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One-Loop Amplitudes

In N=4 Super-Yang-Mills Theory

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Outline

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with Freddy Cachazo and Bo Feng:

hep-th/ 0410179, 0411107,0412103, 0412308

with Freddy Cachazo, Bo Feng and Edward Witten:

hep-th/0501052

Computing One-Loop Amplitudes

At tree-level, $\mathcal{N}=4$ amplitudes of gluons coincide with QCD amplitudes. This is because no fermions or scalars can propagate in the internal lines.

At one-loop, $\mathcal{N}=4$ amplitudes of gluons are part of QCD one-loop amplitudes.

$$A^{\rm QCD} = g = (g+4f+3s)-4(f+s)+s = A^{\mathcal{N}=4}-4A^{\mathcal{N}=1}+A^{scalar}$$

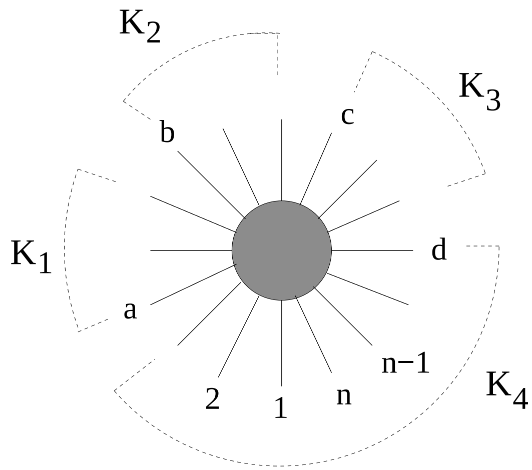
Full QCD one-loop amplitudes are known only up to 5 particles.

$\mathcal{N}=4$ at one-loop:

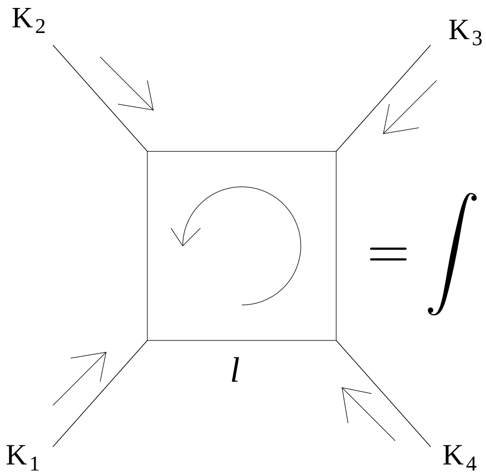
- 1994: MHV (Bern, Dixon, Dunbar, Kosower)
- 1994: 6 gluons (Bern, Dixon, Kosower)
- 2004: 7 gluons (RB, Cachazo, Feng; Bern, Del Duca, Dixon, Kosower)
- 2004: next-to-MHV (Bern, Dixon, Kosower)

One-Loop Amplitudes in $\mathcal{N} = 4$ SYM

- The number of Feynman diagrams required for a direct calculation quickly becomes prohibitive. A clever approach is needed.
- Supersymmetric amplitudes of gluons are four-dimensional cut-constructible. This means that the amplitude is completely determined by its branch cuts and discontinuities. (Bern, Dixon, Dunbar, Kosower 1994)
- All tensor integrals in a Feynman graph calculation of the amplitudes can be reduced to a set of scalar box integrals. (In Dim. Reg: Bern, Dixon, Kosower 1993)



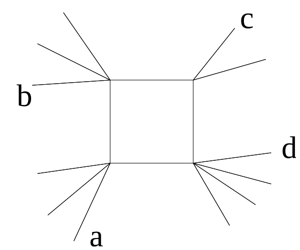
Scalar Box Integrals



$$= \int d^{4-2\epsilon}\ell \frac{1}{\ell^2(\ell-K_1)^2(\ell-K_1-K_2)^2(\ell+K_4)^2}$$

Any n-gluon one-loop amplitude can be written as: (Bern, Dixon, Kosower 1993 & with Dunbar 1994)

$$A_n^{1-\mathrm{loop}} = \sum_{1<a<b<c<d<n} \hat{B}_{abcd} \times$$



I will now present efficient new ways of computing the coefficients B_{abcd}, which are rational functions of $\langle i\ j\rangle$ and $[i\ j]$.

Applications of Collinear and Coplanar Operators

We used collinear and coplanar operators, acting on ordinary unitarity cuts, to derive some box integral coefficients and to prove the coplanarity of the coefficients in NMHV amplitudes.

Collinear Operator: If three points Z_i, Z_j, Z_k are collinear then the vector $V_L = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K$ must vanish. Setting $Z = (\lambda, -i\tilde{\partial})$, we find a spinor-valued first order differential operator:

$$F_{ijk} = \langle i\ j \rangle \tilde{\partial}_k + \langle j\ k \rangle \tilde{\partial}_i + \langle k\ i \rangle \tilde{\partial}_j$$

Coplanar Operator: If four points Z_i, Z_j, Z_k, Z_l are coplanar, then the determinant $K = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L$ must vanish. The second-order differential operator is:

$$\begin{aligned} K_{ijkl} = \quad & \langle i\ j \rangle [\tilde{\partial}_k\ \tilde{\partial}_l] + \langle j\ k \rangle [\tilde{\partial}_i\ \tilde{\partial}_l] + \langle k\ i \rangle [\tilde{\partial}_j\ \tilde{\partial}_l] \\ & + \langle k\ l \rangle [\tilde{\partial}_i\ \tilde{\partial}_j] + \langle i\ l \rangle [\tilde{\partial}_j\ \tilde{\partial}_k] + \langle j\ l \rangle [\tilde{\partial}_k\ \tilde{\partial}_i] \end{aligned}$$

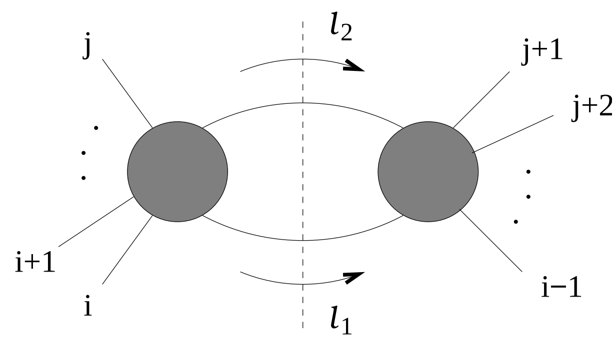
Unitarity Cuts

There are two ways to compute the unitarity cut. One is from the cut integral,

$$C_{i,\ldots,j} = \int d\mu \, A^{\rm tree}(-\ell_1, i, \ldots, j, -\ell_2) A^{\rm tree}(\ell_2, j{+}1, \ldots, i{-}1, \ell_1)$$

where

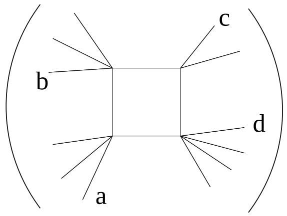
$$d\mu = d^4\ell_1 \; d^4\ell_2 \; \delta^{(4)}(\ell_1 + \ell_2 - P_L) \; \delta^{(+)}(\ell_1^2) \; \delta^{(+)}(\ell_2^2)$$



For two propagators, we drop the principal part, leaving the delta function that places their momenta on-shell.

The second way is by finding the discontinuity Δ of the amplitude across the branch cut of interest. (By unitarity, this is the imaginary part of the amplitude in a certain kinematic regime.)

$$C_{i,\ldots,j} = \Delta A_n^{1-\mathrm{loop}} = \sum \hat{B}_{abcd} \times \Delta$$



Now it is clear that if a given scalar box integral has a nonzero discontinuity across the cut in the P_L channel, then the cut integral has information about its coefficient.

Observation: Let $\mathcal{O}$ be any k-th order differential operator in the spinor variables. If $\mathcal{O}C_{i,i+1,\ldots,j}$ is a rational function, then $\mathcal{O}(\hat{B}) = 0$ for all coefficients $\hat{B}$ whose scalar box integrals participate in this cut.

This is true because the coefficients $\hat{B}$ are rational by construction, and all the discontinuities of box integrals have unique logarithms that cannot conspire to cancel. (Cachazo 10/2004).

Rational Functions from the Holomorphic Anomaly

How do we choose an operator $\mathcal{O}$ such that $\mathcal{O}C_{i,i+1,\ldots,j}$ is a rational function? By exploiting the holomorphic anomaly.

Consider for example:

$$F_{ijk}C_{i,\ldots,j} = \int d\mu F_{ijk} A^{\text{tree}}_{\text{MHV}}(-\ell_1, i, \ldots, j, -\ell_2) A^{\text{tree}}(\ell_2, j{+}1, \ldots, i{-}1, \ell_1)$$

The collinear operator would generically annihilate the MHV amplitude, but there is a delta function contribution from the holomorphic anomaly,

$$d\bar{\lambda}_i \frac{\partial}{\partial \bar{\lambda}_i} \frac{1}{\langle i\, m\rangle} = 2\pi \bar{\delta}(\langle i\, \ell_1\rangle)$$

The delta function is not zero, and moreover it localizes the integral to produce a rational function. (Cachazo, Svrček, Witten 9/2004)

It follows from the observation on the previous page that this collinear operator annihilates certain coefficients in this amplitude.

Thus, by using the holomorphic anomaly, we also obtain information about the twistor space structure of the box integral coefficients.

Coplanarity of NMHV Coefficients

We used the same observation to prove that all scalar box function coefficients in NMHV amplitudes are localized on a plane in twistor space.

The idea is that every nonvanishing term of a unitarity cut takes the form

$$\int d\mu \, A^{\rm tree}_{\rm MHV} A^{\rm tree}_{\rm NMHV}$$

MHV tree amplitudes are collinear (Witten 2003), and NMHV tree amplitudes are coplanar (Cachazo, Svrček, Witten 3/2004).

Every coplanar operator can be shown to annihilate both of these factors, leaving the rational function from the holomorphic anomaly. This implies that all the coefficients are coplanar.

(Alternative approach presented by Bern, Del Duca, Dixon, Kosower 10/2004.)

Computing One-Loop Amplitudes from the Holomorphic Anomaly of Unitarity Cuts

For unitarity cuts where one amplitude factor is MHV, one can use a collinear operator to get information about the coefficients.

For any cut C of any NMHV amplitude, there is some collinear operator F such that $F(B_{abcd}) = 0$ for all coefficients participating in the cut, and therefore

$$F(C) = \sum B_{abcd} \times F(\Delta(\Box_{abcd}))$$

We know the box functions and can do the cut integral, so this is just an algebraic equation for the coefficients.

For the family $A(1^-, 2^-, 3^-, 4^+, ...n^+)$, one can find immediately all the coefficients participating in the cut C_{123}. (Cachazo 10/2004)

For the other NMHV amplitudes, it is still possible to use the operator F to disentangle this equation and find the coefficients systematically. We presented all 35 coefficients of the amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$.

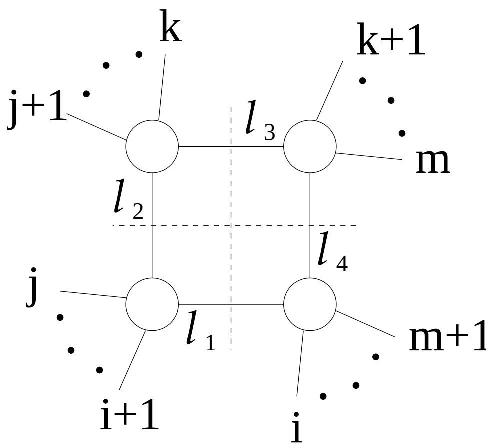
In this procedure, one deals with many coefficients at once and then systematically extracts them.

It is much better to be able to target a given coefficient directly.

While studying NNMHV amplitudes, we discovered how to accomplish this.

Our result is a simple procedure for finding any coefficient in any $\mathcal{N} = 4$ one-loop amplitude.

Quadruple Cuts



There is a notion of “generalized unitarity.” The box diagram has a leading singularity whose discontinuity is given by replacing all four propagators by their on-shell delta functions. This singularity picks out a given box uniquely.

We are in four dimensions, so four delta functions localize the integral completely. This computation is very easy!

For one-loop $\mathcal{N} = 4$ Yang-Mills amplitudes, there are only box functions, so we can get all possible coefficients from quadruple cuts.

The solutions of loop momenta

The coefficients computed from quadruple cuts are given by

$$\hat{B}_{abcd} = \frac{1}{2} \sum_{\mathcal{S}} A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree}$$

$\mathcal{S}$ is the set of all solutions of the on-shell conditions for the internal lines.

$$\mathcal{S} = \{\, \ell \,|\, \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0\}$$

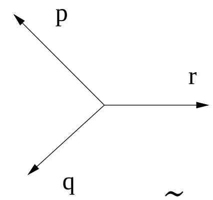
Can these equations always be solved?

No, there are problems at three-gluon vertices, where some $K_i^2 = 0$.

But this is not a problem, unless you try to do the whole calculation in Minkowski signature, with real momenta.

Three-Gluon Vertices

(Witten 2003)



$$r^2 = 0 \Rightarrow p \cdot q = 0$$

$$p \cdot q = \langle \lambda_p \, \lambda_q \rangle [\tilde{\lambda}_p \, \tilde{\lambda}_q] \Rightarrow \langle \lambda_p \, \lambda_q \rangle = 0 \quad \text{or} \quad [\tilde{\lambda}_p \, \tilde{\lambda}_q] = 0.$$

That is, $\lambda_p \sim \lambda_q$ or $\tilde{\lambda}_p \sim \tilde{\lambda}_q$.

In Minkowski signature with real momenta, $\tilde{\lambda} = \bar{\lambda}$, so we must have both $\langle \lambda_p \, \lambda_q \rangle = 0$ and $[\tilde{\lambda}_p \, \tilde{\lambda}_q] = 0$.

$$A(p^+, q^+, r^-) = \frac{[\tilde{\lambda}_p \, \tilde{\lambda}_q]^3}{[\tilde{\lambda}_q \, \tilde{\lambda}_r][\tilde{\lambda}_r \, \tilde{\lambda}_p]} \qquad A(p^-, q^-, r^+) = \frac{\langle \lambda_p \, \lambda_q \rangle^3}{\langle \lambda_q \, \lambda_r \rangle \langle \lambda_r \, \lambda_p \rangle}$$

In Minkowski signature with real momenta, both of these amplitudes vanish on-shell. We could take complex momenta to get nonvanishing amplitudes. Another way out is to work in (2,2) signature, where λ and $\tilde{\lambda}$ are real and independent.

Thus $A(p^+, q^+, r^-)$ is supported where $\lambda_p \sim \lambda_q \sim \lambda_r$, and $A(p^-, q^-, r^+)$ is supported where $\tilde{\lambda}_p \sim \tilde{\lambda}_q \sim \tilde{\lambda}_r$.

The solutions of loop momenta

$$\hat{B}_{abcd} = \frac{1}{2} \sum_{\mathcal{S}} A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$$

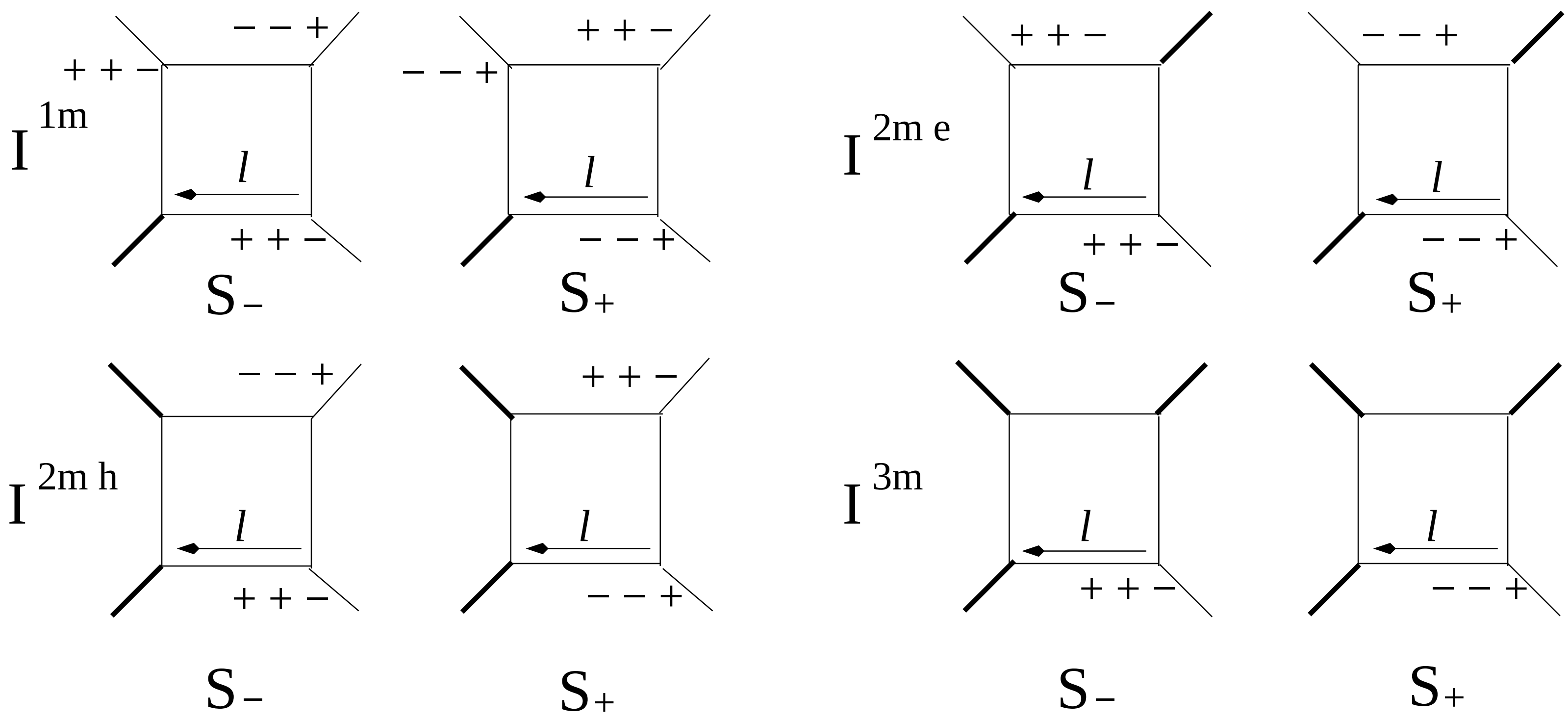
$$\mathcal{S} = \{\, \ell \,|\, \ell^2 = 0, \quad (\ell - K_1)^2 = 0, \quad (\ell - K_1 - K_2)^2 = 0, \quad (\ell + K_4)^2 = 0 \}$$

There are two solutions S_+, S_-. (We presented them explicitly.)

Adding them together produces a rational function. Remember that we must also sum over possible internal helicity assignments and the $\mathcal{N} = 4$ multiplet.

For one-, two-, and three-mass coefficients, S_+ and S_- are individually rational.

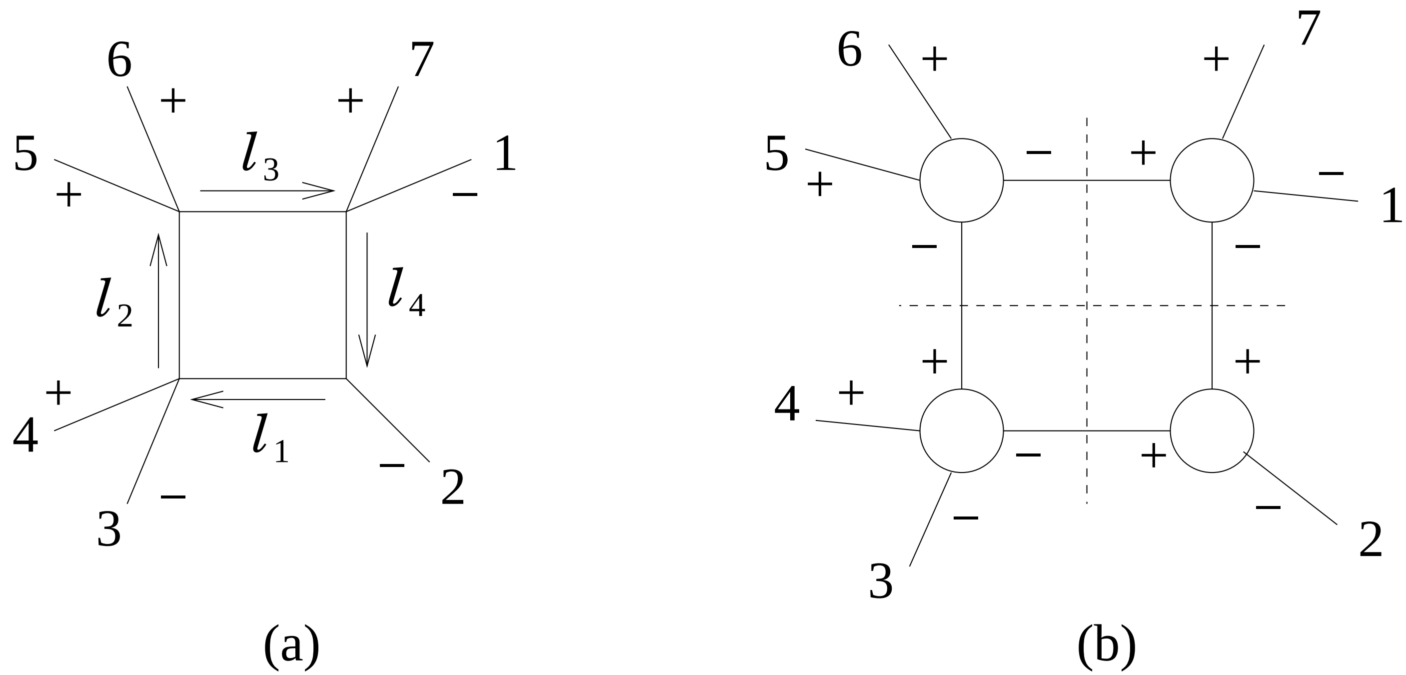
Helicity Assignments at 3-Gluon Vertices



$$+ + - : \quad \lambda_p \sim \lambda_q \sim \lambda_r$$

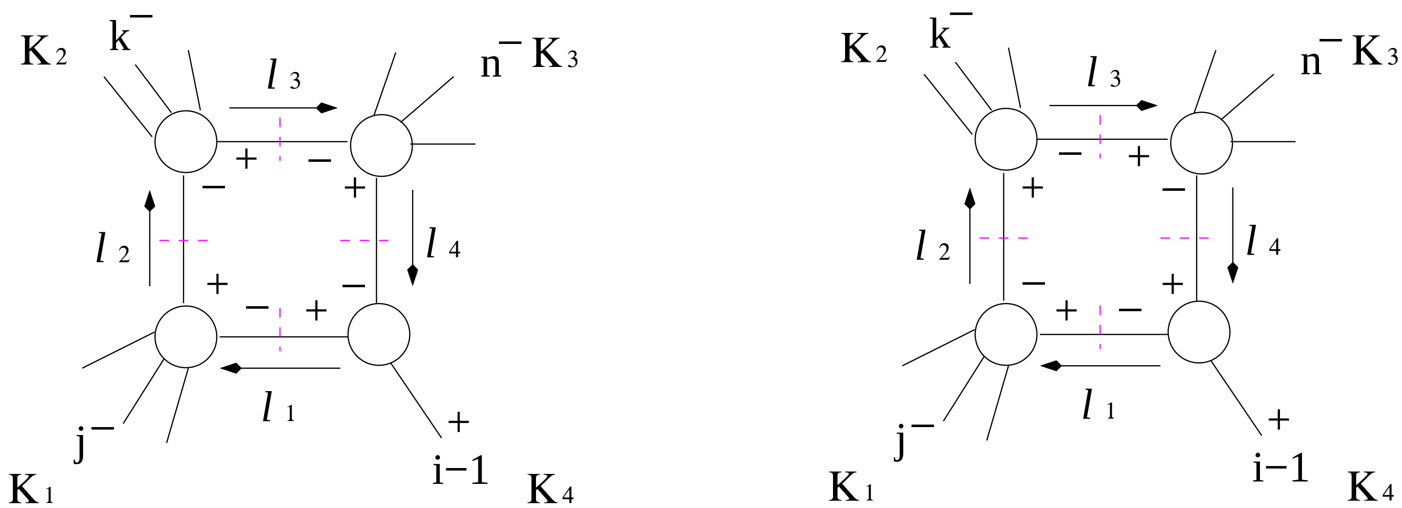
$$- - + : \quad \tilde{\lambda}_p \sim \tilde{\lambda}_q \sim \tilde{\lambda}_r$$

An Example



$$\hat{B}_{2357} = \frac{1}{2} \frac{[\ell_1\, \ell_4]^3}{[\ell_1\, 2][2\, \ell_4]} \frac{[4\, \ell_2]^3}{[\ell_2\, \ell_1][\ell_1\, 3][3\, 4]} \frac{[5\, 6]^3}{[6\, \ell_3][\ell_3\, \ell_2][\ell_2\, 5]} \frac{[\ell_3\, 7]^3}{[7\, 1][1\, \ell_4][\ell_4\, \ell_3]}$$

An Example of a Family of Coefficients



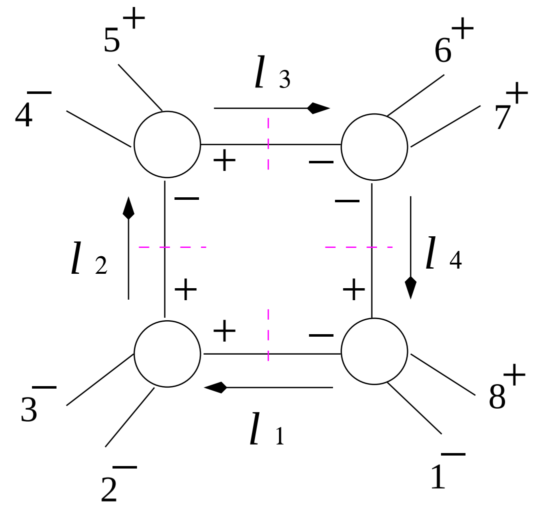
(a) (b)

$$B_{i-1,i,i+r,i+r+r'} =$$

$$\frac{\left(\langle k|K_2\cdot K_3|n\rangle\langle j\; i-1\rangle + \langle k|K_2\cdot(K_1+p_{i-1})|j\rangle\langle n\; i-1\rangle\right)^4}{\langle i+r-1|K_2\cdot K_3|i-1\rangle\langle i+r|K_2\cdot K_3|i-1\rangle\langle i+r+r'-1|K_2\cdot K_1|}$$

$$\times\frac{\langle i+r-1\; i+r\rangle\langle i+r+r'-1\; i+r+r'\rangle}{\langle i+r+r'|K_2\cdot K_1|i-1\rangle K_2^2\prod_{s=1}^{n}\langle s\; s+1\rangle}.$$

A Four-Mass Coefficient

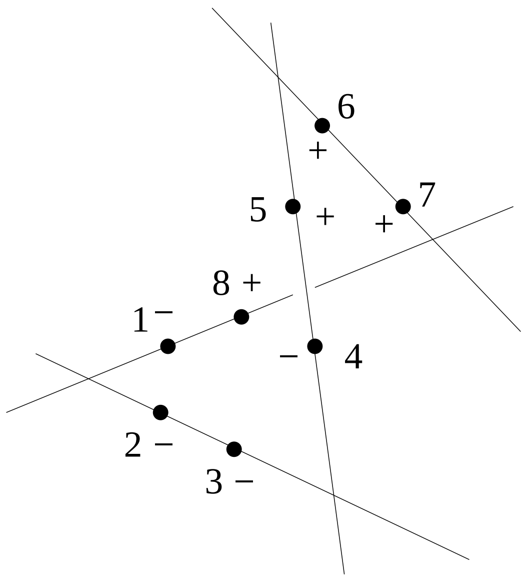


$$B_{2468} = \frac{1}{2} \sum_{S_+, S_-} \frac{[6\,7]^3 \langle 1|\not{\ell}_1\,\not{\ell}_2|4\rangle^3}{\langle 8\,1\rangle [2\,3] \langle 4\,5\rangle \langle 5|\not{\ell}_3\,\not{\ell}_4\,\not{\ell}_1|2] [3|\not{\ell}_2\,\not{\ell}_3|6] \langle 8|\not{\ell}_4|7]}$$

Twistor Space Localization

The formula $\hat{B}_{abcd} = \frac{1}{2} \sum_{\mathcal{S}} A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$ lends itself to twistor space analysis.

In this example, we find the following localization of the coefficient B_{2468}.



$$\hat{B}_{abcd} = \frac{1}{2} \sum_{\mathcal{S}} A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree}$$

Hence $\mathcal{N} = 4$ one-loop amplitudes are easily computed from a knowledge of tree amplitudes.

This very formula played a self-enhancing role, since it also inspired the development of our recursion relation which gives the most compact formulas for tree-level amplitudes!

One-loop $\mathcal{N} = 4$ amplitudes are UV finite but IR divergent.

The IR behavior encodes the corresponding tree-level amplitude.(Giele, Glover 1992; Kunszt, Signer, Trocsanyi 1994) The tree-level amplitude can therefore be written as a sum of box function coefficients.

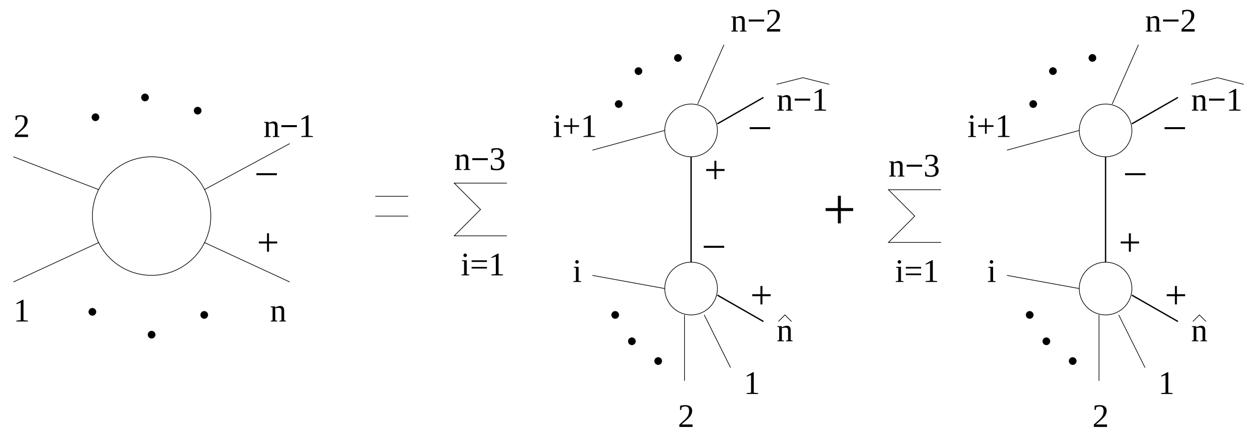
Previously, these relations have been used to obtain hard-to-compute coefficients, or as consistency checks. But they can also be combined to obtain new compact representations of tree-level amplitudes (Bern, Del Duca, Dixon, Kosower 10/2004; Bern, Dixon, Kosower 12/2004; Roiban, Spradlin, Volovich 12/2004).

The box function coefficients are computed from the whole $\mathcal{N} = 4$ multiplet. For a tree-level result, this is more complicated than one would hope for. It turns out that one can do better. We conjectured new recursion relations involving gluons only.

New Recursion Relations for Tree Amplitudes of Gluons

$$A_n(1,2,\ldots,(n-1)^-,n^+) =$$
$$\sum_{i=1}^{n-3}\sum_{h=+,-}\left(A_{i+2}(\hat{n},1,2,\ldots i,-\hat{P}^h_{n,i})\frac{1}{P^2_{n,i}}\right.$$
$$\left.A_{n-i}(+\hat{P}^{-h}_{n,i},i+1,\ldots,n-2,\widehat{n-1})\right)$$

$$\begin{aligned}
P_{n,i} &= p_n+p_1+\ldots+p_i,\\
\hat{P}_{n,i} &= P_{n,i}+\frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\lambda_{n-1}\tilde{\lambda}_n,\\
\hat{p}_{n-1} &= p_{n-1}-\frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\lambda_{n-1}\tilde{\lambda}_n,\\
\hat{p}_n &= p_n+\frac{P^2_{n,i}}{\langle n-1|P_{n,i}|n]}\lambda_{n-1}\tilde{\lambda}_n.
\end{aligned}$$



- This relation is equally valid for any other choice of two external gluons.

- It is like building an amplitude from various factorization limits.

- We computed many previously known examples. In every case, the formula that comes from the recursion relation is the most compact one known. (Use adjacent gluons.)

- Momentum is conserved at each vertex, in contrast to MHV diagrams.

- Iterating the recursion gives an expression for any amplitude in terms of three-gluon amplitudes only. What is the fundamental formulation? Is this related to topological string theory on the quadric hypersurface?

Direct Proof of the Recursion Relations

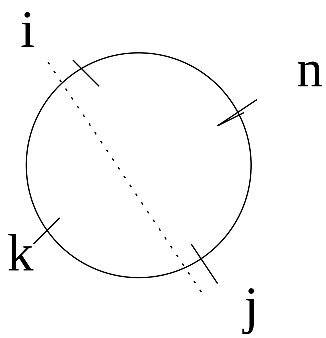
Consider quite generally the following transformation of spinors.

$$\tilde{\lambda}_k \rightarrow \tilde{\lambda}_k - z\tilde{\lambda}_n, \;\; \lambda_n \rightarrow \lambda_n + z\lambda_k.$$

View the amplitude as a function of the complex variable z.

$$A(z) = A(p_1, \ldots, p_{k-1}, p_k(z), p_{k+1}, \ldots, p_{n-1}, p_n(z))$$

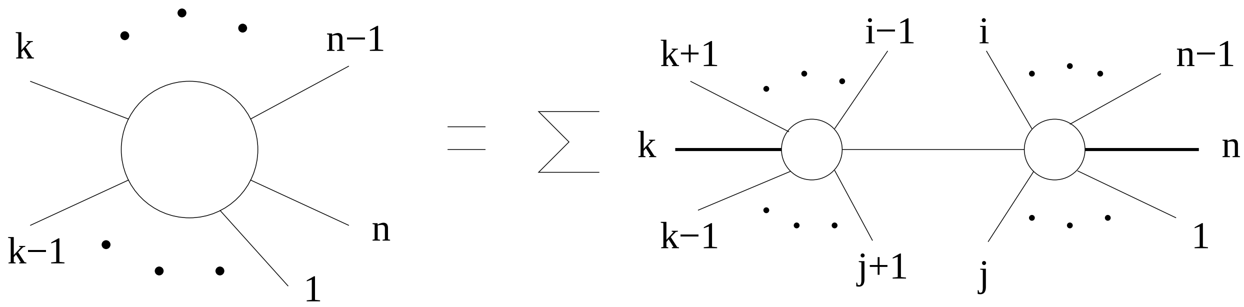
The physical amplitude is $A(0)$. $A(z)$ is a rational function with only simple poles. In particular, the only possible singularities are from propagators of Feynman diagrams, $P_{ij}(z)^2 = P_{ij}^2 - z\langle\lambda_k|P_{ij}|\tilde{\lambda}_n]$.



$$A(z) = \sum_{i,j} \frac{c_{ij}}{z - z_{ij}} + A(\infty)$$

$$z_{ij} = \frac{P_{ij}^2}{\langle\lambda_k|P_{ij}|\tilde{\lambda}_n]}$$

One can show that $A(\infty) = 0$ by following the limiting z behavior using MHV diagrams or Feynman diagrams (for the $+, -$ case).



The pole at at $P_{ij}^2(z) = 0$ comes from this diagram on the right.

The contribution of this diagram is $\sum_h A_L^h(z) A_R^{-h}(z)/P_{ij}(z)^2$.

$P_{ij}(z)^2$ is linear in z, so to obtain $\frac{c_{ij}}{z - z_{ij}}$, just set z equal to z_{ij} in the numerator. The internal line becomes on-shell. The numerator becomes a product $A_L^h(z_{ij}) A_R^{-h}(z_{ij})$ of physical, on-shell scattering amplitudes.

For the physical scattering amplitude, set z to zero in the denominator.

$$A = \sum_{i,j} \sum_h \frac{A_L^h(z_{ij}) A_R^{-h}(z_{ij})}{P_{ij}^2}.$$

This is our recursion relation.

Example

$$A(1^+,2^-,3^+,4^-,5^+,6^-) = \frac{[1\ 3]^4\langle 4\ 6\rangle^4}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle t_1^{[3]}\langle 6|1+2|3]\langle 4|2+3|1]} + \frac{\langle 2\ 6\rangle^4[3\ 5]^4}{\langle 6\ 1\rangle\langle 1\ 2\rangle[3\ 4][4\ 5]t_3^{[3]}\langle 6|4+5|3]\langle 2|3+4|5]} + \frac{[1\ 5]^4\langle 2\ 4\rangle^4}{\langle 2\ 3\rangle\langle 3\ 4\rangle[5\ 6][6\ 1]t_2^{[3]}\langle 4|2+3|1]\langle 2|3+4|5]}$$

The three terms are the contributions of $(1,\hat{2}|\hat{3},4,5,6)$, $(6,1,\hat{2}|\hat{3},4,5)$, and $(5,6,1,\hat{2}|\hat{3},4)$ respectively.

The original formula (Berends, Giele, Mangano, Parke, Xu 1988) had four terms.

Also, notice that all the symmetry of the amplitude is manifest in our new expression!

Example: A New Amplitude

$$A(1^+,2^-,3^+,4^-,5^+,6^-,7^+,8^-) = [I_1 + I_1^{\mathrm{conj.flip}}] + [I_2 + I_2^{\mathrm{conj.flip}}] + I_3$$

$$\text{conjugate flip}: \quad 2 \leftrightarrow 3,\ 1 \leftrightarrow 4,\ 8 \leftrightarrow 5,\ 7 \leftrightarrow 6,\ \langle\ \rangle \leftrightarrow [\ ].$$

$$\begin{aligned} I_1 \quad = & \frac{[1\,3]^4\langle 4\,6\rangle^4\langle 6\,8\rangle^4}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6\,7\rangle\langle 7\,8\rangle\langle 6|K_7^{[2]}|1]\langle 6|K_4^{[2]}|3]\langle 6|K_7^{[2]}K_1^{[3]}|4\rangle\langle 8|K_1^{[3]}K_4^{[2]}|6\rangle} \\ & + \frac{[1\,3]^4[5\,7]^4\langle 4\,8\rangle^4}{[1\,2][2\,3][5\,6][6\,7]t_1^{[3]}t_5^{[3]}\langle 4|K_2^{[2]}|1]\langle 4|K_5^{[2]}|7]\langle 8|K_1^{[2]}|3]\langle 8|K_6^{[2]}|5]} \\ & - \frac{[1\,3]^4\langle 4\,6\rangle^4\langle 8|K_1^{[3]}|7]^4}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle t_1^{[3]}t_4^{[3]}t_8^{[4]}\langle 4|K_5^{[2]}|7]\langle 8|K_1^{[2]}|3][1|K_2^{[2]}K_4^{[3]}|7]\langle 8|K_1^{[3]}K_4^{[2]}|6\rangle} \\ & + \frac{[1\,3]^4[7\,1]^4\langle 4\,6\rangle^4}{[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle[7\,8][8\,1]\langle 4|K_2^{[2]}|1]\langle 6|K_7^{[2]}|1][1|K_2^{[2]}K_4^{[3]}|7][3|K_4^{[3]}K_7^{[2]}|1]} \\ & - \frac{[1\,3]^4\langle 6\,8\rangle^4\langle 4|K_1^{[3]}|5]^4}{[1\,2][2\,3]\langle 6\,7\rangle\langle 7\,8\rangle t_1^{[3]}t_6^{[3]}t_5^{[4]}\langle 4|K_2^{[2]}|1]\langle 8|K_6^{[2]}|5][5|K_6^{[3]}K_1^{[2]}|3]\langle 6|K_7^{[2]}K_1^{[3]}|4\rangle} \\ & + \frac{[1\,3]^4[3\,5]^4\langle 6\,8\rangle^4}{[1\,2][2\,3][3\,4][4\,5]\langle 6\,7\rangle\langle 7\,8\rangle\langle 6|K_4^{[2]}|3]\langle 8|K_1^{[2]}|3][3|K_4^{[2]}K_6^{[3]}|1][5|K_6^{[3]}K_1^{[2]}|3]} \end{aligned}$$

$$
\begin{aligned}
I_2 \quad &= -\frac{\langle 8\ 2\rangle^4[3\ 5]^4\langle 6|K_8^{[3]}|3]^4}{\langle 8\ 1\rangle\langle 1\ 2\rangle\langle 8|K_8^{[3]}|3]t_8^{[3]}[3\ 4][4\ 5]\langle 6\ 7\rangle\langle 7|K_8^{[3]}|3]\langle 6|K_4^{[2]}|3]\langle 2|K_8^{[3]}K_6^{[2]}K_4^{[2]}|3][5|K_6^{[2]}} \\
&\quad -\frac{\langle 8\ 2\rangle^4[5\ 7]^4\langle 4|K_8^{[3]}|3]^4}{\langle 8\ 1\rangle\langle 1\ 2\rangle\langle 8|K_8^{[3]}|3]t_8^{[3]}[5\ 6][6\ 7]t_8^{[4]}t_5^{[3]}\langle 4|K_5^{[3]}K_8^{[3]}|2\rangle[5|K_5^{[3]}K_8^{[3]}|3]\langle 4|K_5^{[3]}|7]} \\
&\quad +\frac{\langle 8\ 2\rangle^4[3\ 7]^4\langle 4\ 6\rangle^4}{\langle 8\ 1\rangle\langle 1\ 2\rangle\langle 8|K_8^{[3]}|3]t_8^{[3]}\langle 4\ 5\rangle\langle 5\ 6\rangle t_4^{[3]}\langle 2|K_8^{[3]}|7]\langle 6|K_4^{[3]}|3]\langle 4|K_4^{[3]}|7]}
\end{aligned}
$$

$$
\begin{aligned}
I_3 \quad &= -\frac{\langle 8\ 2\rangle^4}{\langle 7\ 8\rangle\langle 8\ 1\rangle\langle 1\ 2\rangle\langle 7|K_7^{[4]}|3]}\,\frac{1}{t_7^{[4]}}\,\frac{[3\ 5]^4}{[3\ 4][4\ 5][5\ 6]\langle 2|K_3^{[4]}|6]} \\
&\quad -\frac{[7\ 1]^4\langle 2|K_7^{[4]}|3]^2}{[7\ 8][8\ 1]t_7^{[3]}[3|K_3^{[4]}K_7^{[2]}|1]\langle 2|K_7^{[4]}|7]}\,\frac{1}{t_7^{[4]}}\,\frac{\langle 4\ 6\rangle^4\langle 2|K_3^{[4]}|3]^2}{\langle 4\ 5\rangle\langle 5\ 6\rangle t_4^{[3]}\langle 4|K_5^{[2]}K_7^{[4]}|2\rangle\langle 6|K_3^{[4]}|3]}
\end{aligned}
$$

Symmetries of $A(1^+, 2^-, 3^+, 4^-, 5^+, 6^-, 7^+, 8^-)$

This amplitude has a high degree of symmetry. There are 8 rotations (with possible conjugation) and an overall reflection. Is this manifest in our formula? (The recursion relation breaks almost all of this symmetry.)

Mostly, but not entirely. All except three of our terms are related by these symmetries. These three can be replaced by three others that are “missing” from the rest.

The amplitude can be fully obtained from the symmetry transformations acting on our first three terms,

$$\frac{[1\ 3]^4 \langle 4\ 6\rangle^4 \langle 6\ 8\rangle^4}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle\langle 6\ 7\rangle\langle 7\ 8\rangle\langle 6|K_7^{[2]}|1]\langle 6|K_4^{[2]}|3]\langle 6|K_7^{[2]}K_1^{[3]}|4\rangle\langle 8|K_1^{[3]}K_4^{[2]}|6\rangle},$$

$$\frac{[1\ 3]^4 [5\ 7]^4 \langle 4\ 8\rangle^4}{[1\ 2][2\ 3][5\ 6][6\ 7]t_1^{[3]}t_5^{[3]}\langle 4|K_2^{[2]}|1]\langle 4|K_5^{[2]}|7]\langle 8|K_1^{[2]}|3]\langle 8|K_6^{[2]}|5]},$$

$$-\frac{[1\ 3]^4 \langle 4\ 6\rangle^4 \langle 8|K_1^{[3]}|7]^4}{[1\ 2][2\ 3]\langle 4\ 5\rangle\langle 5\ 6\rangle t_1^{[3]}t_4^{[3]}t_8^{[4]}\langle 4|K_5^{[2]}|7]\langle 8|K_1^{[2]}|3][1|K_2^{[2]}K_4^{[3]}|7]\langle 8|K_1^{[3]}K_4^{[2]}|6\rangle}$$

Conclusions

- Understanding the holomorphic anomaly yields information about the twistor space structure of box function coefficients.

- The box integral coefficients of one-loop $\mathcal{N} = 4$ amplitudes are given in terms of tree amplitudes by using a quadruple cut.

$$\hat{B}_{abcd} = \sum A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree}$$

- We have found new recursion relations for tree-level amplitudes that give the most compact expressions.

$$A_n = \sum_i A_{i+2} \frac{1}{P^2} A_{n-i}$$