Ambitwistors

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"Ambitwistor"

= "Ambidextrous Twistor"

Coined by M. Eastwood ~ '79
[in flat context]

An ambitwistor is a

Complex Light Ray

or, more precisely, a

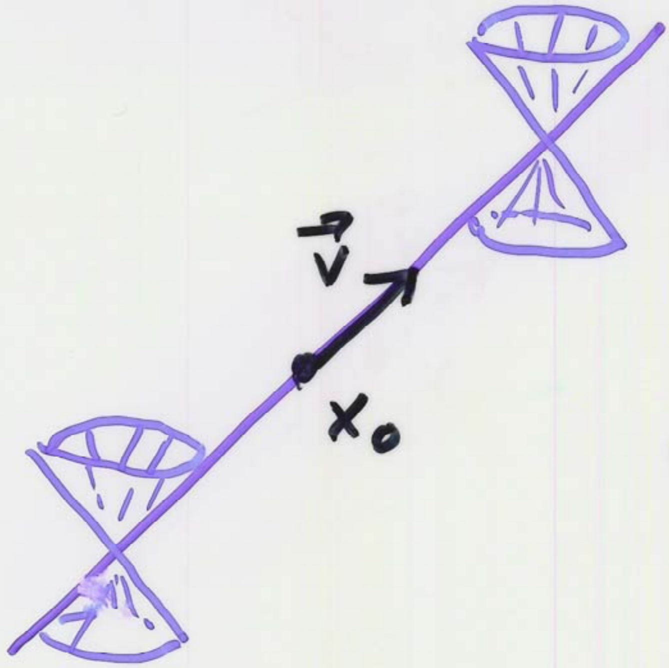
Complex Null Geodesic

in complex space-time.

Prototype: flat $\mathbb{C}^4$

Ambitwistors are straight null complex affine lines and can be parameterized by

$$X^{AA'} = X_0^{AA'} + t\,\lambda^A \mu^{A'}$$



$$g(\vec{v},\vec{v}) = 0$$

$$\Longleftrightarrow$$

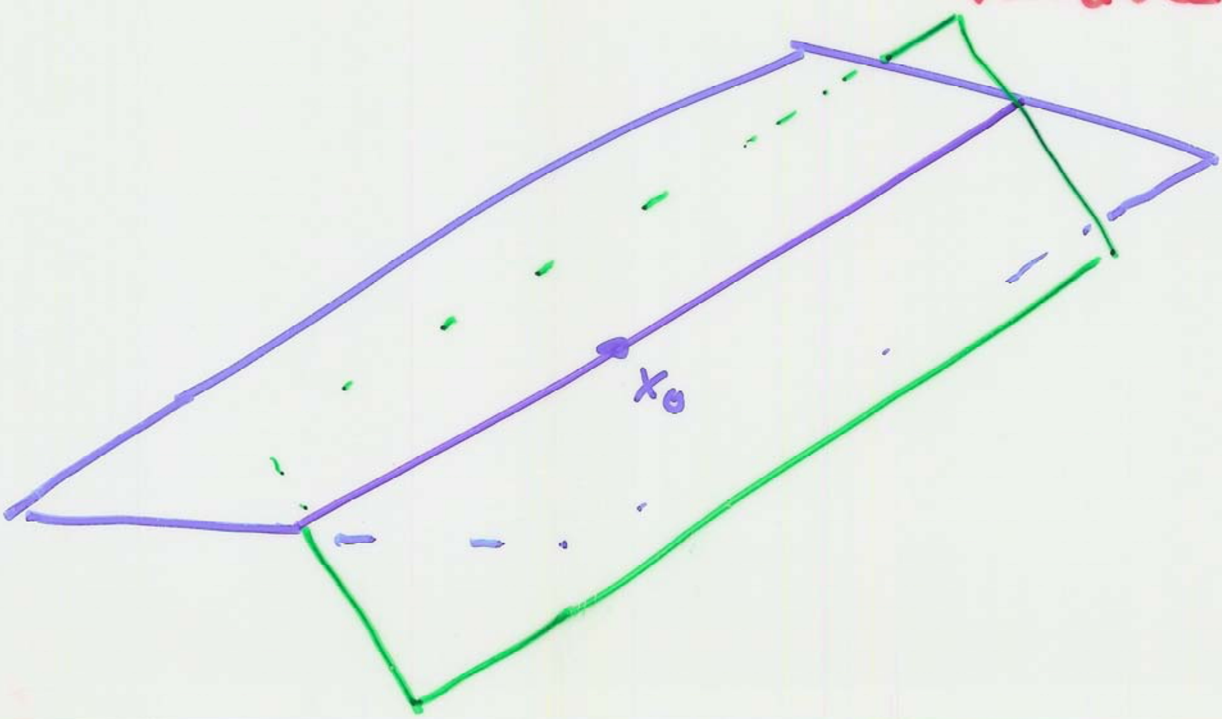
$$V^{AA'} = \lambda^A \mu^{A'}$$

But each such null line is contained in a unique α-plane

$$x^{AA'} = x_0^{AA'} + \underbrace{\xi^A}_{\text{variable}} \mu^{A'}$$

and a unique β-plane

$$x^{AA'} = x_0^{AA'} + \lambda^A \underbrace{\xi^{A'}}_{\text{variable}}$$



However, the generic α-plane and the generic β-plane do not intersect!

In fact, the α-plane

$$i x^{AA'} \pi_{A'} = \omega^A$$

and the β-plane

$$-i x^{AA'} \lambda_A = \mu^{A'}$$

iff

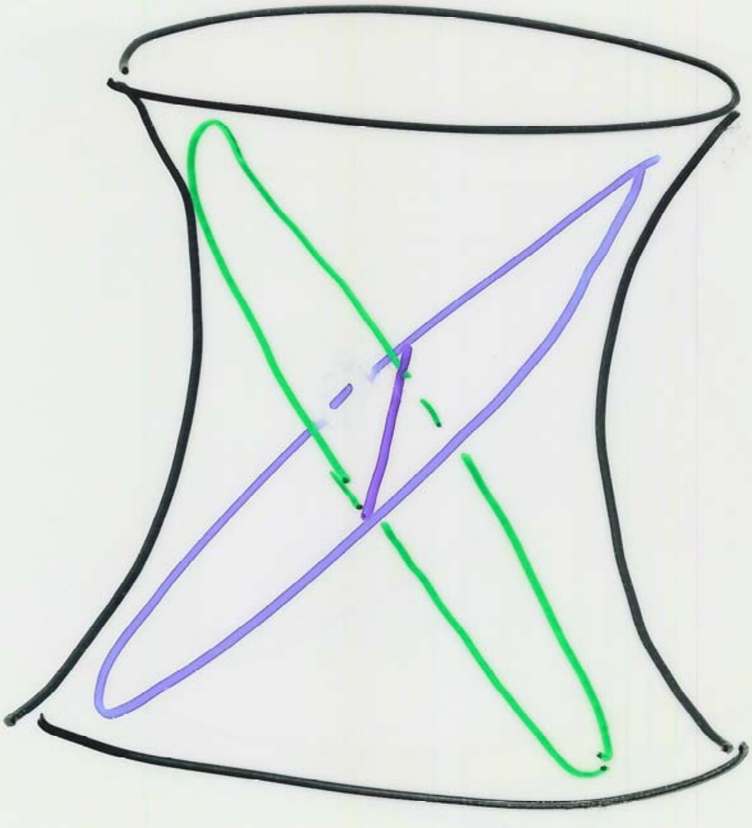
$$\omega^A \lambda_A + \pi_{A'} \mu^{A'} = 0$$

or in other words iff

$$Z^\alpha W_\alpha = 0$$

where $Z = (\omega, \pi)$, $W = (\lambda, \mu)$

By adding some points at infinity we can carry this picture over to the 4-quadric



where the space of α-planes is $\mathbb{CP}_3 = \mathbb{P}(\mathbb{T})$ and the space of β-planes is

$$\mathbb{CP}_3^* = \mathbb{P}(\mathbb{T}^*)$$

thus the ambitwistor space of $\mathbb{M} = Gr_2(\mathbb{T}) = Q_4$ is the hypersurface

$$\mathbb{A} \subset \mathbb{CP}_3 \times \mathbb{CP}_3^*$$

given by

$$\mathbb{A} = \{([Z^\alpha], [W_\beta]) \mid Z^\alpha W_\alpha = 0\}$$

Notice that this the 'complexification' of the real 5-manifold

$$\mathbb{N} = \{[Z] \mid Z^\alpha \bar{Z}_\alpha = 0\} \subset \mathbb{CP}_3$$

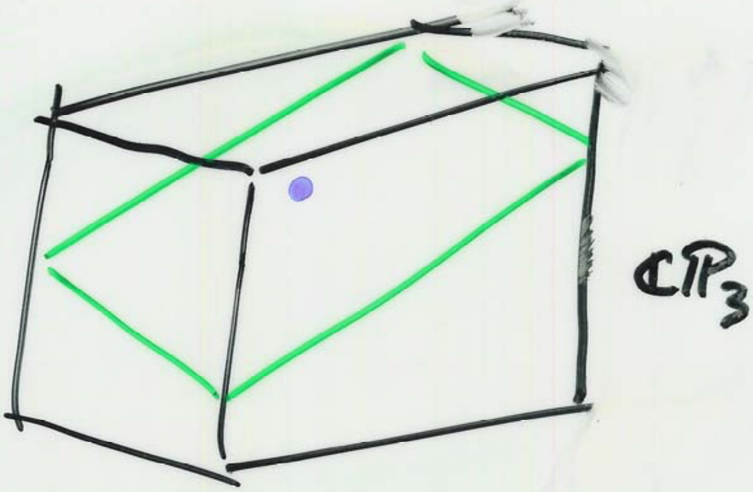
that represents real light rays in compactified Minkowski space.

Picture in twistor space $\mathbb{CP}_3$:

$\mathbb{CP}_3^* = \{[W_\alpha]\}$ is the space of planes in $\mathbb{CP}_3$ where the point $[Z^\alpha]$ is on the plane $[W_\alpha]$ iff

$$Z^\alpha W_\alpha = 0.$$

So $\mathbb{A} = \{(\text{points}, \text{planes}) \mid \text{point is on plane}\}$.



Hence $\mathbb{A} = \mathbb{P}(T^*\mathbb{CP}_3)$!

May also consider the rings of functions

$$\mathcal{O}^{(n)} = \mathcal{O}_{\mathbb{CP}_3 \times \mathbb{CP}_3^*} \big/ \mathcal{I}^{n+1}$$

consisting of functions on $\mathbb{A}$ together with their first n normal derivatives onto $\mathbb{CP}_3 \times \mathbb{CP}_3^*$. The 'ringed space'

$$\mathbb{A}^{(n)} = (\mathbb{A}, \mathcal{O}^{(n)})$$

is called the n^{th} order thickening of flat ambitwistor space.

Thickenings $\mathbb{A}^{(n)}$:

The thing that makes $\mathbb{A}$ a complex manifold is that it's equipped with local holomorphic functions $\mathcal{O} = \mathcal{O}_{\mathbb{A}}$

["Sheaf" – associates functions to their domains]

Related to functions on $\mathbb{CP}_3 \times \mathbb{CP}_3^*$ by

$$\mathcal{O}_{\mathbb{A}} = \mathcal{O}_{\mathbb{CP}_3 \times \mathbb{CP}_3^*} / \mathcal{I}$$

where $\mathcal{I}$ = multiples of $Z^\alpha W_\alpha$

Super Ambitwistors: $\mathbb{A}^{[n]} = \mathbb{A}_{(5|2n)}$

Formally given by locus

$$Z^\alpha W_\alpha + \psi^j \varphi_j = 0$$

in $\mathbb{CP}_{(3|n)} \times \mathbb{CP}^*_{(3|n)}$

Technically

$$\mathbb{CP}_{(3|n)} = \left(\mathbb{CP}_3, \Lambda^\bullet[\mathbb{C}^n \otimes \mathcal{O}(-1)]\right)$$

$$\mathbb{CP}^*_{(3|n)} = \left(\mathbb{CP}^*_3, \Lambda^\bullet[\mathbb{C}^{n*} \otimes \mathcal{O}(-1)]\right)$$

So

$$\mathbb{A}^{[n]} = \left(\mathbb{A}, \Lambda^\bullet[\mathbb{C}^n \otimes \mathcal{O}^{(n)}(-1,0) + \mathcal{O}^{(n)}(0,-1) \otimes \mathbb{C}^{n*}] / f\right)$$

$$f = \langle Z^\alpha W_\alpha + \psi^j \varphi_j \rangle$$

<u>Curved</u> <u>Case</u> :

$(\mathcal{M}_4, \mathscr{g})$ complex

Riemannian 4-manifold

[might be analytic continuation of real-analytic (M^4, g), but not necessary...]

$\exists$ unique connection ∇ associated with $\mathscr{g}$. Holomorphic

in sense that

$$\nabla : \mathcal{O}(T^{1,0}) \to \mathcal{O}(T^{*1,0} \otimes T^{1,0})$$

(Complex)
Null geodesics: holomorphic curves with

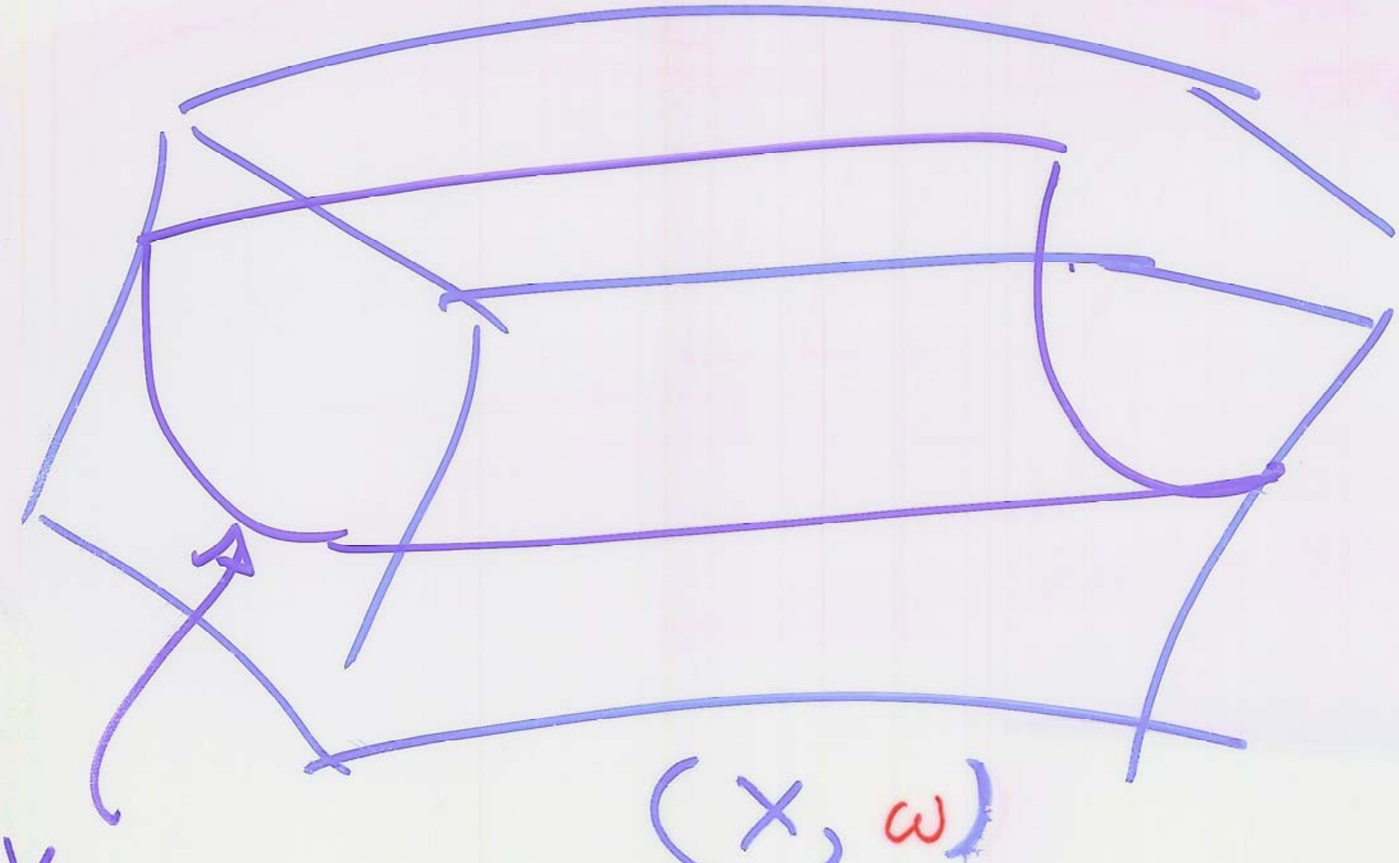
$$g\left(\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\zeta}\right)=0;\ \nabla_{\frac{\partial}{\partial\zeta}}\left(\frac{\partial}{\partial\zeta}\right)=0.$$

Ambitwistors = such curves, modulo reparameterization = "complex light rays"

Remarkable fact:
Conformally invariant!
Unchanged if $g \rightsquigarrow \tilde{g} = u g$

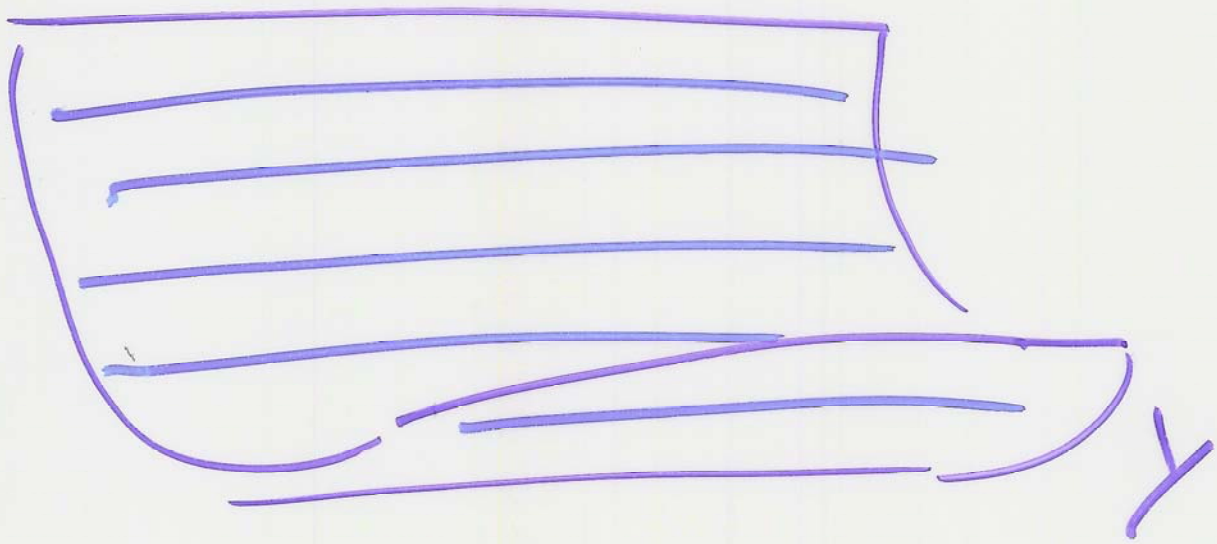
Why? Symplectic Construction.

Also works in pseudo-Riemannian case...



ω closed non-degenerate 2-form
($\mathbb{C}$-valued in holomorphic case...)

Then $\omega|_Y$ has 1-dim'l kernel. Get foliation



by curves. Coincides with flow of any Hamiltonian which is constant on Y.

Application. $(X, \omega) = (T^*M, d\theta)$

$$\theta = \sum p_\mu \, dq^\mu$$

$$\omega = d\theta = \sum dp_\mu \wedge dq^\mu$$

$Y \subset X$ null vectors of g

(conformally invariant!)

1. Null geodesics (ambitwistors) conformally invariant up to reparameterization

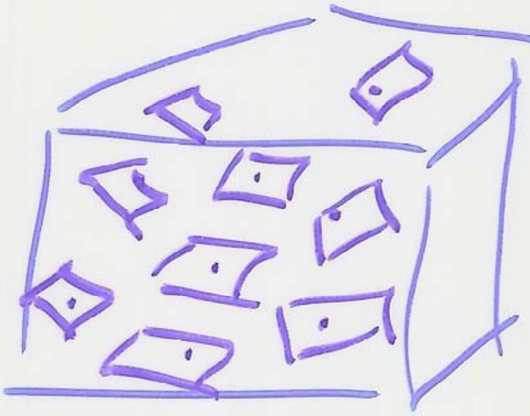
2. Proof works in holomorphic context of $(\mathcal{M}, [g])$

3. Marsden-Weinstein reduction

4. Mod rescaling: Contact structure

Contact Manifold:

(W^{2k+1}, D)



$D \subset TW$ codimension 1

Maximally non-integrable:

$$D \times D \to L = TW/D$$

$$(v, w) \longmapsto [v, w] \bmod D$$

assumed non-degenerate

Restatement: $D = \ker \theta$

θ 1-form

$$\theta \wedge \underbrace{d\theta \wedge \cdots \wedge d\theta}_{k} \neq 0$$

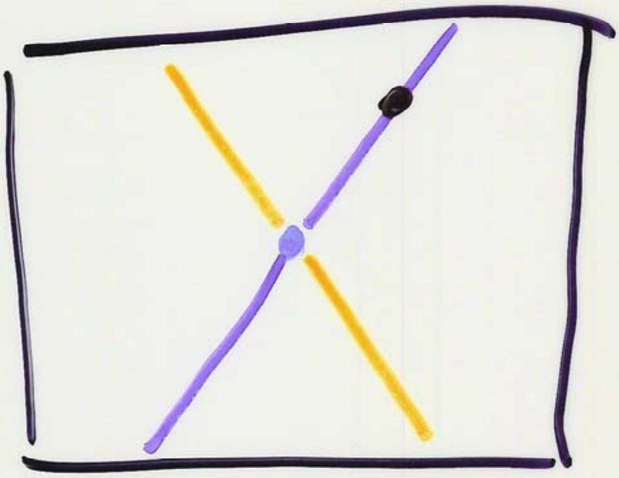
Theorem Assume $(\mathcal{M}_4, g)$ geodesically convex complex Riemannian 4-manifold.

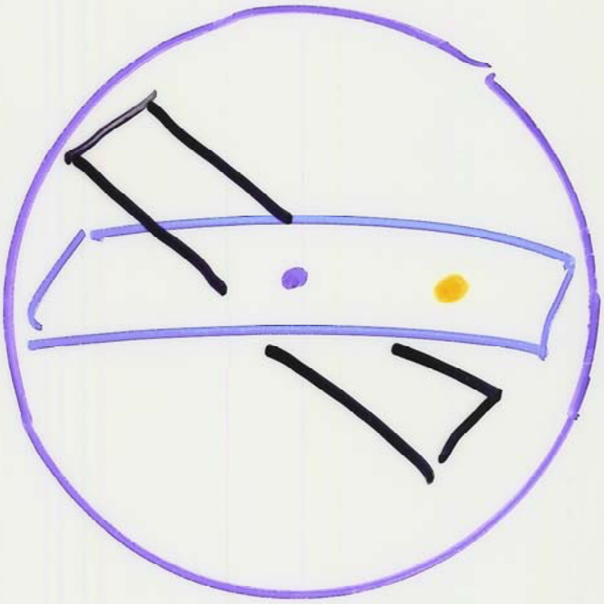
Then corresponding ambitwistor space is a complex contact manifold $(\mathcal{N}_5, D)$ of $\mathbb{C}$-dim $= 5$.

Remark. $\mathcal{M}_m \Rightarrow \mathcal{N}_{2m-3}$ by same proof!

Ambitwistor Correspondence

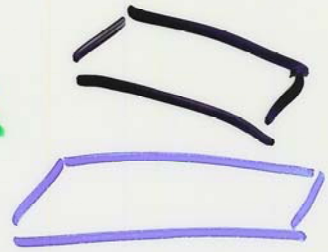
(contact)



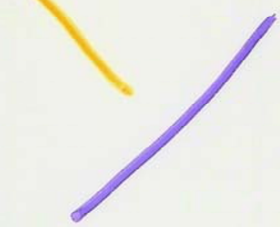


$\mathcal{M}_4$

$\mathcal{N}_5$



Points $\longleftrightarrow$ $\mathbb{CP}_1 \times \mathbb{CP}_1$'s



Null Geodesics $\longleftrightarrow$ points the special $\mathbb{CP}_1 \times \mathbb{CP}_1$'s arising from points of $\mathcal{M}$ are called skies. they are characterized by the following property: the restriction of $c_1(\mathcal{N})$ to any sky is

$$(3,3) \in H^2(\mathbb{CP}_1 \times \mathbb{CP}_1) = \mathbb{Z} \oplus \mathbb{Z}$$

Similar situation in dimensions > 4, but different in dimension 3!

Theorem Suppose $(\mathcal{N}_5, D)$ complex contact 5-manifold. Let $\mathcal{M}$ be the set of embedded holomorphic

$$\mathbb{CP}_1 \times \mathbb{CP}_1 \subset \mathcal{N}$$

such that restriction of $c_1(\mathcal{N})$ to $\mathbb{CP}_1 \times \mathbb{CP}_1$ is

$$(3,3) \in \mathbb{Z} \oplus \mathbb{Z} = H^2(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathbb{Z})$$

then $\mathcal{M}$ is complex 4-manifold, equipped with complex conformal metric $[g]$.

Locally, <u>general</u> $(M_4, [g])$!!!

Theorem Suppose that $(M_4, [g])$ is ASD, with twistor space Z_3. Then its ambitwistor space is

$$\mathcal{N} = \mathbb{P}(T^*Z) \xrightarrow{\mathbb{CP}_2} Z$$

and the skies in $\mathcal{N}$ are the projectivized conormal bundles $\mathbb{CP}_1 \times \mathbb{CP}_1$ of twistor lines $\mathbb{CP}_1 \subset Z$.

<u>Corollary</u> When $(\mathcal{M}_4, [g])$ is ASD, there is a generalization of

$$\mathbb{A} \hookrightarrow \mathbb{CP}_3 \times \mathbb{CP}_3^*$$

given by

$$\begin{array}{ccc} \mathcal{N} = \mathbb{P}(T^*Z) & \hookrightarrow & \mathbb{P}(J^1K) \\ \big\downarrow \mathbb{CP}_2 & & \big\downarrow \mathbb{CP}_3 \\ Z & & Z \end{array}$$

Consequently, can define $\mathcal{N}^{(n)}$, too, in this case.

$\mathcal{N}^{[n]}$ defined if $(\mathcal{M}, [g])$ s.p.m.

Witten and independently Isenberg-Yasskin-Green discovered that holomorphic vector bundles on $\mathbb{A}^{(3)}$ which are trivial on skies exactly correspond to solutions of the full Yang-Mills equations on $\mathbb{M}$ or suitable open subsets. Generalized Ward Correspondence! In fact, the same ideas go through for $\mathcal{N}^{(3)}$ and ASD $(\mathcal{M}_4, [g])$...

Baston - Mason - LeBrun:

<u>Theorem</u> Let $(\mathcal{M}_4, [g])$ be a complex spin space-time with ambitwistor space $\mathcal{N}$. Then there is an analogue $\mathcal{N}^{(4)}$ of $\mathbb{A}^{(4)}$.

This can be extended to $\mathcal{N}^{(5)} \iff [g]$ satisfies conformal gravity equation

$$B_{ab} = 0.$$

If this happens,

$\exists\, \mathcal{N}^{(6)} \iff [g]$ satisfies $E_{abc} = 0$

Here

$$B_{ab} = \left(\nabla^c \nabla^d - \frac{1}{2} \mathring{r}^{cd}\right) W_{acbd}$$

is the Bach tensor arising as the Euler-Lagrange gradient of

$$\int_M |W|^2 d\mu$$

while

$$E_{abc} = W^-_{ABCD} \nabla^{DD'} W^+_{A'B'C'D'} - W^+_{A'B'C'D'} \nabla^{DD'} W^-_{ABCD}$$

is the Eastwood-Dighton tensor.

Both vanish if Einstein

$$\mathring{r} = 0$$

or SD or ASD.

Kozameh-Newman-Tod:

W^+, W^- alg general, $B = E = 0 \Rightarrow$ Conformally Einstein