III. Topological Strings and AdS-CFT (work in progress)

A. Twistors and worldsheet susy

When $D = 3, 4, 6, 10$, $P^m P_m = 0$ (and $P_0 > 0$) $\Rightarrow \boxed{P_m = \lambda \gamma_m \lambda}$

Can replace K-symmetry with worldline/worldsheet/worldvolume susy where

$$\Theta^\alpha = \theta^\alpha + \kappa \lambda^\alpha$$

$$\mathcal{X}^m = x^m + \kappa \psi^m$$

Sorokin, Tkach, Volkov, Zheltukhin '89

$$D\mathcal{X}^m = D\Theta^\alpha \gamma^m_{\alpha\beta} \Theta^\beta \Rightarrow \begin{array}{l} \partial x^m - i\partial\theta\gamma^m\theta = \lambda\gamma^m\lambda \\ \psi^m = \lambda\gamma^m\theta \end{array}$$

Can be used to classically describe superparticle, superstring, super-P-brane as "embedding" of superworldvolume $\rightarrow$ target superspace

Sorokin, Pasti, Tonin, Howe, West, Sezgin, Bergshoeff

Quantization is non-trivial since

$$\lambda^\alpha(y)\, \omega_\beta(z) \rightarrow \frac{\delta^\alpha_\beta}{y-z} \Rightarrow (\lambda\gamma^m\lambda)(y)\, (\lambda\gamma^n\lambda)(z) \rightarrow \text{regular}$$

But $\Pi^m(y)\, \Pi^n(z) \rightarrow \frac{\eta^{mn}}{(y-z)^2}$

However, suppose $\Pi^m = \lambda\gamma^m\lambda + \omega\gamma^m\omega$

$$\Pi^m(y)\, \Pi^n(z) \rightarrow \frac{\eta^{mn}}{(y-z)^2} + \frac{M^{mn}}{y-z} \qquad M^{mn} = \lambda\gamma^{mn}\omega$$

$\Rightarrow$ AdS space

Worldsheet susy possible for AdS target-space?

B. Pure spinor formalism on $AdS_5 \times S^5$

Pure spinor formalism generalizes to any background satisfying low-energy eqns. of motion.

$AdS_5 \times S^5$ background described by $g(x,\theta,\bar{\theta}) \in \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$ as in Metsaev-Tseytlin GS action.

Define left-invariant currents $J^A = (g^{-1}\partial g)^A$

$$\left(\begin{array}{c|c} J^a, J^{(ab)} & J^\alpha + i J^{\hat{\alpha}} \\ \hline J^\alpha - i J^{\hat{\alpha}} & J^{a'}, J^{(a'b')} \end{array}\right) \qquad \begin{array}{lll} \alpha = 1 \text{ to } 16 & a = 1 \text{ to } 5 & (ab) = 1 \text{ to } 10 \\ \hat{\alpha} = 1 \text{ to } 16 & a' = 1 \text{ to } 5 & (a'b') = 1 \text{ to } 10 \end{array}$$

$$S = r^2 \int d^2 z \Big(J^a \bar{J}^a + J^{a'} \bar{J}^{a'} + \tfrac{1}{4} J^\alpha \bar{J}^{\hat{\alpha}} + \tfrac{3}{4} J^{\hat{\alpha}} \bar{J}^\alpha + w_{p\alpha} \bar{\nabla} \lambda_p^\alpha + \bar{w}_{p\hat{\alpha}} \nabla \bar{\lambda}_p^{\hat{\alpha}} \Big)$$

$$\lambda_p \gamma^m \lambda_p = 0, \quad \bar{\nabla}\lambda_p = \bar{\partial}\lambda_p + \bar{J}^{(ab)}(\gamma_{ab}\lambda_p) + \bar{J}^{(a'b')}(\gamma_{a'b'}\lambda_p)$$

$$Q = \int dz\, \lambda_p^\alpha J^{\hat{\alpha}}, \quad \bar{Q} = \int d\bar{z}\, \bar{\lambda}_p^{\hat{\alpha}} \bar{J}^\alpha$$

Action inv. under $\delta g = \Omega g + g \Sigma$, $\delta\lambda_p = \Sigma \lambda_p$, $\delta w_p = \Sigma w_p$

Ω = global $PSU(2,2|4)$

Σ = local $SO(4,1)\times SO(5)$

Can show that action is conf. inv. and BRST inv. at quantum level, but difficult to compute except when $r \to \infty$.

C. Conjecture for topological A-model

Gopakumar + Vafa, Witten, Maldacena, Verlinde, etc. have conjectured that $r \to 0$ limit of $AdS_5 \times S^5$ superstring is a topological string whose open string sector describes $\mathcal{N}=4$ $d=4$ super-YM.

Conjecture based on pure spinor formalism:

Some similarities to Bars twistor-string '04

$$g \in \frac{PSU(2,2|4)}{SU(2,2)\times SU(4)}, \quad J^A = (g^{-1}\partial g)^A = \begin{pmatrix} J^r_s & J_{sj} \\ J^{rj} & J^j_k \end{pmatrix}$$

$r,s = 1$ to 4, $j,k = 1$ to 4

$$S = \int d^2z \left(J^{rj} \bar{J}_{rj} + w^{rj} \bar{\nabla} \lambda_{rj} + \bar{w}_{rj} \nabla \bar{\lambda}^{rj} \right)$$

λ_{rj} is unconstrained, $\bar{\nabla}\lambda_{rj} = \bar{\partial}\lambda_{rj} + J^s_r \lambda_{sj} + J^k_j \lambda_{rk}$

$$Q = \int dz\, \lambda_{rj} J^{rj}, \quad \bar{Q} = \int d\bar{z}\, \bar{\lambda}^{rj} \bar{J}_{rj}$$

$\bar{\lambda}^{rj} = \lambda^\alpha + i\bar{\lambda}^{\hat{\alpha}}$, $\lambda_{rj} = \lambda^\alpha - i\bar{\lambda}^{\hat{\alpha}} \Rightarrow Q + \bar{Q} = \int d\sigma \left(\lambda^\alpha \bar{J}^{\hat{\alpha}} + \bar{\lambda}^{\hat{\alpha}} J^\alpha\right)$

in "zero-tension" limit where $\frac{\partial}{\partial\sigma}$ terms in S are dropped.

No WZW term in this limit $\Rightarrow$ have global $U(1)$ R-symmetry

$PSU(2,2|4)$ isometry $\to$ $SU(2,2|4)$ isometry of free $\mathcal{N}=4$ super-YM

Action is inv. under $\delta g = \Omega g + g\Sigma$ Ω = global $SU(2,2|4)$

$\delta\lambda = \Sigma\lambda$, $\delta w = \Sigma w$ Σ = local $SU(2,2)\times SU(4)$

Can use Σ to gauge away all bosonic components of g

$$g(\theta^{rj},\theta_{rj}) = \begin{pmatrix} 1 & \theta^{rj} \\ \theta_{rj} & 1 \end{pmatrix}$$

$(x^a, x^{a'})$ coordinates for $AdS_5\times S^5$ replaced with unconstrained (λ, w) with 32 parameters

Action has twisted $N=(2,2)$ worldsheet susy

$$Q = \int dz\, \lambda_{rj} J^{rj} \qquad \bar{Q} = \int d\bar{z}\, \bar{\lambda}^{rj} J_{rj}$$

$$b = w^{rj} J_{rj} \qquad \bar{b} = \bar{w}_{rj} \bar{J}^{rj}$$

$N=(2,2)$ chiral and antichiral superfields :

$$\Theta^{rj} = \theta^{rj} + \kappa_+ \lambda^{rj} + \bar{\kappa}_+ \bar{w}^{rj} + \kappa_+\bar{\kappa}_+ h^{rj}$$

$$\Theta_{rj} = \theta_{rj} + \kappa_- w_{rj} + \bar{\kappa}_- \bar{\lambda}_{rj} + \kappa_-\bar{\kappa}_- h_{rj}$$

auxiliary

$$D_-\Theta^{rj} = \bar{D}_-\Theta^{rj} = 0, \quad D_+\Theta_{rj} = \bar{D}_+\Theta_{rj} = 0$$

$$S = \int d^2z\, d^2\kappa_+\, d^2\kappa_- \left(\Theta^{rj}\Theta_{rj} + \tfrac{1}{2}\Theta^{rj}\Theta_{sj}\Theta^{sk}\Theta_{rk} + \dots\right)$$

Twist κ_+ and $\bar{\kappa}_-$ $\Rightarrow$ topological A-model

Bosonic isometries : $\delta\Theta^{rj} = \Lambda^r{}_s\Theta^{sj} + \Lambda^j{}_k\Theta^{rk}$, $\delta\Theta_{rj} = \Lambda^s{}_r\Theta_{sj} + \Lambda^k{}_j\Theta_{rk}$

Fermionic isometries: $\delta\Theta^{rj} = \chi^{rj} + \Theta^{rk}\chi_{sk}\Theta^{sj}$,

$$\delta\Theta_{rj} = \chi_{rj} + \Theta_{rk}\chi^{sk}\Theta_{sj}$$

D. Open String sector and $\mathcal{N}=4$ $D=4$ super-YM

A-model boundary condition: $\Theta^{rj} = \epsilon^{rs}\delta^{jk}\Theta_{sk}$

$z=\bar{z}$
$K_+ = \bar{K}_-$
$K_- = \bar{K}_+$

$\epsilon^{rs} = Sp(4)$ metric, $\delta^{jk} = SO(4)$ metric

$g \in \frac{PSU(2,2|4)}{SU(2,2)\times SU(4)} \longrightarrow g \in \frac{OSp(4|4)}{Sp(4)\times SO(4)}$ on boundary

$J^{rj} = (g^{-1}\partial g)^{rj}$ $\quad r = 1$ to 4, $j = 1$ to 4

$$S_{open} = \int d\tau\, \epsilon_{rs}\delta_{jk}\left(J^{rj}J^{sk} + w^{rj}\nabla\lambda^{sk}\right)$$

$N=2$ on boundary: $Q = \epsilon_{rs}\delta_{jk}\lambda^{rj}J^{sk}$, $b = \epsilon_{rs}\delta_{jk}w^{rj}J^{sk}$

Cohomology is naively trivial, but suppose local $\frac{Sp(4)}{SO(3,1)}$ is used to gauge $\boxed{\lambda^{\alpha j}\lambda^{\dot\alpha j} = 0}$ $\quad \alpha = 1,2$, $\dot\alpha = 1,2$

Constrained $\lambda^{rj} \Rightarrow g(x^{\alpha\dot\alpha}, \theta^{\alpha j}, \theta^{\dot\alpha j}) \in \frac{OSp(4|4)}{SO(3,1)\times SO(4)}$

S_{open} is pure spinor action for superparticle on AdS_4

$\Rightarrow$ g.n.=1 cohom. of Q is $\underline{\mathcal{N}=4\ d=4\ sYM \text{ on } AdS_4}$

Note that $\lambda^{\alpha j}\lambda^{\dot\alpha j} = 0 \Rightarrow \lambda^{\alpha j}\lambda^{k}_{\alpha} = e^{i\varphi}\epsilon^{jklm}\bar\lambda^{\dot\alpha}_{l}\bar\lambda_{\dot\alpha m}$ for some $e^{i\varphi}$

$\Rightarrow$ dim. reduction of $D=10$ pure spinor up to an R-transformation

Conclusion:

Twistors are useful for 1st-quantized description of theories with spacetime susy.

Ex: Superparticle
Superstring
Supermembrane ?