MASS & CURVATURE

ONLY VAGUELY RELATED

TO STRINGS & TWISTORS

OLD STORY

NEW DEVELOPMENTS

MATHEMATICS

$\boxed{\text{"INTUITION" (GUESS WORK)}}$ — COMPUTING

PHYSICS

BOOK BY MANTON & SUTCLIFFE

TOPOLOGICAL SOLITONS

CUP 2004

OUTLINE

1) SKYRME MODEL OF NUCLEUS

SKYRME (1961)

2) ANALOGY WITH MONOPOLES

MANTON (1985 ?)

3) RELATION TO INSTANTONS

ATIYAH MANTON (1989)

4) COMPUTATIONAL RESULTS ON N-SKYRMIONS

MANTON-SUTCLIFFE BATTYE (1997)

5) SKYRMIONS WITH PION MASS

BATTYE SUTCLIFFE (2004)

6) SKYRMIONS ON HYPERBOLIC SPACE

ATIYAH - SUTCLIFFE (2005)

7) RELATION TO INSTANTONS "

SKYRMIONS

SKYRME MODEL IS NON-LINEAR THEORY OF PIONS WHOSE TOPOLOGICAL SOLITON SOLUTIONS GIVE EFFECTIVE DESCRIPTION OF NUCLEI WITH SOLITON NUMBER = BARYON NUMBER

SKYRME FIELD U IS MAP

$$U : R^3 \to SU(2)$$

BDY COND $\quad U(x) \to 1 \quad$ AS $|x| \to \infty$

COMPACTIES TO MAP $\bar{U} : S^3 \to S^3$

HAS TOP INVARIANT INTEGER DEGREE N

RELATED TO PION THEORY BY

$$U = \sigma + i\,\pi \cdot \tau$$

$\pi = (\pi_1, \pi_2, \pi_3)$ PION FIELDS

τ PAULI MATRICES

σ GIVEN FROM CONSTRAINT $\sigma^2 + \pi \cdot \pi = 1$

STATIC SOLUTIONS

ENERGY $= \int \{ E_2 + E_4 + m^2 Tr(1-U) \} dx$

T = DIFFERENTIAL OF MAP U

T^*T POSITIVE SELF-ADJOINT

EIGENVALUES λ_i^2 $(i = 1, 2, 3)$

$E_2 = Tr\, T^*T = \sum \lambda_i^2$ [CONTROLS LENGTH]

$E_4 = Tr(S^*S) = \sum \lambda_i^2 \lambda_j^2$ [CONTROLS AREA]

$S = \Lambda^2 T$

m = MASS = 0 FIRST APPROXN

[SUITABLE PHYSICAL UNITS + CONSTANTS IGNORED]

1) SCALING $x \to \mu x$

$E_2\, dx \to \mu^{-1}(E_2\, dx)$ $E_4 \to \mu\, (E_4\, dx)$

EXPECT SOLUTIONS MINIMIZING ENERGY

2) INEQUALITY $(\lambda_1 \pm \lambda_2 \lambda_3)^2 + \cdot\cdot + \cdot\cdot \geq 0$

$\to$ ENERGY $\geq |B|$ B = BARYON No.

FINDING SOLUTIONS

1. EQUATIONS NOT ANALYTICALLY SOLUBLE

2. SPHERICALLY SYMMETRIC SOLUTIONS

   $\Rightarrow$ ODE

   CAN BE EFFECTIVELY SOLVED NUMERICALLY

3. FOR $N = 1$ THIS GIVES SOLUTION

   BUT FOR $N \geqslant 2$ IT GIVES

   UNSTABLE SOLUTIONS

4. MAJOR PROBLEM

   HOW TO FIND STABLE SOLUTIONS FOR $N \geqslant 2$?

DIRECT NUMERICAL ONSLAUGHT

RELATION WITH INSTANTONS

SU(2) INSTANTONS ON R^4

SOLUTIONS OF $F_A = *F_A$ <u>SDYM</u>

INSTANTON NUMBER N

MODULI SPACE dim $8N$

<u>TWISTOR</u>

<u>METHODS</u>

GIVEN ANY SU(2) GAUGE FIELD ON R^4

(FLAT AT ∞)

WRITE $R^4 = R \times R^3$

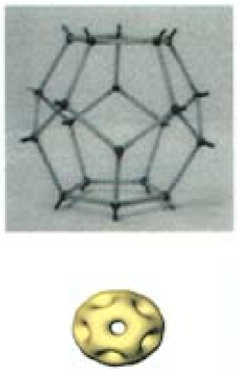
PARALLEL TRANSPORT ALONG R

GIVES <u>SKYRME FIELD</u> ON R^3

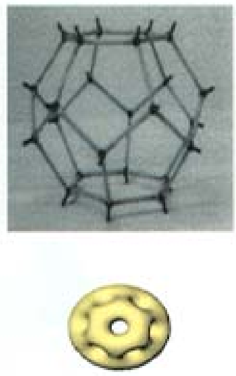
THIS IS A <u>HOMOTOPY EQUIVALENCE</u>

(MOD GAUGE TRANSFNS).

INSTANTON NUMBER $\longleftrightarrow$ SKYRME NUMBER



7:Y_h



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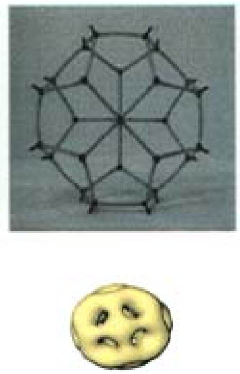

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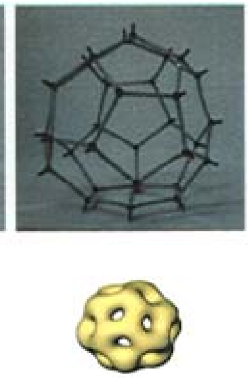

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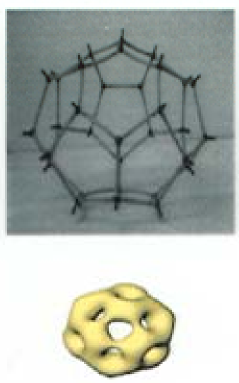

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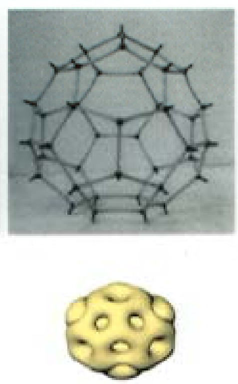

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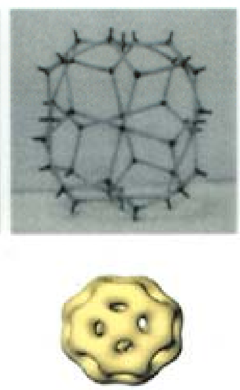

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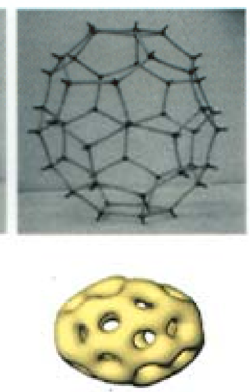

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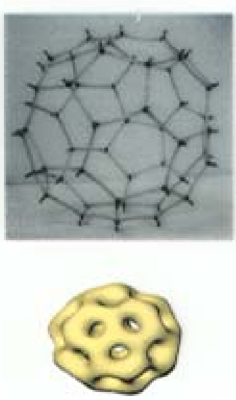

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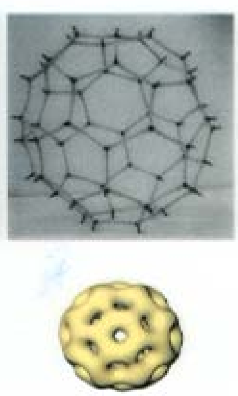

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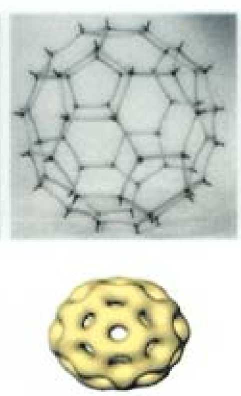

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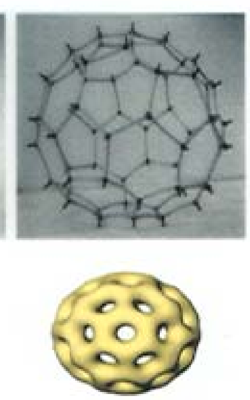

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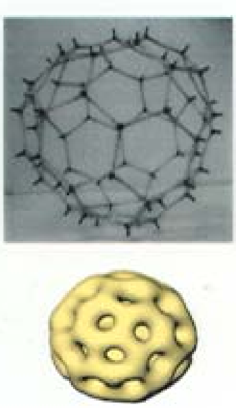

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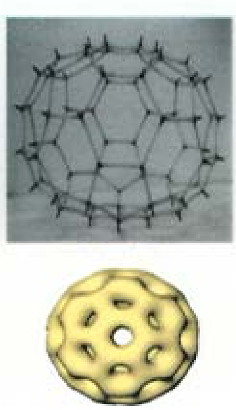

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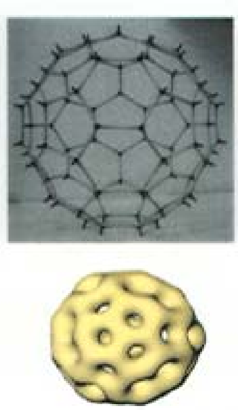

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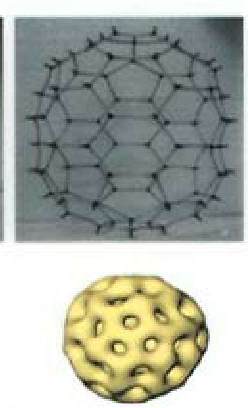

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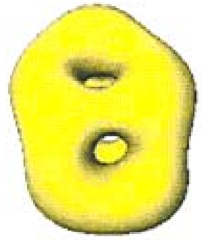
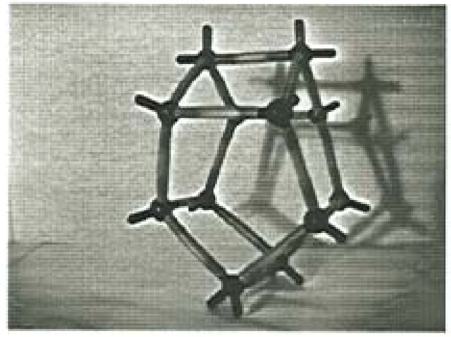
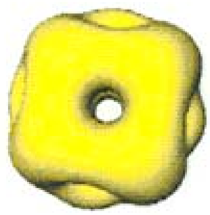
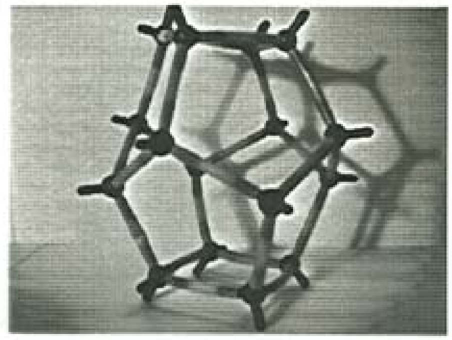
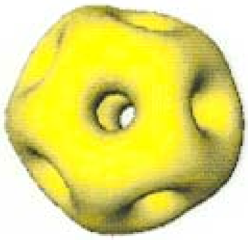
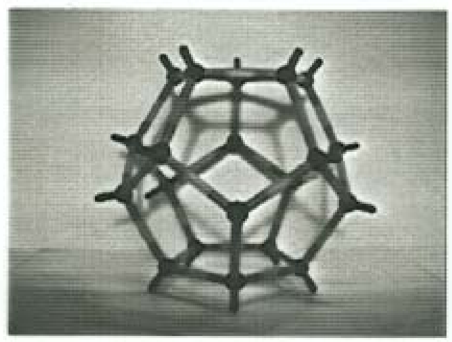


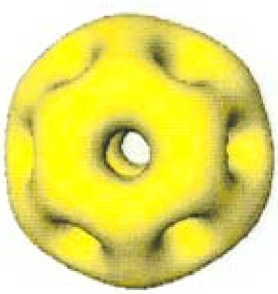


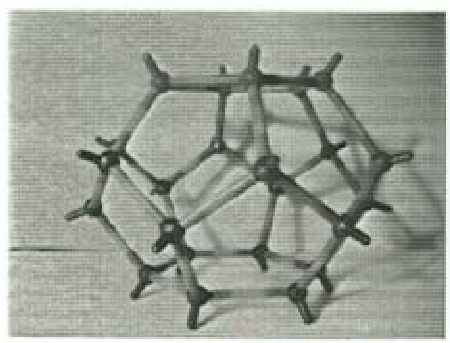


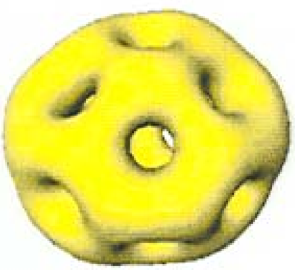


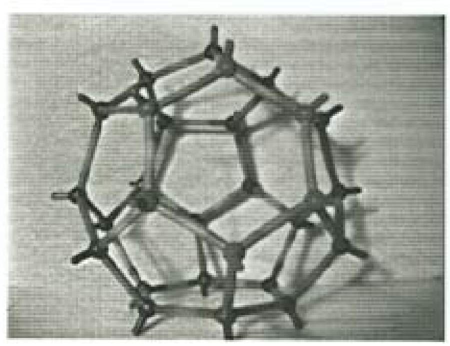


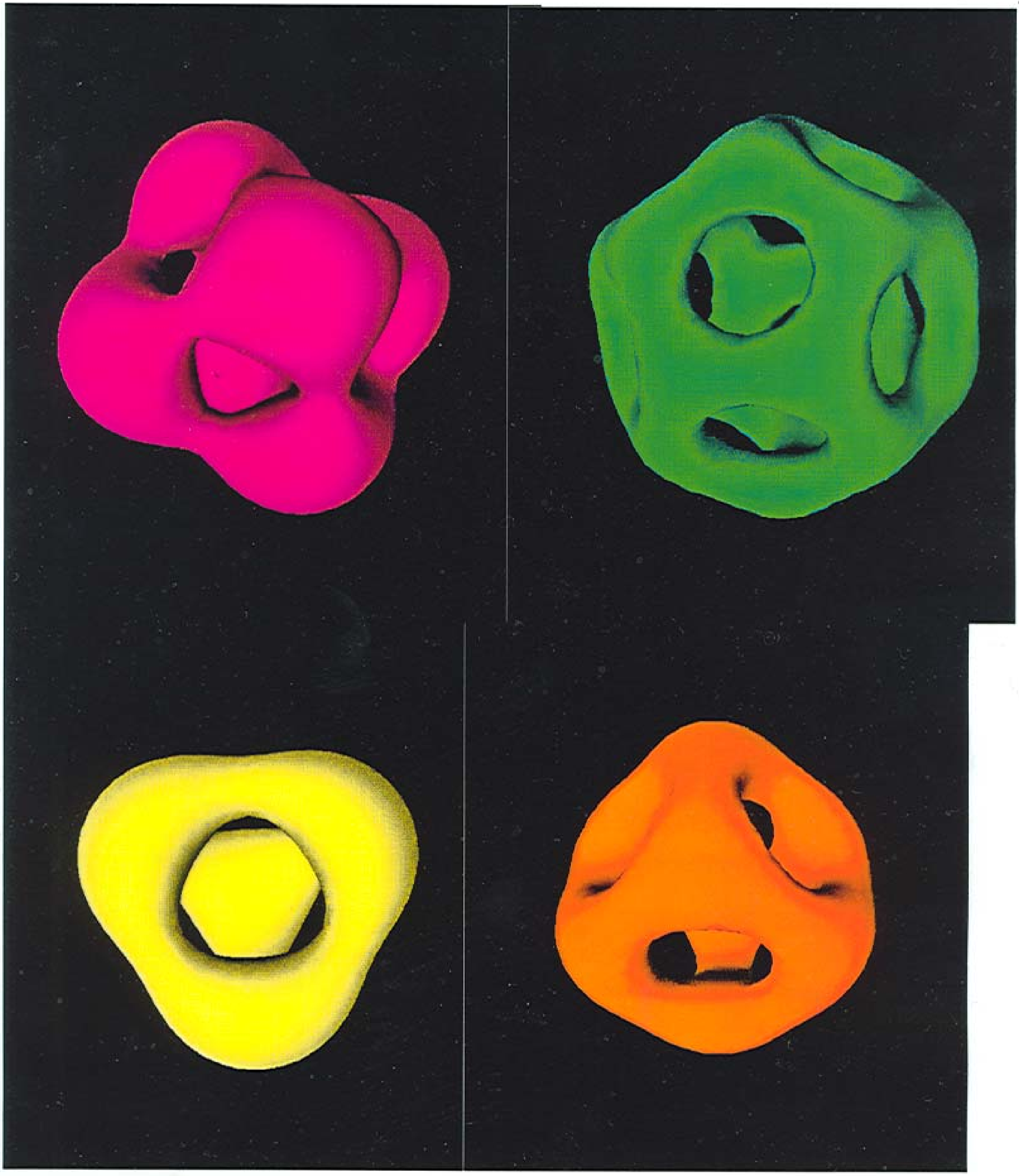












RATIONAL MAP ANSATZ

(BATTYE & SUTCLIFFE)

GIVEN MAP $f: S^2 \to S^2$

CAN SUSPEND $\Sigma f: S^3 \to S^3$

BUT REPARAMETRIZE SUSPENSION

COORDINATE USING AUXILIARY

"PROFILE FUNCTION"

ANSATZ TAKE f RATIONAL DEGREE N

(INVOLVES $4N$ REAL PARAMETERS)

COMPUTE SKYRME ENERGY OF Σf

AND MINIMIZE WITH RESPECT TO

PROFILE FUNCTION & $4N$ PARAMETERS

FIND SOLUTIONS QUALITATIVELY

SIMILAR TO N-MONOPOLES !!

MANTON

1) SKYRMIONS & MONOPOLES BOTH SOLITONS

2) BUT SKYRMIONS ATTRACT SO SOLUTION
UNIQUE [UP TO TRANSLATION & ROTATIONS]
NO MODULI SPACE.

BUT PERHAPS SPECIAL MONOPOLES WITH
EXTRA SYMMETRY MIGHT RESEMBLE SKYRMIONS

VERIFIED COMPUTATIONALLY

EX. N=3 THERE IS A SKYRMION
WITH TETRAHEDRAL SYMMETRY & LOW
ENERGY

ALSO N=7 & ICOSAHEDRAL SYMMETRY

WHAT ABOUT GENERAL N ?

NEED NEW APPROACH

ANALOGY WITH MONOPOLES

BPS MONOPOLES (GROUP $SU(2)$)

GAUGE FIELD A , HIGGS FIELD ϕ

EQNS $D_A\phi = *F_A$ BOGOMOLNY EQN

BDY COND $|\phi| \to 1$ AT ∞

$\phi_\infty : S^2 \to S^2$ DEGREE N

MONOPOLE NUMBER

MODULI SPACE OF SOLUTIONS M_N

$\dim M_N = 4N$

M_N = RATIONAL FUNCTIONS OF z

OF DEGREE N (DONALDSON)

TWISTOR METHODS

CAN FIND MONOPOLES WITH FINITE

SYMMETRIES [SURPRISING]

EX $N=3$ MONOPOLE WITH TETRAHEDRAL

RESTRICT TO INSTANTON MODULI SPACE I_N

GIVES SUBSPACE I'_N OF SKYRME FIELDS OF dim $4N-1$ [APPROXIMATES WHOLE SPACE AS $N \to \infty$]

NOW TRY TO MINIMIZE SKYRME ENERGY ON I'_N

GIVES <u>SURPRISINGLY</u> GOOD RESULTS <u>CLOSE</u> TO REAL SOLUTION

CONCEPTUALLY GOOD APPROACH BUT <u>DIFFICULT TO EXPLOIT</u> BECAUSE I_N IS COMPLICATED SPACE

[OPPOSITE TO USE OF MONOPOLES]

SKYRMIONS WITH NON-ZERO PION MASS m

BATTYE & SUTCLIFFE FIND (BY COMPUTATION) RESULTS WHICH ARE QUALITATIVELY DIFFERENT AND PHYSICALLY MORE REALISTIC THAN WHEN MASS = ZERO

$m = 0$ $\quad N \lesssim 22$ ONE "SHELL"

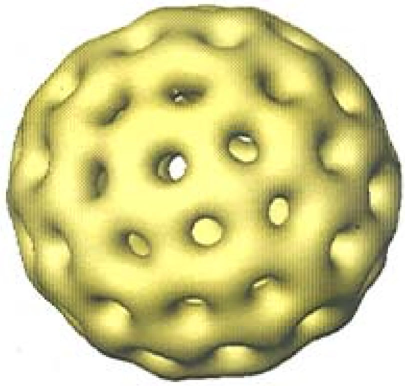
$m \neq 0$ $\quad$ SMALL N SHELL IS FILLED IN

ALSO $N = 5, 8$ NO STABLE

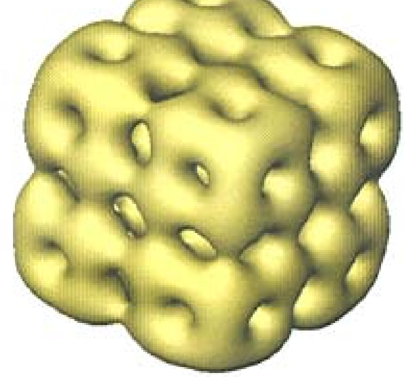
SHELL SOLUTION : $5 \to 3 + 2$

IN AGREEMENT WITH REAL BARYONS

BUT HAVE LOST ANALOGY WITH MONOPOLES



a



b

B = 32

SHELL

CRYSTAL CHUNK

m NEAR ZERO

m LARGE

SKYRMIONS ON HYPERBOLIC SPACE

COMPARE SKYRMIONS ON R^3 MASS m

IDENTIFY R^3 WITH H^3 BY RADIAL MAP

SKYRMIONS ON $H^3(-K^2)$ MASS 0

FIND GOOD MATCH IF $K \sim \frac{m}{2}$

MORE PRECISE FIT $K = K(m)$

SUITABLE FUNCTION OF m

CHECKS WELL FOR SMALL N

USING RATIONAL MAP ANSATZ

WHY IS THIS INTERESTING ?

1) MATHEMATICAL CAN USE GEOMETRY - RELATION WITH INSTANTONS

2) PHYSICS SUGGESTS CONNECTION OF NUCLEAR PHYSICS WITH GRAVITATION ?

3) STUDY $m \to \infty$. LIMIT "INTEGRABLE SYSTEM" ?

INSTANTONS $\to$ HYPERBOLIC SKYRMIONS

FIX $S^1 = SO(2) \subset SO(5)$ ACTING ON S^4

FIXED-POINT SET IS $S^2 \subset S^4$

$S^4 - S^2$ CONFORMAL TO $S^1 \times H^3(-1)$

[OR $S^1(K^{-1}) \times H^3(-K^2)$]

GIVEN $SU(2)$ - GAUGE FIELD ON S^4

(FOR BUNDLE WITH $C_2 = M$)

PARALLEL TRANSPORT ALONG S^1-ORBITS

GIVES SKYRME FIELD ON H^3

[NEED TO FIX ORIGIN ON H^3 AND

USE RADIAL TRANSPORT]

AT FIXED POINTS OF S^1-ACTION

WE GET S^1-ACTION ON FIBRE

WEIGHTS $(p, -p)$

SKYRME NUMBER $N = 2pM$

CAN TAKE $p = 1/2$ (SPIN) $\Rightarrow$ $N = M$

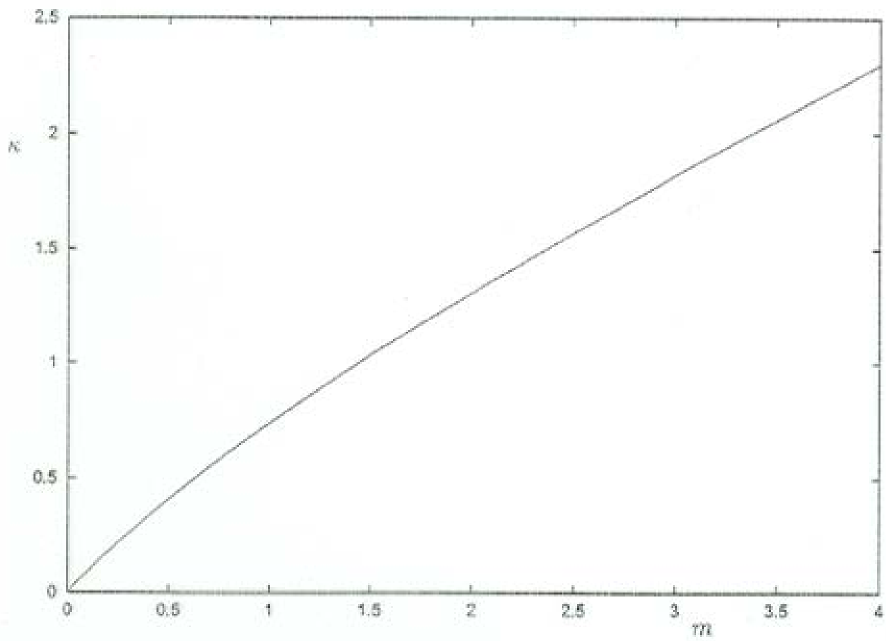


Figure 3: κ as a function of m, obtained by equating the size of a Skyrmion.

A numerical solution of the ordinary differential equation which follows from (2.3) allows the Skyrmion energy to be computed as a function of the pion mass and this is displayed as the solid curve in Fig. 1. The profile functions for $m = 0, 1, 2, 3$ are shown as the solid

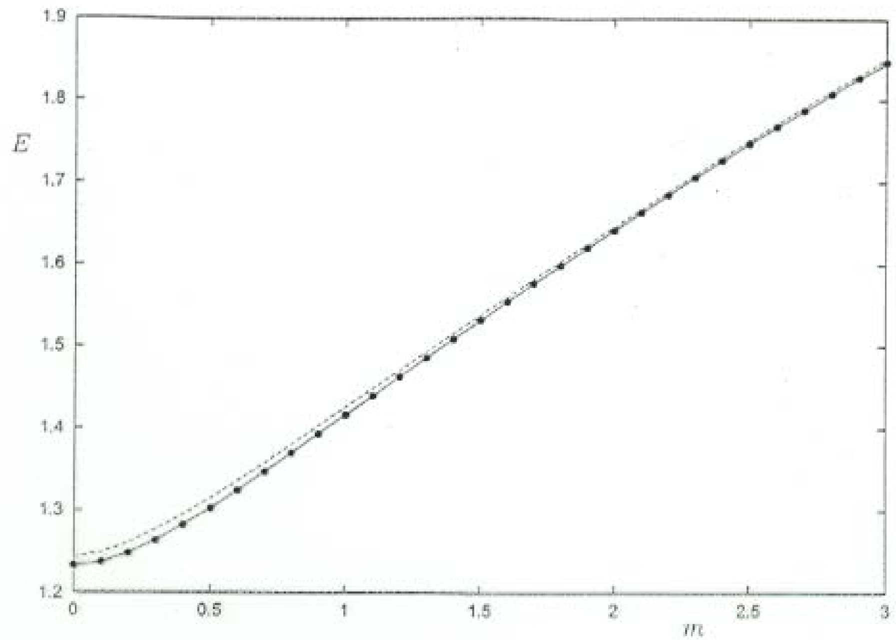


Figure 1: The Skyrmion energy as a function of the pion mass; exact result (solid curve), hyperbolic approximation (circles), instanton approximation (dashed curve).

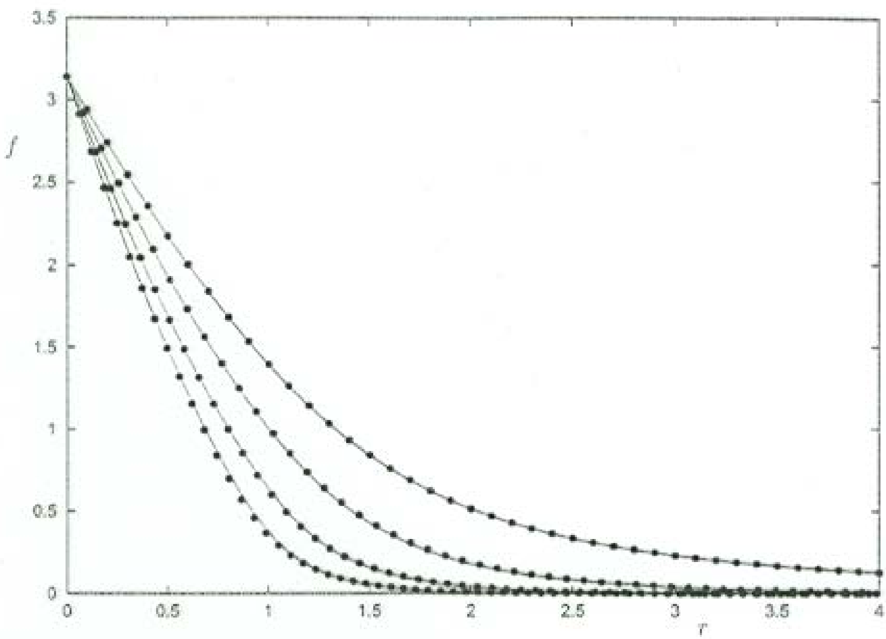


Figure 2: Profile functions for pion masses $m = 0, 1, 2, 3$; exact result (solid curves) and hyperbolic approximation (circles). The curves are more localized for larger m.

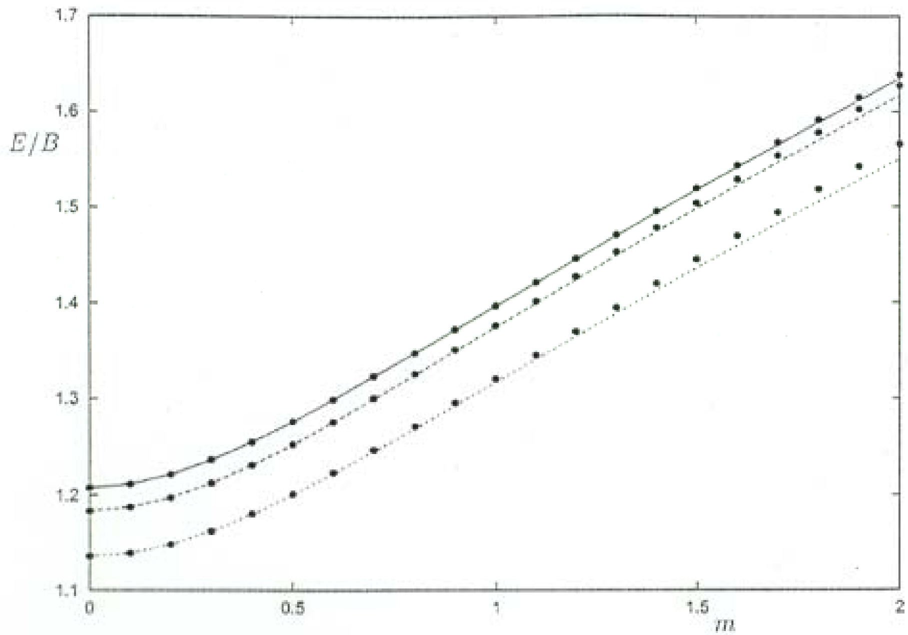


Figure 4: The energy per baryon E/B as a function of the pion mass m for $B = 2$ (solid curve), $B = 3$ (dashed curve), $B = 4$ (dotted curve). The circles denote the energies of the hyperbolic approximations.

CONCLUSION

GAUGE-FIELDS ON S^4 ($c_2 = N$)

$\Downarrow$

SKYRME FIELDS ON H^3 SKYRME NUMBER $= N$

RESTRICT TO INSTANTON MODULI SPACE I_N. GIVES SUBSPACE I'_N OF SKYRME FIELDS ON $H^3(-K^2)$

SKYRME No. N $(\dim I'_N = 8N)$

IN PRINCIPLE COULD BE USED TO FIND APPROX. SOLNS FOR SKYRME EQNS

THEN USE AS APPROX FOR SKYRME EQNS IN R^3 WITH MASS m (RELATED BY $K = K(m)$)

CHECKS WELL FOR $N = 1$ AND VARYING m

RELATION TO MONOPOLES

$F_A = *F_A$ REDUCES TO $D_A\varphi = *F$

ON R^4 — ON R^3

ON S^4 — ON H^3

FOR SOLNS INDEPT OF EXTRA VARIABLE

$\Rightarrow$ MONOPOLE MODULI SPACE M_N (ON H^3)

IS (COMPONENT : $p = 1/2$) OF FIXED-POINT

SET OF S^1-ACTION ON I_N

$\Updownarrow$

$M'_N \subset I'_N$ SKYRMION FIELDS ON H^3

$\uparrow$

"RATIONAL MAP ANSATZ"

$\Omega^2(S^2) \subset \Omega^3(S^3)$ BY SUSPENSION

"EXPLAINS" ANALOGY MONOPOLES ~ SKYRMIONS

PHYSICAL SIGNIFICANCE ?

RELATION TO GR ??

RELATION TO STRING THEORY ???

PHYSICAL SCALES

RELATION $K \sim m/2$ SHOULD BE

$$K \sim \frac{m_\pi c}{2\hbar} \qquad (\text{dim } L^{-1})$$

$\sim$ (COMPTON RADIUS)$^{-1}$

DIFFERS BY ORDER 10^{40}

FROM COSMOLOGICAL CURVATURE

SCALE SIZE Hc^{-1} (H = HUBBLE CONSTANT)

TOTALLY WRONG ORDER OF

MAGNITUDE !

ALSO FOR STATIONARY SOLNS
OF EINSTEIN EQNS 3-dim
SCALAR CURVATURE IS POSITIVE !

BUT IN EXPANDING UNIVERSE WITH
COSMOLOGICAL CONST 3-dim
CURVATURE MIGHT BE NEGATIVE

RIGHT SIGN BUT STILL WRONG
ORDER OF MAGNITUDE!

? OTHER PHYSICAL WAYS TO INTERPRET
USE "GAS OF SKYRMIONS"
OR CONSIDER DISTRIBUTION OF
GALAXIES ??

ANY IDEAS ? OR IS THIS IS
NONSENSE ??

VARYING m OR K

IS $m \to \infty$ (OR $K \to \infty$) EASIER ?

(COULD THIS BE USEFUL ?)

OBSERVATION (ATIYAH & MURRAY 1991)

MONOPOLES ON $H^3(-K^2)$ WITH

$K \to \infty$ RELATED TO INTEGRABLE

SYSTEMS (POTTS - MODEL) FOR

HIGHER GENUS CURVES

NOTE $K \to \infty$ CORRESPONDS TO SHRINKING S^1 IN FIBRATION OVER H^3

KEY FACT DONALDSON MAP

$M_N \to$ RAT. FNS

BECOMES EXPLICIT & EASY IN THIS LIMIT

WORTH EXPLORING

REFERENCES

R.A. BATTYE & P.M. SUTCLIFFE

SKYRMIONS, FULLERENES & RATIONAL MAPS

Rev. Math. Phys. 14, 29 (2002)

SKYRMIONS AND THE PION MASS

SEPT 2004

M.F. ATIYAH & P.M. SUTCLIFFE

SKYRMIONS, INSTANTONS, MASS & CURVATURE

Phys. Lett. B (605) 106-~~115~~ (2005)