

# Twistor Newsletter (no 7: 22, June, 1978)

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## Local $H^1$ 's and Propagation

We have now become accustomed to the fact that, at least for a region  $R \subset \mathbb{C}M$  that is appropriately convex (e.g.  $R = M^+$ , the forward tube), massless free fields of helicity  $s$  in  $R$  are represented, in twistor space, by the elements of the sheaf cohomology group  $H^1(\mathcal{L}, \Omega(-2s-2))$ , where  $\mathcal{L} \subset \mathbb{P}\mathbb{T}$  is the region swept out by all the lines in  $\mathbb{P}\mathbb{T}$  that correspond to the points of  $R$  (e.g.  $\mathcal{L} = \mathbb{P}\mathbb{T}^+$  when  $R = M^+$ , giving the positive frequency fields). (See R.P. in TN 2, N.M.J.W in TN 2, R.S.W. in TN 6, R.P. in TN 5, L.H., M.E. in TN 7 etc.)

There is an asymmetry involved in this correspondence in that  $\mathcal{L}$  is an extended region swept out by lines, whereas  $R$  can be



a local region of  $\mathbb{C}M$ , say a small neighbourhood of a point  $R$  of  $\mathbb{C}M$ . However (as was pointed out to me by C.D. Hill of Stony Brook), non-vanishing  $H^1$ 's can arise also for

local regions  $\mathcal{L}$  of  $\mathbb{P}\mathbb{T}$ . What is the space-time interpretation of these local  $H^1$ 's? I shall attempt to give here a partial answer to this question. There may also be some relation to A.P.H.'s 2<sup>nd</sup> article and M.G.'s article in TN 7.

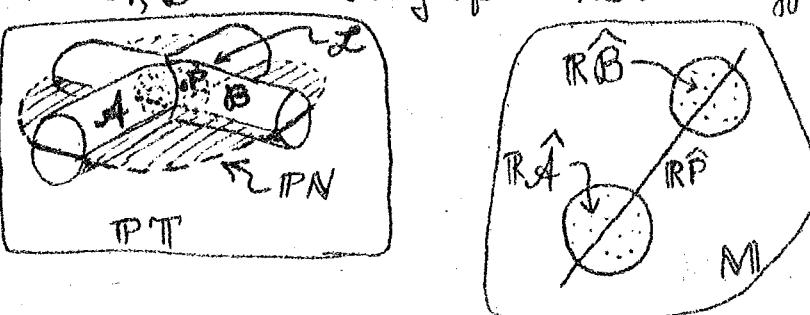
2.

Consider, first, a somewhat more general situation, where an  $n$ -complex-dimensional Stein manifold  $\mathcal{S}$  is divided into two open regions  $\mathcal{S}^+$  and  $\mathcal{S}^-$  by a smooth  $(2n-1)$ -real-dim. hypersurface  $\mathcal{N}$  whose Levi form has constant signature  $(p, q)$ , with  $p+q = n-1$ . Thus,  $\mathcal{S}^+$  has "q degrees of convexity" at  $\mathcal{N}$  and  $\mathcal{S}^-$  has "p degrees of convexity" at  $\mathcal{N}$  (so that  $\mathcal{S}^+$  would be Stein if  $p=0$ , and  $\mathcal{S}^-$  Stein if  $p=n-1$ ). Assume  $p, q > 0$ . Then by results of Andreotti, Hille, etc. there is a tangential Dolbeault complex<sup>for  $\mathcal{N}$</sup> , whose cohomology, which I'll write  $H^r(\mathcal{N}, \mathcal{D})$ , ( $= H^{0,r}$ ) with  $r=0, 1, 2, \dots$ , is given by  $H^0(\mathcal{N}, \mathcal{D}) \cong H^0(\mathcal{S}, \mathcal{O})$ ,  $H^p(\mathcal{N}, \mathcal{D}) \cong H^p(\mathcal{S}^+, \mathcal{O})$ ,  $H^q(\mathcal{N}, \mathcal{D}) \cong H^q(\mathcal{S}^-, \mathcal{O})$ , provided that  $p \neq q$ , and all the other  $H^r(\mathcal{N}, \mathcal{D})$ 's vanish. This is assuming that we are concerned with hyperfunction cohomology on  $\mathcal{N}$ . Had we used, say, just  $C^\infty$  (or even distribution) cohomology on  $\mathcal{N}$  then the above isomorphisms would have to be replaced by inclusions. In the case  $p=q$  the only modification is that now we have  $H^p(\mathcal{N}, \mathcal{D}) \cong H^p(\mathcal{S}^-, \mathcal{O}) \oplus H^p(\mathcal{S}^+, \mathcal{O})$ , so that, in effect,  $H^p(\mathcal{N}, \mathcal{D})$  naturally decomposes into a negative and positive "frequency" part. In each non-zero case, the space  $H^r(\mathcal{N}, \mathcal{D})$  (and  $H^r(\mathcal{S}^\pm, \mathcal{O})$ ) is infinite-dimensional. The Stein manifold  $\mathcal{S}$  can be chosen to be an arbitrarily small (pseudo-convex) neighbourhood of a given point  $P$  of a given real hypersurface  $\mathcal{N}$  in a complex manifold. In this sense, these  $\infty$ -dimensional  $H^r$ 's may be regarded as defined locally on  $\mathcal{N}$ .

We are concerned with the case  $\mathcal{N} = \mathbb{P}\mathbb{N}$ , so  $p=q=1, n=3$ . Thus, we have local  $H^1$ 's on  $\mathbb{P}\mathbb{N}$  which have, in some sense, canonically defined positive and negative "frequency" parts. What do they mean in space-time terms? A point to bear in mind is that since non-analytic cohomology (in fact hyperfunction cohomology) on  $\mathcal{N}$  is involved, the space-time interpretation must refer to non-analytic behaviour in the space-time fields. My contention is that these local  $H^1$ 's refer to propagation of massless fields. Until now, a weakness of the twistor description of fields has been that such questions (e.g. domains of dependence, etc) have been almost totally ignored. We now have the potentiality to remedy this situation. But I can, as yet, give only two partial attempts at interpretation.

The problem is, of course, that  $\mathcal{S} \subset \mathbb{P}\mathbb{M}$ , being Stein, can contain no projective line in  $\mathbb{P}\mathbb{M}$  (since a line is compact and of positive dimension). Thus the "normal" interpretation of  $H^1(\mathcal{S}^\pm, \mathcal{O})$  in terms of massless fields in

$\mathbb{C}M$  must break down. However, let us "approximate"  $\mathcal{S}$  by taking a local neighbourhood  $\mathcal{L}$ , of a point  $P \in \mathbb{P}N$ , where  $\mathcal{L} = A \cap B$ , the open sets  $A, B \subset \mathbb{P}\mathcal{T}$  being of the "normal" type, i.e. swept out by lines in  $\mathbb{P}\mathcal{T}$  corresponding to the points of two disjoint sets  $A$  and  $B$  (dots denote corresponding regions in  $\mathbb{C}M$ ). Since  $P \in A \cap B$ , we have  $\hat{P}$  passing through the two regions  $A$  and  $B$ .



Here  $\hat{P}$  is an  $\mathbb{X}$ -plane in  $\mathbb{C}M$ , but we can equally well think in terms of  $R\hat{P}$ , the corresponding null line in  $M$ , where  $A$  and  $B$  determine subsets of  $M$  through which the null line  $R\hat{P}$  passes. These two real subsets are denoted  $R\hat{A}$  ( $= \hat{A} \cap M$ ) and  $R\hat{B}$  ( $= \hat{B} \cap M$ ). The subset  $\mathcal{L}$  of  $\mathbb{P}N$  cannot be Stein since its boundary is not pseudo-convex (because the boundaries of  $A$  and  $B$  are not). But  $\mathcal{L}$  can be contained in a Stein manifold  $\mathcal{S}$  (at least if  $A$  and  $B$  are "narrow" enough, i.e. if  $A$  and  $B$  are small enough. Thus, if we can interpret  $H^1(\mathcal{L}^\pm, \mathcal{O})$ , we shall have some sort of approximation to  $H^1(\mathcal{S}^\pm, \mathcal{O})$ , where  $\mathcal{L}^\pm = \mathcal{L} \cap \mathbb{P}\mathcal{T}^\pm$ . This supposes that the contributions to  $H^1(\mathcal{S}^\pm, \mathcal{O})$  from the parts of the boundary not on  $\mathbb{P}N$  can be ignored. Now invoke the Mayer-Vietoris sequence for sheaves ( $A$  and  $B$  being open sets):

$$0 \rightarrow H^0(A \cup B) \rightarrow H^0(A) \oplus H^0(B) \rightarrow H^0(A \cap B) \rightarrow H^1(A \cup B) \rightarrow H^1(A) \oplus H^1(B) \rightarrow H^1(A \cap B) \rightarrow H^2(A \cup B) \rightarrow H^2(A) \oplus H^2(B) \rightarrow \dots$$

which is exact, the sheaf being the same (e.g.  $\mathcal{O}$ ) throughout. Let  $A = A^+$ ,  $B = B^+$  or let  $A = A^-$ ,  $B = B^-$ , where  $A^\pm = A \cap \mathbb{P}\mathcal{T}^\pm$ ,  $B^\pm = B \cap \mathbb{P}\mathcal{T}^\pm$ . Then we have:

$$0 \rightarrow H^0(\mathcal{L}^\pm) \rightarrow H^1(A^\pm \cup B^\pm) \rightarrow H^1(A^\pm) \oplus H^1(B^\pm) \rightarrow H^1(A^\pm \cap B^\pm) \rightarrow H^2(A^\pm \cup B^\pm) \rightarrow 0$$

(taking  $A$  and  $B$  to be suitably convex, so that  $H^2(A^\pm) = H^2(B^\pm) = H^0(A^\pm) = H^0(B^\pm) = 0$ ). I am uncertain whether  $H^2(A^\pm \cup B^\pm)$  is likely to be non-zero. But, in any case, it appears to be concerned with "corner" effects at the junction of the boundary of  $A^\pm$  with that of  $B^\pm$ , so it is probably concerned with properties of  $\mathcal{L}$  that would disappear in the case of  $\mathcal{S}$ . So let us ignore  $H^2(A^\pm \cup B^\pm)$  and consider only that part of  $H^1(\mathcal{L}^\pm)$  that arises as the image of the previous map in the sequence i.e.  $\{H^1(A^\pm) \oplus H^1(B^\pm)\} / H^1(A^\pm \cup B^\pm)$ . Now each  $H^1$  in this quotient is a region swept out by lines and can therefore be interpreted back in  $\mathbb{C}M$  in terms of massless free fields. Now  $H^1(A^\pm)$  and  $H^1(B^\pm)$  describe massless fields (of helicity  $s$ , the sheaf now being  $\mathcal{O}(-2s-2)$  in each case) in the respective regions  $A^\pm = A \cap \mathbb{C}M^\pm$  and  $B^\pm = B \cap \mathbb{C}M^\pm$ . Since  $A^\pm$  and  $B^\pm$  are disjoint,  $H^1(A^\pm) \oplus H^1(B^\pm)$  simply describes fields that are defined (and satisfy the field equations) throughout  $A^\pm \cup B^\pm$ . (This is two statements, one for  $A^+ \cup B^+$  and one for  $A^- \cup B^-$ .) One might have thought that this is just what  $H^1(A^\pm \cup B^\pm)$  describes, since  $A^\pm \cup B^\pm$  is precisely the region swept out by the lines in  $\mathbb{P}\mathcal{T}$  that correspond to the points of  $A^\pm \cup B^\pm$ . But

examination of the argument given in R.P. TN 5 shows that this is in fact not the case. Owing to the fact that the overlap region  $\mathcal{L}^\pm = A^\pm \cap B^\pm$  corresponds, in the projection down to  $\mathbb{P}\mathbb{T}$  from  $F$  (= the spin-vector bundle over  $M$ ), to a disconnected portion of  $F$ , we find that local constancy along the fibres of the projection does not imply constancy along the fibres. Instead, we get back to the statement made a moment ago that  $H'(A^\pm) \oplus H'(B^\pm)$  describes the fields in  $A^\pm \cup B^\pm$ . Essentially what is required, in order to obtain  $H'(A^\pm \cup B^\pm)$  rather than  $H'(A^\pm) \oplus H'(B^\pm)$ , is that the field in  $A^\pm \cup B^\pm$  be extendible to a suitable region that connects  $A^\pm$  to  $B^\pm$ , so that the influence of  $A^\pm$  on  $B^\pm$  is correctly taken into account.

This is not, however, very clear in the case of the complex regions  $A^\pm$  and  $B^\pm$  since questions of analytic continuation get confused with questions of propagation. Instead, we shall work with the real regions  $R\hat{A}$  and  $R\hat{B}$  and be concerned with not-necessarily analytic — in fact, hyperfunctional — fields. First consider a more general situation, where we have some open region  $R\hat{C}$  of  $M$ , this being the intersection with  $M$  of some open region  $\hat{C}$  of  $CM$ . We suppose that  $\hat{C}$  corresponds to the system of lines in some open region  $C$  of  $\mathbb{P}\mathbb{T}$ ,  $C$  being, in turn, swept out by these lines. Let  $Z_*(R\hat{C})$  denote the (hyperfunctional) solutions of the massless free-field equations in  $R\hat{C}$  (helicity  $s$ ) which extend as solutions of these equations to the whole of  $M$ . Let  $\mathcal{C}^\pm = C \cap \mathbb{P}T^\pm$  and  $\hat{\mathcal{C}}^\pm = \hat{C} \cap CM^\pm$ . Then it appears to be the case that:

$$0 \rightarrow H'(\mathcal{C}) \xrightarrow{\text{II}} H'(\mathcal{C}^+) \oplus H'(\mathcal{C}^-) \xrightarrow{\text{II}} H'(\mathcal{C} \cap PN, \mathcal{D}(-2s-2)) \rightarrow 0$$

$$0 \rightarrow Z(\hat{\mathcal{C}}) \rightarrow Z(\hat{\mathcal{C}}^+) \oplus Z(\hat{\mathcal{C}}^-) \rightarrow Z_*(R\hat{C}) \rightarrow 0$$

is exact\* (and commutative). (Unmentioned sheaves are  $\mathcal{O}(-2s-2)$ ; and  $Z(\hat{\mathcal{C}})$  denotes massless helicity  $s$  holomorphic fields in  $\hat{\mathcal{C}}$ , etc.) It is possible that some further restrictions on  $\hat{\mathcal{C}}$  are needed, however.

Assuming the validity of the above, we can apply it to the Mayer-Vietoris sequence of the previous page to obtain

$$\begin{array}{ccccccc} & \circ & & \circ & & \circ & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow H^0(\mathcal{L}) & \rightarrow H'(A \cup B) & \longrightarrow H'(A) \oplus H'(B) & \rightarrow H'(\mathcal{L}) & \rightarrow \dots & & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow H^0(\mathcal{L}^+) \oplus H^0(\mathcal{L}^-) & \rightarrow H'(A^\pm \cup B^\pm) & \rightarrow H'(A^\pm) \oplus H'(B^\pm) & \rightarrow H'(\mathcal{L}^\pm) \oplus H'(\mathcal{L}^\mp) & \rightarrow \dots & & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow \frac{H^0(\mathcal{L}^+) \oplus H^0(\mathcal{L}^-)}{H^0(\mathcal{L})} & \rightarrow Z_*(R\hat{A} \cup R\hat{B}) & \rightarrow Z_*(R\hat{A}) \oplus Z_*(R\hat{B}) & \rightarrow H'(\mathcal{L} \cap PN, \mathcal{D}(-2s-2)) & \rightarrow \dots & & \\ & \downarrow & & \downarrow & & \downarrow & \\ & \circ & & \circ & & \circ & \end{array}$$

\* Since the elements of  $Z_*(R\hat{C})$  extend (non-uniquely) to the whole of  $M$ , we can split this extended field into its positive and negative frequency parts. These are holomorphic in  $M^\pm$ , so restricting to  $\hat{\mathcal{C}}^\pm$  we get  $Z(\hat{\mathcal{C}}^+) \oplus Z(\hat{\mathcal{C}}^-)$ . If one part, together with the negative of the other, extends to an element of  $Z(\hat{\mathcal{C}})$ , then we get zero in  $Z_*(R\hat{C})$ . (Thanks and apologies to R.O.Wells Jr.)

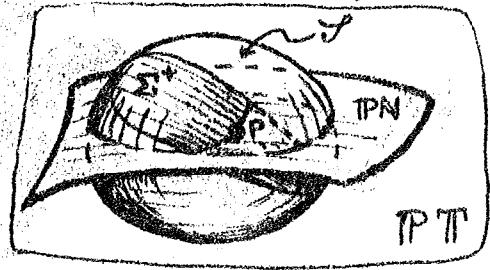
as commutative with exact rows and columns. Note that if  $L$  had been Stein, we should have had  $H^1(L) = 0$ , so the last column would express an isomorphism mentioned at the beginning. In any case, we see that the part of  $H^1(\mathcal{L}_n PN, \mathcal{O}(-2s-2))$  with which we are concerned can be expressed as

$$\{Z_*(R\hat{A}) \oplus Z_*(R\hat{B})\} / Z_*(R\hat{A} \cup R\hat{B}).$$

To put this in words, these are the massless free fields which exist in  $R\hat{A}$  and  $R\hat{B}$  separately modulo those that can be extended to solutions of the field equations in the whole space. This is an expression of the influence that fields in  $R\hat{P}$  have on fields in  $R\hat{B}$  (and vice-versa). We know, after all, that if the field in  $R\hat{A}$  has a discontinuity or non-analyticity along a null geodesic (say  $R\hat{P}$ ) that intersects  $R\hat{B}$  as well as  $R\hat{A}$ , then this non-analyticity must show up also in  $R\hat{B}$ .

Thus, we have an approximate interpretation of the local  $H^1$ 's on  $PN$  in the neighbourhood of a point  $P$  of  $PN$ . As our sets  $A, B$  get narrower and narrower about  $P$ , so also do  $R\hat{A}$  and  $R\hat{B}$  get smaller and smaller. We may consider that in the limiting situation we are concerned simply with propagation along the null line  $R\hat{P}$ , i.e.  $H^1(\mathcal{J}^+) \oplus H^1(\mathcal{J}^-)$ , where  $\mathcal{J}$  is small and Stein, surrounding  $P$  describes the influence that propagates along  $R\hat{P}$ .

There is another way of investigating these local  $H^1$ 's, as suggested to me by H. Grauert. Let  $\mathcal{J}$  be a Stein neighbourhood of a point  $P \in PN$ , with  $\mathcal{J}^\pm = \mathcal{J} \cap P\mathbb{T}^\pm$ ,  $\mathcal{N} = \mathcal{J} \cap PN$ . To construct a non-zero element of  $H^1(\mathcal{J}^+)$  we find some analytic surface  $\Sigma$  in  $\mathcal{J}$



whose intersection  $\Sigma^+$  with  $P\mathbb{T}^+$  is Stein. (Pseudo-convex at  $P$  is sufficient.) We can choose a simple situation, in fact, in which  $\Sigma$  is just the intersection of  $\mathcal{J}$  with a plane, and if this is given by  $A_\alpha Z^\alpha = 0$ , where  $A_\alpha \bar{A}^\alpha \geq 0$ , then  $\Sigma^+$  will indeed be Stein.

Now choose a holomorphic function  $f$  (or twisted function  $f$ , a section of  $\mathcal{O}(-2s-1)$ ) on  $\Sigma^+$  which does not extend meromorphically beyond  $P$  into  $\Sigma^-$  (possible since  $\Sigma^+$  is Stein). Take a Stein covering  $\{\mathcal{U}_i\}$  of  $\mathcal{J}^+$  (necessarily infinite since  $\bigcup \mathcal{U}_i = \mathcal{J}^+$ ) and let  $f_i$  be holomorphic in  $\mathcal{U}_i$  with  $f_i = f$  on  $\Sigma^+$ . Define  $F_{ij} = (f_i - f_j)(A_\alpha Z^\alpha)^{-1}$ .

Then  $F_{ij}$  is holomorphic (section of  $\mathcal{O}(-2s-2)$ ), since the  $f_i$ 's agree on  $\Sigma^+$ , and the cocycle condition is satisfied. Now  $\{F_{ij}\}$  cannot be a coboundary  $F_{ij} = G_i - G_j$ , because  $G_i - f_i(A_\alpha Z^\alpha)^{-1}$  would then be global and meromorphic on  $\Sigma^+$ , but not extendible meromorphically beyond  $P$  — in violation of somebody's theorem,  $\Sigma^+$  being Levi( $+, -$ )-aff.

The twistor function  $F_{ij}$ , having a simple pole near  $P$ , defines a null massless free field (taking  $s \geq 1$ ) defined in the neighbourhood of  $R\hat{P}$  in  $\mathcal{N}$ , and which is non-analytic (possibly hyperfunctional) along  $R\hat{P}$ . Thus, again, we see that  $H^1(\mathcal{J}^+)$  has to do with the propagation of non-analytic behaviour along  $R\hat{P}$ . ~ Roger Penrose

## 6. Some Cohomological Nonsense Applicable to Twistor Theory

§0. Introduction: Recall that if  $Z_n' = \text{self-dual holomorphic solutions of the zero-rest-mass field equations of helicity } n/2 \text{ or } \mathbb{C}M^+$  i.e. in spinor notation :-

$$Z_n' = \left\{ \underbrace{\phi_{AB'...D'}}_n : \mathbb{C}M^+ \rightarrow \mathbb{C} \text{ holomorphic, symmetric in } A', B', \dots, D', \text{ and satisfying } \nabla^{AA'} \phi_{A'B'...D'} \right\}$$

$$\text{then } [RP] \quad Z_n' \cong H^1(\mathbb{C}P^+, \mathcal{O}(-n-2)).$$

The aim of this note is to present this result (also the corresponding isomorphism with unprimed indices  $[RP, RSW]$  and the wave eqns) as special cases of a general machine (Thm 2) which interprets analytic cohomology on  $\mathbb{C}P^+$  in terms of diff<sup>+</sup> eqns over on  $\mathbb{C}M^+$  [cf. Row § 9].

### §1. Some Sheaf Theory: [RG Ch II § 4]

A. Direct Image: Suppose  $\pi: E \rightarrow X$  is a continuous map of topological spaces and that  $\mathcal{S}$  is a sheaf on  $E$ . We may construct the direct image sheaves  $\pi_*^q \mathcal{S}$  on  $X$  for  $q \geq 0$  by means of the presheaves  $V \mapsto H^q(\pi^{-1}(V), \mathcal{S})$  and  $H^q(E, \mathcal{S})$  may be computed (roughly speaking) in terms of the cohomology of these direct image sheaves via the Leray Spectral sequence

$$E_2^{p, q} = H^p(X, \pi_*^q \mathcal{S}) \implies H^{p+q}(E, \mathcal{S}).$$

In §2 the conclusions which follow from this spectral sequence will be provable directly and easily without it so we make no attempt to explain it here.

B. Pull-back: If  $\pi: E \rightarrow X$  is a holomorphic map of complex manifolds and  $\mathcal{A}$  is an analytic sheaf on  $X$  then we may form the pull-back  $\pi^* \mathcal{A}$ , an analytic sheaf on  $E$ . We shall only need the special case where  $\mathcal{A}$  is locally free i.e.  $\mathcal{A}$  is isomorphic to  $\mathcal{O}(V)$  the sheaf of germs of holomorphic sections of a holomorphic vector bundle  $V$  and in this case  $\pi^* \mathcal{O}(V) \cong \mathcal{O}(\pi^* V)$  where  $\pi^* V$  is the usual pull-back of  $V$  as a vector bundle.

C. Resolutions: If  $\mathcal{S}$  is a sheaf on some topological space  $X$  a resolution of  $\mathcal{S}$  is a sequence  $0 \rightarrow \mathcal{R}^0 \rightarrow \mathcal{R}^1 \rightarrow \mathcal{R}^2 \rightarrow \dots$  together with a map  $\epsilon: \mathcal{S} \rightarrow \mathcal{R}^0$  so that the sequence  $0 \rightarrow \mathcal{S} \rightarrow \mathcal{R}^0 \rightarrow \mathcal{R}^1 \rightarrow \dots$  is exact. For shorthand we write  $0 \rightarrow \mathcal{S} \rightarrow \mathcal{R}^*$ . If we know the cohomology of the terms of a resolution of  $\mathcal{S}$  then we may (roughly speaking) compute the cohomology of  $\mathcal{S}$  via a special case of the hypercohomology spectral sequence, namely :

$$E_1^{p, q} = H^q(X, \mathcal{R}^p) \implies H^{p+q}(X, \mathcal{S}).$$

If the resolution is acyclic in the sense that  $H^q(X, \mathcal{R}^p) = 0 \forall p \geq 1$  (e.g. deRham resolution of  $\mathcal{R}$ , Dolbeault resolution of  $\mathcal{O}$ ) then this gives the usual result that a)  $H^p(X, \mathcal{S}) \cong H^p(T(X, \mathcal{R}^*))$ .

In §2 we will encounter a resolution with the property that  $H^q(X, \mathcal{R}^p) = 0 \forall p \geq 2$ . In this case the spectral sequence reduces to an isomorphism  $E^{00} \cong T(X, \mathcal{S})$  and a long exact sequence :-

$$0 \rightarrow E^{10} \rightarrow H^1(X, \mathcal{S}) \rightarrow E^{01} \rightarrow E^{20} \rightarrow H^2(X, \mathcal{S}) \rightarrow E^{11} \rightarrow E^{30} \rightarrow H^3(X, \mathcal{S}) \rightarrow \dots$$

$$\text{where } E^{p,q} = H^p(H^q(X, \mathbb{R}^*)) = \ker : H^q(X, \mathbb{R}^p) \rightarrow H^q(X, \mathbb{R}^{p+1})$$

$$\text{im} : H^q(X, \mathbb{R}^{p-1}) \rightarrow H^q(X, \mathbb{R}^p)$$

7.

This can easily be proved by diagram chasing and in case

$$T(X, \mathbb{R}^p) = 0 \wedge p \geq 0 \text{ we conclude that } \xrightarrow{\text{b)} } H^p(X, \mathbb{A}) \cong H^{p-1}(H^1(X, \mathbb{R}^*))$$

Finally if  $H^q(X, \mathbb{R}^p) = 0$  except for  $(p, q) = (0, 1)$  or  $(2, 0)$

$$\text{then } \xrightarrow{\text{c)} } H^1(X, \mathbb{A}) \cong \ker : H^1(X, \mathbb{R}^0) \rightarrow T(X, \mathbb{R}^2).$$

a), b), c) are easily proved directly and are the only cases needed in §3.

§2. Constructing the Machine: Let  $\mathbb{CM}^+$ ,  $\mathbb{IF}^+$ , and  $\mathbb{PTP}^+$  be as usual with analytic functions  $\alpha, \beta$  and hence the usual diagram

$\mathbb{CM}^+$  may be regarded as a convex open subset of  $\mathbb{C}^{2 \times 2}$ , the space of  $2 \times 2 \mathbb{C}$ -matrices  $\mathbb{M}^{2 \times 2}$ . Explicitly, under  $\mathbb{CM}^+$   $\mathbb{PTP}^+$ , the linear isomorphism  $\mathbb{C}^4 \xrightarrow{\cong} \mathbb{C}^{2 \times 2}$

$$(z^0 + iz^1) = (z^0) \mapsto \begin{bmatrix} z^0 + z^1 & z^2 + iz^3 \\ z^2 - iz^3 & z^0 - z^1 \end{bmatrix},$$

$$\mathbb{CM}^+ = \{(z^0) \text{ s.t. } -y^0 > \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}\} = \mathbb{R}^4 \times \text{past cone}.$$

In particular  $\mathbb{CM}^+$  is Stein.

$\mathbb{IF}^+ \cong \mathbb{CM}^+ \times \mathbb{P}^1$  (holomorphically) and  $\beta$  is then just projection.

We use the standard notation  $T\mathbb{R}' = (\pi_0', \pi_1')$  for homogeneous coords on  $\mathbb{P}^1$  and  $(w^0, \pi_{\mathbb{R}'}) = (w^0, w^1, \pi_0', \pi_1')$  for homogeneous coords on  $\mathbb{PTP}^+$  and in particular  $\mathbb{PTP}^+$  at then has the form  $\alpha(\mathbb{M}^{2 \times 2}, T\mathbb{R}') = (iz^{00}/\pi_0', \pi_{\mathbb{R}'})$ .

To investigate  $\alpha$  we introduce a holomorphic affine bundle  $E$  of rank 2 as follows :-

Let  $V_{\mathbb{O}}^+ = \{(w^0, \pi_{\mathbb{R}'}) \in \mathbb{PTP}^+ \text{ s.t. } \pi_0' \neq 0\}$  These cover  $\mathbb{PTP}^+$

$$V_{\mathbb{I}}^+ = \{(w^0, \pi_{\mathbb{R}'}) \in \mathbb{PTP}^+ \text{ s.t. } \pi_1' \neq 0\}$$

and define  $g_{\mathbb{O}'\mathbb{I}} : V_{\mathbb{I}}^+ \cap V_{\mathbb{O}}^+ \rightarrow \{\text{Invertible affine } f^0 \text{s: } \mathbb{C}^2 \rightarrow \mathbb{C}^2\}$

$$\text{by } g_{\mathbb{O}'\mathbb{I}}(w^0, \pi_{\mathbb{R}'}) \left[ \begin{smallmatrix} u \\ v \end{smallmatrix} \right] = \left[ \begin{smallmatrix} -iw^0 - \sigma\pi_1' \\ \pi_0' \\ -iw^1 - t\pi_1' \\ \pi_0' \end{smallmatrix} \right]. \text{ Use this as a transition function to construct } E.$$

Now we have  $i : \mathbb{F}^+ \hookrightarrow E$  where the inclusion  $i$  is defined by :-

$$i \left( \begin{bmatrix} -iw^0 - \sigma\pi_1' \\ \pi_0' \\ -iw^1 - t\pi_1' \\ \pi_0' \end{bmatrix}, \pi_{\mathbb{R}'} \right) = ((w^0, \pi_{\mathbb{R}'}), \begin{bmatrix} u \\ v \end{bmatrix}) \in V_{\mathbb{O}}^+ \times \mathbb{C}^2 \text{ on } \mathbb{F}^+|_{V_{\mathbb{O}}^+}$$

$$i \left( \begin{bmatrix} u & -iw^0 - \sigma\pi_0' \\ v & -iw^1 - t\pi_0' \end{bmatrix}, \pi_{\mathbb{R}'} \right) = ((w^0, \pi_{\mathbb{R}'}), \begin{bmatrix} u \\ v \end{bmatrix}) \in V_{\mathbb{I}}^+ \times \mathbb{C}^2 \text{ on } \mathbb{F}^+|_{V_{\mathbb{I}}^+}$$

The reason for the above procedure is that calculations on  $\mathbb{F}^+$  can be performed in coordinates on  $E$ . More precisely, we may realize  $\mathbb{F}^+|_{V_{\mathbb{O}}^+}$  explicitly by introducing  $\rho = w^0/\pi_0'$

$$\begin{cases} 2 = w^1/\pi_0' \\ r = \pi_1'/\pi_0' \end{cases}$$

so that  $V_{\mathbb{O}}^+ \cong \{(p, q, r) \in \mathbb{C}^3 \text{ s.t. } \Phi(p, q, r) > 0\}$  where  $\Phi(p, q, r) = p + \bar{p} + \bar{q}r + \bar{r}q$  and then  $\mathbb{F}^+|_{V_{\mathbb{O}}^+} \subset \mathbb{C}^3 \times \mathbb{C}^2 \ni ((p, q, r), \begin{bmatrix} u \\ v \end{bmatrix})$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$V_{\mathbb{O}}^+ \subset \mathbb{C}^3 \ni (p, q, r)$$

$$\text{where } ((p, q, r), \begin{bmatrix} u \\ v \end{bmatrix}) \in \mathbb{F}^+|_{V_{\mathbb{O}}^+}$$

$$\text{when also } \begin{bmatrix} -ip - sr & u \\ -iq - tr & v \end{bmatrix} \in \mathbb{CM}^+.$$

8. Introducing  $F = \{ [ \begin{smallmatrix} \sigma \\ \tau \end{smallmatrix} ] = [ \begin{smallmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{smallmatrix} ] \in \mathbb{P}^2 \text{ s.t. } -4y_2 > |\sigma|^2 \}$ , we have :-

Lemma 1 :

$$U_0^+ \times F \xrightarrow{\text{diffeo}} F \times_{\alpha} U_0^+$$

commutative

where

$$\Psi((p, q, r), [\begin{smallmatrix} \sigma \\ \tau \end{smallmatrix}]) = \left( (p, q, r), \left[ \begin{array}{c} \sigma - \frac{2\bar{\tau}\bar{\sigma}}{\Phi(p, q, r)} + i\bar{\tau} \\ \frac{2\bar{\sigma}}{\Phi(p, q, r)} \end{array} \right] \right)$$

Proof: A messy but straightforward computation (allow 2 days)  $\square$

Note that  $\Psi$  is analytic in  $\sigma$  and  $\tau$  but only smooth in  $(p, q, r)$ . Thus all the fibres of  $\alpha$  are analytically isomorphic (although locally trivial analytically).

Lemma 2: If  $S^{\text{open}} \subset U_0^+$  is Stein then so is  $\alpha^{-1}(S)$ .

Proof: Using coordinates  $(p, q, r, s, t)$  as above we have :-

$$\alpha^{-1}(S) = (S \times \mathbb{C}^2) \cap \{ (p, q, r, s, t) \in \mathbb{C}^5 \text{ s.t. } \begin{bmatrix} -ip - sr & s \\ -iq - tr & t \end{bmatrix} \in \mathbb{C}^{2+} \}$$

= A  $\cap$  B, say.

A is Stein since it's a product of such.

B is Stein since it's an inverse image of a domain of holomorphy,  $\mathbb{C}^{2+}$ , under an analytic map defined on a domain of holomorphy,  $\mathbb{C}^5$ . The lemma follows since an intersection of domains of holomorphy is again a domain of holomorphy.  $\square$

Corollary: If  $\mathcal{F}$  is a coherent analytic sheaf on  $\mathbb{P}^+$  then  $\alpha^* \mathcal{F} = 0 \quad \forall g \geq 1$ .

Proof: If  $x \in U_0^+$  and  $S$  is a Stein neighbourhood of  $x$  in  $U_0^+$  then  $H^2(\alpha^{-1}(S), \mathcal{F}) = 0 \quad \forall g \geq 1$  by the lemma and Cartan's theorem B. Since there are arbitrarily small Stein neighbourhoods  $\alpha^* \mathcal{F}_x = 0$ . If  $x \in U_0^+$  the argument is similar  $\square$

Theorem 1: If  $\mathcal{F}$  is a coherent sheaf on  $\mathbb{P}^+$  then

$$H^2(\mathbb{C}^{2+}, \alpha^* \mathcal{F}) \cong H^2(\mathbb{P}^+, \mathcal{F}) \cong T(\mathbb{C}^{2+}, \beta^* \mathcal{F}).$$

Proof: These isomorphisms follow by plugging information into the Leray spectral sequence of §1.A. The first uses the corollary to lemma 2 and the second uses that  $H^p(\mathbb{C}^{2+}, \beta^* \mathcal{F}) = 0 \quad \forall p \geq 1$  (which follows from theorem B plus a simple case of Grauert's direct image theorem).  $\square$

The machine is constructed by using the above theorem with specific choices of  $\mathcal{F}$  :-

Def<sup>n</sup>: Let  $\Omega_{\alpha}^j$  denote the sheaf of germs of holomorphic sections of the  $j$ th exterior power of the bundle  $T_{\alpha}^*$  on  $\mathbb{P}^+$  where  $T_{\alpha}$  is the subbundle of  $T\mathbb{P}^+$ , the holomorphic tangent bundle, consisting of those vectors tangent to the fibres of  $\alpha$ . Thus  $\Omega_{\alpha}^0 \cong \mathcal{O}_{\mathbb{P}^+}$  the sheaf of germs of holomorphic functions,  $\Omega_{\alpha}^1$  may be described by the exact sequence  $0 \rightarrow \alpha^* \Omega_{\mathbb{C}^{2+}}^1 \rightarrow \Omega_{\mathbb{P}^+}^1 \rightarrow \Omega_{\alpha}^1 \rightarrow 0$ , and then  $\Omega_{\alpha}^2 = \Lambda^2 \Omega_{\alpha}^1$ .  $\Omega_{\alpha}^j$  may be pictured more concretely using the coordinates  $(p, q, r, s, t)$  on  $\mathbb{P}^+ \setminus U_0^+$  as consisting of germs of holomorphic  $j$ -forms involving only the differentials  $dp$  and  $dt$ . The usual exterior derivative induces differential operators  $\Omega_{\alpha}^0 \xrightarrow{d_{\alpha}} \Omega_{\alpha}^1 \xrightarrow{d_{\alpha}} \Omega_{\alpha}^2 \xrightarrow{d_{\alpha}} 0$  ( $\Omega_{\alpha}^3 = 0$  since the fibres of  $\alpha$  are 2  $\mathbb{C}$ -dimensional). On  $\mathbb{P}^+ \setminus U_0^+$  we may use coords  $(\eta_p, \eta_q, \eta_r, u, v)$

and on  $F^+ \setminus U_0^+ \cap U_1^+$  we have  $u = -ip - \alpha r$ . Thus  $du = -rda$  {  
 $v = -iq - \beta r$ }  $dv = -rdb$  {  
9.

so that as a vector bundle  $\Omega_{\alpha}^1$  may be described by a transition f.  
 $t_{10} = \begin{bmatrix} -k & 0 \\ 0 & -l \end{bmatrix}$ . But  $\frac{1}{t} = \frac{-T_{10}}{T_{11}}$  whence  $\Omega_{\alpha}^1 \cong \alpha^* (\mathcal{O}(1))^2$ .  
 $\Omega_{\alpha}^2 \cong \alpha^* (\mathcal{O}(2))$ .

Not<sup>n</sup>: If  $V$  is a holomorphic vector bundle on  $\mathbb{P}\mathbb{M}^+$  we will denote by  
 $\mathcal{G}$  the sheaf of germs of holomorphic sections thereof.

We set  $\mathcal{S}_V = \alpha^* \mathcal{G} \otimes_{\mathcal{O}_{F^+}} \Omega_{\alpha}^1$ . Thus  $\mathcal{S}_V^0 \cong \alpha^* \mathcal{G}$  and we have the  
obvious inclusion  $0 \rightarrow \mathcal{G} \rightarrow \alpha^* \alpha^* \mathcal{G} \cong \alpha^* \mathcal{S}_V^0$ .

Lemma 3: The sequence  $0 \rightarrow \mathcal{G} \rightarrow \alpha^* \mathcal{S}_V^0 \rightarrow \alpha^* \mathcal{S}_V^1 \rightarrow \alpha^* \mathcal{S}_V^2 \rightarrow 0$  is exact  
i.e.  $\alpha^* \mathcal{S}_V$  is a resolution of  $\mathcal{G}$  (§1.C).

proof: Since the statement is local we may restrict attention to a  
polydisc  $\Delta$  in  $U_0^+$  and suppose that  $\mathcal{G}$  is trivial of rank 1.  
It will then suffice to show that  $0 \rightarrow T(\Delta, \mathcal{O}) \rightarrow T(\Delta, \alpha^* \Omega_{\alpha}^1)$   
is exact. We introduce the topological inverse image sheaf  $\alpha^{-1} \mathcal{O}$ ,  
the sheaf of germs of holomorphic functions on  $F^+$  constant on  
the fibres of  $\alpha$ . Then our sequence becomes  $0 \rightarrow T(\alpha^{-1}(\Delta), \alpha^{-1}(\mathcal{O})) \rightarrow T(\alpha(\Delta), \Omega_{\alpha}^1)$ .  
However, the sheaf sequence  $0 \rightarrow \alpha^{-1} \mathcal{O} \rightarrow \Omega_{\alpha}^1$  is exact. To see this  
we use the usual coordinates on  $E$  and note that if we take sections  
over a polydisc within  $F^+$  this is just a parametric version of the  
exact seq:  $0 \rightarrow \mathbb{C} \rightarrow T(P, \Omega^0) \xrightarrow{\text{d}} T(P, \Omega^1) \xrightarrow{\text{d}} T(P, \Omega^2) \rightarrow 0$  for  $P$  a  
polydisc in  $\mathbb{C}^2$  ( $P$  is a topologically trivial domain of holomorphy).

Moreover by lemma 2  $\alpha^{-1}(\Delta)$  is Stein so  $\Omega_{\alpha}^1$  is an acyclic  
resolution of  $\alpha^{-1} \mathcal{O}$  by theorem B. Thus  $0 \rightarrow T(\alpha^{-1}(\Delta), \alpha^{-1}(\mathcal{O})) \rightarrow T(\alpha^{-1}(\Delta), \Omega_{\alpha}^1)$   
is exact if and only if  $H^2(\alpha^{-1}(\Delta), \alpha^{-1} \mathcal{O}) = 0 \quad \forall g \geq 1$ . This statement  
no longer concerns the analytic structure of  $F^+$  and we transfer  
over to an analytic product  $H^2(\alpha^{-1}(\Delta), \alpha^{-1} \mathcal{O}) \cong H^2(\delta^{-1}(\Delta), \delta^{-1} \mathcal{O})$  by lemma 1.  
We may compute  $H^2(\delta^{-1}(\Delta), \delta^{-1} \mathcal{O})$  by using the resolution  
 $0 \rightarrow \delta^{-1} \mathcal{O} \rightarrow \Omega_{\delta}^2$  and since  $\delta^{-1}(\Delta) \cong \Delta \times F$  analytically, the exactness  
of  $0 \rightarrow T(\delta^{-1}(\Delta), \delta^{-1} \mathcal{O}) \rightarrow T(\delta^{-1}(\Delta), \Omega_{\delta}^2)$  follows as a parametric  
version of  $0 \rightarrow \mathbb{C} \rightarrow T(F, \Omega^0) \xrightarrow{\text{d}} T(F, \Omega^1) \xrightarrow{\text{d}} T(F, \Omega^2) \rightarrow 0$  which is  
indeed exact since  $F$  is a topologically trivial domain of holomorphy.  $\square$

The machine which converts analytic cohomology on  $\mathbb{P}\mathbb{M}^+$  into  
differential equations on  $\mathbb{C}\mathbb{M}^+$  is in the form of a spectral sequence:-

Theorem 2: If  $V$  is a holomorphic vector bundle on  $\mathbb{P}\mathbb{M}^+$  then there is  
a spectral sequence  $E^{p, q} = T(\mathbb{C}\mathbb{M}^+, \beta^* \mathcal{S}_V) \implies H^{p+q}(\mathbb{P}\mathbb{M}^+, \mathcal{G})$

(where the differentials are induced by  $d\alpha$ ).

proof: Since by lemma 3  $\alpha^* \mathcal{S}_V$  is a resolution of  $\mathcal{G}$  we have  
(§1.C) a spectral sequence  $E^{p, q} = H^p(\mathbb{P}\mathbb{M}^+, \alpha^* \mathcal{S}_V^q) \implies H^{p+q}(\mathbb{P}\mathbb{M}^+, \mathcal{G})$ .  
But by theorem 1,  $H^2(\mathbb{P}\mathbb{M}^+, \alpha^* \mathcal{S}_V^q) \cong H^2(F^+, \mathcal{S}_V^q) \cong T(\mathbb{C}\mathbb{M}^+, \beta^* \mathcal{S}_V^q)$ .  $\square$

Remarks ①:  $F^+ \cong \mathbb{M}_c^+ \times \mathbb{P}^1 = (\mathbb{M}^+ \times [\mathbb{P}^1 - \text{north pole}]) \cup (\mathbb{M}^+ \times [\mathbb{P}^1 - \text{south pole}])$ ,  
a union of two Stein manifolds. Thus  $E^{p, q} = 0$  for  $q \geq 2$  and  
the spectral sequence may be interpreted as a long exact sequence as in §1.C.  
②: Regions other than  $\mathbb{C}\mathbb{M}^+$ ,  $\mathbb{P}\mathbb{M}^+$  should be amenable to these methods.  
③: One can ask - what has happened to contour integration? Actually it's  
lurking in  $\beta^*$  since this involves (Serre duality) integration over  $\mathbb{P}^1$  [Cf. NJMW].

**10. §3 Applications** : The plan is to substitute the sheaves  $\mathcal{O}(n)$  into theorem 2 and see what happens. Thus we need to calculate :-

**Lemma 4** : Let  $\bigodot^n \mathbb{C}^2$  denote the  $n^{\text{th}}$  symmetric tensor power of  $\mathbb{C}^2$  for  $n \geq 0$  and  $\{\text{point}\}$  for  $n = -1$ . Then :-

$$\begin{aligned} T(\mathbb{C}M^+, \mathbb{B}^* S_{H(n)}^1) &\cong \{\text{analytic functions} : \mathbb{C}M^+ \rightarrow \bigodot^n \mathbb{C}^2\} \\ T(\mathbb{C}M^+, \mathbb{B}^* S_{H(n)}^2) &\cong \{\quad\quad\quad \rightarrow \bigodot^{n+1} \mathbb{C}^2 \otimes \mathbb{C}^2\} \\ T(\mathbb{C}M^+, \mathbb{B}^* S_{H(n)}^3) &\cong \{\quad\quad\quad \rightarrow \bigodot^{n+2} \mathbb{C}^2\} \\ \dots, \mathbb{B}^* S_{H(n)}^4 &\cong \{\quad\quad\quad \rightarrow \bigodot^{n+3} \mathbb{C}^2\} \\ \dots, \mathbb{B}^* S_{H(n)}^5 &\cong \{\quad\quad\quad \rightarrow \bigodot^{n+4} \mathbb{C}^2\} \\ \dots, \mathbb{B}^* S_{H(n)}^6 &\cong \{\quad\quad\quad \rightarrow \bigodot^{n+5} \mathbb{C}^2\} \end{aligned}$$

**proof:**

$$S_{H(n)}^j = \alpha^* \mathcal{O}(n) \otimes \Omega_{\mathbb{P}}^j \cong \begin{cases} \alpha^* \mathcal{O}(n) \otimes \mathcal{O}_{\mathbb{P}^1}^j & \cong \alpha^* \mathcal{O}(n), \quad j=0 \\ \alpha^* \mathcal{O}(n) \otimes \alpha^* \mathcal{O}(1)^j \cong \alpha^* \mathcal{O}(n+1) \otimes \mathbb{C}^2, \quad j=1 \\ \alpha^* \mathcal{O}(n) \otimes \alpha^* \mathcal{O}(2) \cong \alpha^* \mathcal{O}(n+2), \quad j=2. \end{cases}$$

Thus it suffices to show :-

$$T(\mathbb{C}M^+, \mathbb{B}^* \alpha^* \mathcal{O}(n)) \cong \{\text{analytic functions} : \mathbb{C}M^+ \rightarrow \bigodot^n \mathbb{C}^2\}$$

$$\text{and } T(\mathbb{C}M^+, \mathbb{B}^* \alpha^* \mathcal{O}(n)) \cong \{\text{analytic functions} : \mathbb{C}M^+ \rightarrow \bigodot^{n-2} \mathbb{C}^2\}.$$

Now  $\alpha^* \mathcal{O}(n)$  as a bundle on  $\mathbb{P}^1$  is just  $\mathcal{O}_{\mathbb{P}^1}(n)$  on each  $\mathbb{P}^1$  when we write  $\mathbb{P}^1 \cong \mathbb{C}M^+ \times \mathbb{P}^1$  (and trivial in the  $\mathbb{C}M^+$  factor) so :-

$$T(\mathbb{C}M^+, \mathbb{B}^* \alpha^* \mathcal{O}(n)) \cong \{\text{analytic functions} : \mathbb{C}M^+ \rightarrow T(\mathbb{P}^1, \mathcal{O}(n))\}$$

$$\text{and } T(\mathbb{P}^1, \mathcal{O}(n)) \cong \{\text{homogeneous polynomials in } (\tau_{\mathbb{P}^1}) \text{ of degree } n\} \cong \bigodot^n \mathbb{C}^2.$$

$$\text{Similarly } T(\mathbb{C}M^+, \mathbb{B}^* \alpha^* \mathcal{O}(n)) \cong \{\text{analytic functions} : \mathbb{C}M^+ \rightarrow H'(\mathbb{P}^1, \mathcal{O}(n))\}$$

$$\text{and } H'(\mathbb{P}^1, \mathcal{O}(n)) \cong \text{(Serre duality) } T(\mathbb{P}^1, \mathcal{O}(-n-2)) \cong \bigodot^{-n-2} \mathbb{C}^2. \quad \square$$

We must also compute the differentials :  $E_1^{p,q} \rightarrow E_1^{p+1,q}$  induced by  $d\alpha$ .

This is a straightforward computation (omitted here) using the identifications made in lemma 4 and its proof. The result is a 1<sup>st</sup> order linear differential operator which we shall also denote by  $d\alpha$ . Then :-

**Case 1** :  $n = -m-2$ ,  $m \geq 1$ . Here  $E_1^{p,q} = T(\mathbb{C}M^+, \mathbb{B}^* S_{H(n)}^p) = 0$  unless  $p=1$  so (§1.C.(b))  $H'(\mathbb{P}M^+, \mathcal{O}(-m-2)) \cong \ker d\alpha : T(\mathbb{C}M^+, \mathbb{B}^* S_{H(n)}^1) \rightarrow T(\mathbb{C}M^+, \mathbb{B}^* S_{H(n)}^2)$

$$\cong \ker d\alpha : \{\text{analytic } f^{\text{an}} : \mathbb{C}M^+ \rightarrow \bigodot^m \mathbb{C}^2\} \rightarrow \{\text{anal } f^{\text{an}} : \mathbb{C}M^+ \rightarrow \bigodot^{m+1} \mathbb{C}^2 \otimes \mathbb{C}^2\}.$$

$d\alpha$  in this case turns out to be  $\nabla^{\mathbb{P}M^+}$  so  $H'(\mathbb{P}M^+, \mathcal{O}(-m-2)) \cong Z_m'$   $\square$

**Case 2** :  $n = -2$ . Here only  $E_1^{1,1}$  and  $E_1^{2,0}$  are non-zero so (§1.C.(c))  $H'(\mathbb{P}M^+, \mathcal{O}(-2)) \cong \ker : T(\mathbb{C}M^+, \mathbb{B}^* S_{H(n)}^1) \rightarrow T(\mathbb{C}M^+, \mathbb{B}^* S_{H(n)}^2)$ . This is a map on the 2<sup>nd</sup> level of the spectral sequence and so will be a 2<sup>nd</sup> order differential operator (just on analytic functions). In this way we obtain the wave equation  $\square$

**Case 3** :  $n \geq 1$ . Here  $E_1^{p,q} = 0$  unless  $q=0$  so using lemma 4

(and §1.C.(a))  $H'(\mathbb{P}M^+, \mathcal{O}(n)) \cong H'(E_1^{1,0})$

$$\cong \ker d\alpha : \{\text{anal } f^{\text{an}} : \mathbb{C}M^+ \rightarrow \bigodot^{n+1} \mathbb{C}^2 \otimes \mathbb{C}^2\} \rightarrow \{\text{anal } f^{\text{an}} : \mathbb{C}M^+ \rightarrow \bigodot^{n+2} \mathbb{C}^2\}$$

$$\text{in } d\alpha : \{\text{anal } f^{\text{an}} : \mathbb{C}M^+ \rightarrow \bigodot^n \mathbb{C}^2\} \rightarrow \{\text{anal } f^{\text{an}} : \mathbb{C}M^+ \rightarrow \bigodot^{n+1} \mathbb{C}^2 \otimes \mathbb{C}^2\}$$

To interpret this as the zero rest mass fields (anti-self-dual helicity  $n/2$ ) we use diagram chasing with the exact sequence

$$T(\mathbb{C}M^+, \Omega^0) \xrightarrow{d} T(\mathbb{C}M^+, \Omega^1) \xrightarrow{d} T(\mathbb{C}M^+, \Omega^2) \xrightarrow{d} T(\mathbb{C}M^+, \Omega^3) \xrightarrow{d} T(\mathbb{C}M^+, \Omega^4).$$

**References** : RG R. Godement : Topologie Algébrique et Théorie des Faisceaux.

RP R. Penrose : Massless fields and Sheaf Cohomology TN5

RSW R.S. Ward : Sheaf Cohomology and Inverse Twistor F<sup>2</sup> TN6

ROW R.O. Wells Jr : Complex Manifolds and Math. Physics

NJMW N.J.M. Woodhouse : Twistor Cohomology Without Sheaves TN2

## Zero-rest-mass Fields and Topology

11.

Representing solutions of the zero-rest-mass field equations on  $\mathbb{C}M^+$  in terms of  $H^1(\mathbb{P}\mathbb{M}^+, \mathcal{O}(n))$  or  $H^1(\mathbb{P}\mathbb{M}^{*-}, \mathcal{O}(n))$  has been described in various ways [RP, TN2; RP, TN5; RSW, TN6; etc.]. For regions other than  $\mathbb{C}M^+$  the situation is less well understood.

The following is a little speculative and assumes that the representation as described in [MGE, TN7] works for any Stein region  $X$  in  $\mathbb{C}M$ . We describe here the case of the anti-self-dual Maxwell equations.

Let  $\mathbb{P}X$  denote the region in  $\mathbb{P}\mathbb{M}$  corresponding to  $X$  in  $\mathbb{C}M$  (identifying  $\mathbb{C}M$  with  $\mathrm{Gr}(2, \mathbb{C}^4)$ ,  $\mathbb{P}X$  is just those lines within the planes in  $X$ ) and similarly let  $\mathbb{P}X^*$  denote the corresponding region in  $\mathbb{P}\mathbb{M}^*$ .

Let  $\Omega^p(X)$  denote the space of holomorphic  $p$ -forms on  $X$  and recall that the 2-forms split:  $\Omega^2(X) = \Omega_+^2(X) \oplus \Omega_-^2(X)$  where  $\Omega_+^2(X)$  is the space of self-dual 2-forms and  $\Omega_-^2(X)$  the space of anti-self-dual 2-forms. Then trivially we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & 0 & 0 & 0 & 0 & 0 & \\
 \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
 0 \rightarrow \Omega^0(X) & \xrightarrow{d} & \Omega^1(X) & \xrightarrow{d} & \Omega_+^2(X) & \rightarrow 0 & \rightarrow 0 \\
 \parallel & \parallel & \parallel & \uparrow \text{projection} & \uparrow & \uparrow & \\
 0 \rightarrow \Omega^0(X) & \xrightarrow{d} & \Omega^1(X) & \xrightarrow{d} & \Omega^2(X) & \xrightarrow{d} & \Omega^4(X) \rightarrow 0 \\
 \uparrow & \uparrow & \uparrow \text{inclusion} & \uparrow & \uparrow & \uparrow & \\
 0 \rightarrow 0 & \longrightarrow & 0 & \longrightarrow & \Omega_-^2(X) & \xrightarrow{d} & \Omega^3(X) \xrightarrow{d} \Omega^4(X) \rightarrow 0 \\
 \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
 0 & 0 & 0 & 0 & 0 & 0 &
 \end{array}$$

Then it turns out that the first row of this diagram has  $H^*(\mathbb{P}X, \mathcal{O})$  as cohomology ( $H^0(\mathbb{P}X, \mathcal{O}) \cong \ker d: \Omega^0(X) \rightarrow \Omega^1(X)$  etc.) and that  $H^*(\mathbb{P}X^*, \mathcal{O}(-4))$  is the cohomology of the last row ( $H^1(\mathbb{P}X^*, \mathcal{O}(-4)) \cong \ker d: \Omega_-^2(X) \rightarrow \Omega^3(X)$  etc.). Since  $X$  is Stein the middle row has  $H^*(X, \mathbb{C})$  as cohomology. All the columns of the diagram are exact so by standard homological reasoning we obtain

$$H^0(X, \mathbb{C}) \cong H^0(\mathbb{P}X, \mathcal{O}) \text{ and an exact sequence}$$

$$0 \rightarrow H^0(X, \mathbb{C}) \rightarrow H^1(\mathbb{P}X, \mathcal{O}) \rightarrow H^1(\mathbb{P}X^*, \mathcal{O}(-4)) \rightarrow H^2(X, \mathbb{C}) \rightarrow H^2(\mathbb{P}X, \mathcal{O}) \rightarrow \dots$$

In case  $X = \mathbb{C}M^+$  the isomorphism just confirms that there are no non-constant holomorphic functions on  $\mathbb{P}\mathbb{M}^+$  and since  $X$  is topologically trivial we obtain the usual twistor transform isomorphism

$$H^1(\mathbb{P}\mathbb{M}^+, \mathcal{O}) \xrightarrow{\cong} H^1(\mathbb{P}\mathbb{M}^{*-}, \mathcal{O}(-4)).$$

$H^2(\mathbb{P}\mathbb{M}^+, \mathcal{O}) = 0$  by convexity arguments or from arguing as in [RP, TN5] (Thanks here to LPH).

For other  $X$  it would appear that  $H^1(\mathbb{P}X^*, \mathcal{O}(-4))$  represents zero-rest-mass fields directly whereas  $H^1(\mathbb{P}X, \mathcal{O})$  gives a "Hertz potential" representation of those fields which can be so represented. In other words in the sequence  $H^1(\mathbb{P}X, \mathcal{O}) \rightarrow H^1(\mathbb{P}X^*, \mathcal{O}(-4)) \rightarrow H^2(X, \mathbb{C})$  the last map should be called "charge" (Thanks to LPH, RP, RSW, ...).

For the neutrino equations the topology of  $X$  does not enter at all and for higher helicity the above method will give rise to a spectral sequence (which will degenerate to the twistor transform for  $X = \mathbb{C}M^+$ ).

Mike Eastwood.

## 12. Further Remarks on Massless Fields and Sheaf Cohomology

The purpose of this note is to describe a short exact sequence from which it is possible to derive, in a straightforward way, the connection between massless fields and twistor cohomology for both positive and negative helicities. The sequence to be described is, in fact, a special case of R.P.'s sequence A on p. 10 of TN5. It is suggested there, however, that to deal with the negative helicity cases we need a different exact sequence, viz. sequence D on p. 12 of TN5. We shall demonstrate here that, remarkably, sequence A suffices for all helicities.

Consider the following short exact sequence of sheaves:

$$(1) \quad 0 \longrightarrow \mathcal{O}(-2s-2) \xrightarrow{\pi^{R'}} \Phi(-2s-1) \xrightarrow{\pi^{R'}} \Phi(-2s) \longrightarrow 0.$$

Here  $\Phi(-2s-1)$  is the sheaf of functions  $\mathcal{E}^{R'}(x, \pi)$  satisfying  $\nabla_{\pi} \mathcal{E}^{R'}(x, \pi) = 0$ , homogeneous of degree  $-2s-1$  in  $\pi$ . Similarly,  $\Phi(-2s)$  is the sheaf of functions  $\mathcal{E}(x, \pi)$  satisfying  $\square \mathcal{E}(x, \pi) = 0$ , homogeneous of degree  $-2s$  in  $\pi$ . By  $\mathcal{O}(-2s-2)$  we denote the sheaf of holomorphic twistor functions, homogeneous of degree  $-2s-2$ . The helicity  $s$  can assume positive or negative half-integral values. We restrict  $x$  to lie in a finite, holomorphically convex, topologically trivial region  $Q$  of complex Minkowski space (letting  $Q$  denote also the corresponding region in twistor space).

Theorem. Letting  $s$  vary we obtain from the cohomology of sequence (1) above the isomorphism  $H^*(Q, \mathcal{O}(-2s-2)) \cong H^*(Q, \mathbb{Z}_s)$  where  $\mathbb{Z}_s$  is the sheaf of germs of solutions to the zero rest mass equations for helicity  $s$ .

We shall simply outline the proof here by way of illustration with several examples. We require the long exact cohomology sequence obtained from sequence (1):

$$(2) \quad 0 \longrightarrow H^0 \mathcal{O}(-2s-2) \longrightarrow H^0 \Phi(-2s-1) \longrightarrow H^0 \Phi(-2s) \longrightarrow H^1 \mathcal{O}(-2s-2) \\ \longrightarrow H^1 \Phi(-2s-1) \longrightarrow H^1 \Phi(-2s) \longrightarrow H^2 \mathcal{O}(-2s-2) \longrightarrow H^2 \Phi(-2s-1) \longrightarrow \dots$$

For the evaluation of some of these groups we need the following useful resolutions (R.P., TN5, pp. 11-12):

$$(3) \quad 0 \longrightarrow \Phi(-2s-1) \longrightarrow F^{R'}(-2s-1) \xrightarrow{\nabla_{\pi}^{R'}} F_R(-2s-1) \longrightarrow 0,$$

$$(4) \quad 0 \longrightarrow \Phi(-2s) \longrightarrow F(-2s) \xrightarrow{\square} F(-2s) \longrightarrow 0,$$

where  $F'_r$ ,  $F_\alpha$ , and  $F$  are (coherent) sheaves of (unrestricted) holomorphic functions of  $x$  and  $\pi$ , homogeneous in  $\pi$  of the designated degrees. (In what follows,  $Q$  is suppressed.)

$s=0$  Case. From sequence (2) we obtain the segment:

$$(5) \quad H^0 \Phi^{R'}(-1) \longrightarrow H^0 \Phi(0) \longrightarrow H^1 \mathcal{O}(-2) \longrightarrow H^1 \Phi^{R'}(-2).$$

Using the cohomology of resolution (3), together with the fact that  $Q$  is Stein and the  $F$ 's are coherent, we obtain  $H^0 \Phi^{R'}(-1) = 0$  and  $H^1 \Phi^{R'}(-2) = 0$ , and the desired isomorphism between  $H^1 \mathcal{O}(-2)$  and  $H^0 \mathbb{Z}_0$  follows immediately. Note that we must use resolution (4) in order to deduce that  $H^0 \Phi(0) \cong H^0 \mathbb{Z}_0$ .

$s=1/2$  Case. From sequence (2) we obtain the segment:

$$(6) \quad H^0 \Phi(-1) \longrightarrow H^1 \mathcal{O}(-3) \longrightarrow H^1 \Phi^{R'}(-2) \longrightarrow H^1 \Phi(-1).$$

Using resolution (4) we obtain that  $H^0 \Phi(-1)$  and  $H^1 \Phi(-1)$  both vanish. Using resolution (3) we obtain that  $H^1 \Phi^{R'}(-2)$  is the space of spinor fields  $\psi^{R'}(x)$  satisfying  $\nabla_{\mu\nu} \psi^{R'} = 0$ , and we have the desired isomorphism.

$s=-1/2$  Case. From sequence (2) we obtain the segment:

$$(7) \quad H^0 \mathcal{O}(-1) \longrightarrow H^0 \Phi^{R'}(0) \longrightarrow H^0 \Phi(1) \longrightarrow H^1 \mathcal{O}(-1) \longrightarrow H^1 \Phi^{R'}(0).$$

Using resolution (3) we deduce  $H^1 \Phi^{R'}(0) = 0$ . We know that  $H^0 \mathcal{O}(-1)$  must vanish, since it vanishes when restricted down to any line. The group  $H^0 \Phi^{R'}(0)$  consists, by (3), of fields  $\psi^{R'}$  satisfying  $\nabla_{\mu\nu} \psi^{R'} = 0$ . The group  $H^0 \Phi(1)$  is, using resolution (4), the space of polynomials of degree one in  $\pi$  satisfying the wave equation, i.e. functions of the form  $\psi^{R'} \pi_A$  with  $\Box \psi^{R'} = 0$ . Accordingly, we can identify  $H^0 \Phi(1)$  with the space of spinor solutions of the wave equation. This shows us that  $H^1 \mathcal{O}(-1)$  is given by spinor solutions of the wave equation, modulo solutions of the  $s=1/2$  zero rest mass equations. Thus,  $H^1 \mathcal{O}(-1)$  is (as desired) the space of solutions of the  $s=-1/2$  e.r.m. equations (see, e.g., equation R, and the preceding remarks, in L.P.H., TN.4, pp. 12-16). Higher spins, for both positive and negative helicity fields, can be treated similarly.  $\square$

## 14. Compton Scattering for Massless Electrons

We shall find twistor diagrams representing the amplitude for the Compton scattering process. Twistor diagrams are gauge-invariant, because they represent the ingoing or outgoing photons in terms of fields  $\phi_{\alpha c}$  and not in terms of a potential  $A_a$ . This means that there is no way of representing the single Feynman diagram



because this is not gauge-invariant.

But we can translate the gauge-invariant sum of the two diagrams



Only this sum is physically significant, anyway

Particle 1 is an ingoing "electron" with wave function whose Fourier transform is  $\tilde{\psi}_A(k_1)\delta^+(k_1^c)$

Particle 3 is an outgoing "electron" of the same helicity whose complex conjugate has Fourier transform  $\tilde{\psi}_c(k_3)\delta^-(k_3^c)$

Particle 2 is an ingoing photon represented by a potential whose Fourier transform is  $\tilde{A}_{gg}(k_2)\delta^+(k_2^c)$

This photon is not assumed to be in a helicity state, and the potential is not assumed to be in any particular gauge.

Similarly for particle 4, the outgoing photon.

Writing down the Feynman rules in 2-spinor form, one obtains for the sum:

$$\oint d^4 k_1 d^4 k_2 d^4 k_3 d^4 k_4 S(k_1 + k_2 - k_3 - k_4) \tilde{\psi}_A(k_1) \tilde{A}_{gg}(k_2) \tilde{\psi}^c(k_3) \tilde{A}^{00'}(k_4) \\ S^+(k_1^c) S^+(k_2^c) S^*(k_3^c) S^-(k_4^c) \\ \left\{ \frac{\varepsilon_0^A \varepsilon_{c'}^B (k_1 - k_4)^{S'}_0}{(k_1 - k_4)^2} + \frac{\varepsilon_0^{AB} \varepsilon_{c'0'}^{B'} (k_1 + k_2)^{S'}_0}{(k_1 + k_2)^2} \right\}$$

Purely algebraic manipulation of the large bracket expression, making use of the relations

$$\left\{ \begin{array}{l} k_1^2 = k_2^2 = k_3^2 = k_4^2 = 0 \\ k_{3Mc} \tilde{\psi}^c(k_3) = 0 + k_1^A N^A \tilde{\psi}_A(k_1) \end{array} \right.$$

produces the following gauge-invariant expression:

$$\int d^4k_1 d^4k_2 d^4k_3 d^4k_4 \delta(k_1 + k_2 - k_3 - k_4) S^+(k_1^{\epsilon}) S^+(k_2^{\epsilon}) S^-(k_3^{\epsilon}) S^-(k_4^{\epsilon})$$

$$\left\{ \tilde{\psi}_A(k_1) \tilde{\phi}_{E'C'}(k_2) \tilde{\psi}^{e'}(k_3) \tilde{\phi}_F(k_4) k_3^{FE'} \right.$$

$$\left. - \tilde{\psi}_A(k_1) \tilde{\phi}_E(k_2) \tilde{\psi}^{e'}(k_3) \tilde{\phi}_{F'C'}(k_4) k_1^{e'F'} \right\}$$

$$(k_1 + k_2)^2 (k_1 - k_4)^2$$

where  $\tilde{\phi}_{E'C'}(k_2) = k_2^B (\epsilon' \tilde{A}_{C'})_B(k_2)$  etc.

Thus the amplitude splits into two gauge-invariant parts, according to the two helicity parts of the photon. Helicity of the photon is conserved in each part.

Suppose now the incoming photon is of definite helicity, of opposite type to the incoming electron (so the second part of the amplitude vanishes).

The in- and out-states now translate into twistors as follows:

$$\psi_A(x_1) \rightarrow f_1(w_a) w \boxed{ } ; \quad f_1 \text{ homogeneity degree } (-3)$$

$$\phi_{D3}(x_2) \rightarrow f_2(y_a) \boxed{y} \boxed{dY} ; \quad f_2 \text{ of degree } (0)$$

$$\psi''(x_3) \rightarrow f_3(x^a) \boxed{x} \boxed{} ; \quad f_3 \text{ of degree } (-3)$$

$$\phi_{D3}(x_4) \rightarrow f_4(z^a) \boxed{z} \boxed{dz} ; \quad f_4 \text{ of degree } (0)$$

The integral  $\int d^4k_1 d^4k_2 d^4k_3 d^4k_4 \delta(k_1 + k_2 - k_3 - k_4) S^+(k_1^{\epsilon}) S^+(k_2^{\epsilon}) S^-(k_3^{\epsilon}) S^-(k_4^{\epsilon})$

translates into

$$\oint DX W Y Z$$

and finally

$$k_3^{FE'} \rightarrow \boxed{x} \boxed{w} \boxed{d_x}$$

$$((k_1 + k_2)^2 (k_1 - k_4)^2)^{-1} \rightarrow [\boxed{x} \boxed{z} \boxed{d_x} \boxed{d_z}]^T [w \boxed{d_z} \boxed{z} \boxed{d_w}]^T$$

Hence, doing all the contractions correctly, we obtain

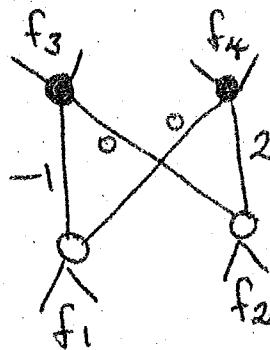
16.  $\oint DXWYZ f_1(w) f_2(y) f_3(x) f_4(z)$

$$\left\{ w \cancel{\partial}_z (\partial_y^{-1} x)^2 \cancel{\partial}_x \cancel{\partial}_z \right\} \left\{ \cancel{x}_2 \cancel{\partial}_x \cancel{\partial}_z w \cancel{\partial}_y \cancel{\partial}_z \sum \cancel{\partial}_w \right\}^{-1}$$

$= \oint DXWYZ f_1(w) f_2(y) f_3(x) f_4(z)$

$$\left\{ \cancel{x}_2 \cancel{\partial}_z \sum \cancel{\partial}_w \right\}^{-1}$$

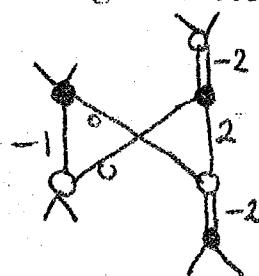
(finite, gauge-invariant and conformally invariant).



Similarly we can deal with the annihilation-creation channel, i.e. + and obtain

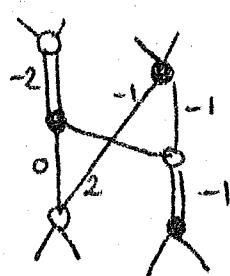


We can if we like put these results into another form by using only functions of homogeneity degrees -3 or -4. Using twistor transforms the first result gives us

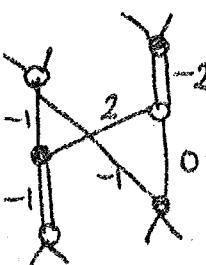


(for the opposite-helicity Compton channel)

and the second result gives us



OR



(for the annihilation-creation channel)

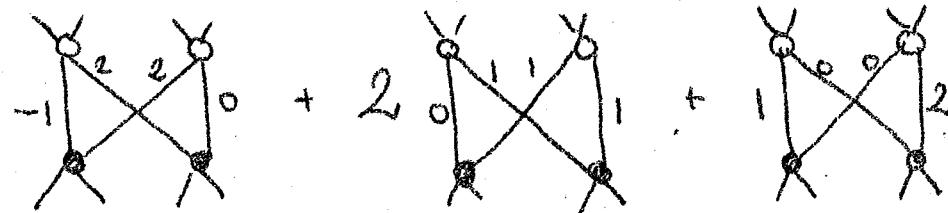
It will be noticed that many of the lines in these diagrams are boundary-prescriptions and not singular factors. Amazingly, all three diagrams reduce to a single integrand:

$$\oint f_{-3}(W) g_{-4}(Y) h_{-3}(X) j_{-4}(Z) DW Y X Z U V$$

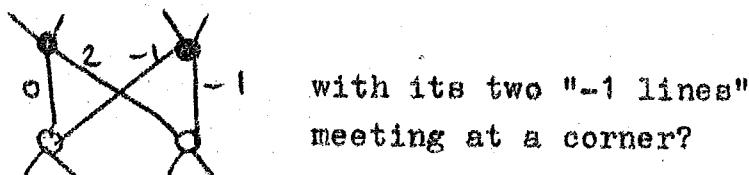
$$\oint \left( \frac{V}{U} \right)^3 Y X$$

different choices of (boundary) contours supplying the different channel amplitudes. Moreover, this form is the same as that derived by R.P. in Physics Reports from a quite different point of view!

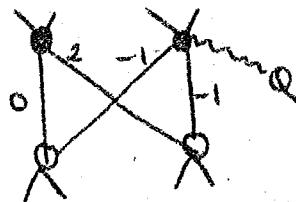
Unfortunately the third channel (same-helicity Compton) does not fit into this neat scheme. It can only be translated into



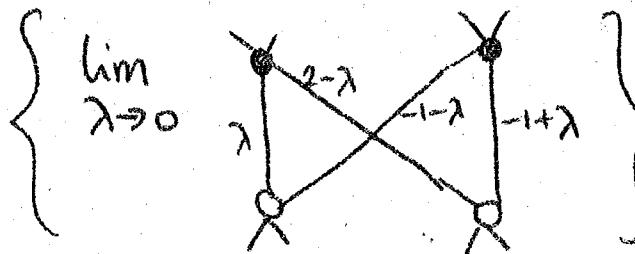
Remark: Is there really a contour for



Yes there is, if we are allowed an extra arbitrary boundary



The answer is then independent of Q, and it also agrees with: —



Andrew Hodges

18.

## Cohomological wave functions

Wave-functions in space-time are  $H^1$ 's in twistor space.

Just for fun, we can write down a converse relation: functions in twistor space are  $H^1$ 's in space-time.

In fact, we can consider the simplest possible function of degree (-2),

$$f(w) = \frac{1}{(w!)^2}. \text{ What does this correspond to in } M?$$

The only sensible interpretation is that it corresponds to

$$\begin{aligned} \phi(x) &= \overbrace{XY}^{x^a} \text{ where } \overbrace{XY}^{x^a} \leftrightarrow x^a \text{ in } M \\ &= \frac{\overbrace{XYZU}^{+++} \overbrace{XY}^{x^a}}{\overbrace{XYZP}^{---} \overbrace{XYZU}^{---}}, \text{ i.e. of form } \frac{(x-p)^2}{(x-q)^2(x-r)^2} \\ &\quad \text{where } (p-q)^2 = (q-r)^2 = (r-p)^2 = 0. \end{aligned}$$

This is an  $H^1$  on  $\{M \sim (\text{plane defined by } U^*)\}$ .

A cohomological wave-function!

Formally, the scalar product of such a "state" with a dual one is

$$U \overbrace{\circ \circ \circ}^{\circ} V = (\overbrace{\circ \circ}^{\circ})^2.$$

It's harder to see what to write down for homogeneities  $\geq 0$ .

Suggestive, half-baked ideas:

- (1) They are something to do with "collapsing wave-functions"
- (2) They are a bridge between the classical photon being a twistor, and the quantum photon being an  $H^1$  in  $\mathbb{PT}^+$ .
- (3) They form a 3-complex dimensional basis for states in a canonical way - necessary for "second quantization?"

pointed out  
by C.M. Patton

Andrew Hodges (incorporating remarks by R.P.)

► TNL 2, correction: "Twistor cohomology without sheaves": In the line following eqn (5), replace "2-sphere" by "hemisphere" and replace the next paragraph by

- "1) The integral is contour independent: By lemma (1),

$$Y \int d(\pi_B, \dots, \pi_C, \alpha \wedge \theta) = 0 = \bar{Y} \int d(\pi_B, \dots, \pi_C, \alpha \wedge \theta) \quad (6)$$

Also,  $Y \int d(\alpha \wedge \theta) = 0 = \bar{Y} \int d(\alpha \wedge \theta)$ . Hence the integrand in eqn (5) projects onto a well defined form on the complex projective space corresponding to  $X$ .

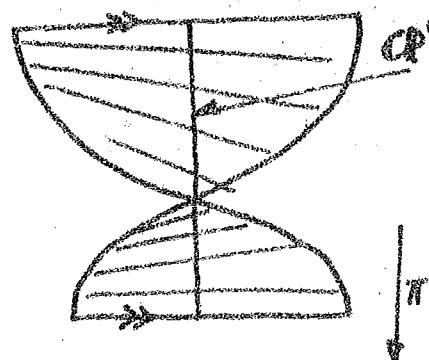
## BLOWING UP THE BOX

19.

This is a short description of the techniques A.P.H and I are using to find twistor counterparts to the three channels in the scattering process represented by the box\*. While we have not yet completely succeeded we both feel that all that remains is a ghastly calculation containing no new ideas.

The homology calculations done so far (by GAT'S) tell us that if the external ears are integrated out first using  $S^1$ 's there do not exist the three contours we require. The idea then is to deform the space in which we are doing the integration to make room for these contours.

The simplest example of blowing up is drawn here. We have replaced a point  $p$  in  $C^2$  by a  $CP^1$  in such a way that different directions at  $p$  are different points on the  $CP^1$ . Let  $C^2$  have coordinates  $(z_1, z_2)$  centred at  $p$  and  $CP^1$  have coordinates  $(\zeta : \eta)$ . Then the blown up space is:



$\{(z_1, z_2; \zeta : \eta) \in C^2 \times CP^1 : z_1 z_2 = \zeta \eta\}$ . It is biholomorphic to  $C^2$  via the projection  $\pi$  except at  $p$ . This blowing up can also be described in terms of the Hopf fibration of  $S^3$  (see RP in TN 4).

To generalise we replace  $C^2$  by a variety  $M$  and  $p$  by a subvariety  $N$  of complex codimension  $m$  in  $M$ . We choose functions  $u_1, \dots, u_m$  whose zeros describe  $N$  (in general this can only be done locally but can be done globally for the non-projective box). Let  $(t_1, \dots, t_m)$  be homogeneous coordinates for  $CP^{m-1}$ . The subvariety  $M'$  of  $M \times CP^{m-1}$  defined by the equations  $t_i u_j = t_j u_i$  ( $i, j = 1, \dots, m$ ) is the blown up version

\* See A.P.H. in TNS p.4.

20.

of  $M$ . Here  $N$  has been replaced by  $N \times \mathbb{C}P^{n-1}$ .

Now for the (nonprojective) box.  $M$  is the space of  $(Y, W, X, Z)$  but there are several different subvarieties  $N$  we could blow up. One of these is presented below as an example - we are still not sure exactly what the correct choice for  $N$  will be.

Let  $N = (Y \propto W) \cap (Y \cdot X = 0) \cap (Y \cdot Z = 0)$ . This is of codimension 5 and is defined by the zeros of:

$$u_1 = \frac{Y}{Z}, \quad u_2 = \frac{Y}{X}, \quad u_3 = \frac{YW}{H}, \quad u_4 = \frac{YW}{AB}, \quad u_5 = \frac{YW}{BC}$$

where  $(ABC)$  is a basis. We choose homogeneous coordinates  $(t_1 : t_2 : t_3 : t_4 : t_5)$  for  $\mathbb{C}P^4$  and write down the equations  $t_i u_j = t_j u_i$  of which only the following four are independent:

$$u_1 t_2 = u_2 t_1, \quad u_1 t_3 = u_3 t_1, \quad u_1 t_4 = u_4 t_1, \quad u_1 t_5 = u_5 t_1.$$

Changing to local coordinates for the  $\mathbb{C}P^4$  these equations become:

$$u_1 \lambda_2 = u_2, \quad u_1 \lambda_3 = u_3, \quad u_1 \lambda_4 = u_4, \quad u_1 \lambda_5 = u_5. \quad \textcircled{O}$$

Now choose functions  $u_6, u_7, u_8$  which make  $(u_1, \dots, u_8)$  a coordinate system for  $(Y, W)$  and consider the coordinates  $(u_1, \dots, u_8, X, Z)$  for the whole space  $M$ . We blow up our subvariety  $N$  by replacing

$(u_1, \dots, u_8, X, Z)$  by  $(u_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, u_6, u_7, u_8, X, Z)$  using  $\textcircled{O}$ . writing the differential form defined by the box diagram in these last coordinates and finding three contours over which to integrate it is the ghastly calculation referred to earlier.

Stephen Huggett

# The Good Cut Equation for Maxwell Fields

21.

Consider a function on  $\mathcal{J}^+$ ,  $A(u, \tilde{s}, \tilde{\tilde{s}})$  of spin wt. 1, which is holomorphic on a thickened region about real  $\mathcal{J}^+$ , i.e. for real  $u$  and for  $\tilde{s} = \tilde{\tilde{s}}$ . Now let  $u = l \equiv x^a l_a(\tilde{s}, \tilde{\tilde{s}})$  with  $l_a = (1+s\tilde{s})^{-1}(1+s\tilde{s}, s+\tilde{s}, -s(\tilde{s}-\tilde{\tilde{s}}), -1+s\tilde{s})$  and consider the differential equation

$$\mathcal{D}F = A(l, \tilde{s}, \tilde{\tilde{s}}) \quad (1)$$

for the spin wt. 0 function  $F(x^a, \tilde{s}, \tilde{\tilde{s}})$ . If we restrict ourselves to the regular solution on the  $S^2$  defined by  $\tilde{s} = \tilde{\tilde{s}}$  then  $F(x^a, \tilde{s}, \tilde{\tilde{s}})$  generates a self-dual source-free Maxwell field by the following method. The potential  $\gamma_a$  is defined by

$$\gamma_a = F_{,a} + \mathcal{D}h l_a - h \mathcal{D}l_a$$

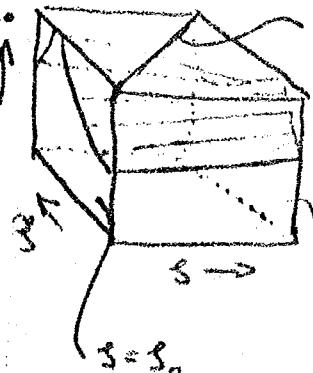
where  $h = l^b \mathcal{D}F_{,b}$ . It can be shown that  $\gamma_a(x^b)$  is independent of  $s$  and  $\tilde{s}$  and furthermore defines by  $F_{ab} = 2 \gamma_{[b,a]}$  a self-dual Maxwell field.

This procedure, which is essentially due to G. Sparling, can be generalized to arbitrary spin, rest-mass zero fields and to arbitrary self-dual Yang-Mills type gauge theories.

A question immediately arises as to the mathematical nature of the "functions"  $A(l, \tilde{s}, \tilde{\tilde{s}})$  and  $F(x^a, \tilde{s}, \tilde{\tilde{s}})$ . (We will actually be more interested in  $G = G(x^a, \tilde{s}, \tilde{\tilde{s}}) \equiv e^F$ .) Should they be thought of as forms or elements of cohomology groups? What is their relationship to the line bundle over twistor space approach to Maxwell theory?

One can in a certain sense understand  $G = e^{F(\tilde{s}, \tilde{\tilde{s}})}$  by considering it first as a surface in a three complex dimensional space  $(\tilde{s}, \tilde{\tilde{s}})$  which is regular on the  $\tilde{s} = \tilde{\tilde{s}}$  diagonal. (See diagram on next page.)

22.



$\tilde{s}$  where both  $G_0$  and  $G_1$  are defined, the mapping from one line to the other

$$G_0 G_1^{-1} = e^{F_0 - F_1} = g(x^a, \tilde{s})$$

which is to be thought of as an element of  $H^1(S^2, \mathcal{O}^*)$  constructed over a line in twistor space. Actually one can show, by integrating Eq. (1) along  $s$  from  $s_0$  to  $s_1$ , that  $F_0 - F_1$  is a twistor function. The decomposition of  $g$  into  $e^{F_0 - F_1}$  is the Spurting splitting of the twistor function.

From this it follows that  $A(l, s, \tilde{s})$  is essentially to be considered as the Dolbeault version of the  $H^1(S^2, \mathcal{O}^*)$  i.e. it is a  $\bar{\partial}$  closed one form. Since the bundle is trivial,  $H^1 = 0$  and the one-form is  $\bar{\partial}$  exact. This is actually the content of equation (1), where  $F$  is essentially (mod factors of  $1+s\tilde{s}$ ) the function such that

$$\bar{\partial} F = A.$$

Ted Newman

Announcement: GAITS has shown that the  $n$ -twistor internal symmetry group is connected (cf. R.P. in TNG).

## Cell Decomposition and Homology of Minkowski space.

The homology of Minkowski space, both real and complexified, can be calculated using standard results for the homology of CW-complexes. The boundaries,  $\mathbb{I}$  and  $\mathbb{C}\mathbb{I}$ , at null infinity, together with the point  $I$  at spatial and timelike infinity can be represented as: an appropriate collection of cells forming the skeleton to which the single cell representing the interior of Minkowski space is attached. In the complex case a twistor argument can be used to identify these cells, while in the real case the result comes more easily using an argument in (real) Minkowski space itself.

A (real)  $n$ -cell  $E^n$  is a topological space homeomorphic to the interior  $\text{Int } I^n$  of the  $n$ -cube  $I^n = [0,1] \times \dots \times [0,1]$  in  $\mathbb{R}^n$ , e.g.  $\mathbb{R}^n$  itself.

A polydisc  $\Delta^n = \{z \in \mathbb{C}^n : |z_i| < 1 \text{ for } i=1, \dots, n\}$  is a  $2n$ -cell and will be referred to as a complex  $n$ -cell.

For a disjoint union of cells  $X = \bigcup_{a \in A} E_a$ , the  $n$ -skeleton of  $X$  is defined to be the space  $K^n(X) = \bigcup_{|a|=n} E_a$ , where  $|a| = \text{dimension}$  of the cell  $E_a$ . A here is simply some index set for the cells forming  $X$ .

Suppose that, for each  $a \in A$ , the maps  $q_a : \text{Int } I^{n_a} \cong E_a \hookrightarrow X$  can be extended to maps  $\bar{q}_a : I^{n_a} \rightarrow X$ . Then the boundary  $\partial E_a$  is defined to be the subspace  $\bar{q}_a(\partial I^{n_a})$  where  $\partial I^n = (\partial I \times I^{n-1}) \cup (I \times \partial I \times I^{n-2}) \cup \dots \cup (I^{n-1} \times I)$ , and  $\partial I = \{0,1\}$ .

If also  $\partial K^n(X) \subseteq K^{n+1}(X)$  for all  $n$ , then  $X$  is said to be a CW-complex. A shorthand notation  $X \approx (a_0, a_1, \dots, a_n)$ , where  $a_i = \# \text{ of cells of dimension } i$ , will occasionally be used for a CW-complex  $X$ . When all of the cells are complex this can be further shortened to  $X \approx (a_0, a_1, \dots, a_n)$ .

Now consider complexified, compactified Minkowski space  $\text{CM}$ , and twistor space  $\text{PT}$ . The point at infinity  $I$  (not to be confused with the interval  $[0,1]$ ) is represented by the infinity twistor  $I^\infty$ , and corresponds to a complex projective line in  $\text{PT}$ , also to be denoted  $I$ . The points of (complexified) null infinity  $\mathbb{C}\mathbb{I}$  correspond in  $\text{PT}$  to the (complex projective) lines which intersect  $I$ . The space of such lines can be divided into cells as follows.

(i) Choose a point  $J$  in  $\text{PT}$  lying on  $I$ . The lines in  $\text{PT}$

24.

passing through  $J$  constitute a  $\mathbb{C}P^2$  which has cell decomposition  $\mathbb{C}P^2 = \Delta_{11}^2 \cup \Delta' \cup \Delta^\circ$ .  $\Delta'$  can be taken to be the line  $I$ , and  $\Delta'$  can be taken to be the set of lines meeting an arbitrarily chosen line  $I'$  in  $PT$  which intersects  $I$  at some point  $J'$  distinct from  $J$ .  $\Delta_{11}^2$  then represents all other lines through  $J$ . That  $\Delta_{11}^2$  and  $\Delta'$  are in fact complex cells satisfying  $\partial\Delta_{11}^2 = \Delta' \cup \Delta^\circ$ , and  $\partial\Delta' = \Delta^\circ$  can be easily verified.

(ii) Having chosen  $I'$  as above, consider the space <sup>$S$</sup>  of lines which intersect both  $I$  and  $I'$ , but not at  $J'$ . These lines rule out a hyperplane, topologically  $\mathbb{C}P^2$ , in  $PT$  except that the point  $J'$  has been removed. Replacing  $J'$ , and adding to  $S$  those lines lying in the hyperplane which pass through  $J'$ , gives a  $\mathbb{C}P^2$ 's worth of lines, of which a  $\mathbb{C}P^1 = \Delta' \cup \Delta^\circ$ 's worth pass through  $J$  and have already been counted. Removing these leaves a set  $S'$  which is a (complex) 2-cell  $\Delta_{11}^2$ .

(iii) For each point on  $I$  except  $J$  - a 1-cell's worth of points - there is a 2-cell's worth of lines which have not yet been counted. These are lines through  $I$  but not intersecting  $I'$ . Collectively these lines constitute a 3-cell  $\Delta^3$ .

The boundary conditions on these cells can easily be checked showing that  $\partial\Delta^3 \cap I \approx (1, 1, 2, 1)_c$  is indeed a CW-complex. Adding the interior cell  $\text{CM} \approx (1, 1, 2, 1, 1)_c$  which agrees with the standard result for the Grassmannian  $G(2, 4)$  of 2-planes in  $\mathbb{C}^4$  using Schubert cycles (see e.g. Griffiths & Adams, Ch. 5).

The case for real Minkowski space is simplified using  $M \times S^3 \times S^1$ . The  $S^3$ 's are 1-point compactifications of hyperplanes with a common tangent null vector,  $v^a$  say.  $S^1$  then represents a time-function  $t$  labelling the hypersurfaces with  $t = \pm\infty$  identified. The  $\pm\infty$  hypersurface corresponds to those points on  $I$  reached by travelling in null directions distinct from  $v^a$ . In shorthand,  $S^3 \approx (1, 0, 0, 1)$  and  $S^1 \approx (1, 1)$  so  $M \times S^3 \times S^1 \approx (1, 1, 0, 1, 1)$ .

Homology: The following facts concerning the singular homology (with integer coefficients) of CW-complexes can be found in various books on elementary algebraic topology.

Let  $X$  be a CW-complex, and  $K''$  its  $n$ -skeleton.

$$(1) \quad H_r(K'', K'''') = 0 \quad r > n \quad (\text{relative homology})$$

$$H_n(K'', K''') = \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{k \text{ times}} \quad (= G_n \text{ say})$$

where  $k = \# \text{ of } n\text{-cells in } K''$ .

The relative homology sequence then gives  $H_r(X) \cong H_r(K'')$  for  $r < n$ , and  $H_r(K'') = 0$  for  $r > n$ .

$$(2) \quad H_n(X) = \frac{\ker(\delta : G_n \rightarrow G_{n-1})}{\text{Im}(\delta : G_{n+1} \rightarrow G_n)}$$

where  $\delta$  is induced from the boundary operator  $\delta : K'' \rightarrow K'''$  considered as a map of singular simplices.

For  $X = CM$  there are only cells in even (real) dimensions, so  $K^{2n+1} = K^{2n}$  and  $G_{2n+1}''' = 0$ ,  $G_{2n} = k\mathbb{Z}$ ,  $k = \# \text{ of complex } n\text{-cells}$ .

Clearly  $\text{Im}(\delta : G_{n+1} \rightarrow G_n) = 0$  for all  $n$ , so  $H_n(X) = G_n$ .

Thus for  $CM$ ,  $H_0 \cong H_2 \cong H_4 \cong H_6 \cong \dots \cong \mathbb{Z}$

$$H_4 \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and all others are zero.}$$

For real Minkowski space, it is necessary to look at the maps

$\delta : K^4 \rightarrow K^3$  and  $\delta : K^3 \rightarrow K^2$  as maps of singular simplices,

$\delta E' = 0$  for any 0-simplex ( $\cong 0\text{-cell = point}$ )

and  $\delta E' = \bar{\varphi}(1) - \bar{\varphi}(0)$  where  ~~$\bar{\varphi}$~~   $\bar{\varphi} : I \times E' \hookrightarrow X$ .

$$\begin{aligned} \text{Hence } \delta S' &= 0. \quad \text{Similarly } \delta S^2 = 0 \quad \text{so } \delta K^4 = \delta M = \delta(S^3 \times S^1) \\ &= (\delta S^3 \times S^1) + (S^3 \times \delta S^1) = 0. \end{aligned}$$

An alternative way to see this result

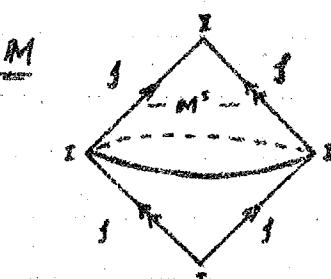
is afforded by the diagram for  $M$ ,

where the identifications on  $I$  are shown.

Tracing a path around the boundary of the interior cell, corresponding parts of  $I$  are traversed in opposite directions, giving a total contribution of zero to the simplicial boundary. Hence, similarly to the complex case  $H_n(M) \cong G_n$ .

$$\text{so that } H_0 \cong H_2 \cong H_4 \cong \mathbb{Z}$$

all others are zero.



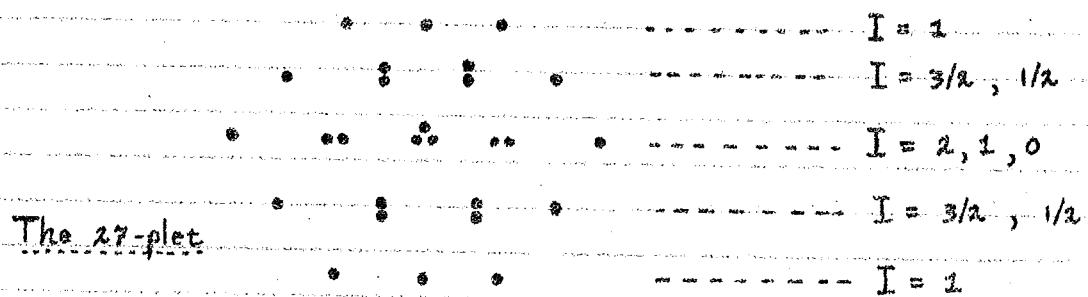
— Ross Moore —

## 26. In Search of the Ninth Vector Meson

In the twistor particle classification scheme, hadrons are represented by functions of three twistors. They fall naturally into multiplets of  $SU(3)$  representations, broadly similar to the conventional classification in the quark model. Hence the vector octet naturally occurs for the mesons. In other words, if the field is written

$$\mathcal{E}_{\alpha i}^{R'j}(x) = \oint p_{\alpha} \pi_i^{R'} \hat{\pi}_R^j f(z_i^*) \Delta \pi , \quad i=1,2,3$$

where  $\hat{\pi}_R^j$  is the operator  $-2/2w_j^*$  with  $z_i^* = (\omega_i^*, \pi_{R'i}^*)$ , and  $p_{\alpha}$  is the restriction operator for the spacetime point  $x^{RR'}$  (cf. TN 6, p. 3), the vector octet is obtained by taking the divergence-free part of  $\mathcal{E}$ . However, in nature there are nine established vector mesons. It is proposed that perhaps one could look for the ninth vector meson in higher dimensional representations, e.g. the 27-plet. The weight diagram for this is shown in the figure below:



In terms of isospin multiplets, we have here one  $I=2$  quintuplet, two  $I=3/2$  quartets, three  $I=1$  triplets, two  $I=1$  doublets, and one  $I=0$  singlet. The idea is that this isosinglet can mix with the isosinglet of the octet representation, giving the  $\omega$  and the  $\phi$ . This mixing can actually explain some unusual properties of the  $\omega$ , such as a large coupling to baryon-antibaryons.

What about the rest of the 27-plet? A number of peaks have been observed in the range 1.5 GeV - 2.0 GeV at Frascati and Orsay in  $e^+e^-$  total cross sections. These must necessarily represent  $J^P = 1^-$  states. The other quantum numbers are largely unknown. Also exotic  $I=2$  states have been observed at CERN around 2 GeV. Although these states have not yet been confirmed, they can conceivably belong to our 27-plet. We can also compare our picture with the conventional quark model, where in order to obtain a higher representation one needs to go to 2 quark-2 antiquark states. The mass spectrum of these  $qq\bar{q}\bar{q}$  states

has been estimated in some specific models, and the mass of the highest members of the 27-plet could be as low as 1.6 GeV. Now the isosinglet, because of the presence of annihilation channels, is expected to be slightly different from the others. According to a specific model calculation its mass is expected to be lower than the other members. Hence the 27-plet isosinglet might conceivably be close enough to the usual mesons for some mixing to occur.

~ Tsun Sheung Tsun and Lane Hughston

## Spin-Statistics and Twistor Theory

We create a massless particle 'localized' (in some appropriate sense) on a geodesic, and annihilate it on another geodesic. Reversing the two geodesics gives a sign change (or not) depending upon whether the particle is a fermion (or not). This is related to the spin-statistics result of classical quantum field theory.

A fairly usual investigation of the connection of spin with statistics begins by considering the quantum field operators  $\varphi(x)$  and  $\varphi(y)$ , with  $(x-y)^2 < 0$ . Local commutativity then asserts that one or the other of

$$[\varphi(x), \varphi(y)]_{\pm} = \varphi(x)\varphi(y) \mp \varphi(y)\varphi(x) \quad (1)$$

vanishes. The content of the spin-statistics theorem is that if  $n=2s$ , where  $s$  is the spin of the field, specifically  $[\varphi(x), \varphi(y)]_{\mp, n+s}$  vanishes.

First, we consider the effect of wedging  $[\varphi(x), \varphi(y)]_{\pm}$  between two vacuum states:

$$\langle 0 | [\varphi(x), \varphi(y)]_{\pm} | 0 \rangle = \langle \varphi^*(x) | 0 \rangle \langle 0 | \varphi(y) | 0 \rangle \pm \langle \varphi^*(y) | 0 \rangle \langle 0 | \varphi(x) | 0 \rangle.$$

Smearing this with test functions  $f(x)$  and  $g(y)$  (with spacelike-separated support), we get

$$\langle \varphi^*(f) | 0 \rangle \langle 0 | \varphi(g) | 0 \rangle \pm \langle \varphi^*(g) | 0 \rangle \langle 0 | \varphi(f) | 0 \rangle.$$

28. Operating on the vacuum state, however, the field operators function only as creation operators; say

$$|\varphi(f)\rangle = |p\rangle \quad \text{and} \quad |\varphi(g)\rangle = |\sigma\rangle. \quad (2)$$

$$\text{Then we have } \langle p^* | \sigma \rangle \pm \langle \sigma^* | p \rangle. \quad (3)$$

The taking of the complex conjugate involves the use of an arbitrary unit timelike vector (as in, say,  $\varphi_A^* = \bar{\varphi}_A \gamma^\mu \gamma_\mu$ ); it might therefore seem that the expression (3) is not Lorentz invariant. This problem can be overcome if we replace  $\varphi(x)$  by  $\varphi^*(x)$  in (1), leading to

$$\langle p | \sigma \rangle \pm \langle \sigma^* | p^* \rangle,$$

each term of which is independent of the timelike vector chosen.

Returning to (3), we choose the sign correctly to conclude that

$$\langle p^* | \sigma \rangle - (-)^n \langle \sigma^* | p \rangle = 0 \quad (4)$$

whenever f and g in (2) are spacelike separated. Now, it turns out that, for elementary states, (4) holds always! (We will show this later.) It is not clear why this should be the case, though - elementary states are certainly not spacelike separated (and hence non-communicating) in any sense.

However, for any elementary state defined using two planes A and B in  $\mathbb{R}M$ , the associated field  $\varphi$  blows up (i.e., has a pole) at all points  $x$  null separated from  $p$ , where  $L_p = A \cap B$ . Furthermore, if  $L_p$  touches  $\mathbb{R}N$ , then those points  $x$  in  $M$  at which  $\varphi$  blows up will lie on a geodesic. If we could increase the degree of the pole in  $\varphi$ , we would, at least in some limiting sense, be concentrating the field on the geodesic. Two such fields should satisfy the relation (4), as they can be thought of as being propagated from distinct (and a spacelike separated choice is always possible) points on their respective geodesics.

With this in mind, we consider the twistor function

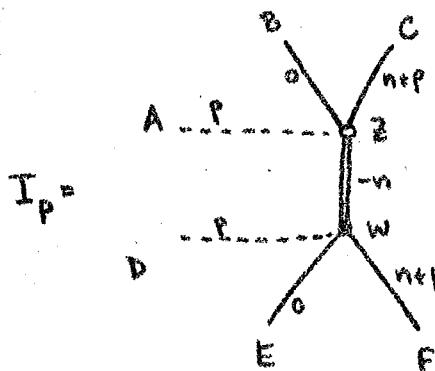
$$f(z^\alpha) = \left(\frac{A}{z}\right)^p \left(\frac{B}{z}\right)^{-1} \left(\frac{C}{z}\right)^{-n-p-1}. \quad (5)$$

If  $A_\alpha = (A_A, A^{A'})$ , we set  $\alpha^{A'} = A^{A'} + i\alpha A^{A'} A_A$ , and define  $\beta^{A'}$  and  $\gamma^{A'}$  similarly. It is easy to see that

$$(A' \dots)_P = \frac{\beta^{A'} \dots \beta_L' (\beta \cdot \alpha)^P}{(\beta \cdot \gamma)^{n+p+1}},$$

exactly as we want.

We are interested in scalar products of states like this, which we draw as



where  $A \cdots \circ Z$  represents  $\begin{pmatrix} A \\ 1 \end{pmatrix}^P$   
and does not enter into the singularity structure.

Starting with the well-known result

$$I_0 = \frac{\begin{pmatrix} B \\ 1 \\ E \end{pmatrix}^n}{(B+E+F)^{n+1}}$$

and differentiating, we get

$$I_p = \left( \frac{\partial}{\partial F} \frac{A}{1} \right)^P \frac{\begin{pmatrix} B \\ 1 \\ E \end{pmatrix}^n}{(B+E+F)^{n+1}} = \begin{pmatrix} B \\ 1 \\ E \end{pmatrix}^n \left( \frac{\partial}{\partial F} \frac{A}{1} \right)^P \frac{1}{(B+E+F)^{n+1}}.$$

Set

$$\alpha = \frac{B \quad C}{E \quad F};$$

now

30.

$$\frac{\partial}{\partial F} \frac{A}{D} \frac{1}{z^{n+1}} = \frac{(n+1)(n+2)}{z^{n+3}} \frac{B}{E} \frac{C}{D} \frac{B}{E} \frac{A}{F} - \frac{n+1}{z^{n+2}} \frac{B}{E} \frac{A}{D}.$$

We are interested in the large  $p$  limit, where we will want all of the derivatives involved to act on ' $\gamma^*$ ', so that the leading constant gets large. Absorbing an overall constant, we get

$$\begin{aligned} I_p &\rightarrow \left(\frac{B}{E}\right)^n \left(\frac{B}{E} \frac{C}{D}\right)^p \left(\frac{B}{E} \frac{A}{F}\right)^p \frac{1}{\left(\frac{B}{E} \frac{C}{F}\right)^{n+2p+1}} \\ &= \frac{\left(\frac{B}{E}\right)^n}{\left(\frac{B}{E} \frac{C}{F}\right)^{n+1}} \left[ \frac{\left(\frac{B}{E} \frac{C}{D}\right) \left(\frac{B}{E} \frac{A}{F}\right)^p}{\left(\frac{B}{E} \frac{C}{F}\right)^2} \right]. \end{aligned} \quad (6)$$

Here  $I_p = \langle p^* | \sigma \rangle$ ; we want  $\lim_{p \rightarrow \infty} I_p$ . (Really,  $\lim_{p \rightarrow \infty}$  of  $\frac{I_p}{(I_p)^{1/(n+1)}}$ , which is what allows us to absorb the  $-p$  dependent constant in (6).) But if this limit is to be interesting, the bracketed term in (6) must be 1 (this amounts to an appropriate choice of  $A$  and  $D$ ), in which case

$$I_p \rightarrow \frac{\left(\frac{B}{E}\right)^n}{\left(\frac{B}{E} \frac{C}{F}\right)^{n+1}} (= I_0). \quad (7)$$

Keeping this in mind, we return to (4). If the field  $P_{A' \dots L'}^*$  comes from the twistor function (5), the field  $P_{A' \dots L'}$  will come from

$$\tilde{f}(z^\mu) = \frac{\left(\frac{\tilde{A}}{z}\right)^p}{\left(\frac{\tilde{B}}{z}\right) \left(\frac{\tilde{C}}{z}\right)^{n+p+1}}.$$

We write  $\tilde{A}_\alpha = T_{\alpha\beta} \bar{A}^\beta$ . For  $A_\alpha = (\omega_\alpha, \pi^A)$ , it is clear from the contour integral representation for  $\rho$  that

$$\tilde{A}_\alpha = T_{\alpha\beta} \bar{A}^\beta = (-\bar{\omega}_\alpha, h_A^{A'} \bar{\pi}^A h_A^{A'}),$$

and therefore

$$T_{\alpha\beta} = \begin{pmatrix} 0 & -h_A^{A'} \\ h_A^{A'} & 0 \end{pmatrix} = -T_{\beta\alpha}.$$

Also, since  $\rho^{*\alpha} = \rho$ ,  $\bar{T}^{\gamma\beta} T_{\beta\alpha} = \delta^\gamma_\alpha$ .

For the second term of (4), we now have, using (7),

$$\langle r^k | \rho \rangle \rightarrow \frac{\left(\frac{-E}{B}\right)^n}{\left(\frac{E}{B} + \frac{F}{C}\right)^{n+1}}.$$

But

$$\begin{aligned} \frac{\bar{E}}{B} &= \frac{\bar{T}^{\alpha\beta} \bar{E}_\beta}{\bar{B}} \\ 1 &= \frac{1}{\bar{T}_{\alpha\gamma} \bar{B}^\gamma} = T_{\alpha\beta} E^\beta \bar{T}^{\alpha\gamma} B_\gamma \\ \frac{\bar{E}}{B} &= -\bar{T}^{\gamma\alpha} T_{\alpha\beta} E^\beta B_\gamma = -\frac{B}{E}. \end{aligned}$$

A similar calculation leads to

$$\frac{\bar{E}}{B} + \frac{\bar{F}}{C} = + \frac{E}{B} + \frac{F}{C},$$

so we do in fact get

$$\langle \rho^k | r \rangle = (-)^n \langle \sigma^k | \rho \rangle,$$

which is to say, (4).

Matt Ginsburg

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### Sheaf Cohomology and Twistor Diagrams

First, we shall identify the cohomological features involved in the diagrams themselves; following this, we will look for cohomological analogs to the calculations they entail.

#### I. The ears of twistor diagrams.

We first examine the single line  $\overset{\circ}{A}$ , representing the function  $(w \cdot A)^{-1}$ . This is an element of

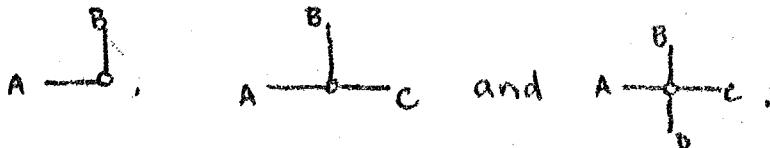
$$H^0(\mathbb{RP} - \{w \cdot A = 0\}; \mathcal{O}(-1)).$$

We can generalize this somewhat. Let  $V$  be a 1 codimensional subvariety in  $\mathbb{RP}$ , and denote  $\mathbb{RP} - V$  by  $X_1$  (or by  $X_k$  if  $V$  is  $k$  codimensional). We shall drop the  $\mathcal{O}$  in the above, for notational convenience, and write

$$H^0(X_1; -1)$$

as the generalized single line. (RP has pointed out that to treat all functions whose singularities are of the same type as  $\overset{\circ}{A}$  we shall have to generalize these domains still further.)

Next, we consider



For the first, we use a two set covering of part of  $\mathbb{RP}$ , namely,  $\mathbb{RP} - A$  and  $\mathbb{RP} - B$ , to obtain an element of

$$H^1(\mathbb{RP} - (A \cup B); \mathcal{O}(-2)),$$

which we generalize to

$$H^1(X_2; -2).$$

Similarly, three lines represent an  $H^2(X_3; -3)$ , and four, an  $H^3(X_4; -4)$ .

#### II. The multiply map.

Let us return to the  $H^1(X_2; -2)$  for a moment. As a function, it is the product of two  $H^0(X_1; -1)$ 's. We can formalize this:

Suppose  $f \in H^0(X_r; -m)$ , represented by  $f_{i_1 \dots i_p}$  on  $U_{i_1} \cap \dots \cap U_{i_p}$ , where  $\{U_i\}$  covers  $X_r$ , and  $g \in H^0(X_s; -n)$ , represented similarly. Now, the covers for  $X_r$  and  $X_s$  together form a cover for  $X_r \cup X_s$ , or  $\mathbb{RP} - ((\mathbb{RP} - X_r) \cap (\mathbb{RP} - X_s))$ . Since the intersection of subvarieties of codimension  $r$  and  $s$  is generally a subvariety of codimension  $r+s$  (this is, essentially a "general position" requirement), we thus have  $X_r \cup X_s = X_{r+s}$ .

Now take the disjoint union of the two covers for  $X_r$  and  $X_s$ , and use it as a cover for  $X_{r+s}$ . On the intersection of  $p+q+2$  of these sets, define a function  $f_{i_1 \dots i_p, j_{p+1} \dots j_{p+q+2}}$ , precisely where  $p+1$  of the sets come from the  $X_r$  cover, and  $q+1$  from the  $X_s$  cover (otherwise, take 0), suitably symmetrized for sign. Because our cover is the disjoint union of the individual covers, the cocycle and coboundary conditions are satisfied trivially, and we have a map

$$\text{mul}: H^p(X_r; -m) \otimes H^q(X_s; -n) \rightarrow H^{p+q+1}(X_{r+s}; -m-n).$$

In a specific case, this gives a map

$$H^0(X_1; -1) \otimes H^0(X_1'; -1) \rightarrow H^1(X_2; -2),$$

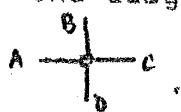
just as we had expected.

ME has suggested that this map can be described as follows:

$$\begin{aligned} H^p(X_r; -m) \otimes H^q(X_s; -n) &\xrightarrow{i^*} H^p(X_r \cap X_s; -m) \otimes H^q(X_r \cap X_s; -n) \\ &\xrightarrow{\cup} H^{p+q}(X_r \cap X_s; -m-n) \xrightarrow{\partial^*} H^{p+q+1}(X_r \cup X_s; -m-n). \end{aligned}$$

Here  $i^*$  (functional restriction) is the dual of the natural injection,  $\cup$  is cup product, and  $\partial^*$  is the Mayer-Vietoris coboundary.

### III. Evaluation of the diagram

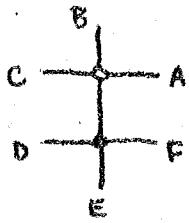


This diagram represents an element of  $H^3(X_4; -4)$ . We have  $X_4 = \mathbb{RP} \cong X$ , and, from Serre duality,

$$H^3(X; -4) \cong H^0(X; 0)^* \cong \mathbb{C},$$

as required.

### IV.

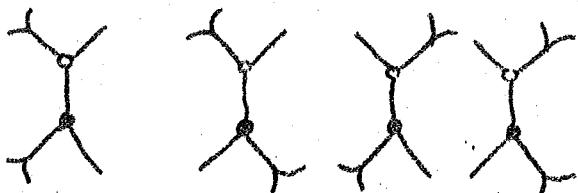


This situation, involving two twistors, is somewhat more complicated. What we are looking for is a map

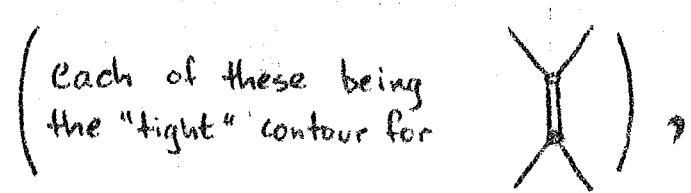
$$H^2(X_3; -3) \otimes H^2(X_3^*; -3) \rightarrow \mathbb{C}$$

(where  $X^*$  is the dual of  $X$ ).

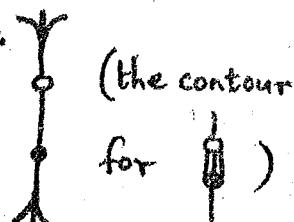
The first thing we consider is: how many maps are we looking for? A homology calculation shows that there are six linearly independent contours for the diagram above; we can sketch them as follows:



(Each of these being the "tight" contour for



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and a final one,



this being the only contour with enough  $S^1$ 's to separate all of the "ears" of the above diagram. In other words, it is the only contour which respects the " $H^2$ -ness" of the incoming states. Thus, we are looking for a single map of the above type.

Now, the integration procedure involves multiplying the two  $H^2$ 's together with the "bracket factor" to get an integrand defined on the product space  $Y \cong X \times X^*$ . (We will denote the result of cutting a subvariety of codimension  $k$  out of  $Y$  by  $Y_k$ , as usual.) The bracket factor, then, appears to be giving us a map

$$bf: H^2(X_3; -3) \otimes H^2(X_3^*; -3) \rightarrow H^6(Y; -4, -4).$$

We can formalize this, realizing that the bracket factor  $b = (W \cdot Z)_{k=1}$  is, for  $k > 0$ , an element of  $H^0(Y_1; -k, -k)$ , where  $Y_1 = Y - \{W \cdot Z = 0\}$ . We now perform two "multiply" operations, as follows: (Note first that  $X_r \times X_s^* = (X_r \times \mathbb{RP}^*) \cap (\mathbb{RP} \times X_s^*)$ ;  $X_r \times \mathbb{RP}^* = Y_r$  and  $\mathbb{RP} \times X_s^* = Y_s$ . Thus  $(X_r \times \mathbb{RP}^*) \cup (\mathbb{RP} \times X_s^*) = Y_{r+s}$ .)

$$H^p(X_r; -m) \otimes H^q(X_s^*; -n) \xrightarrow{*} H^{p+q}(X_r \times X_s^*; -m, -n)$$

$$\xrightarrow{\exists^*} H^{p+q+1}(Y_{r+s}; -m, -n) \xrightarrow{i} H^{p+q+1}(Y_{r+s}; -m, -n) \otimes H^0(Y_1; -k, -k)$$

$$\xrightarrow{\cup} H^{p+q+1}(Y_{r+s} \cap Y_1; -m-k, -n-k) \xrightarrow{\exists^*} H^{p+q+2}(Y_{r+s+1}; -m-k, -n-k).$$

Here  $*$  is the cohomological cross product,  $i$  is the natural injection corresponding to the choice of the bracket factor function  $b$ , and the other maps are as in the multiply construction.

We define the composite map to be the bracket factor

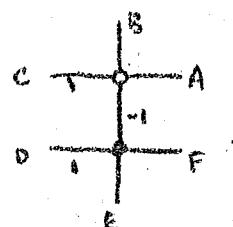
$$bf_k: H^p(X_r; -m) \otimes H^q(X_s; -n) \rightarrow H^{p+q+2}(Y_{r+s+1}; -m-k, -n-k)$$

(for  $k > 0$ ). Using it, we can evaluate this diagram, for we have

$$H^2(X_3; -3) \otimes H^2(X_3^*; -3) \xrightarrow{bf} H^6(Y_7; -4, -4) \cong \mathbb{C}$$

by Serre duality, as  $Y_7 = Y$  and  $Y$  is compact.

V. The final diagram we consider is



This involves a bracket factor type map for  $k \leq 0$  ( $k=0$ , in this case). We could, of course, repeat the construction of the last section, but since the bracket factor  $b$  extends to all of  $Y$ , we would have

$$\begin{aligned} H^p(X_r; -m) \times H^q(X_s^*; -n) &\rightarrow H^{p+q+1}(Y_{r+s}; -m-k, -n-k) \\ &\xrightarrow{\cong} H^{p+q+1}(Y_{r+s}; -m-k, -n-k). \end{aligned}$$

When we apply the  $\partial^*$  map to this, we will get 0, as the Mayer-Vietoris sequence is exact, and  $H^{p+q+1}(Y_{r+s}; -m-k, -n-k)$  is in the image of the preceding map. (Equivalently, the value of the contour integral is zero.)

The solution to this is to do an integration with boundary on  $B \in \{W \cdot Z = 0\}$ . To understand this cohomologically, consider the following exact sequence of sheaves:

$$0 \rightarrow \mathcal{O}(-m-1, -n-1) \xrightarrow{\ast(w \cdot z)} \mathcal{O}(-m, -n) \rightarrow {}_B\mathcal{O}(-m, -n) \rightarrow 0,$$

where  ${}_B\mathcal{O}(-m, -n)$  is the sheaf of germs of homogeneous functions on the subvariety  $B$ . The associated long exact sequence is, in part,

$$\dots \rightarrow H^5(B; -m, -n) \xrightarrow{\partial} H^6(Y; -m-1, -n-1) \rightarrow H^6(Y; -m, -n) \rightarrow \dots$$

The map  $\partial$  can be described as follows: If  $x$  is a closed 5-form on  $B$ , extend it to a 5-form (also called  $x$ ) on  $Y$ . We still have  $dx=0$  on  $B$ , so  $dx = (W \cdot Z)y$ ;  $y = dx/(W \cdot Z)$  is the 6-form on  $Y$  (homogeneous of degree  $-m-1, -n-1$ ) given by the map  $\partial$ .

Also, since the bracket factor maps are, essentially, just multiplication by  $(W \cdot Z)^{-k}$ , the following diagram commutes:

$$\begin{array}{ccc} H^5(Y_6; -4, -4) & & \\ \downarrow bf_1 & \searrow "bf_0" & \\ H^6(B; -4, -4) & \xrightarrow{\partial} & H^6(Y; -5, -5) \rightarrow H^6(Y; -4, -4) \end{array}$$

If  $f \in H^5(Y_6; -4, -4)$  is the function we are trying to integrate,  $bf_0(f) = 0$ , so  $bf_1(f)$  is in the image of  $\partial$ ; say  $bf_1(f) = \partial(g)$ .

In terms of functions,

$$\frac{f}{w} = \partial g = \frac{dg}{z},$$

so  $f = dg$ .

The value we are looking for is

$$\oint f = \oint dg = \oint g,$$

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by the divergence theorem.

Having found  $g$  cohomologically, all that remains is to integrate it. Serre duality does not apply in this situation, as  $\mathcal{O}(-4, -4)$  is not a line bundle over  $B$ , but we can still integrate the 5-form  $g$  over the 5-dimensional compact complex manifold  $B$  to get a map (no longer an isomorphism)

$$H^5(B; -m-4, -m-4) \rightarrow H^0(B; m, m)^*,$$

or, in this case,

$$\int \int \int f = \text{trace } \int \int \int f.$$

Higher order internal lines can be handled similarly, for we have that

$$\int \int \int f = \text{trace } \int \int \int f.$$

The taking of the trace can thus be thought of as a map

$$\text{tr}: H^0(B; m, m)^* \rightarrow H^0(B; m-1, m-1)^*.$$

We therefore handle the general boundary contour as follows:

$$\begin{aligned} H^5(Y; -m-4, -m-4) &\xrightarrow{\text{tr}_1} H^6(Y; -m-5, -m-5) \xrightarrow{\delta^1} H^5(B; -m-4, -m-4) \\ &\xrightarrow{\text{tr}} H^0(B; m, m)^* \xrightarrow{\text{tr}^m} \mathbb{C}. \end{aligned}$$

The work presented here is certainly incomplete in many ways; among the more looming gaps is the fact that we have not justified the assumed correspondence between integrating a function along a contour (as in a twistor diagram) and integrating a form over a manifold (as in Serre duality). In addition, it is possible, for example, for an  $H^1(\mathbb{RP}^4; -m)$  to be inextendible to an  $H^1(X_2; -m)$ . Finally, the natural extensions of our techniques to more complicated twistor diagrams have yet to be achieved. We hope in the future to be able to deal with these points.

Errata.

TN4 p. 12 : condition (b)  
should, of course, read  $\text{Ker } \beta = \text{Im } \alpha$ .

Mark Ginsberg  
Stephen Hawking

TN6 p. 3 ; 2<sup>nd</sup> line : "photon" should be "proton".

p. 5, 1<sup>st</sup> line : "magnetic" should be "magnetic".

p. 5, line 4 in proof of Prop. 2 :  $\frac{\partial f}{\partial z_i}$  should be  $\frac{\partial f}{\partial \bar{z}_i}$ .