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The Grgin-Maslov index.

In compactified Minkowski space, a global solution of the wave equation $\Box u = 0$ must be discontinuous across \mathcal{I} by a factor of -1. (see [1] and references cited there). For some time, this discontinuity has been jokingly referred to as the "Grgin-Maslov index"; my purpose here is to spoil the joke by explaining it.

In the high frequency limit, the wave equation is solved locally by $u = A \exp(-is/t)$ where t is small and, if $K^a = \nabla^a S$,

$$K^a K_a = 0 \quad (1) \quad \text{and} \quad K^a \nabla_a A + \frac{1}{2} A \nabla_a K^a = 0 \quad (2)$$

It follows from (1) that

$$K^a \nabla_a K_b = K^a (\nabla_a K_b - \nabla_b K_a) = 0 \quad (3)$$

and thus that K is tangent to a rotation free null geodetic congruence Γ .

In general, Γ will have caustic singularities where A becomes infinite. However, it is still possible to make sense of the solution even at these singular points by reinterpreting u as a distribution dual to the space of oscillatory test functions of the form $f = \varphi \exp(i\alpha/t)$ (φ and α are smooth and real and φ has compact support). Away from the caustics, $u(f)$ is defined by an integral

$$2. \quad u(f) = (2\pi t)^{-2} \int A \rho \exp(i(\alpha - S)/t) d^4x, \quad (4)$$

the asymptotic behaviour of which as $t \rightarrow 0$ can be found by the method of stationary phase (see [2]). The result is

$$u(f) = \sum_m \left\{ A \rho |\det J|^{-\frac{1}{2}} \exp\left[\pm i\pi \text{sign} J\right] \exp\left[i(\alpha - S)/t\right]\right\}_m (1 + o(t)) \quad (5)$$

where J is the Hessian matrix $\nabla_a \nabla_b (\alpha - S)$ and the summation is over the critical points m of $(\alpha - S)$ in $\text{supp } \rho$.

The critical points correspond to the intersections in the cotangent bundle of space-time of the surfaces $\Lambda_S = \{(x, ds_x)\}$ and $\Lambda_\alpha = \{(x, d\alpha_x)\}$, and the leading term in eq. 5 can be evaluated simply by examining the relative orientations of Λ_S and Λ_α at these points. This geometric technique is used to make sense of $u(f)$ when u itself is undefined. However, if $u(f)$ is to be continuous as a function of f , then the phase function S/t must be made to jump by $\pm i\pi$ times the multiplicity across any caustic C (the multiplicity of C is the number of independent Jacobi fields which vanish on C). This discontinuity originates in the factor $\exp(\pm i\pi \text{sign} J)$ in eq. 5.

Now consider an example: put $A = (p_a x^a)^{-1}$ and $S = \frac{1}{2}(x_b x^b / p_a x^a)$ where p^a is a constant null vector. Then, not only do A and S satisfy (1) and (2), but also u satisfies the wave equation exactly (u is, in fact, a conformally transformed plane wave). In this case, Γ consists of all the null lines intersecting the fixed null line C through O in the direction of p^a . It is easy to see that C is a caustic singularity of multiplicity 2. Thus, thinking of u as a high frequency approximate solution, we must introduce a discontinuity $u \rightarrow -u$ across C and, if u is to be single valued in the large, we must put in a compensating discontinuity at infinity. Hence the Grgin phenomenon.

More generally, if K is tangent to a closed null geodetic congruence in an arbitrary space-time, then we pick up a total phase jump of $\pm i\pi m$ in passing once around the congruence; here m is the Morse index of the closed geodesics. For higher order operators and operators involving external potentials K will not be geodetic. In these cases, the total number of caustics (counted according to multiplicity) encountered in passing round a closed integral curve of K is called the Maslov index (denoted M). Again, by a simple extension of the argument described above, one obtains a total phase jump $\pm i\pi M$ and this must be taken into account in constructing global solutions.

1. D.Lerner: TNL 3, 7 (1976)

2. V. Guillemin and S. Sternberg: Geometric Asymptotics: AMS surveys, 14 (1977).

Baryon Magnetic Moments

1. Introductory Remarks

Measurements have been made for the magnetic moments of several members of the spin $\frac{1}{2}^+$ baryon octet. The magnetic moments of the proton and neutron in particular, are known to great accuracy.

P	n	Λ^0	Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-
27428456	-1.913148	-0.67	2.62		-1.48		-1.85
$\pm .0000011$	$\pm .0000066$	$\pm .06$	$\pm .41$		$\pm .37$		$\pm .76$

Magnetic moments of 8 $\frac{1}{2}^+$ members (in units of $e\hbar/2m_p c$)

In the following discussion we shall show how, within the framework of twistor theory, considerable information can be gained concerning the values of these magnetic moments. Although we are not able to predict precise numerical values, we can with fair accuracy calculate the ratios of the various magnetic moments.

Notation. Triplets of twistors will be denoted Z_i^α , with $i = 1, 2, 3$. The conjugate operators $-\partial/\partial z_i^\alpha$, will be denoted \bar{z}_i^α . By a twistor function $f(z_i^\alpha)$ we mean a cocycle representing a cohomology class in a suitable cohomology group. When $f(z_i^\alpha)$ is specified as satisfying some differential equation we mean (usually) that the cocycle "weakly" satisfies the differential equation, i.e. the cocycle satisfies the equation modulo the production of coboundary terms. We write $Z_j^\alpha = (\omega_j^\alpha, \pi_{AA'}^\alpha)$ for the spinor parts of Z_j^α . We say that Z_j^α is "restricted to the spacetime point x^α " if $\omega_j^\alpha = ix^{AA'}\pi_{AA'}^\alpha$. For any function $F(z_j^\alpha)$ we shall write $\rho_x F(z_j^\alpha) := F(ix^{AA'}\pi_{AA'}^\alpha)$ for " $F(z_j^\alpha)$ restricted to the spacetime point $x^{AA'}$ ". Note, for example, that we have the identity

$$i\nabla_{AA'} \rho_x F(z_j^\alpha) = \rho_x P_{AA'} F(z_j^\alpha),$$

where $P_{AA'} = \pi_{AA'} \hat{\pi}_{AA}^\alpha$ is the momentum operator. ($\hat{\pi}_A^\alpha = -\partial/\partial \omega_A^\alpha$).

Baryon States. The low-lying baryon states can be represented in terms of holomorphic functions $f(z_i^\alpha)$ of three twistors. These functions are required to be in suitable eigenstates of mass, spin, baryon number, electric charge, hypercharge, isospin and the two $SU(3)$ Casimir operators.

By means of a contour integral formula we can associate with each twistor function a spacetime field. Incorporated in each contour integral formula is a certain combination of spinors, this spinor coefficient structure being uniquely determined by the state of the baryon; thus the characteristics of the baryon state are coded in a "dual" way into the spinor coefficient structure. If we denote our three twistors U^α , D^α and S^α ,

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letting u_A' , d_A and s_A denote the respective π -parts of these twistors, then the expressions for the spacetime fields associated with proton and neutron states, for example, are as follows:

$$\Psi_p^A(z) = \int u^A u^B ds' f(z_i) \Delta \pi, \quad \Psi_n^A = \int d^A u^B ds' g(z_i) \Delta \pi,$$

where $f(z_i)$ and $g(z_i)$ are twistor functions appropriate to a proton state and a neutron state, respectively.

2. The Centre-of-Mass Twistor

Any massive system has associated with it a complex centre of mass. For systems of three twistors the complex centre of mass is specified by the skew-symmetric twistor

$$X^{\alpha\beta} = 2m^2 Z_i^\alpha Z_j^\beta \tilde{M}^{ij}.$$

The tensor \tilde{M}^{ij} is defined to be $\tilde{Z}_a^i \tilde{Z}_b^j I^{\alpha\beta}$. Using the fact that $m^2 = M_{ij} \tilde{M}^{ij}$ we easily verify that $X^{\alpha\beta} I_{\alpha\beta} = 2$. Furthermore we have the relations $X^{\alpha\beta} X^{\gamma\delta} = 0$. The twistor $X^{\alpha\beta}$ represents a certain point on the centre of mass line. The remaining points are obtained by propagating from that point in the direction of the momentum of the system.

In what follows we will require the operator $\hat{X}^{\alpha\beta}$ formed from $X^{\alpha\beta}$ by standard twistor quantization ($\tilde{Z}_a^i \rightarrow \tilde{Z}_a^i$). In particular the following result is used:

Proposition 1. If $f(z_i)$ is in a mass eigenstate with eigenvalue m , then:

$$\rho_p \hat{X}^{\alpha\beta} f(z_i) = P^{\alpha\beta} \rho_p f(z_i),$$

where $P^{\alpha\beta}$ is the skew twistor representing the point p .

Proof. First note that if $P^{\alpha\beta}$ represents the π -part of $P^{\alpha\beta}$ with respect to one of its indices then we have the relation $P^{\alpha\beta} = P_{\alpha'}^{\alpha} P_{\beta'}^{\beta}$. Furthermore, if Z_i^α is restricted to the point p , then it must be of the form $P^{\alpha\beta} \tau_{\beta i}$ for some spinor triplet $\tau_{\beta i}$. Thus, we have

$$\begin{aligned} \rho_p \hat{X}^{\alpha\beta} f(z_i) &= 2m^2 \rho_p Z_i^\alpha Z_j^\beta \tilde{M}^{ij} f(z_i) \\ &= 2m^2 P^{\alpha A'} P^{\beta B'} \rho_p \tau_{A i} \tau_{B j} \tilde{M}^{ij} f(z_i) \\ &= m^{-2} P_{\alpha'}^{\alpha} P_{\beta'}^{\beta} \epsilon^{A'B'} \tau_{A i} \tau_{B j} \tilde{M}^{ij} f(z_i) \\ &= m^2 P^{\alpha\beta} \rho_p M^2 f(z_i), \end{aligned}$$

where $M^2 = \tau_{A i} \tau_{A j} \tilde{M}^{ij}$ is the mass-squared operator, and the desired result immediately follows.

3. The Magnetic Moment Operator

The magnetic moment of a baryon can be calculated by taking the expectation value of a certain component of the operator

$$\mathcal{M}^{\alpha\beta} = \mu_0 \epsilon_{ij} Z_j^{(\alpha} \hat{X}^{\beta)} \hat{Z}_i ,$$

where ϵ_{ij} is the matrix diag. $(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$.

The component of $\mathcal{M}^{\alpha\beta}$ we require is the operator given by $\hat{\rho} = \sigma_{\alpha\beta} \mathcal{M}^{\alpha\beta}$, where $\sigma_{\alpha\beta}$ is a constant symmetric twistor satisfying $\sigma_{\alpha\beta} I^{\beta\gamma} = 0$.

We require the following result:

Proposition 2. If $|f(z_i^*)\rangle$ is in a mass eigenstate with eigenvalue m , then:

$$\rho_p \hat{\rho} f(z_i^*) = \hat{\rho}_n \rho_p f(z_i^*) ,$$

where $\hat{\rho}_n$ is the operator defined by

$$\hat{\rho}_n = -\mu_0 \epsilon_{ij} T_{jA'} \sigma^{A'B'} \frac{\partial}{\partial T_{iB'}} ,$$

and σ_{AB}' is the only non-vanishing spinor part of $\sigma_{\alpha\beta}$.

Proof. We have :

$$\begin{aligned} \hat{\rho}_n \rho_p f(z_i^*) &= \hat{\rho}_n f(i p^{AA'} T_{Ai}, T_{Ai}) \\ &= -\mu_0 \epsilon_{ij} T_{jA'} \sigma^{A'B'} \frac{\partial}{\partial T_{iB'}} f(i p^{AA'} T_{Ai}, T_{Ai}) \\ &= -\mu_0 \epsilon_{ij} T_{jN} \left\{ \sigma^{A'B'} i p^{B'B} \rho_p \frac{\partial f}{\partial \omega_i^B} + \sigma^{A'B'} \rho_p \frac{\partial f}{\partial \omega_i^B} \right\} \\ &= -\mu_0 \epsilon_{ij} \sigma_{\alpha\beta} Z_j^{(\alpha} \hat{X}^{\beta)} \rho_p \frac{\partial f}{\partial z_i} \\ &= \mu_0 \epsilon_{ij} \sigma_{\alpha\beta} \rho_p Z_j^{(\alpha} \hat{X}^{\beta)} \hat{Z}_i f(z_i^*) \\ &= \rho_p \hat{\rho} f(z_i^*) , \end{aligned}$$

where in going from the fourth line to the fifth line we have used Proposition 1.

4. The Proton-Neutron Magnetic Moment Ratio

Although we do not know from first principles what the value of μ_0 is, we note that in a ratio such as

$$\frac{\rho_p}{\rho_n} = \langle f | \hat{\rho} | f \rangle / \langle g | \hat{\rho} | g \rangle ,$$

where $|f\rangle$ is a proton state and $|g\rangle$ is a neutron state, the value of μ_0 cancels out. Thus, we can calculate ρ_p/ρ_n exactly.

If $|f\rangle$ corresponds to a particular proton state, then $\hat{\rho}|f\rangle$ corresponds to a superposition of that proton state with a certain A^+ decuplet state $|f^*\rangle$;

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i.e. we have $\hat{\rho}|f\rangle = \rho_\alpha |f\rangle + \rho_\beta |f^*\rangle$ (with $\langle f|f\rangle = 1$ and $\langle f^*|f^*\rangle = 1$) for some values of the coefficients α and β . Inasmuch as $\langle f|f^*\rangle = 0$ (since $|f\rangle$ and $|f^*\rangle$ have distinct quantum numbers) we have $\langle f|\hat{\rho}|f\rangle = \rho_\alpha$. Similarly, for the neutron, we obtain $\hat{\rho}|g\rangle = \rho'_\alpha |g\rangle + \rho'_\beta |g^*\rangle$, where $|g^*\rangle$ is a certain Δ^0 decuplet state, and where α' and β' are distinct from α and β . The ratio ρ_p/ρ_n accordingly, is given by the number α'/α .

The numbers α, β, α' and β' can all be readily calculated. To see this we observe that

$$\oint (\pi_i^{(1)} \dots) \rho_p \hat{\rho} f(z_i^1) \Delta n = \oint \{ \rho_n (\pi_i^{(1)} \dots) \} \rho_p f(z_i^1) \Delta n ,$$

which follows from Proposition 2, integration by parts and making use of the fact that σ_{AB} is symmetric. (Thus $\hat{\rho}$ and ρ_n are adjoint with respect to the inner product operation which the contour integral defines between the twistor function and the spinor coefficients.) If we now take $f(z_i^1)$ to be in a definite eigenstate then there will be a unique spinor coefficient structure $\{\Pi_f\}$ for which the contour integral is non-vanishing. In this sense $\{\Pi_f\}$ can also be considered to be in an eigenstate. If for example,

$|f(z_i^1)\rangle$ represents a proton state and $\{\Pi_f\}$ is the coefficient structure appropriate to that proton state, then $\oint \{ \Pi_f \} \hat{\rho} f(z_i^1) \Delta n$ gives the "proton component" of the superposition of states represented by $|\hat{\rho} f(z_i^1)\rangle$. Using the relation above we have that:

$$\oint \{ \Pi_f \} \hat{\rho} f(z_i^1) \Delta n = \oint \{ \rho_n \{ \Pi_f \} \} f(z_i^1) \Delta n .$$

Now, $\hat{\rho}_n \{ \Pi_f \} = \alpha \{ \Pi_f \} + \sum$ where α is a number and \sum denotes the sum of the spinor coefficient structures associated with states distinct from the original proton state. Thus we may write

$$\oint \{ \Pi_f \} \hat{\rho} f(z_i^1) \Delta n = \alpha \oint \{ \Pi_f \} f(z_i^1) \Delta n .$$

This gives us a general method for calculating the various numerical coefficients $\alpha, \alpha', \beta, \beta'$ etc. If we apply the procedure outlined above to a state $|f\rangle$ representing a proton in a spin $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ state and a state $|g\rangle$ representing a spin $(\frac{1}{2}, \frac{1}{2})$ neutron and where $\hat{\rho}$ is taken as the 'z' component of magnetic moment then we obtain for the ratio ρ_p/ρ_n the value $-\frac{3}{2}$. This result is in reasonably close agreement with the data in the table in section 1.

In a forthcoming article we shall give the details of the calculation described above, and discuss a number of related matters.

An Example of an \mathcal{X} -space

7.

I wish to describe an example of an \mathcal{X} -space due chiefly to Sparling. The good-cut equation [1] is $\mathfrak{Z}^* Z = \sigma^*(z, \xi, \bar{\xi})$ and was solved by Sparling when σ^* has the form $\sigma^* = \lambda f(\xi)(1 + \xi\bar{\xi})^{-1} u^{-3}$, $\lambda \in \mathbb{C}$, where f is a quartic polynomial in ξ so that σ^* is purely $l=2$ spherical harmonics. The simplest case (leading to a type N \mathcal{X} -space) is when f has the form of a fourth power and wlog we can take it to be one. A solution can also be found when f is a perfect square (type D) or simply when f has a repeated root.

To solve $\mathfrak{Z}^* Z = \lambda(1 + \xi\bar{\xi})^{-1} Z^{-3}$... (1) we make the ansatz $Z^1 = \mathfrak{z}^2 + \lambda s^2$, ... (2) where $\mathfrak{z} = \mathfrak{z}^a \ell_a$; $s = s^a \ell_a$; $\ell_a = (1 + \xi\bar{\xi})^{-1}(1, \xi, \bar{\xi}, \xi\bar{\xi})$; $\mathfrak{z}^a = (u, x, y, v)$. Then the solution depends on the four parameters \mathfrak{z}^a and (1) serves to fix s^a . In fact, since ℓ_a is null in the flat metric η_{ab} , the quantity $\eta_{ab}(\mathfrak{z}^a \mathfrak{z}^b + \lambda s^a s^b)$ is undetermined in (2). We fix it, and thereby s^a , by requiring s^a to be null with respect to η_{ab} . Substituting (2) in (1) now leads to $\mathfrak{z}^a = (uv - xy)^{-1}(y, v, 0, 0)$.

Twistor methods can be used to find the metric of \mathcal{X} -space. First the displacement $d\mathfrak{z}^a$ is null if $d\mathfrak{z}^a Z_{,a}$ is zero for some fixed ξ and all $\bar{\xi}$, [2]. With an obvious notation this means $\mathfrak{z}^a d\mathfrak{z}^b + \lambda s^a ds = 0$... (3) for all $\bar{\xi}$ at fixed ξ . The left hand side of (3) is quadratic in ξ so this gives three conditions involving \mathfrak{z}^a , ds and ξ . The aim is to eliminate ξ from these leaving a quadratic relation between the $d\mathfrak{z}^a$, which is the conformal metric. The result is

$$2du dv - 2dx dy - 2\lambda(uv - xy)^{-1}(y dv - v dy)^2 = 0$$

To fix the conformal factor it is necessary to use Penrose's construction [2] for the contraction of two null vectors. The result for the whole metric is

$$ds^2 = 2du dv - 2dx dy - 2\lambda(uv - xy)^{-1}(y dv - v dy)^2 \quad (4)$$

This is in Kerr-Schild form and corresponds in linear theory to a type N elementary state, [3]. It is regular everywhere except on the light-cone of the origin. It is possible to displace this singularity to the light-cone (in the background flat-space) of the point $i\mathfrak{t}^a$, so that the \mathcal{X} -space is regular on the (background) future tube, by the translation $u \mapsto u - i\mathfrak{t}^a \ell_a(\xi, \bar{\xi})$ of u where \mathfrak{t}^a is time-like and future-pointing. This translation makes σ^* positive frequency, and is an indication of how the two notions of positive frequency should be connected.

Finally the ansatz (2) can be used when f is a perfect square to obtain a type D elementary state for the \mathcal{X} -space.

1: Newman GRG 7 (1976) 107

2: Penrose GRG 7 (1976) 31

3: Penrose and MacCallum Phys. Rep. 6C (1973) 242

K.P.Tod

*Here $\eta_{ab} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and the first two terms of (4) are $\eta_{ab} d\mathfrak{z}^a d\mathfrak{z}^b$.

8.

Plebanski-ising the Self-dual Yang-Mills Equations.

In a suitable gauge, the self-dual Yang-Mills equations in a right-flat space can be written in terms of a multiplet of scalar potentials. The equations then simplify and bear a strong resemblance to Plebanski's Θ -formulation of the general right-flat metric, [1].

The self-dual Yang-Mills potentials $A_{AA';}^j$ satisfy

$$\nabla_{A(A'} A_{B)}^{A';j} + A_{A'(A}^k A_{B)}^{A';j} = 0 \quad (1)$$

And the fields are then

$$\phi_{A'A';}^j = \nabla_{A(A'} A_{B)}^{A';j} + A_{A(A'}^k A_{B)}^{A';j} \quad (2)$$

For a constant spinor O^A it is possible to find a gauge such that

$$O^A A_{AA';}^j = 0 \quad (3)$$

(since if $A_{AA';}^j$ is in a general gauge, (3) requires a gauge transformation with matrix g_i^j such that

$$O^A A_{AA';}^j g_k^j + O^A \nabla_{AA'} g_k^j = 0 \quad (4)$$

The integrability condition for (4) is just (1).)

Thus

$$A_{AA';}^j = O_A \beta_{A'}^j \quad (5)$$

Substituting (5) in (1) and transvecting with O^A gives $S^{A'} \beta_{A';}^j = 0, S_{A'} = O^A \nabla_{AA'}$

so $\beta_{A';}^j = S_{A'} \Theta_{;}^j$ for scalar potentials $\Theta_{;}^j$. (6)

The remainder of (1) is

$$\square \Theta_{;}^j + 2 S^{A'} \Theta_{;}^k S_{A'} \Theta_{k;j} = 0 \quad (7)$$

which is just the Yang-Mills wave-equation

$$(S_{;}^j \nabla_a + A_{a;j}) (S_j^{k'} \nabla^{a'} + A^{a';k'}) \Theta_{k';}^m = 0$$

Equation (7) is to be compared with Plebanski's second Heavenly equation:

$$\square \Theta + S_c S_{c'} \Theta S^{c'} S^{c''} \Theta_{c''} = 0 \quad (8)$$

where Θ is the potential determining the right-flat space, [1].

Introducing Hertz potentials $B_{AA';}^j$ by

$$B_{AA';}^j = O^A O^B \Theta_{;}^j$$

$$A_{AA';}^j = \nabla_{AA'} B_{AA';}^j = O^A S_{A'} \Theta_{;}^j$$

$$\phi_{A'A';}^j = \nabla_{A(A'} A_{B)}^{A';j} = S_{A'} S_{B'} \Theta_{;}^j$$

so that the fields have a simple expression in terms of the scalar potentials. The analogues of these equations are the expressions for metric, connection and curvature of a right-flat space, [2]

$$H^{AB}_{AA'} = O^A O^B S_{A'} S_{B'} \Theta ; \Gamma^{AB}_{AA'C'} = \nabla_{A(A'} H^{AB}_{B)}^{C'} = O^A S_{A'} S_{B'} S_{C'} \Theta$$

$$\tilde{\Psi}_{A'A'C'D'} = \nabla_{A(A'} \Gamma^{AB}_{B)}^{C'D'} = S_{A'} S_{B'} S_{C'} S_{D'} \Theta$$

The analogy goes further in the $SL(2, \mathbb{C})$ case when we can identify $\Theta_{;}^j$ with $S_{A'A';}^j \Theta$ (in Plebanski's coordinates this means $\Theta_{;}^j = (\Theta_{xy}, \Theta_{xz})$) since (8) gives

$$\square S_c S^{c'} \Theta + 2 S^A S_{c'} S^{c'} \Theta S_{A'} S_{c'} S^{c''} \Theta_{c''} = 0$$

which is just (7).

INTEGRALS FOR STRAND NETWORKS

SU(2) diagrams from Spin Networks.

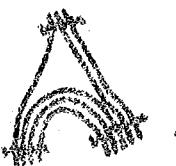
We can evaluate the norm of a closed spin network by calculating the value of the following $SU(2)$ invariant diagram. The typical spin network vertex



where the lines represent ϵ^A , ϵ_{AB} , and ϵ^{AB} so that the original spin network becomes a contracted symmetrised product of spinor epsilon and deltas. These diagrams are explained in TN 4 by John Mousoulis. Rather than calculate their values directly we make use of certain reduction formulae (analogous to those for binors described by R.P. in his article in "Quantum Theory and Beyond"). The diagrams can be simplified a little if we draw



instead of



Integrals for $SU(2)$ diagrams.

George Sparling discovered another way of evaluating an $SU(2)$ diagram. We associate a contour integral with each symmetriser. For example the diagram

$\begin{array}{c} \diagup \\ a \end{array} \begin{array}{c} \diagdown \\ b \end{array}$ has spinors attached to it as follows



The corresponding integral is then

$$\frac{1}{2\pi i} \int \frac{(\alpha \cdot s)^a (\beta \cdot s)^b (\gamma \cdot r)^c (\delta \cdot r)^d (ab\bar{r})!}{a! b! c! d! (q \cdot s)^{ab+cd}} \{s\} \{r\} q \cdot dr$$

10.

where the indices of the spinors α, \dots, β have been omitted and the contour is an S^2 . An adjacent part of the diagram may be

 Here the spinors α and β are also integrated out - indeed in a closed diagram all the spinor variables are integrated.

These diagrams obey certain identities some of which are listed below. Equations one to four are obvious and six follows from five and one. Equation five is proved using an integration by parts argument which is messy but not hard.

$$\textcircled{1} \quad \cancel{\int} = 0$$

$$\textcircled{4} \quad \cancel{\int} \cancel{\int} = \cancel{\int}$$

$$\textcircled{2} \quad \cancel{\int}_a^b = \frac{(a+b)!}{a! b!} \cancel{\int}_0^a$$

$$\textcircled{5} \quad \cancel{\int}_a^b = \sum_{r=0}^b \cancel{\int}_a^r \cancel{\int}_{a+r}^b$$

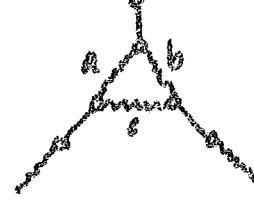
$$\textcircled{3} \quad \cancel{\int} P_b = \frac{(a+b+1)!}{b!(a+1)!} \cancel{\int}_a^b$$

$$\textcircled{6} \quad \cancel{\int}_a^b = b \cancel{\int}_a^a \cancel{\int}_a^b$$

The point is that those identities are precisely the reduction formulae (already referred to) for $SU(2)$ diagrams.

As an example (to which we shall return) consider the diagram  the corresponding integral can be

 represented by the diagram

 where vertices are spinor variables being integrated and straight (wavy) lines are denominator (numerator) factors in the integrand.

Integrals for $SU(1,1)$ diagrams.

As part of the general program of extending the spin network results to other groups we can try to find the

analogous reduction formulae between strand networks representing $SU(1,1)$ coupling coefficients. John Mousouris has pointed out that to obtain some of these coupling coefficients it is enough to simply exchange numerator and denominator factors in our integrals (and to adjust the new numerator until the homogeneity of the integrand is zero again). We shall integrate the triangle in the centre of the $SU(1,1)$ version of the $SU(2)$ integral mentioned above:

$$\text{Diagram} = \int \frac{(\beta \cdot \beta)^{abc-2} (\eta \cdot \gamma)^{ab-c} (\beta \cdot \alpha)^{bc-a} \gamma \cdot d \beta \cdot \eta \cdot d \gamma \cdot \beta \cdot d \beta}{(\beta \cdot \eta)^a (\eta \cdot \beta)^b (\beta \cdot \beta)^c}$$

We can do the β integration first using an S^1 contour:

$$\int \frac{(\beta \cdot \beta)^{abc-2} \beta \cdot d \beta}{(\beta \cdot \eta)^a (\beta \cdot \beta)^c} = \frac{(abc-2)!}{a! c!} \int (\beta \cdot \beta)^{a-1} (\eta \cdot \beta)^{-1} \beta \cdot d \beta = \frac{2\pi i (abc-2)! (\beta \cdot \beta)^{a-1}}{a! c! (\eta \cdot \beta)^{abc-1}}$$

Instead of doing the remaining integral (which has an S^2 contour) we notice that it corresponds to an $SU(2)$ diagram which we then evaluate using the $SU(2)$ reduction formulae:

$$\int \frac{(\beta \cdot \alpha)^{b+c-2} (\beta \cdot \beta)^{a-1} (\eta \cdot \gamma)^{ab-2} (\eta \cdot \beta)^{c-1} \eta \cdot d \gamma \cdot \beta \cdot d \beta}{(\eta \cdot \beta)^{ab+c-1}} = \text{Diagram}$$

$$\text{from } ⑤ \sum_{r=0}^{ab-2} \text{Diagram} = \sum_{r=0}^{ab-2} \frac{(\beta \cdot \beta)^{ab-2-r} (\beta \cdot \beta)^{a-1-r} (\beta \cdot \beta)^{b-1-r}}{(\eta \cdot \beta)^{ab+r}} = (\eta \cdot \beta)^{ab-1} (\beta \cdot \beta)^{a-1} (\beta \cdot \beta)^{b-1}$$

because the loop  on β has $a-1-r$ on it and ⑤ implies that $r=a-1$ is the only non-zero term in the sum. It is not clear whether this example is sufficient to describe all the integrals arising in the $SU(1,1)$ case. Stephen ... with help from R.P., G.A.T.S., and T.M. suggests

12.

A 2-TWISTOR FUNCTION FOR THE ELECTROMAGNETIC DIPOLE

We can write the field due to an electromagnetic dipole in the form

$$\Phi_{AB} = \mu^{A'} \nu^{B'} \nabla_{AA'} \nabla_{BB'} \left(\frac{1}{r} \right), \quad (1)$$

Our purpose here is to show how to describe such a dipole by means of a 2-twistor function. The following twistor prescription for the electromagnetic potential $\varphi_{AA'}$ and the field Φ_{AB} is used:

$$\varphi_{AA'} = \frac{1}{(2\pi i)^4} \oint \left(Z_A \frac{\partial}{\partial Z_A} - X_A \frac{\partial}{\partial X_A} \right) D(\bar{z}, \bar{x}) d^2 z \wedge d^2 x \quad (2)$$

$$\Phi_{AB} = \frac{1}{(2\pi i)^4} \oint i \bar{Z}_k \left(\frac{\partial^2}{\partial Z^A \partial X^B} + \frac{\partial^2}{\partial X^A \partial Z^B} \right) D(\bar{z}, \bar{x}) d^2 z \wedge d^2 x. \quad (3)$$

Here $D(\bar{z}, \bar{x})$ is homogeneous of degree $(-2, -2)$ and the contour of integration in (2) and (3) is $S^4 S^4$. It follows from (3) that Maxwell's equations amount to

$$\oint \left(Z^{A'} \frac{\partial}{\partial Z^B} - X^{A'} \frac{\partial}{\partial X^B} \right) \bar{Z}_k \frac{\partial}{\partial Z^A} \frac{\partial}{\partial X^B} D(\bar{z}, \bar{x}) d^2 z \wedge d^2 x = 0, \quad (4)$$

The twistor functions

$$\bar{Z}_X / \bar{Z}_X^2 \bar{Z}_X \quad \text{and} \quad \bar{Z}_X / \bar{Z}_X \bar{Z}_X^2$$

where (a) $R = \begin{pmatrix} -2 \\ m^2 \end{pmatrix} \begin{pmatrix} 0 & P_A^{B'} \\ P_{A'}^B & 0 \end{pmatrix}$, ($P_A = (m, 0, 0, 0)$) is an inverse

angular momentum twistor (see TN4) and (b) A satisfies $|A| = 0$, can be shown to satisfy (4). Now consider the function

$$D_1(\bar{z}, \bar{x}) = \bar{Z}_X / \bar{Z}_X^2 \bar{Z}_X^2 \quad \text{where } A = \begin{pmatrix} 0 & r_A^{B'} \\ t_A^B & 0 \end{pmatrix}.$$

Substituting this into (2) we obtain a (one-sided) potential, which can be written in the form

$$\varphi_{AA'} = \frac{m}{8} (r_A^{B'} - t_A^{B'}) p_{c'} \nabla_{A'}^c \left(\frac{1}{r} \right).$$

This gives $\Phi_{AB} = \nabla_B^B \varphi_{AA'} = 0$ ((2) $\Rightarrow \nabla^2 \varphi_a = 0 \Rightarrow \nabla_B^B \varphi_{AA'}$ is automatically symmetric) and choosing $r_A^{B'} - t_A^{B'} = (-2)(\frac{8}{m^2})(\frac{1}{m}) \bar{\mu}_A \bar{\nu}^B p_0^{B'}$ we have

$$\tilde{\Phi}_{A'B'} = \nabla_{B'}^B \varphi_{AA'} = \bar{\mu}^A \bar{\nu}^B \nabla_{AA'} \nabla_{BB'} \left(\frac{1}{r} \right).$$

To obtain the Φ_{AB} part of the field, consider the twistor function

$$D_2(\bar{z}, \bar{x}) = \frac{\bar{Z}_X}{\bar{Z}_X^2 \bar{Z}_X} - \frac{\bar{Z}_X}{\bar{Z}_X \bar{Z}_X^2} \quad \text{where } \Pi = \begin{pmatrix} 0 & 0 \\ 0 & \frac{8}{m} \mu^{A'} \nu^{B'} \end{pmatrix} \text{ and } \tilde{A} = \begin{pmatrix} 0 & \tilde{r}_A^{B'} \\ \tilde{t}_A^B & 0 \end{pmatrix}.$$

This gives rise, via (2), to a potential which we can write in the form $\varphi_{AA'} = \left[\frac{2}{m^2} P_{AA'} \mu^{c'} \nu^{c'} + \frac{m}{8} (r_A^{c'} - t_A^{c'}) \epsilon_{A'}^{c'} \right] p_{c'}^c \nabla_{c'} \nabla_{c'} \left(\frac{1}{r} \right)$.

Now choose

$$\tilde{r}_A^{c'} - \tilde{t}_A^{c'} = \left(\frac{-16}{m^3} \right) P_{AE'} \mu^{c'} \nu^{c'}$$

then $\tilde{\Phi}_{A'B'} = 0$ and $\Phi_{AB} = \mu^{A'} \nu^{B'} \nabla_{AA'} \nabla_{BB'} \left(\frac{1}{r} \right)$.

We therefore have shown that the twistor function

$D(\bar{z}, \bar{x}) = D_1(\bar{z}, \bar{x}) + D_2(\bar{z}, \bar{x})$ will generate the dipole field.

Nicholas P. Jell.

13.

Sheaf Cohomology and an Inverse Twistor Function.

In TN5, R.P. discussed massless fields and sheaf cohomology for both negative and positive homogeneity twistor functions. This note provides an alternative treatment of the positive homogeneity case.

We work in the future tube $\mathbb{C}M^+$. Let F^+ be the primed spin-bundle over $\mathbb{C}M^+$, with coordinates (x^α, π_α) ; F^+ is a bundle over both $\mathbb{C}M^+$ and \mathbb{T}^+ (see TN5). Let $\mathcal{H}(n)$ ($n \geq -1$) be the sheaf of holomorphic functions on F^+ , homog. of degree n in π_α .

Define the differential operator D_A by $D_A = \pi^\alpha \nabla_{A\alpha}$; then the subsheaf $\mathcal{O}(n) = \{f \in \mathcal{H}(n) \mid D_A f = 0\}$ is in effect the sheaf of holomorphic functions on \mathbb{T}^+ , homog. of degree n . Define a third sheaf $\mathcal{Q}_A(n)$ by $\mathcal{Q}_A(n) = \{q_n(x^\alpha, \pi_\alpha) \mid D^\alpha q_n = 0, q_n \text{ homog. deg. } n \text{ in } \pi_\alpha\}$.

It is not difficult to check that these three sheaves fit into a short exact sequence

$$0 \rightarrow \mathcal{O}(n) \rightarrow \mathcal{H}(n) \xrightarrow{D_A} \mathcal{Q}_A(n+1) \rightarrow 0.$$

This leads to a long exact sequence of cohomology groups:

$0 \rightarrow H^0(\mathcal{O}(n)) \rightarrow H^0(\mathcal{H}(n)) \rightarrow H^0(\mathcal{Q}_A(n+1)) \rightarrow H^1(\mathcal{O}(n)) \rightarrow H^1(\mathcal{H}(n)) \rightarrow \dots$,
 where the base space in each H^* is understood to be F^+ . Let us investigate each of the groups in (i).

(i) $H^0(\mathcal{O}(n))$ is the space of twistor polynomials of degree n . In space-time it can be represented as

$$T_n = \left\{ \underbrace{M_{\alpha_1 \dots \alpha_n}}_n(x) \mid \nabla_{A(\alpha_1} M_{\alpha_2 \dots \alpha_n)} = 0, M_{\alpha_1 \dots \alpha_n} = M_{\alpha'_1 \dots \alpha'_n} \right\}.$$

(ii) $H^0(\mathcal{Q}_A(n))$ is clearly isomorphic to

$$A_n = \left\{ \underbrace{\lambda_{\alpha_1 \dots \alpha_n}}_n(x) \mid \lambda_{\alpha_1 \dots \alpha_n} \text{ holom. on } \mathbb{C}M^+, \lambda_{\alpha'_1 \dots \alpha'_n} = \lambda_{(\alpha_1 \dots \alpha_n)} \right\}.$$

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(iii) An element of $H^0(\mathcal{Q}_n(n+1))$ has the form $\Psi_{AA'\dots c} \pi^{A'} \dots \pi^c$, where $\Psi_{AA'\dots c}(x)$ is holomorphic on $\mathbb{C}M^+$, symmetric in its $n+1$ primed indices, and satisfies

$$\nabla_A(A' \Psi_{A'\dots c}) = 0. \quad (2)$$

Denote the group of such $\Psi_{AA'\dots c}$ by \mathcal{V}_n .

(iv) $H^1(\mathcal{O}(n))$ is just $H^1(\mathbb{P}\mathbb{T}^+, \mathcal{O}(n))$.

(v) $H^1(\mathcal{H}(n)) = 0$.

For $n=-1$, take $T_n=0$, $\Lambda_n=0$.

Putting all this together, the sequence (1) becomes

$$0 \rightarrow T_n \xrightarrow{i} \Lambda_n \xrightarrow{\sigma_1} \mathcal{V}_n \xrightarrow{\sigma_2} H^1(\mathbb{P}\mathbb{T}^+, \mathcal{O}(n)) \rightarrow 0, \quad (3)$$

where i is the obvious injection, the map σ_1 is given by

$$\sigma_1 : \lambda_{A'\dots c} \mapsto \Psi_{AA'\dots c} = \nabla_A(A' \lambda_{B'\dots c}), \quad (4)$$

and σ_2 will be described below. But first we observe that if we define Φ_n to be the group of helicity $-\frac{1}{2}(n+2)$ massless fields on $\mathbb{C}M^+$, i.e.

$$\Phi_n = \left\{ \varphi_{\underbrace{A\dots c}_{n+2}}(z) \mid \varphi_{A\dots c}(z) \text{ holom. on } \mathbb{C}M^+, \nabla^{AA'} \varphi_{A\dots c} = 0 \right\},$$

then the sequence

$$0 \rightarrow T_n \xrightarrow{i} \Lambda_n \xrightarrow{\sigma_1} \mathcal{V}_n \xrightarrow{\sigma_2} \Phi_n \rightarrow 0 \quad (5)$$

is exact, the map σ_2 being defined by

$$\sigma_2 : \Psi_{AA'\dots c} \mapsto \varphi_{AB\dots D} = \underbrace{\nabla_{(A}^{A'} \dots \nabla_{D)}^{D'} \Psi_{A)B\dots c}}_{n+1 \text{ derivatives}}. \quad (6)$$

[Note that the sequence (5) is conformally invariant.]

Comparing (3) and (5) yields the theorem

$$\boxed{\Phi_n \cong H^1(\mathbb{P}\mathbb{T}^+, \mathcal{O}(n)).}$$

So we see that the map σ_2 is in effect an inverse twistor function: given a massless field

$\varphi_{A\dots c}$, first find a potential $\psi_{AA'\dots c}$ and then apply σ_2 to give a twistor cocycle. One can describe σ_2 explicitly using a technique pioneered by G.A.J. Sparling: the construction is as follows.

Given $\psi_{AA'\dots c}$ satisfying (2), we want a cover $\{U_j\}$ of PT^+ and twistor functions f_{jk} on $U_j \cap U_k$. Let $\{U_j\}$ be a locally finite cover with the property that for each U_j there exists a point Y_j^* in PT^+ such that

$Z^* \in U_j \Rightarrow$ the line joining Y_j^* and Z^* lies entirely in PT^+ . — (7)

Such a cover certainly exists (one can construct it explicitly).

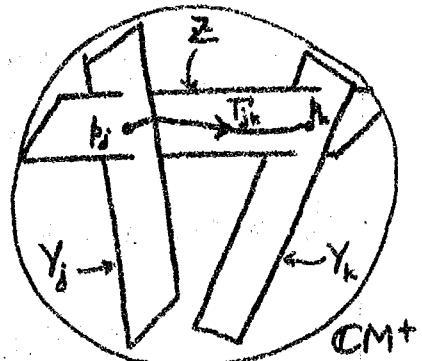
Suppose now that $Z^* \in U_j \cap U_k$. In CM^+ the setup is that Y_j^* , Y_k^* and Z^* are represented by totally null 2-planes Y_j , Y_k and Z ; the points $p_j = Y_j \cap Z$ and $p_k = Y_k \cap Z$ lie in CM^+ (because of (7)). Choose some contour T_{jk} in Z from p_j to p_k and put

$$f_{jk}(Z) = \int_{T_{jk}} \psi_{AA'\dots c} \pi^{A'} \dots \pi^c dx^{AA'}$$

One can now check that

- (a) f_{jk} is a holomorphic 1-cocycle and hence determines an element F of $H^1(\text{PT}^+, \Omega(n))$;
- (b) the choice of T_{jk} doesn't affect f_{jk} ;
- (c) a gauge transformation $\psi_{AA'\dots c} \mapsto \psi_{AA'\dots c} + \nabla_A(A'\lambda^{A'\dots c})$ changes f_{jk} but does not change F .
- (d) If $\sigma_4 : H^1(\text{PT}^+, \Omega(n)) \rightarrow \mathbb{E}_n$ is the usual evaluation map, then $\sigma_4 \circ \sigma_2 = \sigma_3$.

Richard Ward.



16. Complex pp-Waves

These are the simplest half-flat spacetimes; the metrics can be put into the form $ds^2 = du dv - dy dz + g(u, z) du^2$, where g is an arbitrary holomorphic function. They can all be generated explicitly from the non-linear graviton construction. To see this, take $PT := U \cup \hat{U}$, where $U = \{[z^\alpha] : z^2 \neq 0\}$ and $\hat{U} = \{[z^\alpha] : z^3 \neq 0\}$; then any element of $H^1(PT, \Omega(2))$ can be represented by a function homogeneous of degree 2 and holomorphic over $U \cap \hat{U}$. Such a function generates a deformation given by $\hat{w}^A := \exp\left\{e^{CD} \frac{\partial f}{\partial w^C} \frac{\partial}{\partial w^D}\right\} w^A$ provided that the series converges. Under the assumption that f is independent of Z^1 , this becomes

$$\hat{w}^0 = w^0; \quad \hat{w}^1 = w^1 + \frac{\partial f}{\partial w^0}; \quad \hat{\pi}_{A1} = \pi_{A1}. \quad (1)$$

A cross-section of the resulting fiber space $\mathcal{F} \rightarrow \mathbb{C}^2 - \{0\}$ will be given by a collection $\{\hat{w}^A = \hat{g}^A(\pi_B); w^A = g^A(\pi_B)\}$ satisfying the transition relation in (1) where g^A (resp. \hat{g}^A) are homogeneous of degree 1 and holomorphic for $\pi_{A1} \neq 0$ (resp. $\pi_{A1} \neq 0$).

So $g^0 = \hat{g}^0$, and by Hartog's theorem $\{g^0 \cup \hat{g}^0\}$ is entire \Rightarrow

$$g^0 = \hat{g}^0 = u\pi_{01} + j\pi_{11} \quad (2)$$

with u and j arbitrary constants. For the second components, we have

$$\hat{g}^1(\pi_{A1}) = g^1(\pi_{A1}) + \frac{\partial f}{\partial w^0}(g^0(\pi_{A1}); \pi_{A1}), \quad \pi_{01}, \pi_{11} \neq 0. \quad (3)$$

Now fix u and j ; then $\frac{\partial f}{\partial w^0}(u\pi_{01} + j\pi_{11}; \pi_{A1})$ is holomorphic and homogeneous of degree 1 over an annular region of $P_1(\mathbb{C})$. So it can be "split"; and each different way of writing it as a coboundary will give rise to a different pair $\{g^1, \hat{g}^1\}$.

To see that there is precisely a 2-parameter family of such splittings for each fixed (u, j) , is not difficult: Let D (resp. \hat{D}) be the region of $P_1(\mathbb{C})$ given by $|\pi_{11}/\pi_{01}| < b$ (resp. $|\pi_{11}/\pi_{01}| > a$) with $a < b$, and let

$\Pi, \tilde{\Pi}$ be the boundaries of these regions (both positively oriented in the chart 17. $\Pi_0 \neq \emptyset$). Let δ^{A1}, δ^{B1} be any two origins such that $[\delta^{A1}], [\delta^{B1}] \notin \partial \Pi \cup \tilde{\Pi}$ and define

$$g^1(u, s, \delta, \bar{s}; \pi_{A1}) := (\delta^{A1} \pi_{A1})(\delta^{B1} \pi_{B1}) \left\{ \frac{1}{2\pi i} \oint_{\Gamma} \frac{\frac{\partial f}{\partial w_0}(u_{\rho}, s_{\rho}; p_{cl}) \Delta p}{(\delta^{A1} p_{cl})(\delta^{B1} p_{cl})(\pi_{A1} p_{cl})} \right\} \quad \dots \quad (4)$$

$$g^1(\quad) = " \quad \left\{ \oint_{\Gamma} " \quad \right\}$$

The expressions for the g 's depend only on the ratios $\gamma = \delta^1/\delta_0$, and $\bar{\gamma} = \delta^{11}/\delta_0$, so we have a 4-parameter family of cross-sections, although it turns out that $\delta + \bar{\delta}$ are not the most useful coordinates.

Computing the metric: For simplicity, choose $\delta^{A1} = (t, -1)$, $\delta^{B1} = (\bar{t}, -1)$ with $|t|, |\bar{t}| > a$, so that $(\delta^{A1} p_{cl})^{-1}$ and $(\delta^{B1} p_{cl})^{-1}$ are holomorphic over int Π .

Write $\frac{\partial f}{\partial w_0}(u_{\rho}, s_{\rho}; p_{cl}) = p_{cl} \frac{\partial f}{\partial w_0}(u + s\rho; 1, \rho) = p_{cl} \sum_{n=-\infty}^{+\infty} a_n(u, s) \rho^n$. Plugging this series into (4), it is not difficult to see that g^1 must have the form

$$g^1 = r(u, s, \delta, \bar{\delta}) \pi_{01} + s(u, s, \delta, \bar{\delta}) \pi_{11} - \sum_{n=2}^{\infty} a_n(u, s) \frac{(\pi_{11})^n}{(\pi_{01})^{n-1}} \quad \dots \quad (5)$$

Making the coordinate transformation $(u, s, \delta, \bar{\delta}) \rightarrow (u, s, v = s(u, s, \delta, \bar{\delta}), \tilde{s} = r(u, s, \delta, \bar{\delta}))$ the cross-sections take the form (in the first chart):

$$\begin{aligned} g^0 &= u \pi_{01} + \bar{s} \pi_{11} \\ g^1 &= \tilde{s} \pi_{01} + v \pi_{11} - \sum_{n=2}^{\infty} a_n(u, s) \frac{(\pi_{11})^n}{(\pi_{01})^{n-1}} \end{aligned} \quad \dots \quad (6)$$

A tangent vector with components $(du, dv, ds, d\tilde{s})$ at the point (u, v, s, \tilde{s}) corresponds to the cross-section

$$\begin{aligned} V^0 &= \pi_{01} du + \pi_{11} dv \\ V^1 &= \pi_{01} d\tilde{s} + \pi_{11} dv - \left[\sum_{n=2}^{\infty} \frac{\partial a_n}{\partial u}(u, s) \frac{(\pi_{11})^n}{(\pi_{01})^{n-1}} \right] du - \left[\sum_{n=2}^{\infty} \frac{\partial a_n}{\partial s} \frac{(\pi_{11})^n}{(\pi_{01})^{n-1}} \right] ds \end{aligned} \quad \dots \quad (7)$$

18 (together with the appropriate V^0, V^1) of the normal bundle of the section labelled by (u, \bar{v}, \bar{z}, v) . Fixing $(du, d\bar{v})$ arbitrarily, V^0 will vanish if $(\pi_0, \pi_{11}) \sim (-d\bar{z}, du)$. Substituting this into V^1 , we see that $V^1 = 0 \Leftrightarrow dv, d\bar{z}$ satisfy

$$0 = du dv - d\bar{v} d\bar{z} - \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\partial a_n}{\partial u} \frac{(du)^{n+1}}{(d\bar{v})^{n-1}} - \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\partial a_n}{\partial \bar{z}} \frac{(dv)^n}{(d\bar{z})^{n-2}}$$

$$\text{or } 0 = du dv - d\bar{v} d\bar{z} + \frac{\partial a_2(u, \bar{z})}{\partial \bar{z}} du^2 - \sum_{n=2}^{\infty} (-1)^{n+1} \left[\frac{\partial a_n}{\partial u} - \frac{\partial a_{n+1}}{\partial \bar{z}} \right] \frac{(du)^{n+1}}{(d\bar{z})^{n-1}}$$

Due to the functional form of $\frac{\partial f}{\partial w^0}(u+jp; l, p)$, the power series vanishes identically, and we are left with

$$ds^2 = du dv - d\bar{v} d\bar{z} + \frac{\partial a_2(u, \bar{z})}{\partial \bar{z}} du^2 \quad \dots \quad (8)$$

(Strictly speaking, this is only the conformal metric, but it is not difficult to see that the conformal factor is 1.)

So, given $f(z^a)$, independent of z^1 , the metric of the pp-wave is given by

$$ds^2 = du dv - d\bar{v} d\bar{z} + \left\{ \frac{1}{2\pi i} \oint \frac{dp}{p^2} \left[\frac{\partial^2 f}{\partial w^0 \partial \bar{z}}(u+jp; l, p) \right] \right\}_{p=0} du^2 \dots \quad (9)$$

It is not difficult to see that any monomial $u^a \bar{v}^b \bar{z}^c$ can be generated in this fashion by choosing $f(z^a)$ to have the form $\frac{(z^0)^a}{(z^4)^b (z^3)^c}$ with $a-b-c=2$. By taking linear combinations, an arbitrary holomorphic function of $u+\bar{v}$ can be generated, so we get all of the pp-waves.

W.D. Curtis, D. Kerner, & F.R. Millar
December, 1977

(D. Kerner acknowledges an extremely useful conversation with K.P. Tod)

Note on the n -twistor internal symmetry group

Consider a system of n twistors Z_1, Z_2, \dots, Z_n with complex conjugates $\bar{Z}_1, \bar{Z}_2, \dots, \bar{Z}_n$, where we write $\phi = Z_n$ and $\bar{\phi} = \bar{Z}_n$ ($n = 1, \dots, n$).

The kinematic twistor (i.e. "angular momentum twistor")

$$A^{\alpha\beta} = U \text{ is } A^{\alpha\beta} = 2 \sum_{n=1}^n Z_n I^{(\beta)} \bar{Z}_n, \text{ i.e. } U = \begin{bmatrix} & \phi \\ \bar{\phi} & \end{bmatrix}.$$

The n -twistor internal symmetry group (or, simply, the n -twistor group) is defined to be the group of complex linear transformations of $\phi, \bar{\phi}$ (i.e. linear transformations on $T \oplus \dots \oplus T \oplus T^* \oplus \dots \oplus T^{**}$) that preserve U ($= A^{\alpha\beta}$) and the relation of complex conjugacy between ϕ and $\bar{\phi}$.

For some time it has been known that the transformations

$$\text{g: } \begin{pmatrix} Z_n \\ \bar{Z}_n \end{pmatrix} \mapsto \begin{pmatrix} U_n^s \delta_\alpha^\alpha & \Lambda^s \bar{U}_n^s I^{*\beta} \\ \bar{\Lambda}^s U_n^s I_{\alpha\beta} & U_n^s \delta_\alpha^\beta \end{pmatrix} \begin{pmatrix} Z_s \\ \bar{Z}_s \end{pmatrix} \quad \begin{cases} \text{with } U_n^s \text{ (nn)-unitary} \\ \text{and } \Lambda^s \text{ (nn)-complex skew} \\ \bar{U}_n^s = U_n^s, \bar{\Lambda}^s = \Lambda^s \end{cases}$$

belong to the n -twistor group, and it has been suspected that they constitute the entire n -twistor group. Recently, we were able to show that, indeed, they constitute the entire connected component of the identity of the n -twistor group (so it remains only to show that, the n -twistor group is connected — or not, as the case may be). Writing U for U_n^s , U^* for its transpose and Λ for Λ^s , the U, U^* multiplication law becomes

$$(U, \Lambda) \circ (U', \Lambda') = (UU', \Lambda + U\Lambda'U^*) \quad (\dots \text{an "inhomogeneous" } U(n))$$

20.

This group acts as

$$\phi \mapsto \phi - \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array}, \quad \phi \mapsto \phi - \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} \quad \text{where } \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \text{ are } U, \bar{U}, A, \bar{A}.$$

The argument consists of showing that the infinitesimals of the n -twistor group have the form of infinitesimal T 's:

$$\phi \mapsto \phi + i \epsilon \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} - \epsilon \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} \quad \text{with } \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} \text{ Hermitian, } \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} \text{ skew.}$$

Certainly the infinitesimal n -twistor group elements have the form $\phi \mapsto \phi + \epsilon (\begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} + \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array})$, and preservation of U leads to:

$$\begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} + \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} = 0, \quad \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} + \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} = 0 \quad (2)$$

(reflection in the horizontal mass complex conjugation)

By first symmetrizing (1) over the first three free arms and then applying Π to the last arm of (1) we obtain expressions which, by the use of simple lemmas give

$$\begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} = \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} \quad \text{for some } \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array}, \text{ which is readily shown to be skew.}$$

By applying somewhat similar operations to (2) and using rather more involved lemmas we finally derive $\begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array} = i \begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array}$ for some Hermitian $\begin{array}{c} \text{square} \\ \text{with } \frac{1}{2} \text{ loops} \end{array}$. \square

Note : The quantized twistor operators obtained from these infinitesimal generators define the algebra which gives the classification of n -twistor particles (quantization: $\mathbb{Z}_{\ell^{\infty}}^{n(n-2)}$). For $n=3$ this algebra is the same as that of Z.P.'s inhomogeneous $SU(3)$.

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