Chromatic Evaluation of Strand Networks

This note describes R.P.'s "chromatic" method for evaluating strand networks, which was mentioned in his papers on spin networks,* but never published. We consider the example shown in fig. 1; in the next Newsletter, we will show that evaluation of this network in fact amounts to a calculation of the familiar Clebsch-Gordan coefficients.

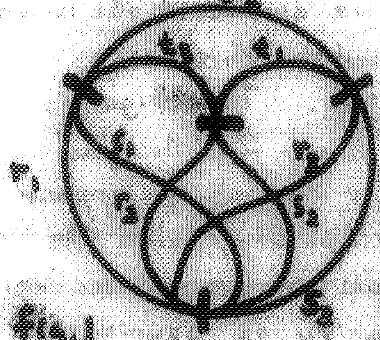

fig. 1

Recall that each edge labelled by an integer e represents a bundle of e strands, and each bar b signifies the alternating sum of all cross-connections (subtracting for odd parity). Since there are no open edges, the network becomes a polynomial P in closed loops when all the bars are expanded. The <u>value</u> of the network is defined as $P(-2)$, divided by $\prod_{\text{all edges}} e!$.

A polynomial is completely determined by its behavior at integers $N \geq 0$. Moreover when closed loops are given positive values N, $P(N)$ can be interpreted as a scalar in an ordinary N-dimensional Cartesian space: each bar in the strand network is regarded as a generalized delta $\delta^{j_1 j_2 \cdots j_r}_{i_1 i_2 \cdots i_r}$ ($= \pm 1$ for $i_1 \cdots i_r$ an $\begin{smallmatrix}\text{even}\\\text{odd}\end{smallmatrix}$ permutation of $j_1 \cdots j_r$, 0 otherwise) and each edge as a bundle of indices to be contracted. Since

$$\delta^{j_1 j_2 \cdots j_r}_{i_1 i_2 \cdots i_r} = r!\, \delta^{[j_1}_{i_1} \delta^{j_2}_{i_2} \cdots \delta^{j_r]}_{i_r} \quad \text{and} \quad \delta_i^{\ j} \delta_j^{\ k} = \delta_i^{\ k}$$

the contractions reduce to a polynomial in the closed loop $\delta_i^{\ i} = N$ — precisely $P(N)$ as defined above. The chromatic method relates $P(N)$ for $N \geq 0$ to the number of N-colourings of a set of graphs.

Expanding in components, we regard the index values from 1 to N as a set of N distinct colors. Then each non-zero term in the contraction of generalized δ's corresponds to an N-coloring of the strands such that for each strand entering a bar, there is exactly one strand of the same color leaving the opposite side of the bar. Hence we may follow a strand through consecutive bars until it closes into a <u>cycle</u> — that is, a sequence $e_1 b_1 e_2 b_2 \cdots e_m b_m e_{m+1} = e_1$ of edges and bars such that e_i and e_{i+1} enter opposite sides of b_i and no two bars are the same. Cycles that have a bar in common (<u>incident</u> cycles) must have distinct colors

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* In particular, at the end of "Applications of Negative Dimensional Tensors" in <u>Combinatorial Theory and Applications</u>, ed. D. Welsh (Academic Press, 1971). See also: "Angular Momentum: An Approach to Combinatorial Space-Time" in <u>Quantum Theory and Beyond</u>, ed. Ted Bastin (Cambridge Univ. Press, 1971) and "On the Nature of Quantum Geometry" in <u>Magic Without Magic</u>, ed. J.R. Klauder (Freeman, 1972)

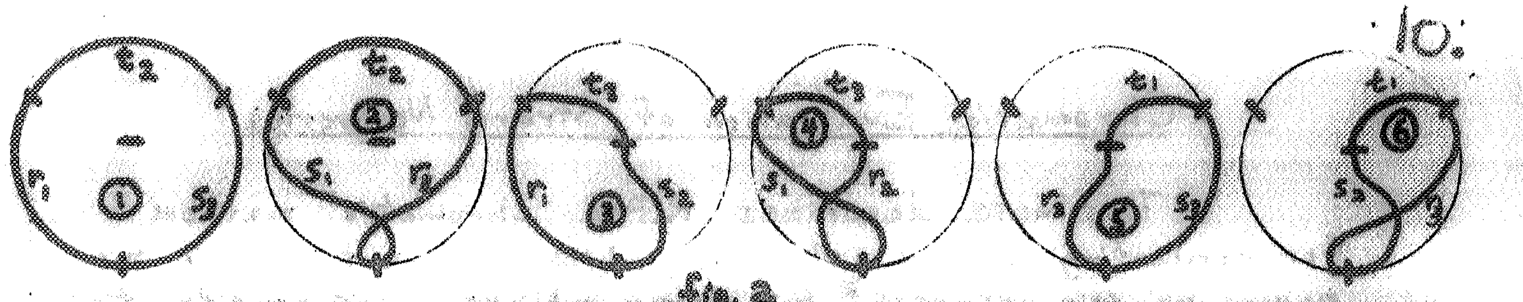

fig. 2

For example, the network in fig. 1 has only six kinds of cycle (shown in fig. 2), all of which are incident (since the bottom bar is common).

Now in each ± 1 term in the δ-contraction we can count the number of cycles of each type, say c_i of the i^{th} cycle. We notice that

1) <u>All terms with cycle numbers c_i have the same sign</u>. This sign is the parity of the number of crossings that occur at bars. But the total number of crossings between different cycles is even: they always appear and disappear in <u>pairs</u> as we move cycles apart. For simplicity, invoke the convention that the individual strands within each edge bundle are non-intersecting. Then the total parity of bar-crossings must equal the parity of "spurious" intersections of different cycles which occur away from the bars (e.g. in fig 2 cycles 2 and 3 have a spurious intersection where edge r_2 crosses s_2). But the number of spurious intersections is simply

$$\sum_{e_i \cap e_j} e_i \cdot e_j - \sum_{\text{self-intersecting cycles}} c_k^2$$

where the first term involves products of strand numbers of spuriously intersecting edges, and the second eliminates self-intersecting cycles. Hence the overall parity of the term is completely determined by $\{c_k\}$. (In our example, the parity is $s_1 r_2 + s_1 r_3 + r_3 s_2 + c_2 + c_4 + c_6 \pmod 2$.)

2) <u>For each allowed colouring of the cycles the number of terms in the sum is precisely $\prod e_j!$</u> An allowed colouring assigns c_i colors to the i^{th} cycles, with incident cycles having distinct colours. Given any term with this coloring, $\prod e_j!$ terms are generated by permuting strands within each edge. We have proven:

<u>Theorem</u>: The loop polynomial $P(N)$ of a strand network is given by

$$\frac{P(N)}{\prod_{\text{all edges}} e_j!} = \sigma \sum_{\{c_i\}} \epsilon \; K(\{c_i\}; N)$$

$$\text{where} \quad \sigma = (-1)^{\sum_{e_i \cap e_j} e_i \cdot e_j}, \qquad \epsilon = (-1)^{\sum_{\text{self-intersecting}} c_k},$$

$K(\{c_i\}; N)$ is the number of allowed N-colorings of the $\{c_i\}$ cycles, and the sum is over all non-negative cycle numbers $\{c_i\}$ such that for each edge $e_i = \sum_{\text{cycles through } e_i} c_j$.

From this formula we obtain the value of the strand network by simply setting $N = -2$. An immediate corollary is that this value is an integer, since any polynomial integer-valued at positive integers is integer-valued at negative integers as well (Express $P(N)$ in the form $\sum_{k=0}^{m} b_k \binom{N}{k}$, where $b_0 = P(0), \ldots, b_k = P(k) - \sum_{j=0}^{k-1} b_j \binom{k}{j} \ldots$ are all integers, as are $\binom{-N}{k} = \frac{(-N)(-N-1)\cdots(-N-k+1)}{k!} = (-1)^k \binom{N+k-1}{k}$, so $P(-N)$ is integral also

In fig. 1 there are 9 edges and 4 bars (across which sums of strand numbers must be equal), leaving 5 independent parameters. Hence we expect that if one cycle number is fixed the other 5 will be determined. Indeed if we set $c_1 = z$, then since $t_2 = c_1 + c_2$ we have $c_2 = t_2 - z$, and similarly for the other cycles $c_3 = r_1 - z$, $c_4 = t_3 - r_1 + z$, $c_5 = s_3 - z$, $c_6 = t_1 - s_3 + z$. Hence in this case the chromatic sum is simply over the range of z which gives non-negative values for all c_i. Recall $\epsilon = (-1)^{c_2+c_4+c_6} = (-1)^{t_1+t_2+t_3-r_1-s_3+z}$ and $\sigma = (-1)^{s_1r_2+s_1r_3+r_3s_3}$

Finally, since all cycles are incident, $K(\{c_i\}; N)$ is simply the number of ways of distributing N objects into six bins: $\frac{N!/(N-J)!}{c_1!\,c_2!\,c_3!\,c_4!\,c_5!\,c_6!}$. Here $J = \sum c_i = t_1 + t_2 + t_3$ is independent of z, so the numerator $N!/(N-J)!$ factors out of the chromatic sum. Setting $N = -2$, we simply get $(-2)(-3)\cdots(-J-1) = (-1)^J (J+1)!$. Putting all this together we obtain for the value of our strand network

$$(-1)^q (J+1)! \sum_z \frac{(-1)^z}{z!\,(t_2-z)!\,(r_1-z)!\,(t_3-r_1+z)!\,(s_3-z)!\,(t_1-s_3+z)!}$$

where $q \equiv s_1r_2 + s_1r_3 + r_3s_3 + r_1 + s_3 \pmod 2$.

Although this method can be applied to any spin network, it becomes more complicated to evaluate $K(\{c_i\}; N)$ when not all cycles are incident. Define a graph G_c whose points are the cycle types of the network, with lines joining incident cycles. Then $K(\{c_i\}; N)$ is the number of N-colorings of G_c which assigns precisely c_i colors to the ith point, with distinct colors for adjacent points. Equivalently we may define a graph $G\{c_i\}$ obtained from G_c by expanding the ith point into a complete subgraph of c_i points and connecting all points in adjacent subgraphs. Then

$$K(\{c_i\}; N) = \frac{C(G\{c_i\}; N)}{\prod c_i!},$$

where $C(G\{c_i\}; N)$ is the "chromatic polynomial*", giving the number of ordinary N-colorings of $G\{c_i\}$.

— John Moussouris.

* See N. Biggs Algebraic Graph Theory (Cambridge Univ. Press, 1974) for extensive results on chromatic polynomials.