*} H^0(\mathcal{N}, \Psi^A) \xrightarrow{\beta^*} H^0(\mathcal{N}, \Phi_A) \xrightarrow{\delta^*} H^1(\mathcal{N}, \Phi^A).$$

Since $H^0(\mathcal{N}, S) \cong \Gamma(\mathcal{N}, S)$, it follows that $H^1(\mathcal{N}, \Phi^A)$ is the "obstruction" for the existence of Hertz potentials for e.r.m. fields defined on \mathcal{N} , i.e. any field on \mathcal{N} which doesn't admit a Hertz potential defined globally on \mathcal{N} will have as its image under δ^* some non-vanishing element of $H^1(\mathcal{N}, \Phi^A)$.

- 7) In the cases of $H^1(\mathcal{L}, \mathcal{O}(-3))$ and $H^1(\mathcal{L}, \mathcal{O}(-2))$, the representative cocycles are fed directly into the basic contour integral formulas (using a 'branched contour'; see R.P.'s account in TN2). However, in the case of $H^1(\mathcal{L}, \mathcal{O}(-2))$ what actually goes into the contour integral formula is the image of an element of $H^1(\mathcal{L}, \mathcal{O}(-1))$ under the operation ∂_A . This gives an element of $H^2(\mathcal{L}, \partial_A \mathcal{O}(-2))$. In fact, $H^1(\mathcal{L}, \mathcal{O}(-1)) \cong H^2(\mathcal{L}, \partial_A \mathcal{O}(-2))$, as follows from the exact sheaf sequence $0 \rightarrow \mathcal{O}_{\mathcal{N}}(-1) \xrightarrow{i} \mathcal{O}(-1) \xrightarrow{\pi} \partial_A \mathcal{O}(-2) \rightarrow 0$, using the exact cohomology sequence, and the fact that $H^t(\mathcal{L}, \mathcal{O}_{\mathcal{N}}(-1)) \cong 0$ for all t .

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SPIN NETWORKS AND THE VECTOR COUPLING COEFFICIENTS

Spin networks are a graphical method for dealing with Young symmetrizers for $SU(2)$ — that is, with operations of combining and splitting angular momenta in non-relativistic quantum mechanics.

Recall that if $| \frac{1}{2} \pm \frac{1}{2} \rangle = u^A \equiv \delta$ and $| \frac{1}{2} - \frac{1}{2} \rangle = d^A \equiv \delta$ are the "up" and "down" states of a spin $\frac{1}{2}$ system, then basis states $| jm \rangle$ for an arbitrary spin j system can be constructed from symmetrized products :

$$| jm \rangle = | rs \rangle \equiv \underbrace{u^{(A_1}_r u^{(A_2}_s \dots u^{(A_N}_r d^{B_1}_s \dots d^{B_M}_s)}_{\text{symmetrized}} = \overbrace{\delta \dots \delta}^r \overbrace{\delta \dots \delta}^s = r \overbrace{\delta \dots \delta}^{2j} s$$

$$\text{where } j = \frac{r+s}{2}, m = \frac{r-s}{2}.$$

Antisymmetrical products represent composite states with cancellation of total spin (for example, $\epsilon^{AB} \epsilon_{AB} \epsilon^{CD} = \square$ is the scalar part of ϵ^{AE}). Here we are using the invariance of the totally antisymmetric 2-spinor ϵ_{AB} (i.e., the "S" in $SU(2)$).

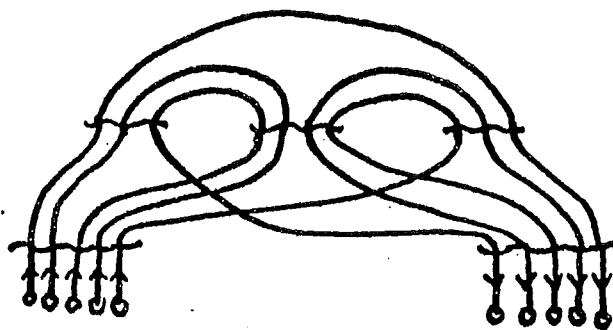
In order to compute the vector coupling coefficient or Wigner coefficient $\{ j_1, j_2, j_3 \}_{m_1, m_2, m_3}$ giving the amplitude for three spins to combine to zero, we simply construct the product state

$$| j_1 m_1 \rangle | j_2 m_2 \rangle | j_3 m_3 \rangle = \begin{array}{c} 2j_1 & 2j_2 & 2j_3 \\ \overbrace{\delta \dots \delta}^{r_1} & \overbrace{\delta \dots \delta}^{r_2} & \overbrace{\delta \dots \delta}^{r_3} \\ \text{where } j_i = \frac{r_i + s_i}{2}, m_i = \frac{r_i - s_i}{2} \end{array}$$

$$J = \sum_i j_i = \sum_i r_i = \sum_i s_i, \quad O = \sum_i m_i$$

18.

and then apply antisymmetrizers (i.e., ϵ 's) to reduce it to a scalar. For example, $\left\{ \begin{smallmatrix} \frac{1}{2} & 2 & \frac{3}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{smallmatrix} \right\}$ corresponds to

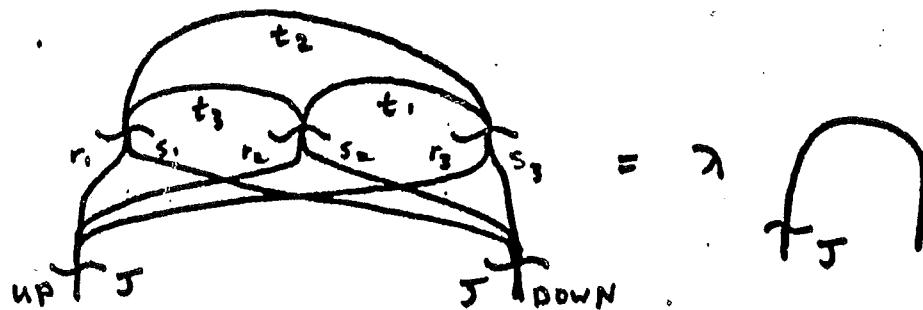


Here the strands in the diagram represent

$$\epsilon^{AB} = \underbrace{\quad}_{B}, \quad \delta_A^B = \underbrace{\quad}_A, \text{ etc.}$$

(by the usual convention, upper and lower indices are indicated as up and down pointing vertical end-segments of an expression).

We now note that in the general $\left\{ \begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix} \right\}$ expression



since every ϵ_{AB} -term that has an index at the UP end on the left must have the other at the DOWN end (otherwise it would be symmetrized). Note also that

$j_1 = j_2 + j_3 - j_4$, etc. are determined by j 's.

Hence the coupling computation reduces to finding \mathcal{I} , and introducing proper normalization.

First we must examine our graphical method more closely. One motivation for this notation is that it captures identities such as

$$\delta_A^B \epsilon_{BC} = \epsilon_{AC} : \quad \text{Diagram} = \text{Diagram} .$$

However, certain sign problems arise if we insist that graphical expressions retain their identity under more general deformations. In particular, note that

$$\text{a) } \text{Diagram}^D = -\text{Diagram} \quad \text{and b) } \text{Diagram}^D = -\text{Diagram}$$

$$\delta_A^D \epsilon_{CD} \delta_B^C = -\epsilon_{AB} \quad \epsilon_{AD} \epsilon_{BC} \epsilon^{CD} = -\epsilon_{AB} .$$

But we can eliminate these problems by simply adopting the convention of introducing an extra negative sign into our strand diagram for each appearance of

A) a crossing and B) a contravariant $\cup = \epsilon^{AB}$.

It is not difficult to show, in fact, that these two sign conventions are sufficient to ensure that all topologically equivalent strand diagrams correspond to the same invariant spinor expression (provided crossings are simple, vertical tangents isolated, and that free ends occur at the extreme top and bottom of the diagram).

20.

An immediate consequence of B) is that the closed loop $\textcircled{0} = -\epsilon_{ABC}\epsilon^{ABC} = -2$. Hence we may interpret our result as an isomorphism between invariant spinor algebra and the algebra of a "-2 dimensional" Cartesian system of binors. Because of A), the spinor symmetrizers become antisymmetrizers in binor algebra, while the antisymmetric ϵ_{ABC} becomes the symmetric binor $\Pi = \rho^A_B$. In binor algebra all the sign problems of spin combinations are isolated in the negative trace of this binorial "Kronecker delta".

Returning finally to the vector coupling network, we determine λ by contracting the free ends of the diagram and replacing symmetrizers with binor antisymmetrizers (bars):

$$\begin{array}{c} t_2 \\ \textcircled{r}_1 \quad \textcircled{s}_1 \quad \textcircled{t}_3 \\ \textcircled{r}_3 \quad \textcircled{s}_2 \quad \textcircled{s}_3 \\ \textcircled{s}_4 \end{array} = \lambda + \textcircled{\lambda}$$

The network on the left is precisely the one we evaluated by the chromatic method in the last T.N.L. article. The result gives Racah's famous formula for the vector coupling coefficients, modulo a factor which can be computed by normalizing the ingoing states and requiring unitarity.

These methods, which were invented by R.P., can clearly be applied to computation of higher-order Clebsch-Gordan coefficients as well.

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