TWISTORS IN SUPERSTRING THEORY

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Motivation: Describe 1st-quantized theories with spacetime supersymmetry

I. Twistors and the superparticle
   A. "Brink-Schwarz" versus "Ferber"
   B. "Open" Twistor-String
   C. κ-symmetry and super-YM eq's in $D=10$

II. Pure spinors and the superstring
   A. RNS and GS formalisms
   B. Pure spinor formalism
   C. Multiloop amplitude prescription

III. Topological strings and AdS-CFT (work in progress)
   A. Twistors and worldsheet susy
   B. Pure spinor formalism on $AdS_5 \times S^5$
   C. Conjecture for topological A-model
   D. Open string sector and $\mathcal{N}=4$ $d=4$ sYM

I. Twistors and the superparticle

A. "Brink-Schwarz" versus "Ferber"

Massless rel. particle: $S = \int d\tau \,(P_m \dot{x}^m + e P^2)$ ⇒ massless Klein-Gordon equation

Massless spinning particle: $S = \int d\tau \,(P_m \dot{x}^m + \psi_m \dot{\psi}^m + e P^2 + \chi P \cdot \psi)$ ⇒ massless Dirac eqn.

Local worldline susy: $\delta x^m = \epsilon \psi^m$, $\delta \psi^m = \epsilon P^m$, $\delta e = \epsilon \chi$, $\delta \chi = \dot{\epsilon}$

Brink-Schwarz superparticle: $S = \int d\tau \,(P_m(\dot{x}^m - i\dot{\theta}\gamma^m\theta) + e P^2)$ Brink, Schwarz '81

Global spacetime supersymmetry: $\delta\theta^\alpha = \epsilon^\alpha$, $\delta x^m = i\theta\gamma^m\epsilon$

Local κ-symmetry: $\delta\theta^\alpha = (\not{P}\kappa)^\alpha$, $\delta x^m = -i\theta\gamma^m\delta\theta$, $\delta e = \dot{\theta}\kappa$

Siegel '83

$\frac{\partial L}{\partial \dot{\theta}^\alpha} = -i(\not{P}\theta)_\alpha$ ⇒ Dirac constraint

$$d_\alpha \equiv p_\alpha + i(\not{P}\theta)_\alpha = 0$$

$\{d_\alpha, d_\beta\} = i P_m \gamma^m_{\alpha\beta}$, $P^2 = 0$ ⇒ First and second-class constraints

⇒ Difficult to covariantly quantize

Can compute spectrum in light-cone gauge.

When $D=10$, physical $SO(8)$ vector and $SO(8)$ spinor

⇒ $D=10$ super-YM spectrum (massless states of open superstring)

Twistor description:

D=4 Rel. particle: Penrose '76

$$S = \int d\tau \left(\bar{w}_\alpha \dot{\lambda}^\alpha + \bar{\lambda}_{\dot{\alpha}} \dot{w}^{\dot{\alpha}} + e\left(\bar{w}_\alpha \lambda^\alpha + \bar{\lambda}_{\dot{\alpha}} w^{\dot{\alpha}} - n\right)\right)$$

$$= \int d\tau \left(Y_A \dot{Z}^A + e\left(Y_A Z^A - n\right)\right)$$

$$Z^A = (\lambda^\alpha, w^{\dot{\alpha}}), \quad Y_A = (\bar{w}_\alpha, \bar{\lambda}_{\dot{\alpha}})$$

$Z^A Y_B = SO(4,2)$ conformal generators

Quantization $\Rightarrow$ massless D=4 particle of spin $\frac{n-2}{2}$

$$\varphi^{\beta_1 \ldots \beta_{n-2}}(x) = \int d\lambda^\alpha \lambda_\alpha \tilde{\varphi}(\lambda^\alpha, w^{\dot{\alpha}} = (x\lambda)^{\dot{\alpha}}) \lambda^{\beta_1} \ldots \lambda^{\beta_{n-2}}$$

Ex: $\tilde{\varphi} = \dfrac{A_1 \cdot A_2}{(A_1 w)(A_2 w)} \Rightarrow \varphi(x) = \dfrac{1}{x^2}$ D=4 Green's function

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D=4 superparticle Ferber '78

$$S = \int d\tau \left(\bar{w}_\alpha \dot{\lambda}^\alpha + \bar{\lambda}_{\dot{\alpha}} \dot{w}^{\dot{\alpha}} + \bar{\eta}_j \dot{\eta}^j + e\left(\bar{w}_\alpha \lambda^\alpha + \bar{\lambda}_{\dot{\alpha}} w^{\dot{\alpha}} + \bar{\eta}_j \eta^j - n\right)\right)$$

$j = 1 \ldots \mathcal{N}$

$$= \int d\tau \left(Y_I \dot{Z}^I + e\left(Y_I Z^I - n\right)\right)$$

$$Z^A = (\lambda^\alpha, w^{\dot{\alpha}}, \eta^j), \quad Y_A = (\bar{w}_\alpha, \bar{\lambda}_{\dot{\alpha}}, \bar{\eta}_j)$$

$Z^I Y_J = (P)SU(2,2|\mathcal{N})$ superconformal generators

Quantization $\Rightarrow$ massless D=4 supermultiplet of superspin $\frac{n-2}{2}$

$$\Phi^{\beta_1 \ldots \beta_{n-2}}(x + \tfrac{i}{2}\theta\sigma\bar{\theta}, \theta) = \int d\lambda^\alpha \lambda_\alpha \tilde{\Phi}\left(\lambda^\alpha, w = (x + \tfrac{i}{2}\theta\sigma\bar{\theta})^m (\sigma_m \lambda)^{\dot{\alpha}}, \eta^j = \lambda_\alpha \theta^{\alpha j}\right) \lambda^{\beta_1} \ldots \lambda^{\beta_{n-2}}$$

Covariant quantization in D=4 is easy

Quantization above D=4? D=6 Cederwall, Bengtsson
D=10 ? NB

Interactions? Twistor-string

I.B. "Open" D=4 Twistor-String

$$\mathcal{S} = \int d^2z \left(Y_I \bar{\nabla} Z^I + \bar{Y}_I \nabla \bar{Z}^I \right) + S_c$$ NB '2004

$(\nabla, \bar{\nabla})$ includes both worldsheet spin and $U(1)$ connection.

S_c contains left and right-moving current algebra.

After gauge-fixing $U(1)$ and reparam. invariances,

$$\mathcal{S} = \int d^2z \left(Y_I \bar{\partial} Z^I + \underset{\substack{\uparrow \\ \text{Virasoro ghosts}}}{b \bar{\partial} c} + \underset{\substack{\uparrow \\ U(1)\text{ ghosts}}}{u \bar{\partial} v} + \text{complex conj.} \right) + S_c$$

Virasoro constraint : $T = Y_I \partial Z^I + T_c + b\partial c + \partial(bc) + u\partial v$

$U(1)$ constraint : $J = Y_I Z^I$

No anomalies $\Rightarrow$ T_c has $c = 28$, $\mathcal{N} = 4$

NB + Witten, '04

Open string theory describes $\mathcal{N}=4$ super-YM coupled to $\mathcal{N}=4$ conformal supergravity

Open string vertex op's :

$V_f = c Y_I f^I(Z)$

$V_g = c \partial Z^I g_I(Z)$

$f^I(Z), g_I(Z)$ describe conf. supergravity

$V_\varphi = c K^r \varphi_r(Z)$ — $\varphi_r(Z)$ describes super YM

K^r are currents from S_c satisfying $K^r(y) K^s(z) \to \frac{k \delta^{rs}}{(y-z)^2} + \frac{f^{rs}_{\ \ t} K^t}{y-z}$

Tree-level amp's can be easily computed by summing over worldsheets of different $U(1)$ instanton number.

Loop amplitudes? Restrictions on S_c? Closed string sector?

I.C. K-symmetry and super YM eqns in D=10

D=10 Brink-Schwarz superparticle:

$$S = \int d\tau \left(P_m (\underbrace{\dot{x}^m - i \dot{\theta}^\alpha \gamma^m_{\alpha\beta} \theta^\beta}_{\Pi^m}) + e P_m P^m \right)$$

$\alpha = 1$ to 16, $\gamma^m_{\alpha\beta} = \gamma^m_{\beta\alpha}$, $\gamma^{(m}_{\alpha\beta} \gamma^{n)\beta\gamma} = 2\delta^\gamma_\alpha$ $\quad \Gamma^m = \begin{pmatrix} 0 & \gamma^{m\alpha\beta} \\ \gamma^m_{\alpha\beta} & 0 \end{pmatrix}$

K-symmetry ⇒ "super-light-like" integrability ⇒ D=10 sYM eqn

Witten '86

$$x^m \sim x^m + P^m \tau - i \theta \gamma^m \not{P} K$$
$$\theta^\alpha \sim \theta^\alpha + (\not{P} K)^\alpha$$

Integrability ⇒ $\lambda^\alpha \nabla_\alpha \Phi = 0$, $P^m \nabla_m \Phi = 0$ $\quad \lambda^\alpha = \not{P} K$

$$\nabla_\alpha = D_\alpha + A_\alpha(x,\theta) = \left(\frac{\partial}{\partial \theta_\alpha} - i (\gamma^m \theta)_\alpha \partial_m \right) + A_\alpha(x,\theta)$$

$$\nabla_m = \partial_m + A_m(x,\theta)$$

$$\{ \lambda^\alpha \nabla_\alpha, \lambda^\beta \nabla_\beta \} \Phi = [P^m \nabla_m, \lambda^\alpha \nabla_\alpha] \Phi = 0$$

$$\Rightarrow F_{\alpha\beta} = D_\alpha A_\beta + D_\beta A_\alpha + \{ A_\alpha, A_\beta \} - \gamma^m_{\alpha\beta} A_m = 0$$

⇒ super-YM eqn's $\gamma^{\alpha\beta}_{m_1 \ldots m_5} F_{\alpha\beta} = 0$ and $A_m = \gamma^{\alpha\beta}_m (D_\alpha A_\beta + A_\alpha A_\beta)$

"conventional constraint"

$\gamma^{\alpha\beta}_{m_1 \ldots m_5} F_{\alpha\beta} = 0$ ⇒ can gauge $A_\alpha(x,\theta)$ s.t.

Siegel '79

$$A_\alpha(x,\theta) = (\gamma^m \theta)_\alpha a_m(x) + (\gamma^m \theta)_\alpha (\theta \gamma_m \chi(x)) + \ldots$$

where $a_m(x)$ and $\chi^\alpha(x)$ are on-shell gluon and gluino.

"Pure spinor" integrability $\Leftrightarrow$ D=10 sYM eqn's Howe '91

$X^m \sim X^m - i\,\theta\gamma^m\lambda_p$ where $\lambda_p\gamma^m\lambda_p = 0$ "D=10 pure spinor"

$\theta^\alpha \sim \theta^\alpha + \lambda_p^\alpha$

Integrability $\Rightarrow \lambda_p^\alpha \nabla_\alpha \Phi = 0$

$\{\lambda_p^\alpha \nabla_\alpha, \lambda_p^\beta \nabla_\beta\}\Phi = 0 \Rightarrow \lambda_p^\alpha \lambda_p^\beta F_{\alpha\beta} = 0 \Rightarrow \gamma^{m_1\ldots m_5}_{\alpha\beta} F^{\alpha\beta} = 0$

Pure spinor version of superparticle NB '2000

$$S = \int d\tau\, (P_m \dot{x}^m - \tfrac{1}{2}P_m P^m + p_\alpha \dot{\theta}^\alpha + \omega_{p\alpha}\dot{\lambda}_p^\alpha)$$

$$= \int d\tau\, (P_m \Pi^m - \tfrac{1}{2}P_m P^m + d_\alpha \dot{\theta}^\alpha + \omega_{p\alpha}\dot{\lambda}_p^\alpha)$$

$\Pi^m = \dot{x}^m - i\dot{\theta}^\alpha \gamma^m_{\alpha\beta}\theta^\beta$, $d_\alpha = p_\alpha + i(\not{P}\theta)_\alpha$

Action is already gauge-fixed.

Physical states defined by g.n.=1 cohomology

of nilpotent "BRST" operator $\boxed{Q = \lambda_p^\alpha d_\alpha}$ $\{d_\alpha, d_\beta\} = \not{P}_{\alpha\beta}$

$V = \lambda_p^\alpha A_\alpha(x,\theta)$

$QV = 0 \Rightarrow \lambda_p^\alpha \lambda_p^\beta D_\alpha A_\beta = 0 \Rightarrow$ linearized D=10 sYM eqn

$\delta V = Q\Omega \Rightarrow \delta A_\alpha = D_\alpha \Omega \Rightarrow$ linearized gauge transformation

Can easily generalize to superstring

II. Pure Spinors and the Superstring

A. Other formalisms for the superstring

RNS: Ramond '75, Neveu-Schwarz '75

$$S = \int d^2z \left(\partial x^m \bar\partial x_m + \psi^m \bar\partial \psi_m + b\bar\partial c + \beta \bar\partial \gamma + \text{complex conj.} \right)$$

$Q = \int dz \left(cT + \gamma G + \text{ghosts} \right)$ $\qquad T = \frac{1}{2}(\partial x \cdot \partial x + \psi \cdot \partial \psi)$

$\qquad G = \partial x \cdot \psi$

N=1 super-Virasoro

Physical states $\Leftrightarrow$ g.n.=1 cohomology of Q

ψ^m can be $\left\{ \begin{array}{l} \text{periodic} \\ \text{anti-periodic} \end{array} \right\} \Rightarrow$ spacetime $\left\{ \begin{array}{ll} \text{fermion} & \text{"Ramond"} \\ \text{boson} & \text{"Neveu-Schwarz"} \end{array} \right.$

g-loop amplitude: $\displaystyle \mathcal{A}_{g,N} = \sum_{s=1}^{2^{2g}} \int DS_{g,s} \int Dx\, D\psi\, Db\, Dc\, D\beta\, D\gamma\, e^{-S}\, V_1 \dots V_N$

Need to sum over 2^{2g} spin structures to obtain amplitudes with spacetime supersymmetry

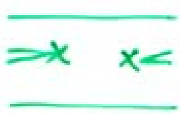
$\Rightarrow$ difficult to prove finiteness properties.

Not known how to use RNS formalism in R-R backgrounds.

GS: Spacetime susy is manifest Green + Schwarz '84

$$S = \int d^2z \left(\Pi_m \bar\Pi^m + (\theta \gamma^m \bar\partial \theta) \Pi_m - (\theta \gamma^m \partial \theta) \bar\Pi_m + \dots \right)$$

Only known how to quantize in light-cone gauge.



Light-cone interaction point operators make it difficult to do computations.

B. Pure Spinor Formalism

NB '2000

$$S = \int d^2z \left(\partial X^m \bar\partial X_m + p_\alpha \bar\partial \theta^\alpha + w_{p\alpha} \bar\partial \lambda^\alpha_p + \text{complex conj.} \right)$$

$$= \int d^2z \left(\Pi^m \bar\Pi_m + d_\alpha \bar\partial \theta^\alpha + w_{p\alpha} \bar\partial \lambda^\alpha_p + \text{complex conj.} \right)$$

$\lambda_p \gamma^m \lambda_p = \bar\lambda_p \gamma^m \bar\lambda_p = 0 \rightarrow \lambda^\alpha_p$ has 11 complex parameters

After Wick rotation, $\lambda^\alpha_p \in \mathbb{C} \times \frac{SO(10)}{U(5)}$

$w_{p\alpha} \simeq w_{p\alpha} + \Lambda^m (\gamma_m \lambda_p)$ for any Λ^m

$\Rightarrow w_{p\alpha}$ only appears in gauge-inv. comb's

$$\boxed{\begin{aligned} N_{mn} &= \tfrac{1}{2} \lambda_p \gamma_{mn} w_p \\ J &= \lambda^\alpha_p w_{p\alpha} \end{aligned}}$$

$N_{mn}(y) N_{pq}(z) \rightarrow -3 \frac{\eta_{m[q}\eta_{p]n}}{(y-z)^2} + \ldots, \quad \frac{1}{2}(\theta\gamma_p)(y)\, \frac{1}{2}(\theta\gamma p)(z) \rightarrow \frac{+4\,\eta\eta}{(y-z)^2} + \ldots$

$\Rightarrow N_{mn} + \frac{1}{2}\theta\gamma_{mn} p$ has same central charge as $M_{mn} = \psi_m \psi_n$ in RNS

$J(y)\, T(z) \rightarrow -\frac{8}{(y-z)^3} + \ldots \Rightarrow$ ghost. number anomaly $= -8$

$T(y)\, T(z) \rightarrow \frac{10 - 32 + 22}{2(y-z)^4} + \ldots \Rightarrow$ no conformal anomaly

Spacetime supersymmetry is manifest.

Quantization is easy.

$$d_\alpha = p_\alpha + i(\gamma^m\theta)_\alpha \partial x_m + \frac{1}{8}(\gamma^m\theta)_\alpha (\theta\gamma_m\partial\theta)$$

$$\Pi_m = \partial x_m - i\,\partial\theta\gamma_m\theta$$

OPE's: $d_\alpha(y)\, d_\beta(z) \to \frac{\gamma^m_{\alpha\beta}\Pi_m}{(y-z)}$

$d_\alpha(y)\,\Pi_m(z) \to \frac{\gamma_{m\alpha\beta}\partial\theta^\beta}{(y-z)}$

Siegel '86

Physical states for open superstring defined as. g.n.=1 cohomology of $\boxed{Q = \int dz\, \lambda^\alpha_p d_\alpha}$

$V = \lambda^\alpha_p A_\alpha(x,\theta)$ $\quad QV=0 \Rightarrow$ super-YM

$V = \lambda^\alpha_p\,[\partial x^m B_{\alpha m}(x,\theta) + \partial\theta^\beta B_{\alpha\beta}(x,\theta) + d_\gamma B^\gamma_\alpha(x,\theta) + \dots]$

$QV=0 \Rightarrow$ massive spin-2 multiplet

⋮

Cohomology reproduces open superstring spectrum

Physical states for closed superstring defined as g.n.=2 cohom. of $\boxed{Q+\bar{Q} = \int dz\, \lambda^\alpha_p d_\alpha + \int d\bar{z}\, \bar{\lambda}^\alpha_p \bar{d}_\alpha}$

$V = \lambda^\alpha_p \bar{\lambda}^\beta_p A_{\alpha\beta}(x,\theta,\bar{\theta})$ $\quad QV=0 \Rightarrow$ Type II supergravity

⋮

Cohomology reproduces closed superstring spectrum

C. Multiloop amplitude prescription

$$\mathcal{A}_{g,N} = \int DS_g \int Dx\, D\theta\, Dp\, D\lambda_p\, Dw_p\, e^{-S}\, V_1 \ldots V_N$$

Functional integration over (λ_p, w_p) non-zero modes is straightforward using OPE's.

To functionally integrate over (λ_p, w_p) zero modes,

use $d\lambda_p^{\alpha_1} \wedge \ldots d\lambda_p^{\alpha_{11}} = [D\lambda_p]\, \lambda^\beta \lambda^\gamma \lambda^\delta\, T^{\alpha_1 \ldots \alpha_{11}}_{\beta\gamma\delta}$

$$T^{\alpha_1\ldots\alpha_{11}}_{\beta\gamma\delta} = \epsilon^{\alpha_1\ldots\alpha_{16}}\, \gamma^m_{\alpha_{12}\beta}\, \gamma^n_{\alpha_{13}\gamma}\, \gamma^p_{\alpha_{14}\delta}\, (\gamma_{mnp})_{\alpha_{15}\alpha_{16}}$$

$[D\lambda_p]$ is a Lorentz scalar measure of g.n. = 8

Similarly, $dN^{m_1n_1} \wedge \ldots \wedge dN^{m_{10}n_{10}} \wedge dJ = [Dw_p]\, (\lambda\gamma^{(5)}\lambda)(\lambda\gamma^{(5)}\lambda)(\lambda\gamma^{(5)}\lambda)(\lambda\gamma^{(5)}\lambda)$

$[Dw_p]$ is a Lorentz scalar measure of g.n. = −8

$[D\lambda_p]$ is also useful for defining D=10 Penrose transform

$$\varphi^{\beta_1\ldots\beta_{n-8}}(x) = \int [D\lambda_p]\, \tilde{\varphi}(\lambda_p, w_p = \not{x}\lambda_p)\, \lambda_p^{\beta_1} \ldots \lambda_p^{\beta_{n-8}}$$

NB + Cherkis '04

e.g. $\tilde{\varphi} = \dfrac{(T^{-1})^{\beta\gamma\delta}_{\alpha_1\ldots\alpha_{11}}\, A_1^{\alpha_1} \ldots A_{11}^{\alpha_{11}}\, w_{p\beta} w_{p\gamma} w_{p\delta}}{(A_1 w_p) \ldots (A_{11} w_p)} \Rightarrow \varphi(x) = \dfrac{1}{x^8}$

D=10 Green's function

Also generalizes to pure spinors in any even dimension as proposed by Hughston

$$\varphi^{\beta_1\ldots\beta_{n+2-d}}(x) = \int [D\lambda_p]\, \tilde{\varphi}(\lambda_p, w_p = \not{x}\lambda_p)\, \lambda_p^{\beta_1} \ldots \lambda_p^{\beta_{n+2-d}}$$

where $[D\lambda_p]$ has g.n = 2 − d

Functional integration over $\mathcal{D}x\,\mathcal{D}\theta\,\mathcal{D}p$ is straightforward.

Finiteness properties come from integration over

$16g + 16$ fermionic zero modes of p_α and θ^α.

Since x^m, θ^α, λ_ρ^α carry zero conf. weight,

Partition function $= (\mathrm{Det}\, D_0)^{-10} (\mathrm{Det}\, D_0)^{32} (\mathrm{Det}\, D_0)^{-22} = 1$

Note that Partition function $\neq 1$ for bosonic or RNS string

In twisted $N=2$ topological strings, Partition function $= 1$.

Does the pure spinor formalism have a twisted

$N=2$ worldsheet susy?

Matone, Mazzucato, Oda, Tonin, Sorokin '03

NB + Vafa '94

NB '92

Tonin '91

Sorokin, Tkach, Volkov, Zheltukhin '8