Essentials of 4-dimensional Riemannian Geometry:

(M^4, g) $\qquad g \quad ++++$

$\underline{\text{oriented}}$ $\qquad$ Riemannian

$$\star : \Lambda^2 \to \Lambda^2$$

$$\varphi_{ab} \longmapsto \frac{1}{2}\varepsilon_{ab}{}^{cd}\varphi_{cd}$$

$$\star^2 = +1$$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

$$\Lambda^+ = \{\text{self-dual 2-forms}\}$$
$$= \{\varphi_{ab} \mid \star\varphi = \varphi\}$$

$$\Lambda^- = \{\text{anti-self-dual 2-forms}\}$$
$$\star\varphi = -\varphi$$

Decomposition of the Riemann Curvature tensor $R^a{}_{bcd}$

$$R : \Lambda^2 \to \Lambda^2$$

$$\varphi_{ab} \longmapsto \tfrac{1}{2} R^{cd}{}_{ab}\, \varphi_{cd}$$

"curvature operator"

$$\begin{array}{cc} & \begin{array}{ccc} \Lambda^+ & \oplus & \Lambda^- \end{array} \\ R = & \left[\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array}\right] \begin{array}{c} \Lambda^+ \\ \oplus \\ \Lambda^- \end{array} \end{array}$$

$s = R^{ab}{}_{ab}$ = scalar curvature

$\mathring{r}_{ab} = R^c{}_{acb} - \frac{s}{4} g_{ab}$

= trace-free Ricci

W_+ = self-dual Weyl curvature

Similarly

W_- = anti-self-dual Weyl curvature

Important facts:

(1) W_+ and W_- are

conformally invariant

that is, rescaling the metric by function

$$g \rightsquigarrow u(x)\, g = \tilde{g}$$

leaves $W_\pm$ unchanged:

$$(\tilde{W}_+)^a{}_{bcd} = (W_+)^a{}_{bcd}$$

(2) Orientation of M crucial.

Changing orientation interchanges W_+ and W_-

Definition. An oriented Riemannian 4-manifold (M^4, g) is said to be anti-self-dual (ASD) if

$$\boxed{W_+ \equiv 0}$$

Actually clearer to apply this concept to

$$(M, [g])$$

where

$$[g] = \{ u g \}$$

is conformal class of metrics

Other settings:

① Split Signature

$(M^4, [g])$

oriented conformal pseudo-Riemannian of metric signature $++--$

In this case, we again have

$$\star^2 = +1$$

on Λ^2, so $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$

and

$$\mathcal{R} \sim s \oplus \mathring{r} \oplus W_+ \oplus W_-$$

as before. Define ASD again to mean $W_+ \equiv 0$.

If $(M^4, [g])$ is split signature,
a 2-dim'l sub-space

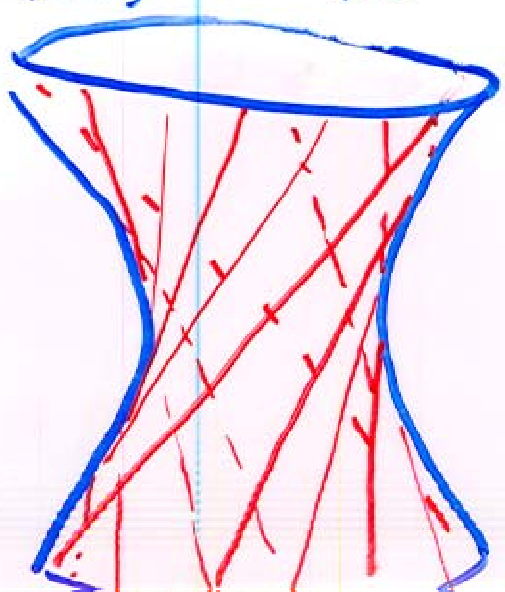
$$P \subset T_x M$$

is said to be isotropic if
it is totally null:

$$g(v, w) = 0 \quad \forall v, w \in P.$$

Two flavors: $\begin{cases} \alpha\text{-planes} \\ \beta\text{-planes} \end{cases}$

In $\mathbb{P}(T_x M) \cong \mathbb{RP}^3$ picture is:



Lines on quadric:
C. Wren.

P is an α-plane

$$\Longleftrightarrow \quad \Lambda^2 P \subset \Lambda^+$$

P is a β-plane

$$\Longleftrightarrow \quad \Lambda^2 P \subset \Lambda^-$$

<u>Definition</u> A surface S in an oriented split signature 4-manifold is an α-surface if its tangent space is everywhere an α-plane.

(2) Complex (i.e. holomorphic) metrics

$\mathcal{M}$ complex 4-manifold

g holomorphic tensor field

where locally

$$g = \sum_{j,k=1}^{4} \underbrace{g_{jk}(z^1, \ldots, z^4)}_{\text{holomorphic functions!}} dz^j \otimes dz^k$$

$$g_{jk} = g_{kj}, \quad \det[g_{jk}] \neq 0$$

Curvature of such an object in $\Lambda^{2,0} \otimes \Lambda^{2,0}$ and is holomorphic

$$\Lambda^{2,0} = \Lambda^+ \oplus \Lambda^-$$

(after passing to cover if necessary)

$$\mathcal{R} \underset{\text{as before}}{\sim} s \oplus \mathring{r} \oplus W_+ \oplus W_-$$

As before, the termilogy

anti-self-dual (ASD)

means $W_+ \equiv 0$.

Note decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

really comes from structure of Lie algebras:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$$
$$\mathfrak{so}(2,2) \cong \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$$
$$\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$$

By contrast,

$$\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$$

is simple — no Lorentzian analogue!

We can also talk about α-planes in a complex-Riemannian 4-manifold, but now they are 2-dim'l complex subspaces of $T^{1,0}$.

An α-surface would thus be a complex 2-manifold in this context.

Riemannian case: $P \subset \mathbb{C} \otimes T_x M$

$P = T^{1,0}$ for g-compatible almost-complex structure

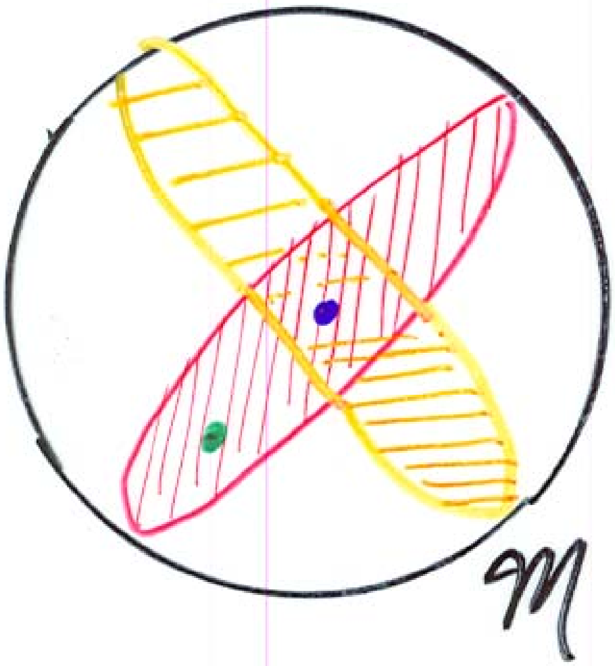
Penrose twistor correspondence:

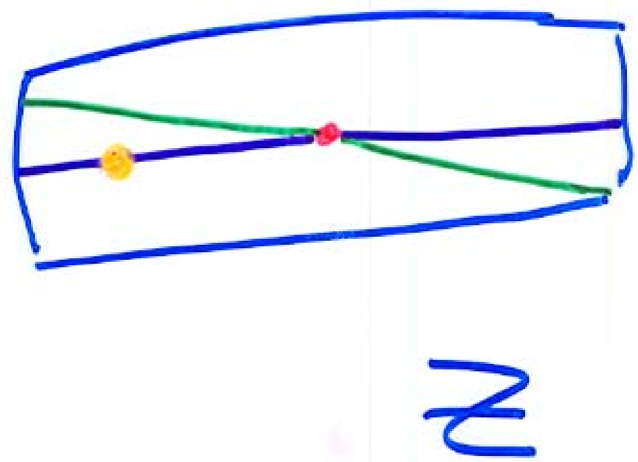
ASD 4-manifolds $\longleftrightarrow$ certain complex 3-folds "twistor spaces"

__Lemma__ (Penrose) A complex Riemannian 4-manifold $(\mathcal{M}, [g])$ is anti-self-dual iff every α-plane is tangent to an α-surface, meaning a complex 2-manifold $\mathcal{S}$ such that $T_p\mathcal{S}$ is an α-plane $\forall p \in \mathcal{S}$.

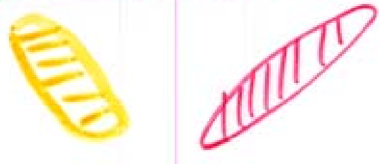
Proposition Let $(\mathcal{M}^4, [g])$ be an anti-self-dual complex conformal Riemannian 4-manifold. [By restriction to a neighborhood of an arbitrary point, assume that $\mathcal{M}$ is geodesically convex for some $g \in [g]$.] Then the space of α-surfaces in $\mathcal{M}$ is a complex 3-manifold Z, called the twistor space of $(\mathcal{M}, [g])$.

Penrose Twistor Correspondence

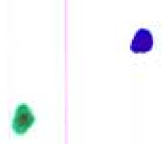


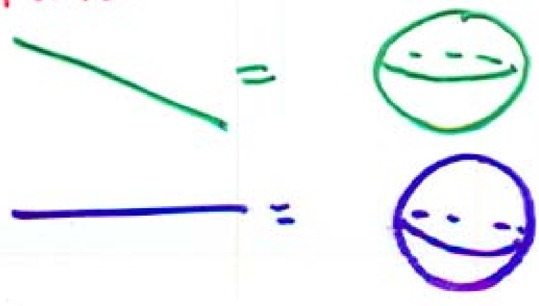


α-surfaces $\longleftrightarrow$ Points of Z



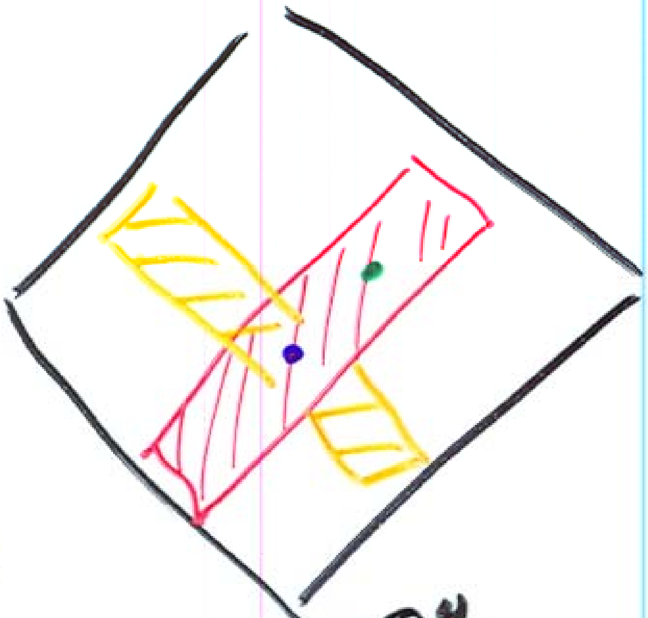
Points of $\mathcal{M}$ $\longleftrightarrow$ $\mathbb{CP}_1$ "Twistor lines"





Prototype

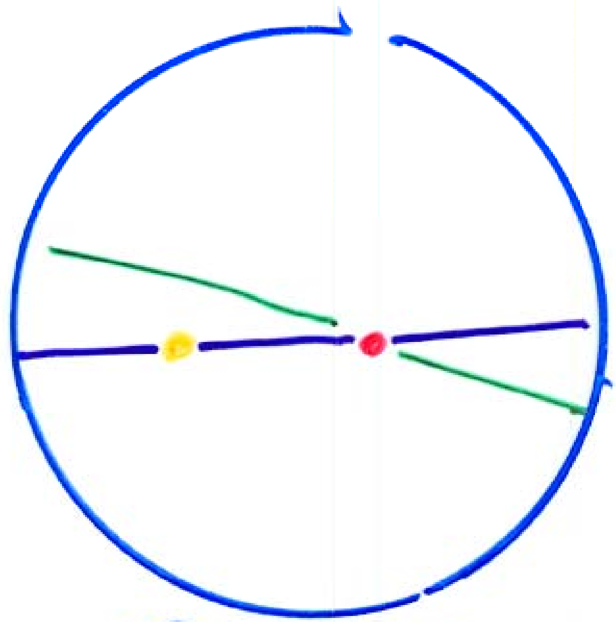
[Linear Non-Graviton !]



$\mathbb{C}^4$

$\cap$

Q_4



$\mathbb{CP}_3 - \mathbb{CP}_1$

$\cap$

$\mathbb{CP}_3$

Proposition. In the above situation, each twistor line C is an embedded $\mathbb{CP}_1 \subset Z$, and has normal bundle

$$N := \left[T^{1,0}Z\big|_C\right] \Big/ T^{1,0}C$$

isomorphic to $\boxed{\mathcal{O}(1) \oplus \mathcal{O}(1)}$

where $\mathcal{O}(1)$ is the degree 1 hol. line bundle on $\mathbb{CP}_1$

Theorem (Kodaira) Suppose $Y \subset X$ is compact complex submanifold of a complex manifold, with normal bundle

$$N = T^{1,0}X / T^{1,0}Y.$$

If
$$\boxed{H^1(Y, \mathcal{O}(N)) = 0}$$

then the set of cpt complex submanifolds $Y' \subset X$ near Y is a complex manifold $\mathcal{M}$ with tangent space

$$\boxed{H^0(Y, \mathcal{O}(N))}$$

at Y.

Theorem (Penrose et al.)

Let Z be a complex 3-fold, and let $\mathcal{M}$ be the 'moduli space' of all holomorphically embedded compact genus 0 curves in Z with normal bundle

$$N \cong \mathcal{O}(1) \oplus \mathcal{O}(1).$$

Then $\mathcal{M}$ is a complex 4-fold, and carries a canonical ASD complex metric. Locally, this is the general solution of ASD equations.

Warning: For most choices of Z, this $\mathcal{M}$ will be empty!!!

However, Kodaira's result immediately also tells us the following:

Theorem (Penrose)

If Z has $\mathcal{M} \neq \emptyset$, small deformations Z_t of Z also have $\mathcal{M}_t \neq \emptyset$.

Penrose was primarily interested in finding solutions of Einstein equation

$$\boxed{\mathring{r} = 0}$$

which were just incidentally also ASD.

<u>Theorem</u> (Penrose, Hitchin, Ward)

If $(M, [g])$ is ASD, with twistor space Z, then Einstein metrics $g \in [g]$ correspond to non-zero elements

$$\theta \in H^0(Z, \Omega^1(K^{-1/3}))$$

By analytic continuation, these ideas give a rough outline of how to understand real analytic $++++$ or $++--$ ASD manifolds locally.

But for global results with less regularity perhaps assumed, new ideas have proved necessary.

Riemannian Case: $++++$

Atiyah-Hitchin-Singer constructed the twistor space Z of an ASD Riemannian $(M^4, [g])$ as an S^2-bundle over M, equipped with an integrable almost-complex structure [cf. Newlander-Nirenberg]

$$\begin{array}{ccc} S^2 & \to & Z = S(\Lambda^+) \\ & & \downarrow \\ & & M \end{array}$$

Main Problems:

① Which smooth compact 4-manifold M admit ASD Riemannian metrics.

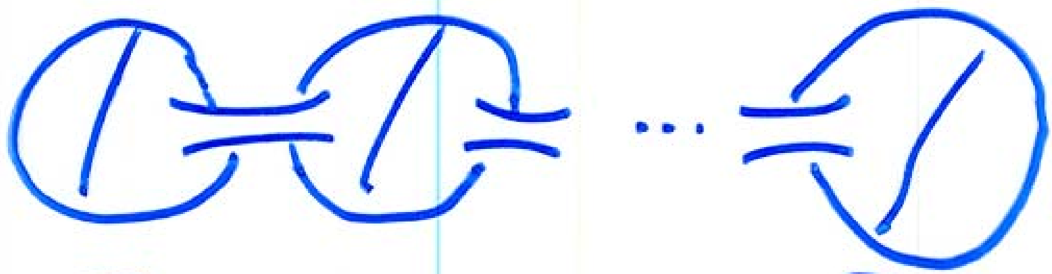
② Understand the geometry & moduli of these metrics.

③ Understand their twistor spaces.

Major progress on this set of problems was made during the period 1985–1995.

[Poon – Donaldson – Friedman – Floer – L – Joyce – M. Singer – Campana – Taubes – Kim – Pontecorvo –]

Theorem (LeBrun) There are ASD metrics with $s > 0$ on any connected sum $k\overline{\mathbb{CP}}_2$



$$\overline{\mathbb{CP}}_2 \# \overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2$$

of reverse oriented complex projective planes. But, aside from S^4, no other simply connected possibilities with $s>0$.

In fact, corresponding twistor spaces can be constructed algebraically, but are never projective or Kähler if $k>1$.

Theorem (Taubes) Let X be any compact oriented smooth 4-manifold. then there exists some k_0 such that

$$M_k = X \# k\overline{\mathbb{CP}_2}$$

admits ASD metrics for any $k \geq k_0$.

Corresponding twistor spaces are the weirdest compact complex manifolds currently known to us!

Split Signature Case

[Joint with L. Mason]

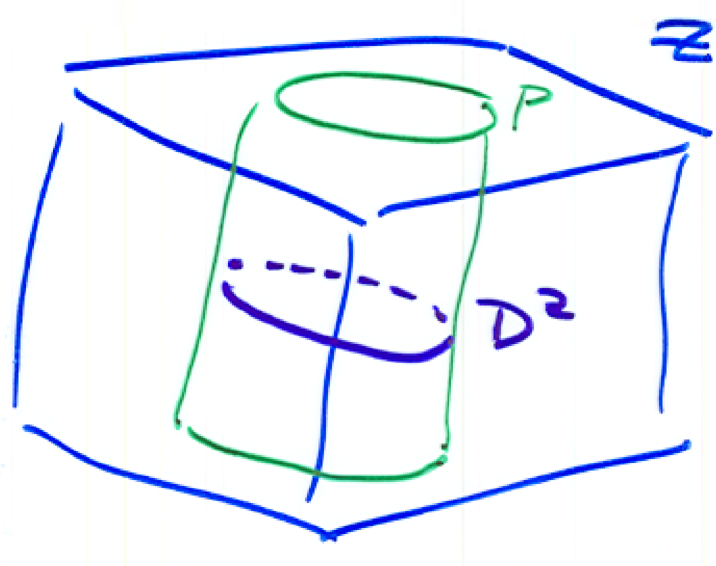
$$P \subset Z$$

smooth 3-manifold — complex 3-manifold

Totally real: $TP \cap J(TP) = 0.$

$$\begin{array}{ccc} D^2 & \subset & Z \\ \cup & & \cup \\ \partial D^2 & \subset & P \end{array}$$

holomorphic disk



Doubled Normal Bundle

$$\begin{array}{ccc} \hat{N} & \to & \mathbb{CP}_1 \\ & & \parallel \\ N \cup_{N_P} \bar{N} & \to & D \cup_{S^1} \bar{D} \end{array}$$

$\hat{N}$ rank 2 hol. vector bundle

Theorem (L–M) Suppose that Z complex 3-fold and $P \subset Z$ smooth totally real 3-dim'l submanifold. Let M be the set of holomorphic disks $D^2 \subset Z$ with $\partial D^2 \subset P$ and with $\hat{N} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$

Then M is a smooth 4-manifold, and carries a unique ASD conformal metric $[g]$ for which the α-surfaces are all disks through some point of P.

Theorem (L–M) The moduli space of split signature ASD conformal structures on

$$M = S^2 \times S^2$$

is infinite dimensional. Near standard structure, null geodesics are always automatically periodic, and the moduli space of ASD metrics corresponds to deformations of standard $\mathbb{RP}^3 \subset \mathbb{CP}_3$