INTEGRABILITY IN YANG-MILLS THEORY

- $D = 4$, $\mathcal{N} = 4$, $SU(N)$ Superconformal Yang-Mills Theory, in the Planar Limit $N \to \infty$, fixed $g^2 N$

- $PSU(2,2|4)$ Integrable Spin Chain

- Yangian Symmetry Algebra of $PSU(2,2|4)$

- Local Commuting Hamiltonians of the Periodic Chain

- Hamiltonians Annihilate Chiral Primary States

- Higher-Loop Corrections $\mathcal{O}(g^2 N)^2$

hep-th/0308089, 0401243 $\{$*LD, C.Nappi, E.Witten*$\}$, *hep-th/0411020*$\{$*LD, CN*$\}$

How is it that a 4-dimensional field theory, such as $\mathcal{N} = 4$ SYM, yields to integrable methods of $1+1$-dimensional quantum field theories?

Large N gauge theory can be described by a spin chain, a version of $1+1$ QFT where the space dimension has been discretized. These lattice integrable models are quantum spin models of integrable magnetic chains.

With spacetime signature $+ - --$, the superconformal group is $PSU(2,2|4) \supset SU(2,2) \times SU(4)_R$.

$$\mathcal{L} = \frac{1}{g_{YM}^2} \mathrm{Tr} \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^I D^\mu \phi^I - \frac{1}{2} [\phi^I, \phi^J][\phi^I, \phi^J] + \mathrm{fermions} \right) .$$

Massless gluon multiplet: $(\pm 1, 4(\pm \frac{1}{2}), 6(0))$ in adjoint of $SU(N)$

Large N planar graphs - ’tHooft 1974

RADIAL QUANTIZATION ON $\mathbb{R} \times S^3$

· Dilation generator is the Hamiltonian $D \sim U(P^0 + K^0)U^{-1}$

· States are in $1 \leftrightarrow 1$ correspondence with local operators $\mathcal{O}(x)$,
$\lim_{|x|\to 0} \mathcal{O}(x)|0\rangle = |\mathcal{O}\rangle$.

· Local operators $\sim$ trace of product of letters

· Letters are elementary fields

$$\Phi_{ij}^{I,\alpha,\mu,\ldots}\{ \quad \begin{aligned} \phi^I &= \phi^{IA}(x)T_{ij}^A \\ \psi_\alpha &= \psi_\alpha^A(x)T_{ij}^A \\ F_{\mu\nu} &= F_{\mu\nu}^A(x)T_{ij}^A \\ &D_\mu\phi^I, \ldots \end{aligned}$$

· Single trace operators survive in large N,
$\mathcal{O}(x) = \Phi^{(1)}(x)_{ij}\Phi^{(2)}(x)_{jk}\ldots = \mathrm{Tr}\,\Phi^{(1)}(x)\Phi^{(2)}(x)\ldots\Phi^{(L)}(x)$
represents a state of a chain of L spins (partons),
and this 1-dimensional chain suggests why planar SYM is integrable.

$PSU(2,2|4)$ SPIN CHAIN IS THE PLANAR 1-LOOP DILATATION OPERATOR

· Local spin variables J_i^A satisfy $[J_i^A, J_j^B\} = f_C^{AB} J_i^C \delta_{ij}$.
· Symmetry generators at $g^2 N = 0$ are $J^A = \sum_i J_i^A$, $[J^A, J^B\} = f_C^{AB} J^C$
· Exact symmetry generators: $\tilde{J}^A = J^A + (g^2 N)\delta J^A + \ldots$

· The dilation operator D to one-loop: $\tilde{D} = D + g^2 N \delta D + \ldots$ was computed from one-loop feynman graphs, and was seen to be a nearest-neighbor interaction (Minahan, Zarembo hep-th/0212208, Beisert /0307015, Staudacher /0307042):

$$\delta D \equiv H = \sum_{i=1}^{L} H_{i,i+1}$$

$$H_{i,i+1} = 2(\psi(J_{ij}+1) - \psi(1)) = \sum_{j=0}^{\infty} 2h(j) P_{i,i+1;j}$$

$h(j) = \sum_{n=1}^{j} \frac{1}{n}$ are the harmonic numbers, $h(0) = 0$, $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$.

· All letters span V_F, the infinite-dimensional module of one-particle states.
· The two-particle Casimir, with $J^A = J_1^A + J_2^A$, is $J^A J^A = J_{12}(J_{12}+1)$.
· The tensor product decomposition is $V_F \otimes V_F = \bigoplus_{j=0}^{\infty} V_j$.

$P_{i,i+1;j}$ projects the fields at positions $i, i+1$ to the module V_j.

The eigenvalue of J_{12} takes values $j = 0, 1, 2, \ldots \infty$.

The modules V_j, for $j = 0, 1, 2, \ldots \infty$ label the irreducible representations of $PSU(2,2|4)$ which describe the two-particle system in free $N = 4$ super Yang-Mills theory. For each j, the lowest component is the superconformal primary $|\psi(j)\rangle \in V_j$:

$$V_0 : \ |\psi(0)\rangle \sim \phi^I \phi^J + \phi^J \phi^I - \frac{1}{3}\delta^{IJ}\phi^K\phi^K$$
$$V_1 : \ |\psi(1)\rangle \sim \phi^I \phi^J - \phi^J \phi^I .$$

where ϕ^I, $I = 1, \ldots, 6$ are the elementary scalars of $\mathcal{N} = 4$ SYM, in the vector representation of the R symmetry group $SU(4) \cong SO(6)$. The primaries in V_0 and V_1 have dimension two and non-trivial R-symmetry quantum numbers.

The primaries $|\psi(j)\rangle$ in V_j, $j \geq 2$, are singlets of the R symmetry, and have dimension j:

$$V_j : \ |\psi(j)\rangle \sim \sum_{I=1}^{6} \sum_{k=0}^{j-2} c_k^{(j-2)} \partial^k \phi^I \partial^{j-2-k} \phi^I + \ldots$$

with coefficients $c_k^{(j-2)} = (-1)^k / k!^2 (j-k-2)!^2$.

Only V_0 is a *chiral* primary representation, which has no anomalous dimension.

·We can also derive H from an R-matrix:

$$R_{ik}(u) = (-1)^J \frac{\Gamma(J_{ik}+1+u)}{\Gamma(J_{ik}+1-u)} = \sum_{j=0} (-1)^j \frac{\Gamma(j+1+u)}{\Gamma(j+1-u)} P_{ik;j}$$

which satisfies a Yang-Baxter relation $\mathcal{R}_{12}\mathcal{R}_{32}\mathcal{R}_{31} = \mathcal{R}_{31}\mathcal{R}_{32}\mathcal{R}_{12}$:

$$R_{ik}(u-v)R_{im}(\omega-v)R_{km}(\omega-u) = R_{km}(\omega-u)R_{im}(\omega-v)R_{ik}(u-v)$$

where $R_{im}(u) = uI_i \otimes I_m + 2J_i^A \otimes T_{A\,m}$,

J_I^A acts on V_F, and $T_{A\,m}$ is the $4|4$ of $U(4|4)$.

· Forming the transfer matrix, $F_k(u) = \mathrm{tr}_k R_{L\,k}(u) R_{L-1\,k}(u) \ldots R_{1k}(u)$,

the trace of the monodromy, we can find the Hamiltonian again,

$$H = \sum_{i=1}^{L} 2(\psi(J_{i,i+1}+1) - \psi(1)) \sim \frac{d}{du} \ln F_k(u)|_{u=\frac{i}{2}} \,.$$

·For a single site i, let $J_b^a \equiv J_i^A (T_A)_b^a$, where a, b label the matrix indices at the auxiliary site m. Use the property that the representation V_F satisfies the criterion at a given site i, $J_b^a J_c^b = \alpha J_c^a$ modulo the identity δ_{ac}, for some proportionality constant α. This occurs for any representation M, when $M \otimes \bar{M}$ contains the adjoint only once, such as the n-dimensional representation of $SU(n)$. It will also be needed to form representations of the Yangian.

We can also calculate H directly from the

YANGIAN SYMMETRY GENERATORS:

Bilocal Yangian generators at $g^2 N = 0$ are:

$$Q^A = f^A_{BC} \sum_{i<j} J_i^B J_j^C$$

where $[J^A, Q^B\} = f^A_{BC} Q^C$.

Require that $[H, Q^A] = q^A \equiv J_1^A - J_{L+1}^A$
as $[H, J^A] = 0$.

$$[H, Q^A] = \sum_{i=1}^{L} [H_{i,i+1}, Q^A_{i,i+1}] = \sum_{i=1}^{L} q^A_{i,i+1} = q^A$$

Use the identity $Q^A \equiv \frac{1}{4}[J^D J^D, q^A]$.

Prove $H_{12}(J)$ is some function of the Casimir $J = J_{12}$, for any state $|\kappa(j)\rangle \in V_j$:

$$\begin{aligned}
&[H_{12}(J), Q_{12}^A]\ |\kappa(j)\rangle \\
&= \frac{1}{4}[H_{12}(J),\ [(J_1^D + J_2^D)(J_1^D + J_2^D),\ q_{12}^A]]\ |\kappa(j)\rangle \\
&= \frac{1}{2}\ (j\ (H_{12}(j) - H_{12}(j-1))\ |\chi^A(j-1)\rangle \\
&+(j+1)(H_{12}(j+1) - H_{12}(j))|\rho^A(j+1)\rangle) \\
&= |\chi^A(j-1)\rangle + |\rho^A(j+1)\rangle\ =\ q_{12}^A\,|\kappa(j)\rangle \qquad\qquad *
\end{aligned}$$

where the action of q_{12}^A on a state in V_j is

$q_{12}^A|\kappa(j)\rangle = |\chi^A(j-1)\rangle + |\rho^A(j+1)\rangle$,
where $|\chi^A(j-1)\rangle \in V_{j-1}$ and $|\rho^A(j+1)\rangle \in V_{j+1}$.

Solving $*$ requires $j\,(H_{12}(j) - H_{12}(j-1)) = 2$.

$\frac{1}{2}(H_{12}(j) - H_{12}(0)) = \sum_{n=1}^{j} \frac{1}{n}$

Again $H_{12}(J) = 2(\psi(J+1) - \psi(1))$.

In any case, the news is, however you find the Hamiltonian, that the

dilatation eigenvalues of the states $|\mathcal{O}\rangle$,

the one-loop anomalous dimensions in the planar limit,

can be found from an integrable local operator H with nearest neighbor interactions, using Bethe Ansatz techniques.

Related integrable features in gauge theory were seen earlier by Belitsky, Braun, Derkachov, Kormchemsky, Lipatov, Manashov.

Spin model integrability and Yangians are reviewed by L. Faddeev, hep-th/9605187; N. Reshetikhin, Nucl. Phys. B**251** 565 (1985); P. Kulish, N. Reshetikhin, E. Sklyanin, Lett. Math. Phys. **5** 393 (1981).

Yangians and monondromy matrices in spin models are reviewed by D. Bernard, O. Babelon, hep-th/9111036, 9211133.

YANGIAN SYMMETRY

We describe an infinite-dimensional non-abelian symmetry algebra that is found in planar superconformal Yang-Mills theory. A basis for the generators: $\mathcal{J}_n^A$ where $\mathcal{J}_0^A = J^A$, $\mathcal{J}_1^A = Q^A$, and $\mathcal{J}_n^A$, $n = 0, 1, 2, \ldots$.

The higher charges arise from commutators of the Q's. The algebra, called a Yangian $Y(G)$, is an associative Hopf algebra (Drinfeld 1985) that satisfies:

$$[J^A, J^B] = f_C^{AB} J^C \, , \qquad [J^A, Q^B] = f_C^{AB} Q^C \, , \tag{1}$$

and the Serre relations

$$\begin{aligned} &[Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]] \\ &= \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\} \, , \end{aligned} \tag{2}$$

$$\begin{aligned} &[[Q^A, Q^B], [J^C, Q^D]] + [[Q^C, Q^D], [J^A, Q^B]] \\ &= \frac{1}{24} (f^{AGL} f^{BEM} f^{KFN} f_{LMN} f_K^{CD} \\ &\qquad + f^{CGL} f^{DEM} f^{KFN} f_{LMN} f_K^{AB}) \{J_G, J_E, J_F\} \, . \end{aligned} \tag{3}$$

·In 2D integrable sigma models for Lie-algebra valued matrix fields $g(x,t)$ with a family of flat connections, the Kac-Moody symmetry

$$g(x,t) \longrightarrow g(x,t) + \delta_n^A g(x,t), \tag{4}$$

for $m, n \geq 0$, can be used to define generators (LD 1981)

$$M_n^A \equiv \int d^2x \delta_n^A g(x,t) \frac{\delta}{\delta g(x)} \tag{5}$$

which close a partial Kac-Moody loop algebra

$$[M_n^A, M_m^B] = f_C^{AB} M_{n+m}^c \,. \tag{6}$$

These transformations are related to those generated via Poisson brackets by the non-local charges, *eg.*

$$\{Q^A, g(x,t)\} = -\delta_1^A g(x,t) + \frac{1}{2} f_{ABC} J^B \, \delta_0^C (g(x,t)). \tag{7}$$

·The 4D self-dual Yang-Mills equations also have Kac-Moody symmetry, and are classically integrable.

· Our Yangian construction is motivated by the the existence of non-local symmetries in the AdS/CFT dual worldsheet theory of the type IIB superstring on $AdS_5 \times S^5$, a coset sigma model, as found Bena, Polchinski and Roiban, where the gauge theory parameters $g^2 N$ and N
are related to the string parameters g_s and $\frac{R^2}{\alpha'}$ as
$g^2 N = \frac{1}{4\pi} \frac{R^4}{\alpha'^2}$ and $g^2 = g_s$.

They found $Y(PSU(2,2|4))$ for $N \to \infty$ in the supergravity limit
$g^2 N \to \infty$, the classical sigma model.

· Together with the tree level $g^2 N = 0$ expressions, this motivates us to conjecture that $\mathcal{N} = 4$ SYM has the Yangian symmetry $Y(PSU(2,2|4))$ in the planar limit for all values of $g^2 N$:

$$\tilde{J}^A = J^A + (g^2 N)\delta J^A + \mathcal{O}((g^2 N)^2)$$

$$\tilde{Q}^A = Q^A + (g^2 N)\delta Q^A + \mathcal{O}((g^2 N)^2).$$

· Confining gauge theory in the large N limit is a free theory, $g_s = 0$.
We might expect a free theory to be integrable.

NON-LOCAL CURRENTS AS NOETHER CURRENTS

Noether current for $so(2,4)$:

Classically, $j^{A\mu}(x) = \kappa^A_\nu \theta^{\mu\nu}(x)$, κ^A_μ are the conformal Killing vectors

$$\begin{aligned} \theta^{\mu\nu} &= 2\mathrm{Tr} F^{\mu\rho} F^\nu_{\ \rho} + 2\mathrm{Tr} D^\mu \phi^I D^\nu \phi^I - g^{\mu\nu} \mathcal{L} \\ &\quad - \frac{1}{3} \mathrm{Tr}(D^\mu D^\nu - g^{\mu\nu} D_\rho D^\rho) \phi^I \phi^I \ + \ \mathrm{fermions}\,. \end{aligned}$$

$\partial_\mu j^{A\mu}(x) = 0$ for any $g^2 N$. If we set $g^2 N = 0$, then the *untraced* matrix

$$\begin{aligned} (\theta^{\mu\nu})_i^{\ j} &= F^{\mu\rho} F^\nu_{\ \rho} + F^{\nu\rho} F^\mu_{\ \rho} + \partial^\mu \phi^I \partial^\nu \phi^I + \partial^\nu \phi^I \partial^\mu \phi^I \\ &\quad - g^{\mu\nu} \left(\frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} + \partial_\mu \phi^I \partial^\mu \phi^I \right) \\ &\quad - \frac{1}{3} (\partial^\mu \partial^\nu - g^{\mu\nu} \partial_\rho \partial^\rho) \phi^I \phi^I + \mathrm{fermions}\,, \end{aligned}$$

is also conserved, as is $\kappa^A_\nu (\theta^{\mu\nu})_i^{\ j}$. Then
$Q_0^{AB\ldots} = \int_M \kappa^A_\nu (\theta^{0\nu})_i^{\ j} \int_M \kappa^B_\rho (\theta^{0\rho})_j^{\ k} \ldots$, at $g^2 N = 0$.

TREE LEVEL CONSTRUCTION OF YANGIAN GENERATORS

$J^A = \sum_i J_i^A$

$Q^A = f^A_{BC} \sum_{i<j} J_i^B J_j^C$

To close the Serre relation,

$$[Q^A, [Q^B, J^C]] + [Q^B, [Q^C, J^A]] + [Q^C, [Q^A, J^B]]$$
$$= \frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\}$$

J_i^A must be in a representation R such that the adjoint only appears once in the decomposition of $R \otimes \overline{R}$.

For a single site representation, $J^A = J_1^A$, $Q^A = 0$. Then $\frac{1}{24} f^{ADK} f^{BEL} f^{CFM} f_{KLM} \{J_D, J_E, J_F\} = 0$, which holds when $J_b^a \equiv J_i^A (T_A)_b^a$ satisfies $J_b^a J_c^b = \alpha J_c^a$ modulo a c-number.

LOCAL COMMUTING HAMILTONIANS OF THE PERIODIC CHAIN

$$H_1 = \sum_{i=1}^{L} 2\left(\psi(J_{i,i+1}+1) - \psi(1)\right) \equiv H$$

$$H_2 = \sum_{i=1}^{L} [\psi(J_{i,i+1}+1), \psi(J_{i+1,i+2}+1)] \equiv U$$

with periodic boundary conditions $J_i^A = J_{i+L}^A$. Then

$$[H, J^A] = 0\,, \qquad [U, J^A] = 0.$$

Derive H, U as Casimirs of the Yangian:

$$[\bar{H}, Q^A] = J_1^A - J_{L+1}^A = q^A$$
$$[\bar{U}, Q^A] = [q^A_{1\,L+1}, (\psi(J_{12}+1) + \psi(J_{L+1,L+2}+1))]$$

where $\bar{H}, \bar{U}$ are defined for the open chain .

Derive H, U by imposing periodic boundary conditions on $\bar{H}, \bar{U}$.

HAMILTONIANS ANNIHILATE CHIRAL PRIMARY STATES

Chiral primaries are representations of $PSU(2,2|4)$ with lowest weight states $|\lambda_L\rangle$, built by symmetric traceless products of the scalar fields ϕ^I.

For $L=2$, $\;(J_1^A + J_2^A)^2|\lambda_2\rangle = 0, \quad j=0$.

For $L>2$, since the chiral primary state is symmetric and traceless in all indicies, it follows that it is symmetric and traceless in each pair. States on a pair of sites have definite j as described by $V_F \otimes V_F = \bigoplus_{j=0}^{\infty} V_j$; since each pair is symmetric and traceless in the scalar fields then $j=0$, so $P_{i,i+1,j}|\lambda_L\rangle = 0$ for $j \neq 0$. Then

$$H|\lambda_L\rangle = \sum_{i=1}^{L}\sum_{j=0}^{\infty} 2\,h(j) P_{i,i+1,\,j}\,|\lambda_L\rangle = \sum_{i=1}^{L} 2\,h(0) P_{i,i+1,\,0}\,|\lambda_L\rangle = 0\,.$$

$$
\begin{aligned}
U|\lambda_3\rangle &= \sum_{i=1}^{L} [h(J_{i,i+1}), h(J_{i+1,i+2})]\,|\lambda_3\rangle \\
&= \sum_{i=1}^{L} \sum_{j,j'=0}^{\infty} h(j)h(j')[P_{i,i+1,\,j}, P_{i+1,i+2,\,j'}\,]\,|\lambda_3\rangle = 0\,.
\end{aligned}
$$

· H, U annihilate all the states in the chiral primary module, since $[H, J^A] = [U, J^A]$.

· We conjecture that all the Casimirs of the Yangian annihilate the chiral primaries. This would explain why we see only the ordinary $PSU(2,2|4)$ symmetry in the supergravity Lagrangian of the AdS/CFT dual theory.

HIGHER LOOP CORRECTIONS in $g^2 N$

$$[\tilde{J}^A, \tilde{J}^B] = f^{AB}_C \tilde{J}^C\,, \qquad [\tilde{J}^A, \tilde{Q}^B] = f^{AB}_C \tilde{Q}^C$$

Expanded to 1-loop, for $J^A = D$,

the integrable Hamiltonian δD is a Casimir of the bare Yangian:

$$[\delta D, J^B] = 0, \qquad [\delta D, Q^B] \sim 0.$$

Is the *exact* anomalous dimension operator $\Delta D \equiv \tilde{D} - D$ also an integrable Hamiltonian?

Since $[\tilde{D}, \tilde{Q}^B] = \lambda^B \tilde{Q}^B$ and $[D, \tilde{Q}^B] = \lambda^B \tilde{Q}^B$,
then $[\Delta D, \tilde{Q}^B] = 0$, together with $[\Delta D, \tilde{J}^B] = 0$.

So ΔD is a Casimir of the exact Yangian.

Can we use twistors to sum the higher loops?