TWISTOR DIAGRAMS AND GAUGE-THEORETIC SCATTERING AMPLITUDES

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These pages reproduce the material shown as transparencies in the course of my presentation. I have added a number of further comments, which arose from answering the stimulating questions raised during the talk or put to me immediately afterwards. I have also added a few references.

§0. The twistor diagram programme: We have twistor representations for free z.r.m. fields, thought of as in-states (positive frequency) and out-states (negative frequency). There are multilinear functionals of these fields yielding (Feynman) scattering amplitudes in Minkowski space.

The aim of the twistor diagram programme has been to express these functionals *directly and entirely* in twistor space.

We would like to do this in way that expresses:—

▼ manifest gauge invariance

▼ manifest finiteness

▼ manifest decomposition into atomic elements like the propagators in Feynman diagrams, with the hope of finding a new dynamical principle in twistor space analogous to the Lagrangian in space-time.

All the basic elements in the theory were identified by Roger Penrose in about 1970, and scattering amplitudes for massless QED were studied as helicity amplitudes.

Perhaps no-one guessed how important helicity amplitudes would be...

Comment: in particular it would hardly have been expected in 1970 that Feynman diagrams for the strong interactions *would be needed for interpreting actual collision experiments, and that the helicity amplitude approach to simplifying the Feynman predictions for SU(3) scattering would be vital in practical computation.*

Comment: by manifest finiteness I mean finiteness of the S-matrix for all finite-normed states in the Hilbert space. This is a much more stringent criterion than the usual practice of computing functions of momenta, without worrying much about the singularities exhibited by these functions for special external momentum values.

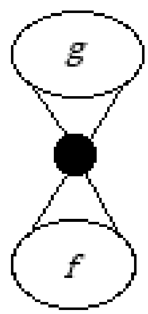
Momentum states are obviously very important for comparison with collision experiments, but for real consistency with the principles of quantum mechanics, the S-matrix should be completely well-defined for all states.

§1. Zeroth order: The simplicity of the inner product for z.r.m. fields (i.e. without any interaction) gave an elegant starting-point for this programme.

If $\psi(x), \phi(x)$ are fields of helicity $n/2$, with corresponding twistor representations $f_{-n-2}(Z^\alpha)$, $g_{n-2}(Z^\alpha)$, then the inner product $\langle\psi|\phi\rangle$ is given simply by a contour integral:

$$\langle\psi|\phi\rangle = \oint f_{-n-2}(Z^\alpha)\, g_{n-2}(Z^\alpha)\, DZ$$

This is gauge-invariant and manifestly finite. We make a start on an analogue with Feynman diagrams by thinking of the Z as a 'vertex' and writing the integral as:



Comment: throughout, out-states are written as negative-frequency states, rather than as the complex-conjugates of positive-frequency states. This is the best convention with which to express crossing symmetry. The symbol $\langle\psi|\phi\rangle$ implies linearity in both the ψ and the ϕ, rather than being antilinear in the ψ as in standard notation.

For the definition of these standard quantum field 'inner products', which go back to 1930s work of Fierz, and the correspondence to this twistor integral, see the original Penrose and MacCallum paper (See References). The simplicity of this basic formula deserves to be more widely known.

Comment: the black vertex implies a contour integration over the twistor variable Z, analogous to a Feynman vertex meaning integration over a momentum. I have written the form DZ deliberately vaguely. In the original projective diagrams this would mean the 3-form, but it could also be the 4-form (naturally with one extra dimension for the contour). For the non-projective diagrams to be introduced shortly, it must be the 4-form.

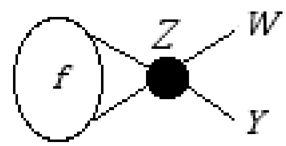
Comment: some readers may expect the word 'contour' to imply a 1-dimensional path. However, the contours we need (i.e. representatives of an appropriate homology class) will be p-real-dimensional spaces in a p-complex-dimensional space. They will, typically, have a topology involving higher spheres, and should not be thought of as always reducible to the usual residue calculus.

§2. Representing the momentum space δ-function:

A prerequisite for scattering amplitudes is that we have a twistor equivalent of

$$\int d^4x \ \text{(x-space)} \qquad \text{or} \qquad \delta\left(p_1 + p_2 + p_3 + \ldots + p_r\right) \ \text{(p-space)}$$

To obtain such an equivalent, consider first *scalar* fields (other helicities can then be considered by spin-raising operations), and note the *evaluation* diagram:

$\psi(x) = \qquad I^{\alpha\beta}W_\alpha Y_\beta$ 

where $I^{\alpha\beta}$ is the 'infinity twistor', and W, Y are any two dual-twistors spanning the twistor line corresponding to the point x. The lines in the twistor diagram joining the vertex Z to W and Y simply represent *simple poles* $(W_\alpha Z^\alpha)^{-1}$, $(Y_\alpha Z^\alpha)^{-1}$

Comment: these first steps use a great deal of twistor geometry which, though elementary, may not be familiar. They follow from the correspondence of lines in projective twistor space PT to points in complexified compactified Minkowski space M.

Thus the line spanned by WY corresponds to a varying space-time point x. The effect of the simple poles can be described as confining Z to WY.
The line defined by the infinity twistor $I^{\alpha\beta}$ *corresponds to the vertex I of the null cone at infinity in M.* $I_{\alpha\beta}X^\alpha Z^\beta = 0$ *is the condition for WY to represent an infinite point; the factor* $I^{\alpha\beta}W_\alpha Y_\beta$ *is the conformal factor which enters into the compactification of M.*

The representation of the f(Z) by two 'ears' suggests the two separated singularities needed to define it in its original naive 'function' form. More abstractly, it is implicit in this and all the contour integrals to be defined that the integration structure is to be consistent with the nature of f(Z) as a first-cohomology element in one half of twistor space. Some of this has been made rigorous for the simplest projective twistor diagrams by Huggett, Singer, Eastwood et al., but virtually nothing has been done for non-projective diagrams. In practice, we assume that a contour integral construction which 'sees' the two singularities correctly is one that could, in principle, be described correctly as a functional on the first-cohomology space.

Now note that integration over real (compactified) Minkowski space can be translated into a compact contour integral in the product of the W and Y projective spaces. Essentially, this is an integration over the *lines* in twistor space parametrised by WY.

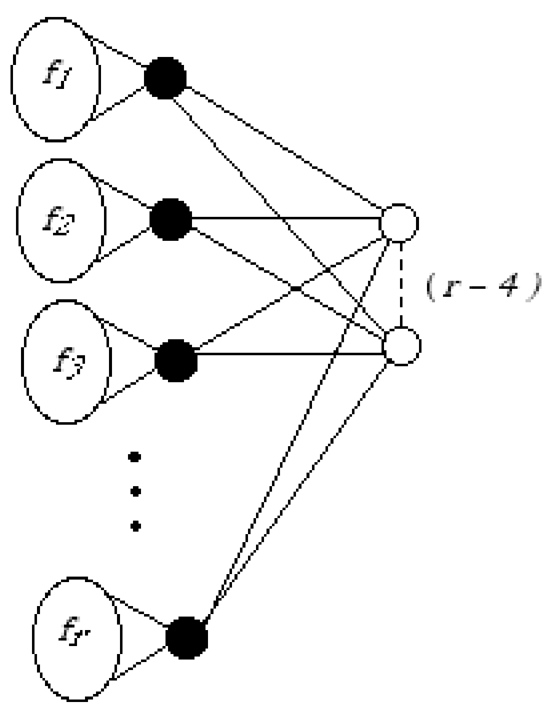
The form d^4x corresponds to the form $(I^{\alpha\beta}W_\alpha Y_\beta)^{-4}\, DW \wedge DY$

The contour has topology $S^3 \times S^1 \times S^1 \times S^1$.

Comment: crucially, this is compact, hence the integral is manifestly finite. My remarks are intended only to give a rough idea of how the x-space integral corresponds to the twistor integral. The central idea is that the simple poles have the effects of constraining the external twistors all to lie on the same line WY — readers may compare this with Witten's construction.

We use an *unfilled* vertex ○ to represent an integration over a dual-twistor variable.

Then the product of r scalar fields over Minkowski space can be given by the twistor integral:



Here the *dashed* line represents the *numerator* factor of $(I^{\alpha\beta}W_\alpha Y_\beta)^{r-4}$ which doesn't affect the singularity structure. In the case $r = 4$, which we now consider, this factor is absent, showing the conformal invariance of the functional.

Comment: One advantage of twistor representation is that full conformal invariance, and the breaking of conformal invariance, is made manifest. Thus an expression which uses $I^{\alpha\beta}$ as a fixed element can't be conformally invariant. If $r \neq 4$ one sees this conformal breaking directly. It will be scale invariant *if it is homogeneous of degree 0 in $I^{\alpha\beta}$. It will be Poincaré-invariant if it involves $I^{\alpha\beta}$ but no other fixed twistors.*

Historically, Penrose's early evaluation of twistor diagrams began with the $r = 4$ case, thought of as representing first-order ϕ^4 scattering.

Crossing symmetry emerges as the following principle: for different allocations of positive and negative frequency fields, there will be different contours for the same integrand.

Comment: An interesting fact, which may not be generally known, is that the three amplitudes in the three channels, when analytically continued in their parameters to a common region, sum to zero. This must reflect a linear dependence of the relevant contours. This should not be confused with the other relationship involving the sum of three expressions which will be noted later in connection with gauge fields.

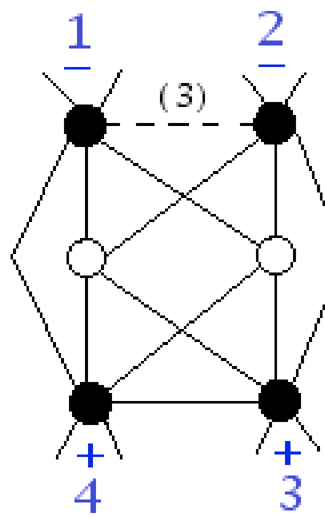
§3. First order: the interaction of four gauge fields

We will go immediately to pure gauge-field theory, using the Parke-Taylor formula.

It is immediate to write down a (formal) twistor-integral expression for the four-gauge-field scattering amplitude. The Parke-Taylor formula

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}\quad\delta\left(p_1+p_2+p_3+p_4\right)$$

can be translated into



The new lines represent those new singular terms $(I_{\alpha\beta}Z_1^{\alpha}Z_2^{\beta})^{-1}$ etc.

The point of the above was to indicate that the transcription of the kernel of the Parke-Taylor formula is very simple and direct: the infinity twistor $I^{\alpha\beta}$ picks out and combines exactly those spinor parts of the twistors which are indicated by the angle-bracket notation.

However, one should also remember the four external functions, which are left as blank 'ears' in the twistor diagram. In Witten's approach these are twistor-momentum states in (2+2) space. There doesn't seem to be an analogue for this in (1+3) space.

Note that you can read off whether the external function is to be positive or negative helicity from the homogeneities. The symbols + and – are redundant.

This integral is gauge-invariant but does not otherwise meet the demands of the twistor diagram programme.

(A) there are many different elements in the integral and it doesn't resemble the simple Feynman diagram formalism.

(B) for finiteness there must be a *contour* avoiding all those new singularities

Comment: and just as importantly, (C) this integral does not show the conformal invariance which is of such importance, and which a twistor representation ought to be making manifest! All we can see is the scale-invariance.

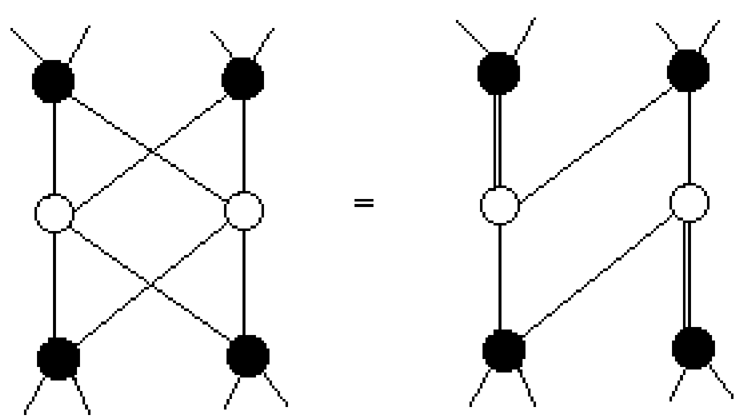
We address (A) by noting that there are *other twistor integral representations of the δ-function.*

As a general point: twistor geometry is not restricted to z.r.m. fields. Such fields arise particularly simply from 1-functions of *one twistor*. But more general fields (off-shell or massive, for instance), can be expressed by functions of *two or more twistors*. The algebra governing such representations has been studied by twistor theorists from the earliest days and it plays a role now.

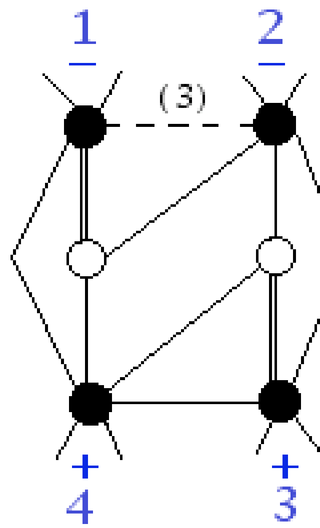
In particular, for four scalar fields, the completely symmetric formalism of evaluating all four fields can be replaced by an asymmetric formalism that only evaluates two of them. This is connected with the *spin eigenstates of 2-twistor functions*.

Comment: Roger Penrose's second talk also emphasised the n-twistor functions and gave a definition.

We need a new diagram element: the *double* line for a *double* pole like $(W_\alpha Z^\alpha)^{-2}$
Then we have



With this new version of the δ-function, the translation of the four-field amplitude is effected by:



But it is now necessary to allow the contour to have a *boundary* corresponding to the boundary of compactified Minkowski space where in- and out-states are defined, viz. a boundary on hypersurfaces of the form

$$I_{\alpha\beta} X^{\alpha} Z^{\beta} = 0, \quad I^{\alpha\beta} W_{\alpha} Y_{\beta} = 0$$

These boundaries will not be marked in what follows. They form part of the general problem (only partially solved) of determining contours for the resulting integrals.

The advantage of this asymmetric formalism is that it allows an *integration by parts* to reduce the number of elements in the integrand. The poles like $(I_{\alpha\beta} Z_1^{\alpha} Z_2^{\beta})^{-1}$ will be eliminated. We just need a natural extension to:

▼ triple lines for a triple pole like $(W_{\alpha} Z^{\alpha})^{-3}$

▼ quadruple lines for a quadruple pole like $(W_{\alpha} Z^{\alpha})^{-4}$

and most importantly,

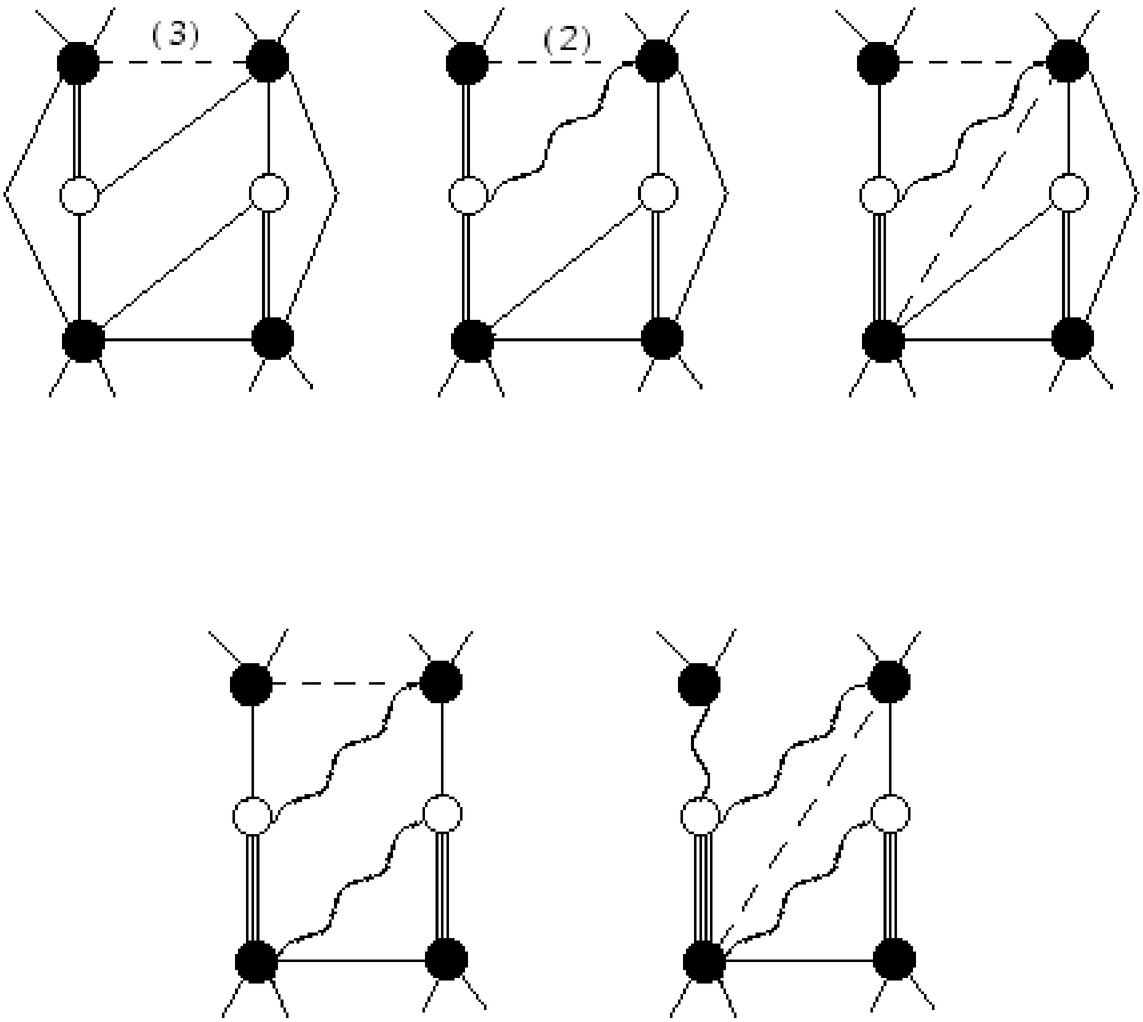
▼ a new element for the *antiderivative* of a single pole, i.e. some new element which satisfies the formal relation

$$\partial / \partial Z^{\alpha} \qquad = \quad W_{\alpha} \;\; (W_{\beta} Z^{\beta})^{-1}$$

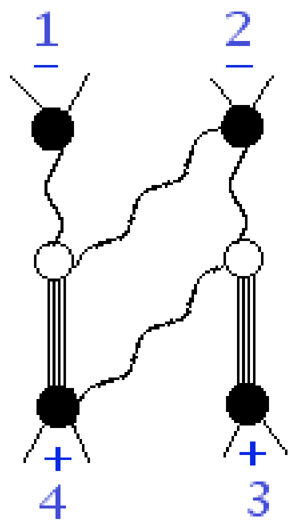
The natural solution is to choose to be the condition that the contour have a *boundary* on $W_{\alpha} Z^{\alpha} = 0$.

Comment: in the history of the subject several other apporaches were tried, e.g. involving logarithms. However, I now regard the purely geometrical idea of boundaries (i.e. relative homology) as being the most fundamental. Note that this definition implies that corresponding poles are always Cauchy poles, i.e. can be thought of restrictions to a submanifold.

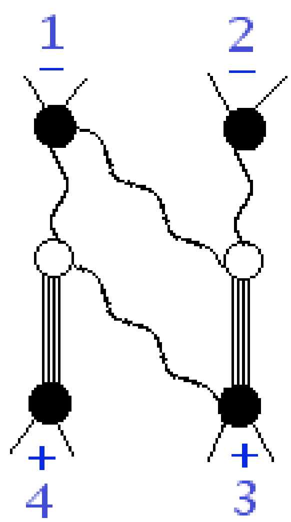
Use the diagram notation to integrate by parts: formally we obtain



ending with



We could equally well have arrived at



and hence these must be equal. This is a first example of the *non-uniqueness* of the diagrams, to which we will return in §4.

In both cases the ordering of the vertices round the central polygon is that of (1234), the ordering of the external gluons that went into the Parke-Taylor formula.

The unmarked boundaries on subspaces of the type $I_{\alpha\beta}X^{\alpha}Z^{\beta} = 0$, $I_{\alpha\beta}X^{\alpha}Z^{\beta} = 0$, are now the only elements in the formalism which break the conformal invariance.

Comment: This conformal breaking is very important. Roughly speaking, the twistor diagrams just dscribed can be regarded as conformally invariant versions of the Parke-Taylor formulas. But it is only the integrand *that is conformally invariant; the boundaries are not. As we shall see now, this is just a foretaste of the problem that forces itself on our attention through the demand for complete finiteness.*

Now (B): is there a genuine contour for manifestly finite amplitudes?

No.

This is not because of any defect in the twistor-geometric approach. It is because the amplitudes, as so far defined, are in general not finite. The integral we have written down does not, in general, exist. There is in general an *infra-red divergence* in all these processes arising where the exchanged virtual gluon has zero momentum.

The underlying problem is with the perturbation expansion: the first-order amplitude cannot really be distinguished from the zeroth order: i.e. no interaction. This is the well-known divergence in the forward direction for Coulomb-type interactions.

Three attitudes one could take:

▼ treat the resulting expressions formally, interpreting them for momentum states and not for finite-normed states.

▼ consider only those exceptional cases where there is no infra-red divergence (this happens when the in-states and out-states are such that they cannot occur without an interaction)

Comment: I mention these first two options as they reflect how, in practice, much development has been done. But they are obviously unsatisfactory strategies for anything claiming to be a fundamental theory.

▼ modify the theory. We can define a regularised amplitude as follows. Take the virtual gluon to have mass m instead of zero: this removes the IR divergence. Subtract $\log(m^2/M^2)$ x {the zeroth order amplitude}, where M is some fundamental mass.

Now let $m \to 0$. The limit is finite. In momentum space it is exactly the same as the original amplitude, except for the forward direction. There it is now finite, but depends on M.

There is a twistor-geometric approach which yields exactly this regularised amplitude. We modify all the poles and boundaries as so far defined.

Instead of the singularities being $(W_\alpha Z^\alpha)$ to negative powers, and boundaries being on $W_\alpha Z^\alpha = 0$, we employ $(W_\alpha Z^\alpha - k)$ to those negative powers, and boundaries on $W_\alpha Z^\alpha = k$

There is a natural value $k = e^{-\gamma}$ (the Euler constant γ arises rather as it does in dimensional regularisation, as the derivative of the gamma function).

The length scale M comes in by modifying the boundaries so that:

$$I^{\alpha\beta}W_\alpha Y_\beta = 0, \quad I_{\alpha\beta}X^\alpha Z^\beta = 0 \quad \text{become} \quad I^{\alpha\beta}W_\alpha Y_\beta = M, \quad I_{\alpha\beta}X^\alpha Z^\beta = M$$

This modification leaves the δ-function formula and all previously finite amplitudes unaffected but now contours exist for the 'divergent' amplitudes too.

This is not the same as a cut-off. The k and M are not to be thought of as small.

Comment: It is very striking that in the redefined integrals, the twistors Z etc are actually now constrained NOT to meet lines WY etc — and yet the results of the integrals remain the same.

It uses geometry which isn't equivalent to Minkowski space — it uses an extra dimension.

Comment: This idea is the nearest thing I know to Roger Penrose's original idea of fuzzing out points, while keeping null-cone structure.

As a general point: twistor theory is actually based on *non-projective* twistor space and its deformations (emphasised by Roger Penrose, e.g. in connection with getting a *complete* description of gravity.) This gives a chance for putting in something new that could not be expressed in space-time.

Hope: this twistor-geometric modification will deal consistently with all divergences.

Hope: it is connected with the proper twistor description of mass, gravity, and an underlying non-perturbative theory of scattering.

Comment: it will be noted that twistor diagrams involve 'loops' even at this simplest level of Feynman tree diagrams. Evaluation of these integrals, for elementary states, actually arrives at the same logarithm and dilogarithm functions of complex points, as arise in the 'box function' of momenta in standard Feynman diagram theory. It remains to be seen whether some use can be made of this.

Comment: In this presentation I motivated this development by the problem of the elementary infra-red divergence. But there are actually other good reasons to adopt this change to 'inhomogenous' elements; e.g. the consistency requirement that simple poles should always be treated as Cauchy poles leads naturally to it.

Comment: Obviously we have now definitely broken conformal symmetry, though we have confined the breaking to the boundaries, leaving the integrand conformally invariant. It should be noted that it is not just the IR problem which demands that conformal symmetry should be broken. In the Baston & Bailey book (see references) I gave an explicit example of a finite 5-field amplitude which is scale-invariant but not conformally invariant.

Comment: As Roger Penrose raised, the m and k thus introduced might show up in some testable way at a higher level of the theory. Also, the choice of k as positive rather than negative involves a subtle time-asymmetry.

§4. Emergent form of twistor diagrams

The simple and double pole lines can be thought of as Cauchy poles which pull all the twistor functions on to the same space-time point for evaluation.

Boundary lines do this 'pulling' together but they also absorb the content of the Feynman propagators which are inverse differential operators. Normally these are left as inverse operators, i.e. as differential equations still requiring solution. The claim is that the contours with these boundaries actually specify the correct, complete finite, solutions for in- and out-states in the Hilbert space of free z.r.m.

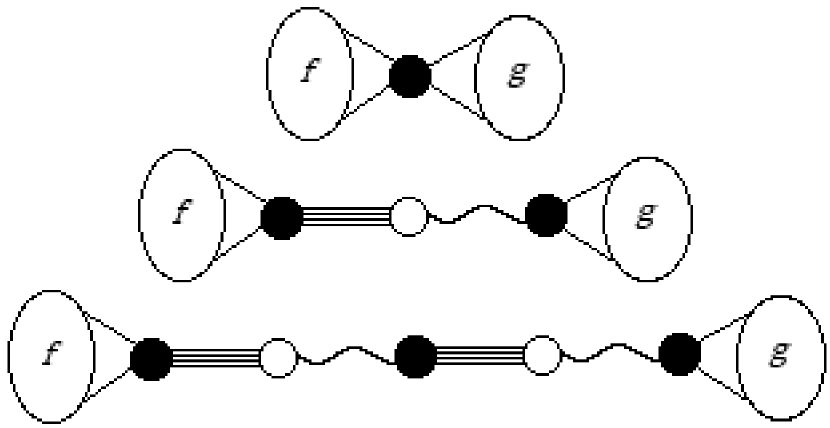
Comment: the properties of these boundary lines is closed related to the properties of the Feynman propagator. Questions of coefficents of logarithms, and discontinuities across cuts, which others use in evaluating momentum-space integrals, relate directly to the 'period contours' that arise from leaving out one boundary line.

If we studied other parts of the standard model we would find single, double and triple lines appearing in a natural way in diagrams which express the propagation of spin-0 and spin-1/2 fields

But in pure gauge-field theory we only get the boundary lines and the quadruple lines.

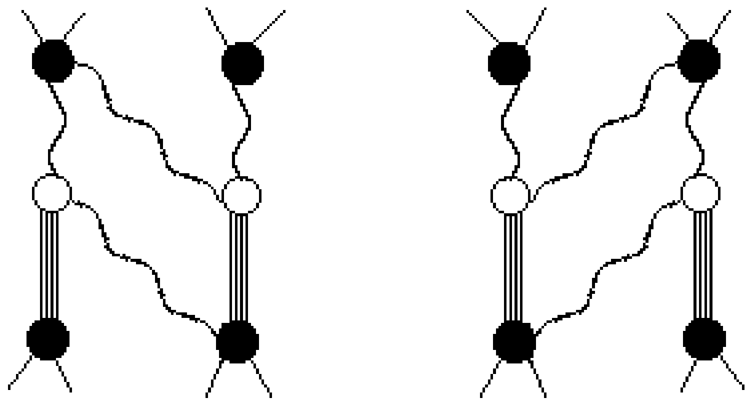
Come back to the question of *non-uniqueness:*

Even for *zeroth* order, the following integrals are equivalent:

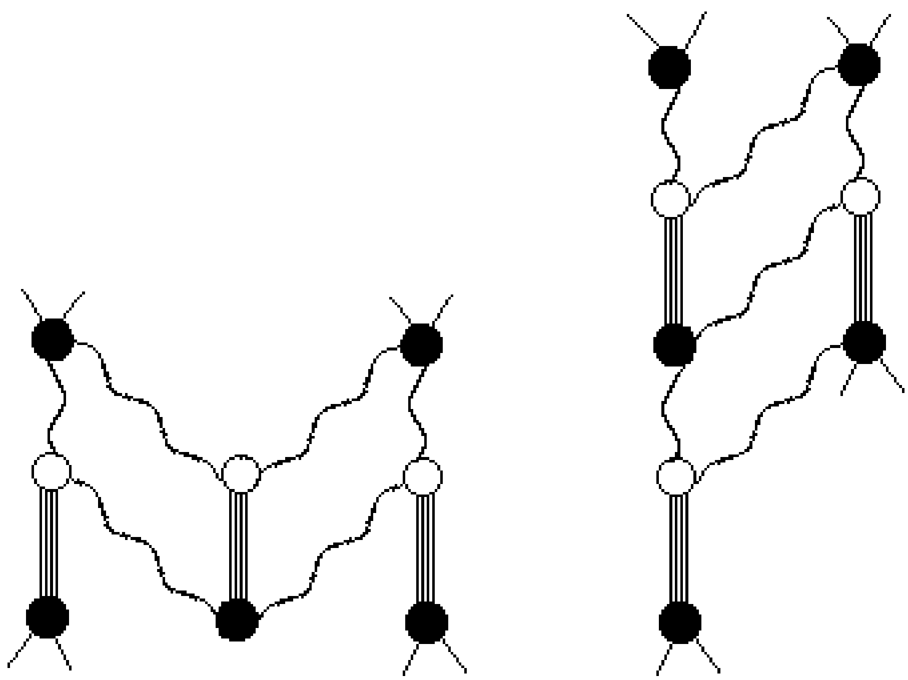


so these elements are not like Feynman diagram elements. These integrals should be considered as *representatives* of a more abstract twistor object which embodies this functional of the fields.

For first order we have already, for the (+ + – –) case, noted the equivalence of:

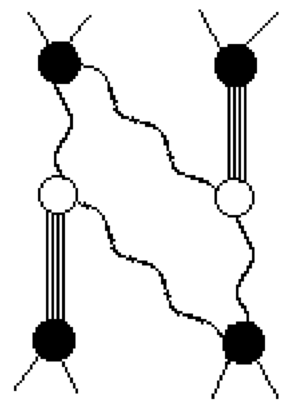


But both of these are also equivalent to two ‘double-box’ representations:

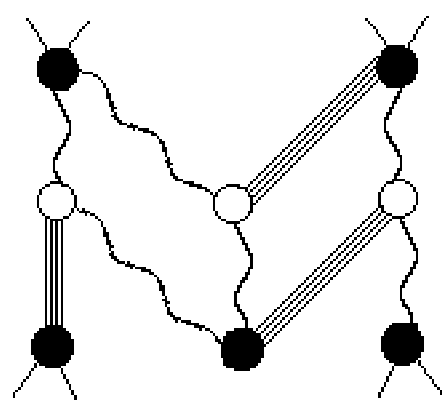


which (it appears) can be extended to chains of any length

For the remaining type (+ – + –) the simplest representation is:



but it can also be represented as:



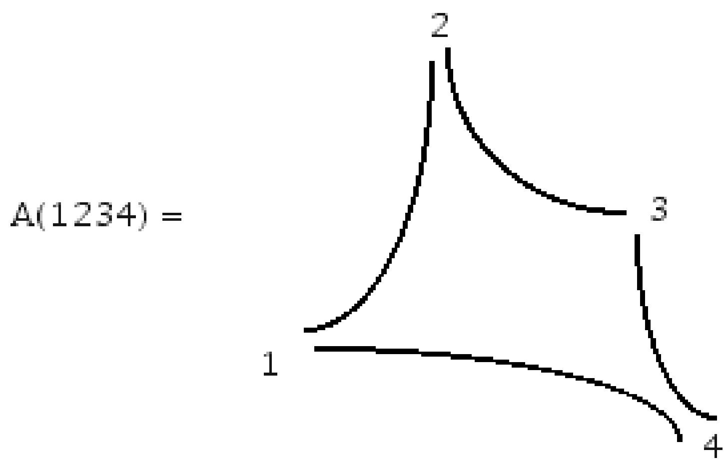
and again, it appears this can be extended in the obvious way as a chain of any length.

In every case the diagram can be considered as giving a *planar* polygon, and the gauge-theoretic trace always follows the edges of this polygon.

Note also that the difference between the (+ + – –) and the (+ – + –) integrals lies only in the boundary lines; the integrands are the same.

As the integrals are not unique they should be seen as *representatives* of some more abstract object. We can get some ideas of these more abstract objects by studying properties of these integrals and the amplitudes they represent.

If the four fields are labelled by (1, 2, 3, 4), write these ('colour-stripped') amplitudes as **A**(1234), **A**(1324), **A**(1423). Guided by the 'polygons' found above, and the related trace property, which is the same whatever integral representation we use, we might picture these objects as 2-surfaces with edges:



The identity $\quad \mathbf{A}(1234) + \mathbf{A}(1342) + \mathbf{A}(1423) = 0$

supports this notation.

Then the complete amplitude is given by

$$\mathbf{A} \;=\; \mathrm{Tr}(1234)\,\mathbf{A}(1234) \;+\; \mathrm{Tr}(1343)\,\mathbf{A}(1342) \;+\; \mathrm{Tr}(1423)\,\mathbf{A}(1423)$$

In the integral representations given, the integrand can be taken to be the same for all the **A**($i\,j\,k\,l$). The differences between them lie entirely in the boundaries chosen.

Thus the *complete* regularised amplitude **A** can be written

$$\int \frac{f_1(Z_1)f_2(Z_2)f_3(Z_3)f_4(Z_4)}{(Z_3.W_3-k)^4(Z_4.W_4-k)^4}DZ_1DZ_2DZ_3DZ_4DW_3DW_4$$

with the contour consisting of

$$\sum_{\Pi}\mathrm{tr}(\,i\,j\,k\,l\,)\ \ \mathrm{V}(\,i\,j\,k\,l\,)$$

where V($i\,j\,k\,l$) is a homology class (relative to the allowed boundary subspaces), with the linear dependence:

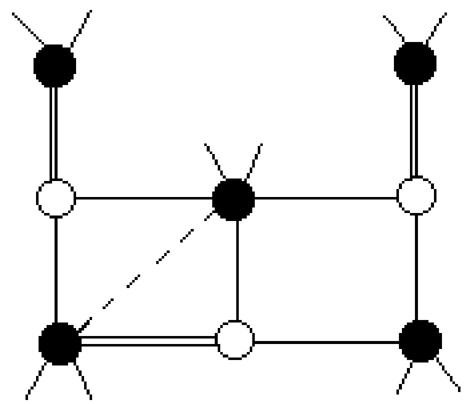
$$\mathrm{V}(1234)+\mathrm{V}(1342)+\mathrm{V}(1423)=0$$

Comment: more precisely, there is such a linear dependence for each channel; and then a linear dependence for the sum over channels (when analytically continued) for each cyclic ordering!

All these features persist for 5 and 6 fields.

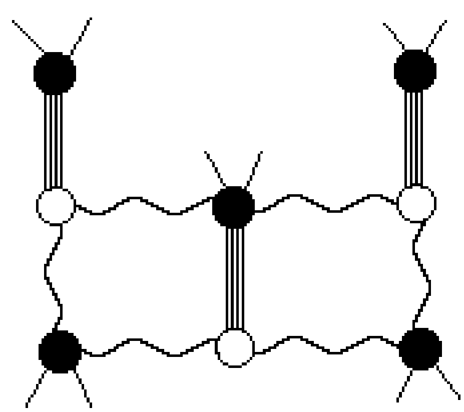
§5 Five- and six-field MHV amplitudes

Here again we can make no progress using the original 1-twistor transcription of the δ-function for five scalar fields. Instead, use this representation, which again arises from the algebra of 2-twistor functions:



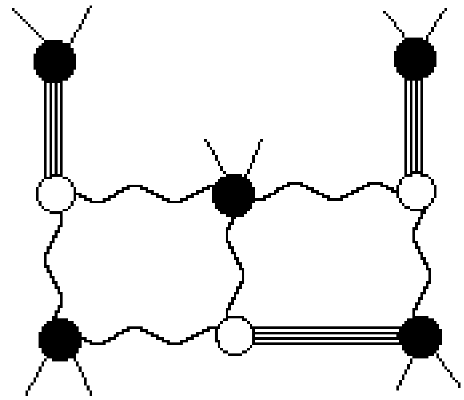
Integration by parts leads to representations of the two different kinds of 5-field MHV amplitudes.

For (+ + + – –) we obtain:.



and other alternative forms which have the same integrand but a different disposition of boundary lines and the same feature of an internal polygon in the right vertex order.

For the type (+ − + + −) there is only one simple representation:



A complete amplitude then takes the form over of a sum of 12 terms, six of one type and six of the other:

The ‘disgusting mess’ that Dr Bern discussed, boils down to:

$$\int \frac{f_1(Z_1)f_2(Z_2)f_3(Z_3)f_4(Z_4)f_5(Z_5)}{(Z_3.W_3-k)^4(Z_4.W_4-k)^4(Z_5.W_5-k)^4} DZ_1DZ_2DZ_3DZ_4DZ_5DW_3DW_4DW_5$$

with all the content going into the choice of homology classes V($i\ j\ k\ l\ m$)

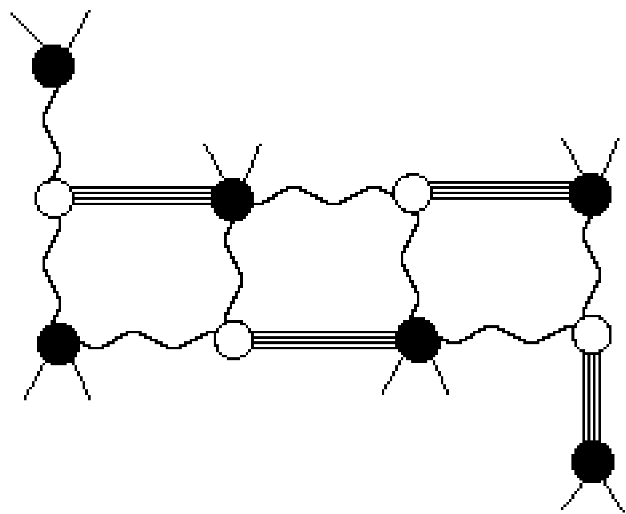
There is one new feature: a limit in which one of the external + fields (i.e. of homogeneity 0) is dropped. At least formally, the diagrams then become the appropriate 4-field diagrams.

Comment: Can these integrals be evaluated? Note that even for the simplest four-field diagrams, evaluated on elementary states, the results involve dilogarithms. Five and six-field amplitudes will require more complicated polylogarithmic expressions. Instead of making such ‘evaluation’ a focus, it looks more sensible to develop methods to (1) show the completeness and finiteness of the prescriptions and (2) show how the results agree with usual momentum-space expressions.

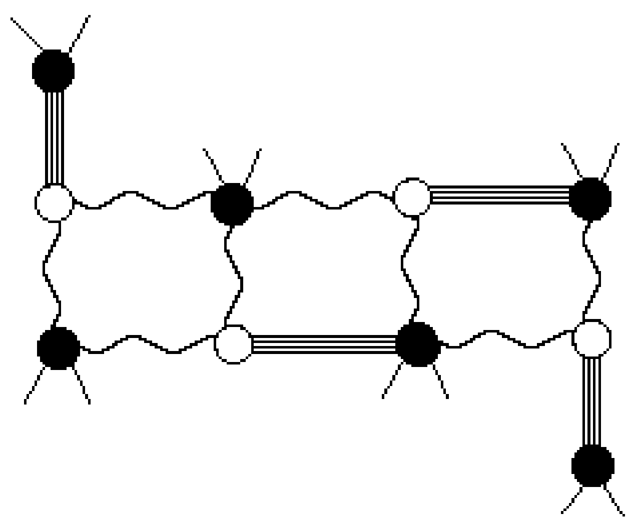
For *six* fields the situation is similar, with an amplitude made up of

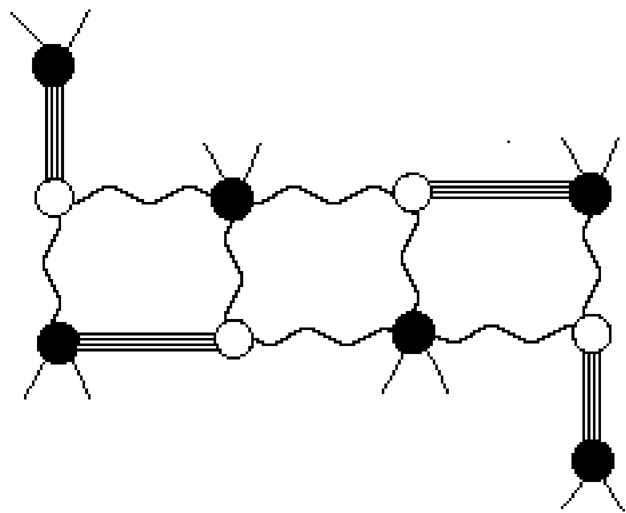
24 terms of type (+ + + + – –)
24 terms of type (+ + + – + –)
12 terms of type (– + + – + +)

The first type has three equally good representations, of which one is



and the other types have just one:





Again we have polygons formed in the correct order of the external vertices.

Again we have integrals which differ only through their boundaries.

The ‘dropped photon’ limit going from 4→2 to 3→2 makes sense.

The pattern for all MHV processes is clear: an extended ladder of these boxes making up a simple polygon.

It looks very like the BCF representation....

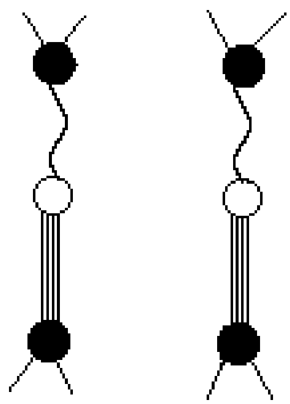
§6. Conjectured diagrams for six fields, non-MHV cases

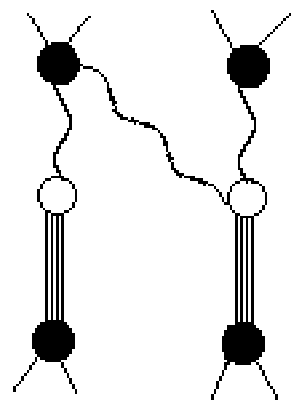
This section presents a conjecture for the (+ + + − − −), (+ + − + − −), and (+ − + − + −) amplitudes.

It is based on the idea of building up from sub-processes which are MHV.

A very early idea of Roger Penrose suggested the first-order diagrams before they were verified analytically.

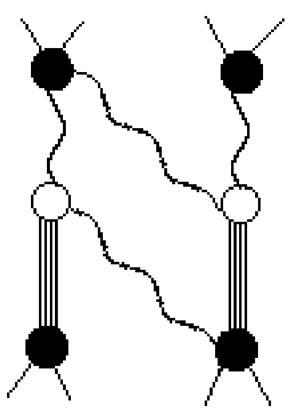
Given the zeroth order (non-interaction) amplitude, written as:

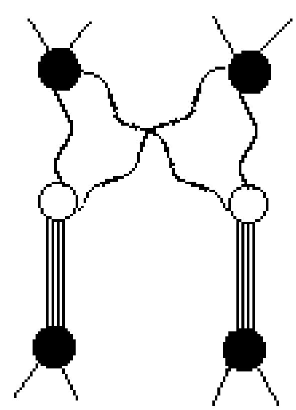




a first guess is that a spin-1 line joining them will create a first-order interaction.

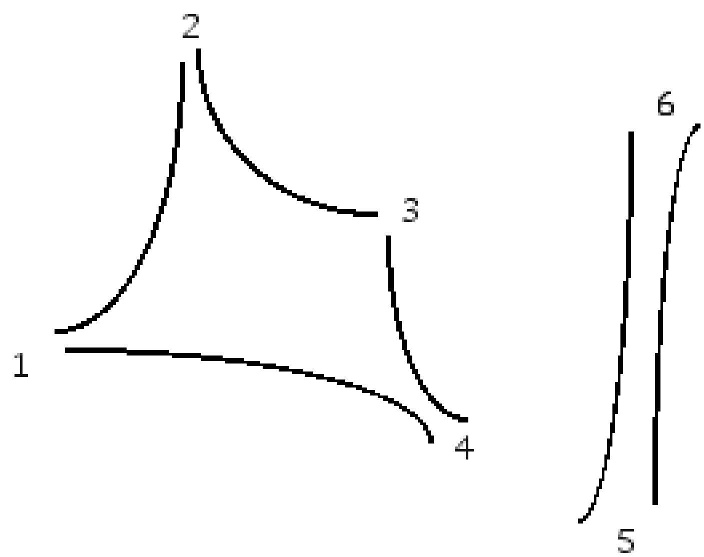
However, a *single* line corresponds to an 'on-shell' field, not to the 'off-shell' field. For this, it is necessary to connect with *two* lines, which can represent a function which is *not zero-mass*.

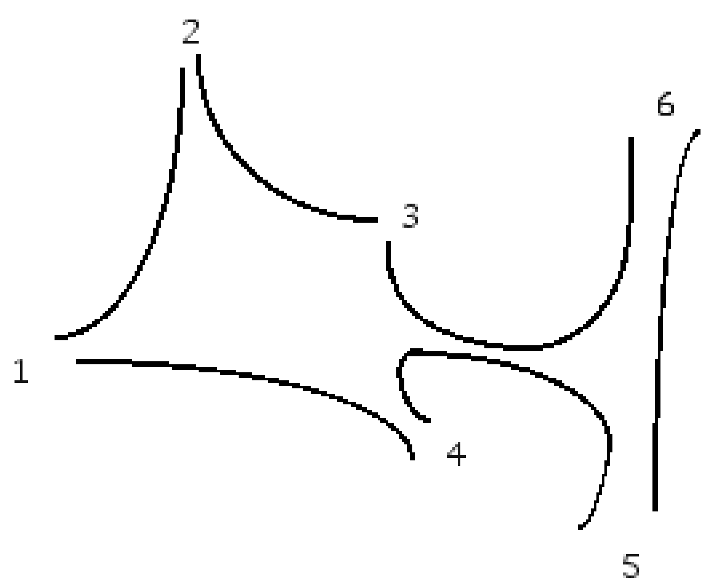




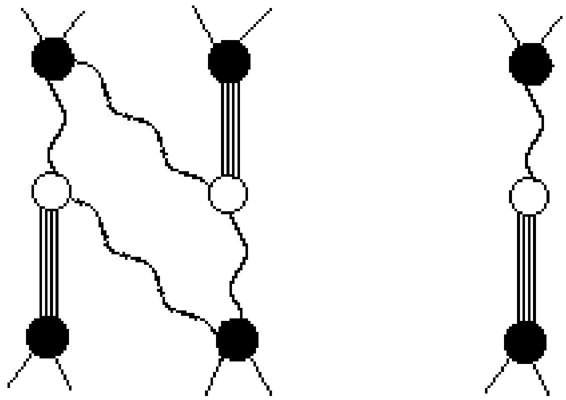
This argument doesn't account for details but is essentially right. The *pair* of boundary lines joining the two zeroth-order processes is a *two-twistor function* translating the Feynman propagator.

Now apply this idea to building 3 → 3 by 'joining' 2 → 2 to 1 → 1

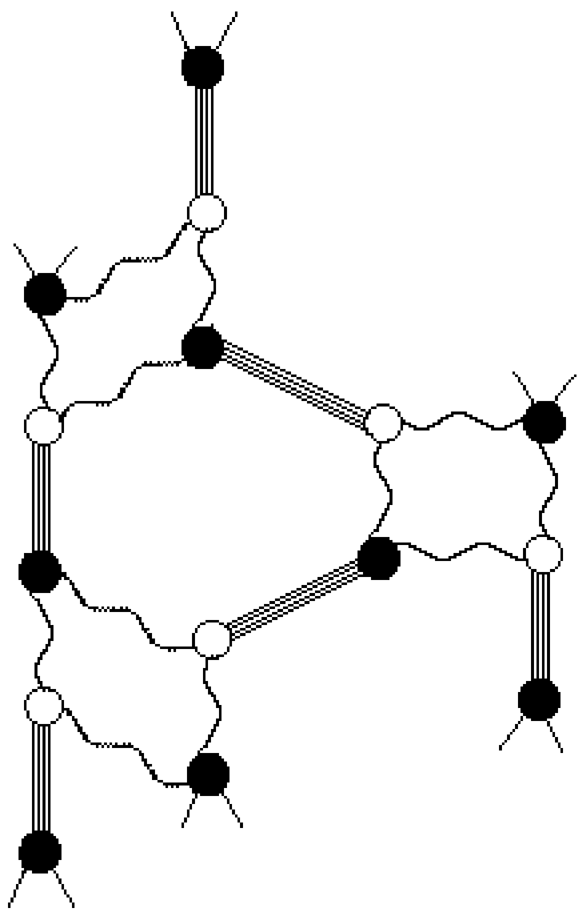




Consider the (+ – + – + –) case. We should somehow 'join' with a pair of boundary lines, the diagrams:



The following procedure achieves this in a way that has a three-fold symmetry.



As before, this is not unique. Similar arguments lead to similar conjectured diagrams for the other two types of $3 \to 3$ amplitudes, also non-unique.

These conjectured diagrams satisfy some strong constraints:

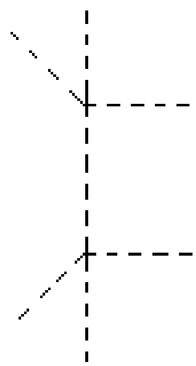
(1) duality: complex-conjugating to exchange + and – yields a consistent result

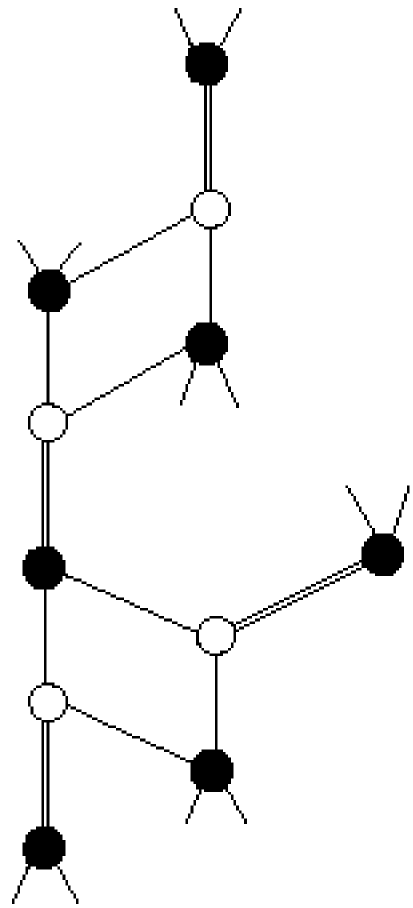
(2) infra-red divergent parts are consistent

(3) $2 \to 3$ diagrams obtained by the 'soft photon'limit are consistent.

There's another important piece of evidence in favour of diagrams with this structure.

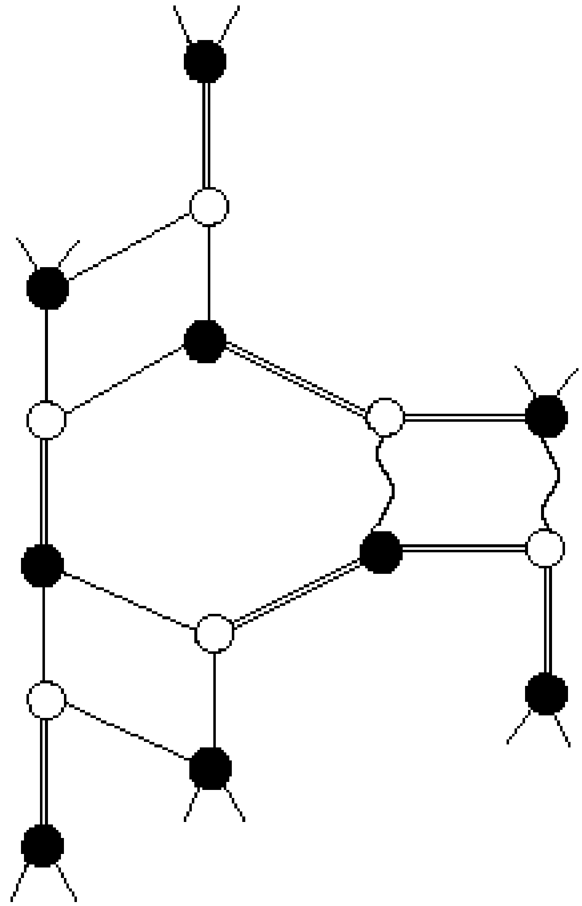
For *scalar* fields with the ϕ^4 interaction, an early conjecture was that there would be a correspondence between Feynman and twistor diagrams like this:





But the central double pole here would be represent an *on-shell* field and not the desired Feynman propagator.

A correct representation is *proved* (non-trivially!) to be given by



Why *two* extra lines? The *three* connecting lines correspond to a *three*-twistor representation of the scalar Feynman propagator, and this diagram seems to be intimately connected with the algebra of *three-twistor functions*.

Guess: the same is true for the (3 + 3) gluon processes when understood as a whole.

Guess: 2-twistor functions for MHV
3-twistor functions for 'next to MHV'
....

Comment: the general idea of 'MHV subprocesses' being treated as vertices, and joined together through the CSW argument, has obviously proved of great value. This must now be related properly to the proposed joining together of corresponding twistor diagrams. My guess, based on the remarkable simplicity of the CSW results, is that this should be quite straightforward.

Twistor diagram representations will not necessarily give something more computationally tractable. In this scalar case we already know the ‘answer’, viz.

$$\frac{1}{(p_1 + p_2 + p_3)^2}\delta(p_1 + p_2 + p_3 + p_4 + p_5 + p_6)$$

However, this approach

▼ should connect with ideas of MHV vertices, but bringing in *n*-twistor algebra

▼ ... so might give a new way of looking at loop diagrams

▼ gives a new possibility for regularising divergences geometrically

▼ shows conformal invariant elements and makes conformal breaking explicit

▼ help to connect pure gauge-field theory with all massless field theories *(including gravity, as suggested by Dr Nair’s results)*

▼ ... help to suggest a fully twistor-geometric theory of fundamental physics.

Comment: I should like to express my gratitude to the organisers and the other speakers at this remarkable conference. The results and ideas communicated have already stimulated new developments of the theory presented here.

References:

R. Penrose and M. A. H. MacCallum: Twistor theory: an approach to the quantisation of fields and space-time, *Phys. Reports* **6** 241-315 (1972)

This classic paper contained all the essential ideas of twistor diagram theory, although the contours used for the first-order ϕ^4 and massless QED amplitudes were not quite correct. As a result, the discussion missed the IR divergence problem which, when identified, stymied progress until 1983. The identification of contours for different channels was also unresolved in this paper, and this question remained a major difficulty for the theory.

A. P. Hodges: A twistor approach to the regularisation of divergences. *Proc. Roy. Soc. Lond.* **A 397** 341-374 (1985)

first introduced inhomogeneous twistor diagram elements to deal with this divergence (and also to describe massive fields).

S. A. Huggett and K. P. Tod: *An introduction to twistor theory,* London Mathematical Society Student Texts, 4 (1985)

is a fine exposition of the geometry of twistor space and zero-mass free fields. It does not extend to any discussion of twistor diagrams.

A. P. Hodges: Twistor diagrams, in T. N. Bailey and R. J. Baston (eds.), *Twistors in Mathematics and Physics,* London Mathematical Society Lecture Note Series 156, CUP (1990)

this review included twistor diagrams for the four-pure-gauge-field interaction amplitude.

A. P. Hodges: The Twistor Diagram Programme, in *The Geometric Universe*, OUP (1998)

first discussed five-field amplitude diagrams amongst other new lines of enquiry.