Outline:

I Localization in Twistor Space

S.G., L. Motl, A. Neitzke

hep-th/0404085

II Topological M-theory

R. Dijkgraaf, S.G., A. Neitzke, C. Vafa

hep-th/0411073

Witten:

perturbative $\mathcal{N}=4$ super Yang-Mills	$\Longleftrightarrow$	Topological String (B-model) in twistor space $\mathbb{CP}^{3\|4}$

- $\mathbf{Z}^{A} := (\lambda^1, \lambda^2, \mu^1, \mu^2 \,|\, \psi^1, \psi^2, \psi^3, \psi^4) \in \mathbb{C}^{4|4}$

holomorphic $4|4$ - form :

$$\Omega = d\lambda^1 \, d\lambda^2 \, d\mu^1 \, d\mu^2 \, d\psi^1 \, d\psi^2 \, d\psi^3 \, d\psi^4$$

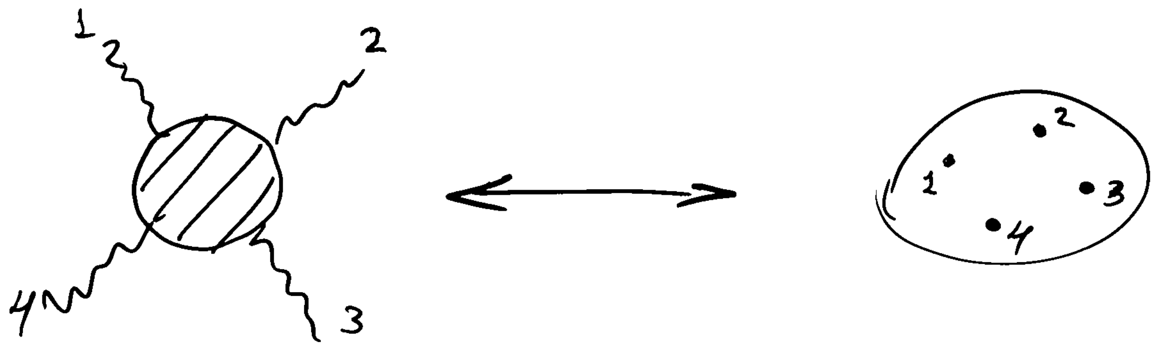
identification :

$$(\lambda, \mu \,|\, \psi) \sim (t\lambda, t\mu \,|\, t\psi) \qquad t \in \mathbb{C}^*$$

$\Longrightarrow$ Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$

$$\mathcal{A}(\underbrace{+++\overbrace{---}^{q}}_{n}) = \int \omega$$

moduli space of hol. curves
in $\mathbb{CP}^{3|4}$ of degree $d = q-1$
with n marked points

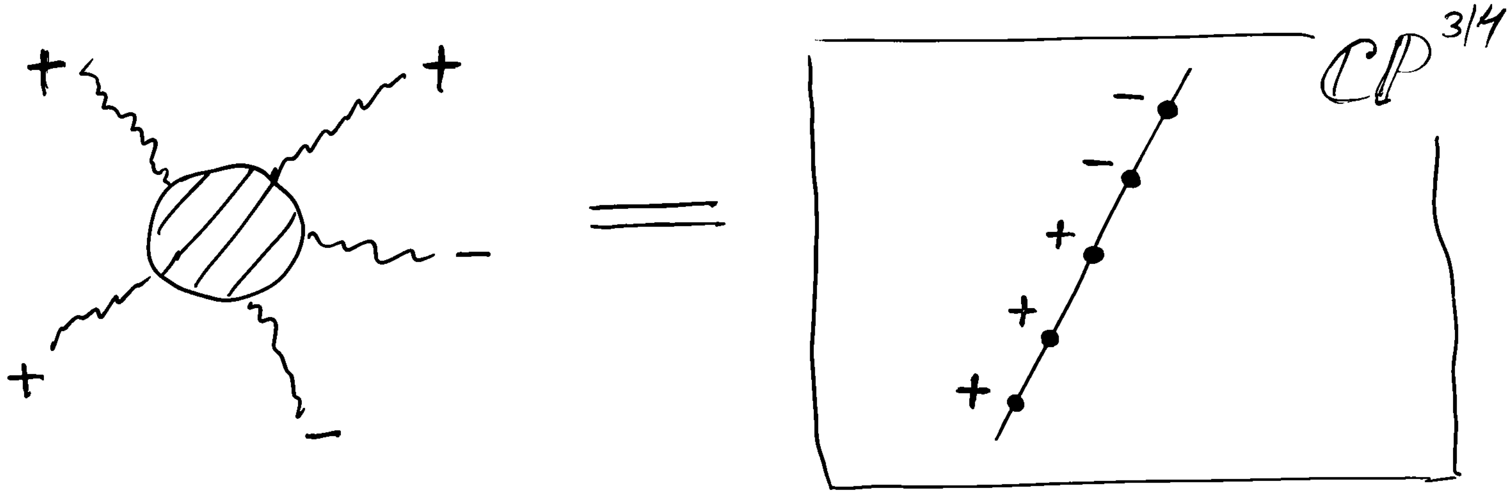


<u>Remark:</u> in the B-model, hol. curves are interpreted as D1-brane instantons.

Example 1:

$\mathcal{A}(+++--)$

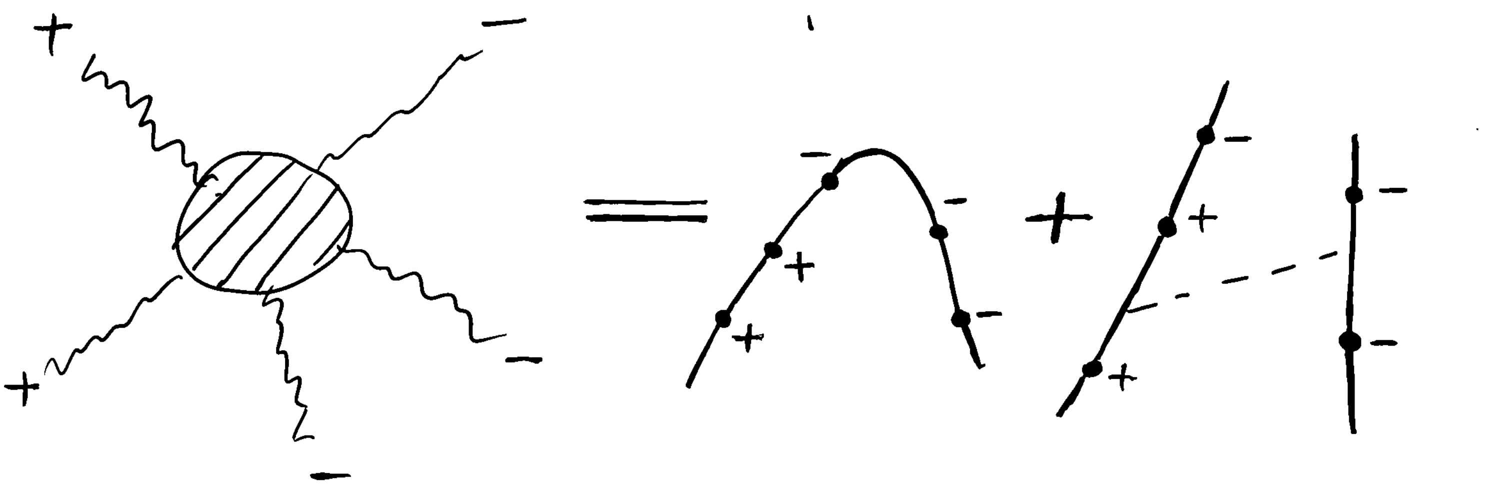
$d = q - 1 = 1$



Example 2:

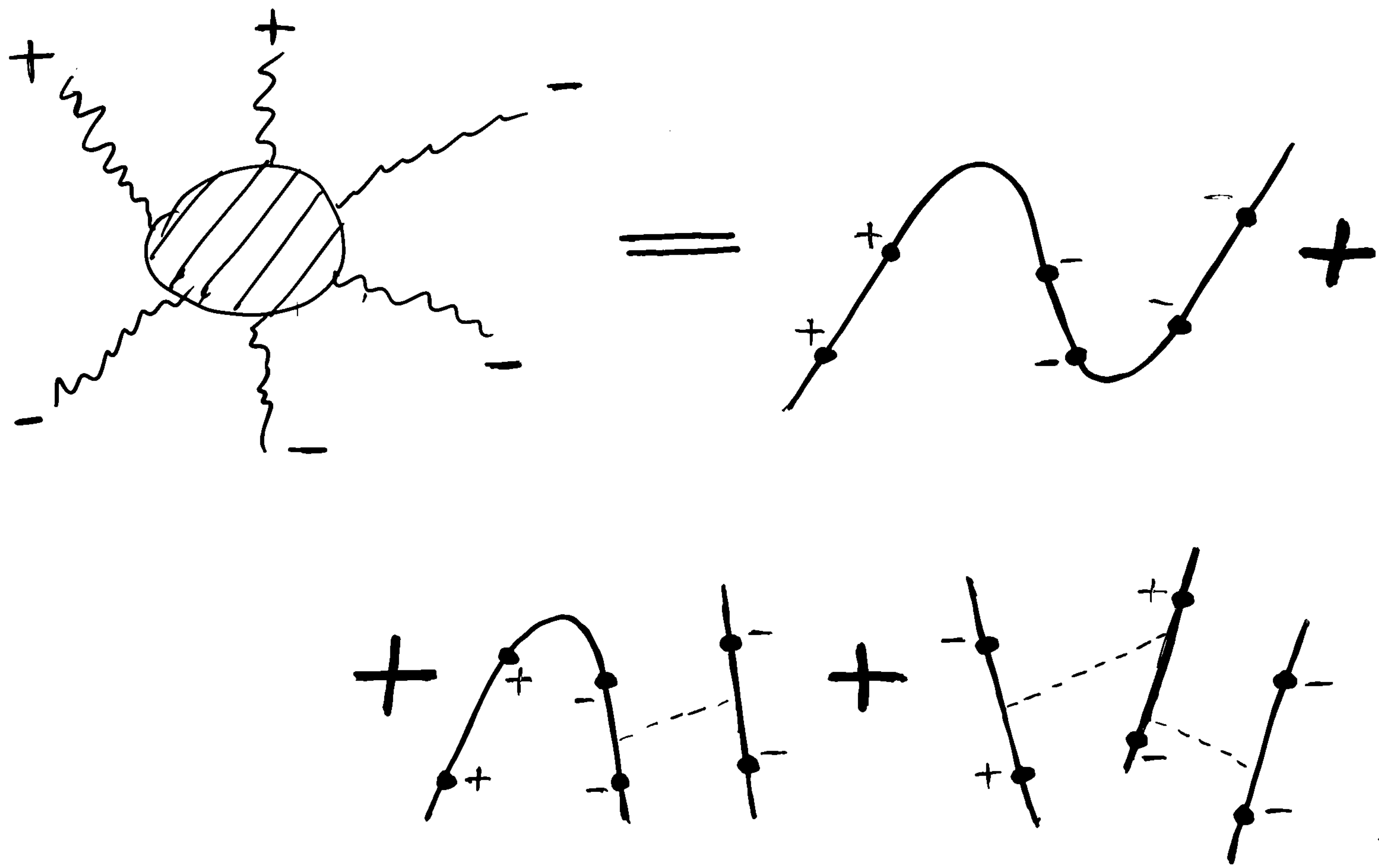
$\mathcal{A}(++---)$

$d = 2$



Example 3:

$\mathcal{A}(++----)$ $\qquad$ $\underline{d = 3}$



Connected Instantons:

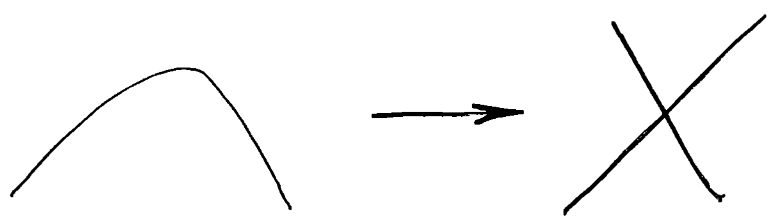
- $\mathcal{M}_{0,n,d}(\mathbb{CP}^{3|4})$ moduli space of genus 0, n-pointed curves of degree d in $\mathbb{CP}^{3|4}$

- $\dim \mathcal{M}_{0,n,d}(\mathbb{CP}^{3|4}) = (4d+n) \,|\, (4d+4)$

- Following [Witten], we realize it as the space of automorphism classes of maps

$$P : \mathbb{CP}^1 \to \mathbb{CP}^{3|4}$$

- $\mathcal{M}_{0,n,d}(\mathbb{CP}^{3|4})$ is non-compact due to deg. curves, e.g.



$\Rightarrow$ moduli space of stable maps

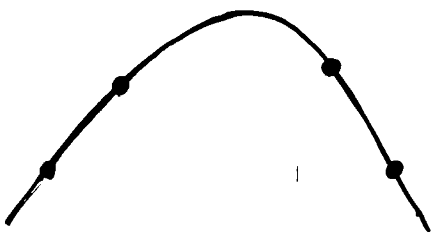
$$\overline{\mathcal{M}}_{0,n,d}(\mathbb{CP}^{3|4})$$

$$\begin{cases} P: \mathbb{CP}^1 \to \mathbb{CP}^{3|4} \\ P^{\mathbb{A}}(\sigma) = Z^{\mathbb{A}} = \sum_{k=0}^{d} \beta_k^{\mathbb{A}} \, \sigma^k \end{cases}$$

<u>moduli $\beta_k^{\mathbb{A}} \in \mathbb{C}^{4d+4|4d+4}$</u>

natural measure

$$\mu_d = \prod_{k,\mathbb{A}} d\beta_k^{\mathbb{A}}$$



$$\mathcal{A}_{YM}^{conn} = \int_{\overline{\mathcal{M}_{0,n,d}}} \prod_{k,\mathbb{A}} d\beta_k^{\mathbb{A}} \cdot \underbrace{\prod_{i=1}^{n} \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}}_{\substack{\text{hol. } n\text{-form on } (\mathbb{CP}^1)^n \\ \omega(\sigma_i)}} \prod_{i=1}^{n} ev_i^*(\phi_i)$$

- ϕ_i $\bar{\partial}$-closed $(0,1)$-form on $\mathbb{CP}^{3|4}$

- $GL(2,\mathbb{C})$: $\sigma \mapsto \sigma' = \dfrac{a\sigma + b}{c\sigma + d}$

$$ad - bc \neq 0$$

$$\Rightarrow \mathcal{A}_{YM}^{conn} = \int_{\overline{\mathcal{M}_{0,n,d}}} \frac{\prod_{k,A} d\beta_k^A \prod_{i=1}^{n} \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}}{GL(2,\mathbb{C})} \wedge \prod_{i=1}^{n} ev_i^*(\phi_i)$$

$$= \int_{\overline{\mathcal{M}_{0,n,d}}} \frac{\mu_d \wedge \omega(\sigma_1, \dots, \sigma_n)}{GL(2,\mathbb{C})} \wedge \Phi$$

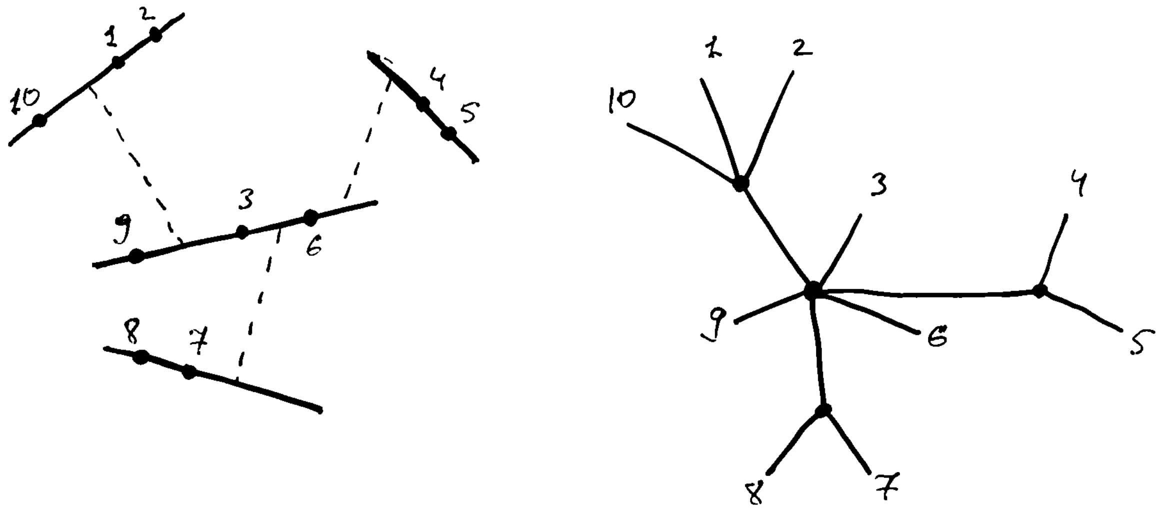
* **Surprise:**

$$\mathcal{A}_{YM}^{conn} = \mathcal{A}_{YM}$$

R. Roiban,
M. Spradlin,
A. Volovich

Completely Disconnected Instantons

$\underline{\underline{\Gamma}}$:



- $Q_i : \mathbb{CP}^1 \to \mathbb{CP}^{3|4}$

$$Q_i^{\mathbb{A}}(\sigma) = \sum_{K=0}^{1} \beta_{K,i}^{\mathbb{A}} \, \sigma^K$$

- moduli $\beta_{K,i}^{\mathbb{A}} \in \mathbb{C}^{8d|8d}$

- $GL(2,\mathbb{C})^d \Longrightarrow 4d \,|\, 8d$

$$\mathcal{M}_{lines}^{\Gamma} = \prod_{i=1}^{d} \mathcal{M}_{0,n,1}\left(\mathbb{CP}^{3|4}\right)$$

(Holomorphic) Measure:

i) $\mu_{lines} = \prod_{k,\mathbb{A},i} d\beta^{\mathbb{A}}_{k,i}$

ii) $\omega_i := \omega(\sigma_1, \ldots \sigma_{n_i})$

iii) $\Phi = \prod_i ev_i^* (\phi_i)$

iv) $D(x,y)$ (0,2)-form on $\mathbb{CP}^{3|4} \times \mathbb{CP}^{3|4}$

"holomorphic CS propagator"

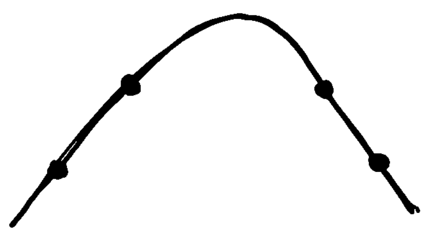
$$\boxed{\mathcal{A}^{disc}_{YM} = \sum_{\Gamma} \int_{\mathcal{M}^{\Gamma}_{lines}} \frac{\mu_{lines} \wedge \left(\prod_{i=1}^{d} \omega_i\right) \wedge \Phi \wedge D}{(GL(2,\mathbb{C}))^d}}$$

* Yet another surprise:

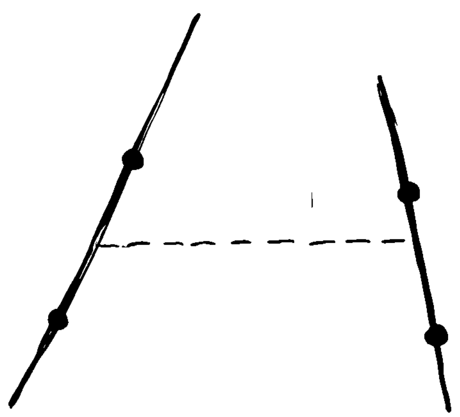
$$\mathcal{A}^{disc}_{YM} = \mathcal{A}_{YM}$$

F. Cachazo,
P. Svrcek,
E. Witten

Puzzle:

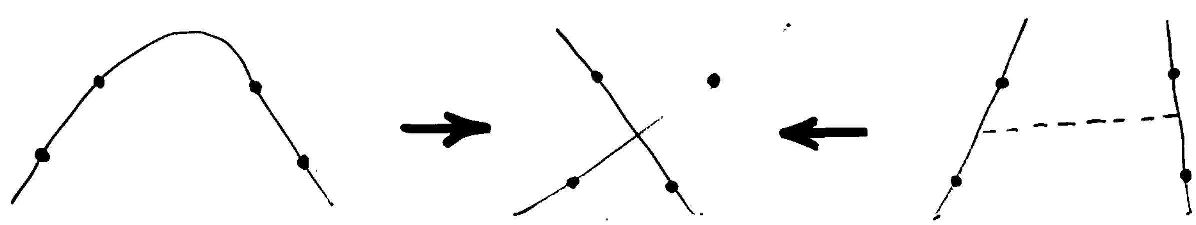


$$\mathcal{A}_{YM} = \int_{\mathcal{M}_{0,n,d}} \omega_{conn}$$



$$\mathcal{A}_{YM} = \int_{\mathcal{M}_{lines}} \omega_{disc}$$

- both integrals have a simple pole along $\underline{\underline{\mathcal{M}_{int}}}$:



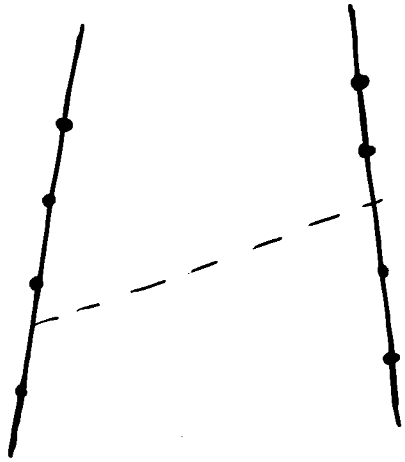
- same residue $\Rightarrow$

$$\int_{\mathcal{M}_{0,n,d}} \omega_{conn} = \int_{\mathcal{M}_{lines}} \omega_{disc}$$

- ## Residue from ω_{disc}

F. Cachazo, P. Švrček, E. Witten

$$\mu^{\dot a} + x^{a\dot a}\lambda_a = 0$$

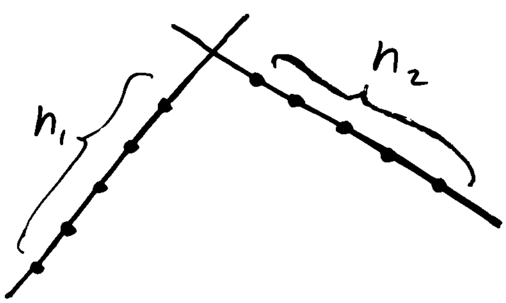


$$\mu^{\dot a} + x'^{a\dot a}\lambda_a = 0$$

$$y^{a\dot a} := x'^{a\dot a} - x^{a\dot a} \Rightarrow$$

$$\omega_{disc} \propto \frac{d^4 y}{y_{a\dot a}\, y^{a\dot a}} = \frac{dy^{1\dot 1}\, dy^{2\dot 2}\, dy^{1\dot 2}\, dy^{2\dot 1}}{2\left(y^{1\dot 1} y^{2\dot 2} - y^{1\dot 2} y^{2\dot 1}\right)}$$

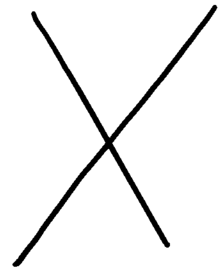
$$\Rightarrow \sum_\Gamma \int_{\mathcal{M}_{int}^\Gamma} \frac{\mu_{int} \wedge \prod_{i=1}^{n_1} \frac{d\sigma_i}{(\sigma_i - \sigma_{i+1})} \wedge \prod_{i=1}^{n_2} \frac{d\sigma_i}{(\sigma_i - \sigma_{i+1})}}{\mathrm{vol}\,(GL(2,\mathbb{C}))^2} \wedge \Phi$$



- Residue from ω_{conn}

ε deformation parameter

$\varepsilon = 0$:



$$\boxed{xy = \varepsilon}$$

- $\omega_{conn} = \dfrac{\mu_d \wedge \omega(\sigma_1, \dots, \sigma_n)}{vol(GL(2,\mathbb{C}))} \wedge \Phi$

- $\mu_d \sim \dfrac{d\varepsilon}{\varepsilon^3}$

- $\omega(\sigma_1, \dots, \sigma_n) \sim \varepsilon^2$

$$\Rightarrow \quad \boxed{\omega_{conn} \sim \frac{d\varepsilon}{\varepsilon}}$$

- simple pole if the distribution of marked points into two groups preserves the cyclic order

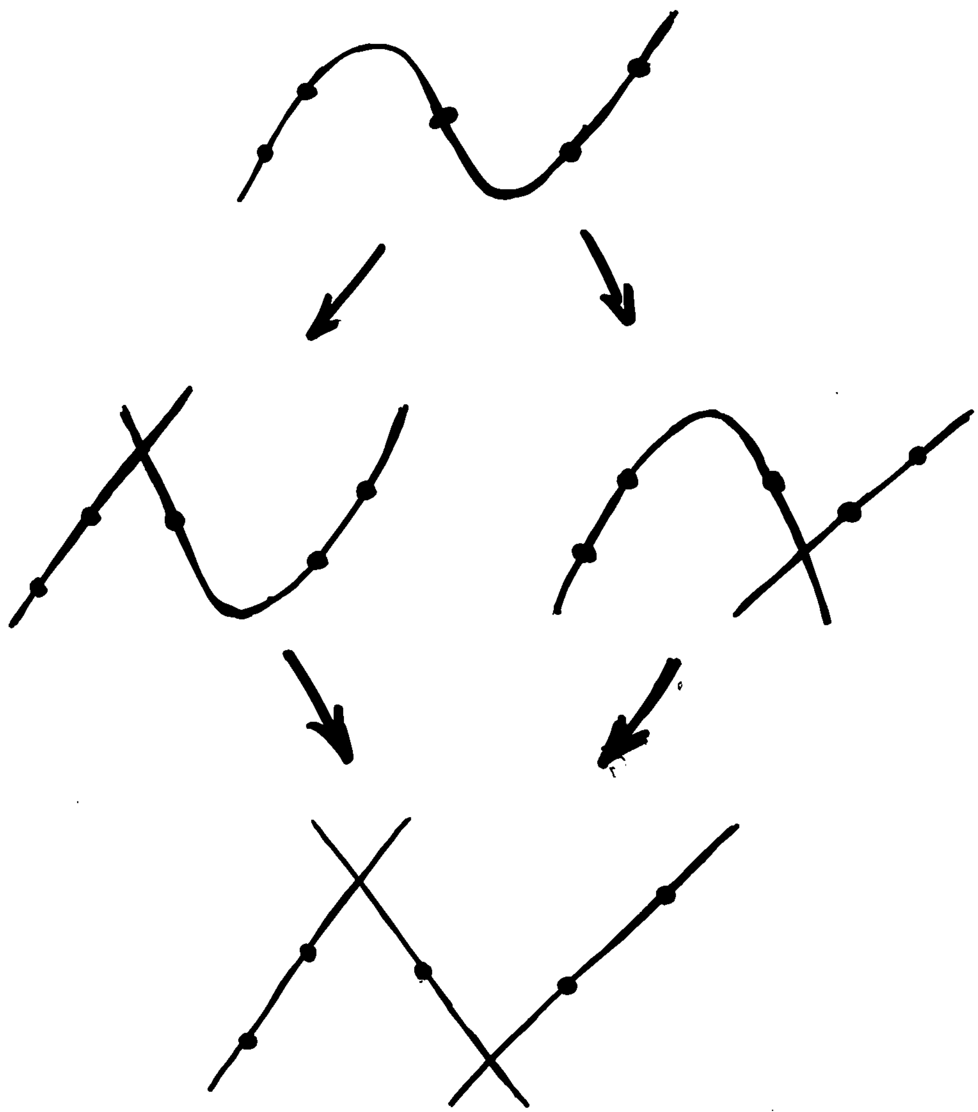
- same residue as in the disconnected case

Higher Degree Curves and Intermediate Prescriptions

$\overline{\mathcal{M}}_{0,n,d}$

$\mathcal{M}^{\Lambda_1}_{int} \cup \mathcal{M}^{\Lambda_2}_{int}$

$\mathcal{M}^{\Lambda_1}_{int} \cap \mathcal{M}^{\Lambda_2}_{int}$



Open Questions

- Choice of contour / integral
- External wavefunctions
- Loops and higher genus
- Choice of prescriptions
- Recursion relations

⋮

- Physical Interpretation ?
- Dual String Theory ?

Many Faces of Topological M-theory

- Geometry of $\mathcal{N}=1$ String Vacua
- Form Theories of Gravity in Diverse Dimensions
- Black Hole Entropy & Attractors
- Topological String Theories and Dualities Between Them

Special Holonomy Manifolds

X n-dimn'l Riemannian manifold

$$Hol(g_X) \subseteq SO(n)$$

Metric	Holonomy	n	SUSY $\nabla\xi = 0$	Invariant p-forms
Calabi-Yau	$SU(3)$	6	$1/4$	$p=2$: K $p=3$: Ω
Exceptional	G_2	7	$1/8$	$p=3$: Φ $p=4$: $G = *\Phi$
	$Spin(7)$	8	$1/16$	$p=4$: Ψ

Invariant Forms

ξ covariantly constant spinor $(\nabla \xi = 0)$

$$\omega^{(p)} = \xi^{+} \Gamma_{i_1 \ldots i_p} \xi$$

$\omega^{(p)}$ covariantly constant
invariant under $Hol(g_X)$

- S minimal (supersymmetric) cycle
  $\dim(S) = p$

$$\Rightarrow \quad Vol(S) = \int_S \omega^{(p)}$$

Example:

$$\mathrm{Hol}(g_X) = G_2 \iff \begin{cases} d\Phi = 0 \\ d*\Phi = 0 \end{cases}$$

$$\Rightarrow \quad ds^2 = \sum_{i=1}^{7} e^i \otimes e^i$$

$$\Rightarrow \Phi = \frac{1}{3!} \psi_{ijk}\, e^i \wedge e^j \wedge e^k$$

$$\sigma_i \sigma_j = -\delta_{ij} + \psi_{ijk} \sigma_k \,, \qquad \sigma_i \in \mathrm{Im}(\mathbb{O})$$

$$\Leftarrow \quad B_{jk} = -\frac{1}{144} \Phi_{ji_1i_2} \Phi_{ki_3i_4} \Phi_{i_5i_6i_7} \varepsilon^{i_1 \dots i_7}$$

$$g_{ij} = \det(B)^{-1/9} B_{ij}$$

$$\rightsquigarrow {}^*\Phi \,, \text{ etc.}$$

<u>Form Theories of Gravity</u>

<u>in D dimensions</u>

<u>$D=2$:</u> 2D Topological Gravity

<u>$D=3$:</u> Chern-Simons Gravity

<u>$D=4$:</u> 2-form Gravity
(self-dual sector of loop quantum gravity)

<u>$D=6$:</u>
$$\begin{cases} \text{Kähler Gravity (A-model)} & dK=0 \\ \text{Kodaira-Spencer Gravity (B-model)} & d\Omega=0 \end{cases}$$

<u>Hitchin:</u>
$$\begin{cases} \text{4-form theory} & dK=0 \\ \text{3-form theory} & d\Omega=0 \end{cases}$$

4D 2-form Gravity

M 4-manifold

A $SU(2)$ gauge connection

$\Sigma \in \Omega^2(M) \otimes adj(SU(2))$ $\qquad k = 1, 2, 3$

$$\boxed{S_{4D} = \int_M \Sigma^k \wedge F_k - \frac{\Lambda}{24} \Sigma^k \wedge \Sigma_k}$$

$$\begin{cases} D_A \Sigma = 0 \\ F_k = \frac{\Lambda}{12} \Sigma_k \end{cases}$$

$$\Rightarrow \sqrt{g}\, g_{ab} = -\frac{1}{12} \Sigma^i_{aa_1} \Sigma^j_{ba_2} \Sigma^k_{a_3a_4} \varepsilon^{ijk} \varepsilon^{a_1 \dots a_4}$$

self-dual Einstein:

$$R_{ab} = \Lambda g_{ab}, \qquad W^{(+)}_{abcd} = 0$$

3-form theory in $D=7$

X 7-dimn'l manifold N. Hitchin

$\Phi \in \Omega^3(X)$

- action:

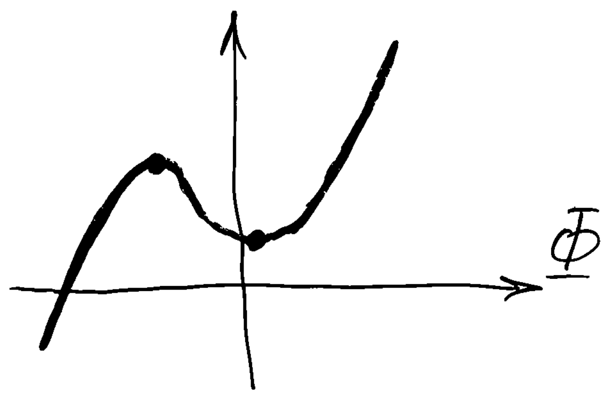
$$S(\Phi) = \frac{1}{2}\int_X \Phi \wedge *_\Phi \Phi = \int_X \sqrt{g_\Phi} =$$

$$= \int_X (\det B)^{1/9}$$

homog. of degree $7/3$ in Φ

- assume $d\Phi = 0$, fix $[\Phi] \in H^3(X)$

vary $\Phi = \Phi_0 + dB$

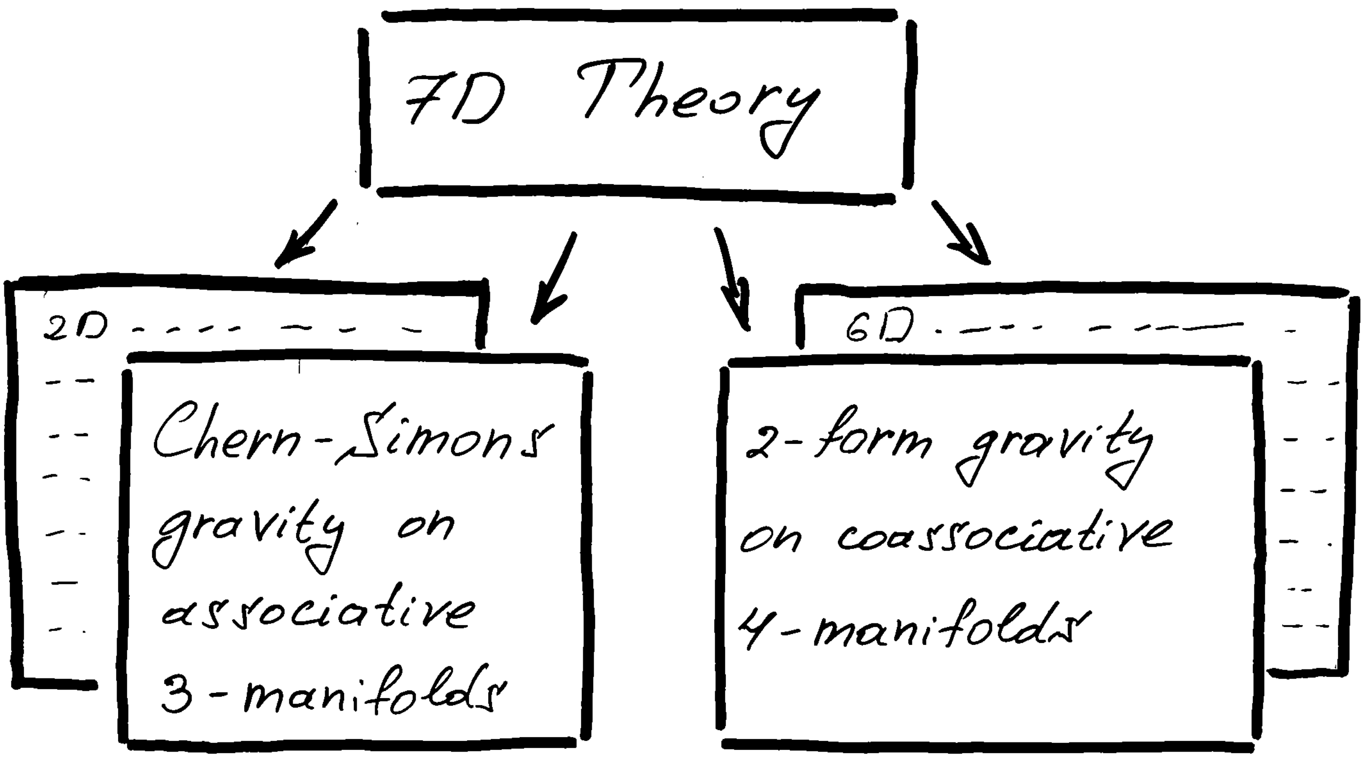


- crit. pts: $\begin{cases} d\Phi = 0 \\ d*\Phi = 0 \end{cases}$

$$\Rightarrow \sqrt{g}\, g_{jk} = -\frac{1}{144} \Phi_{j i_1 i_2} \Phi_{k i_3 i_4} \Phi_{i_5 i_6 i_7} \varepsilon^{i_1 \dots i_7}$$

metric with G_2 holonomy!

- 7D theory on X reduces to n-dimensional form theory of gravity on a calibrated cycle $M \subset X$, $\dim(M) = n$



"local model":

$$\begin{array}{ccc} \mathbb{R}^m & \to & X \\ & & \downarrow \\ & & M^n \end{array} \qquad m+n=7$$

- $SO(m)$ - invariant ansatz:

y^i local coordinate on $\mathbb{R}^m$

$$\alpha^i := D_A y^i = dy^i + (Ay)^i$$

Example ($m=3$, $n=4$):

$$\Phi = \alpha^{123} + \alpha^1 \wedge \Sigma^1 + \alpha^2 \wedge \Sigma^2 + \alpha^3 \wedge \Sigma^3$$

$$\begin{cases} d\Phi = 0 \\ d * \Phi = 0 \end{cases} \Rightarrow \begin{cases} D_A \Sigma = 0 \\ F = \frac{\Lambda}{12} \Sigma \end{cases}$$

$\sqrt{g}\, g_{ij}$ along M reduces to

$$\sqrt{g}\, g_{ab} = -\frac{1}{12} \Sigma^i_{aa_1} \Sigma^j_{ba_2} \Sigma^k_{a_3 a_4} \varepsilon^{ijk} \varepsilon^{a_1 \dots a_4}$$

- $$\begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & X \\ & & \downarrow \\ & & M^4 \end{array}$$

bundle of self-dual 2-forms on M^4, $X = \Lambda^2_+(M^4)$

∂X = bundle of self-dual 2-forms of unit norm

$= T(M^4)$

- In fact, g_X is asymptotic to a cone on $T(M)$

$T(M^4)$ has integrable cplx. structure $\longleftrightarrow$ M^4 self-dual

Penrose,

Atiyah-Hitchin-Singer

$T(M^4)$ Kähler $\longleftrightarrow$ M^4 Einstein

Hitchin

<u>Conj:</u> topological gauge theory on M^4 $\simeq$ topological gravity ("M-theory") on $X \setminus M^4$

$\cong$

line bundle over $T(M^4)$

$$\boxed{Z_{YM}(M) \overset{?}{=} Z_A^{top}(T(M^4))}$$