GRAVITON AMPLITUDES & ...

1. SOME OBSERVATIONS ABOUT MHV GRAVITON AMPLITUDES

2. USING CONSTRAINTS IN $\mathcal{N}=4$ YM FOR MHV + ...

YANG-MILLS MHV AMPLITUDE

$$\mathcal{A}(1_-^{a_1}, 2_-^{a_2}, 3_+^{a_3} \cdots\cdots n_+^{a_n})$$

$$= \underbrace{\mathrm{Tr}(t^{a_1} t^{a_2} \ldots t^{a_n})}_{\text{CHAN-PATON}} \; \delta^{(4)}\Big(\sum_i p_i\Big) \, \mathcal{M}$$

$$\mathcal{M} = \langle 1 2 \rangle^4 \frac{1}{\langle 12 \rangle \langle 23 \rangle \cdots\cdots \langle n1 \rangle}$$

$$p_{\dot{A}A} = \bar{\pi}_{\dot{A}} \pi_A$$

$$\langle 12 \rangle = \epsilon^{AB} \pi_{1A} \pi_{2B}$$

$$[12] = \epsilon^{\dot{A}\dot{B}} \bar{\pi}_{1\dot{A}} \bar{\pi}_{2\dot{B}}$$

helicity $= -\frac{1}{2}$ (degree of homogeneity in π's in $\mathcal{M}$)

$\langle 12\rangle^4$ FROM SUPERSYMMETRY

$$\int d^8\theta \prod_{\alpha=1}^{4} \theta_A^\alpha \pi_1^A \prod_{\beta=1}^{4} \theta_B^\beta \pi_2^B \sim \langle 12\rangle^4$$

$$\delta^8(\pi\eta) = \int d^8\theta \; e^{i\,\theta_A^\alpha \pi_i^A \eta_{\alpha i}}$$

η_α = SUSY PARTNERS OF SPINOR MOMENTA

$$\delta(\Sigma p)\,\delta(\Sigma\pi\eta) = \int d^4x\, d^8\theta \prod_i e^{i\bar{\pi}_i x \pi_i + i\theta_A^\alpha \pi_i^A \eta_{\alpha i}}$$

<u>DENOMINATOR</u>

$$\langle J(1)\, J(2) \cdots J(n)\rangle = \frac{1}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}$$

(NAIR)

$$J(1) = \alpha(1)\,\beta(1) \longleftarrow \text{FREE FERMIONS}$$

$$\langle \beta(1)\,\alpha(2)\rangle = \frac{1}{\langle 12\rangle}$$

$$\mathcal{A} = \mathrm{Tr}(t^{a_1} \cdots t^{a_n}) \int d^4x\, d^8\theta \,\langle V(1) \cdots V(n)\rangle$$

$$V(1) = J(1)\, \exp\left[i\,\bar{\pi}_1 x \pi_1 + i\,\theta^\alpha \pi_1 \eta_{\alpha 1}\right]$$

$$\tilde{\mathcal{A}} = \delta(\omega_{\dot{A}i} - x_{\dot{A}A}\pi_i^A)\,\delta(\psi_i^\alpha - \theta_A^\alpha \pi_i^A)\,\langle J(1)\cdots J(n)\rangle$$

(WITTEN)

$$\mathcal{A} = \int d^4x\, d^8\theta \int d^2\omega\, d^4\psi \quad \tilde{\mathcal{A}}\; e^{i\omega_i\bar{\pi}_i} e^{i\psi_i\eta_i^\alpha}$$

MHV GRAVITON AMPLITUDES

(BERENDS et al)

$$\mathcal{M}(1-, 2-, 3+, \cdots n+) = \left(\frac{\kappa}{2}\right)^{n-2} \langle 12\rangle^8 \times$$

$$\left\{ \frac{[12]\,[n-2\ \ n-1]}{\langle 1\ n-1\rangle} \frac{1}{N(n)} \right.$$

$$\left. \times \prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle i\, j\rangle \prod_{l=3}^{n-3} [l\, K_{l+1,n-1}\, n\rangle + \text{perm's}\,(2,3,\ldots,n-2) \right\}$$

$$K_{l+1,n-1} = P_{l+1} + \cdots + P_{n-1}$$

$$[l\ K\ n\rangle = \bar{\pi}_l^{\dot{A}}\, K_{\dot{A}A}\, \pi_n^A$$

$$N(n) = \prod_{\substack{i,j \\ i<j}}^{n} \langle i\, j\rangle \qquad \kappa = \sqrt{32\pi G}$$

$$\text{HOLOMORPHIC} \times \underbrace{\text{ANTI HOLOMORPHIC POLYNOMIAL}}_{\text{CAN REPLACE WITH DERIVATIVES W.R.T. } \omega}$$

SIMPLIFY FOR 4 GRAVITONS

$$\mathcal{M}(1-, 2-, 3+, 4+) = \left(\frac{\kappa}{2}\right)^2 \frac{\langle 12\rangle^8 \; [12]}{N(4)\;\langle 34\rangle}$$

$$N(4) = \langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle$$

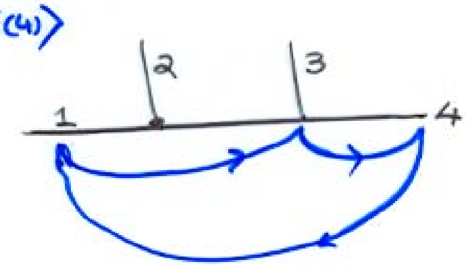
$$= -\; C(4)\;\langle 13\rangle\langle 24\rangle$$

$$C(4) = \langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle$$

$$\frac{[2\,P\,4\rangle}{\langle 24\rangle} = \frac{[2\,(p_2+p_3+p_4)\,4\rangle}{\langle 24\rangle} = -\frac{[2\;p_1\;4\rangle}{\langle 24\rangle}$$

$$= -\frac{[12]\langle 41\rangle}{\langle 24\rangle}$$

$$\mathcal{M} = \left(\frac{\kappa}{2}\right)^2 \underbrace{\langle 12\rangle^8}_{\text{SUPERSYMMETRY}} \underbrace{\left[\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right]}_{\text{CAN ARISE FROM } \langle J(1)\,J(2)\,J(3)\,J(4)\rangle} \left\{\frac{[2\,P\,4\rangle}{\langle 24\rangle}\; \underbrace{\frac{1}{\langle 13\rangle\langle 34\rangle\langle 41\rangle}}_{\text{cyclic symmetry for 3 particles}}\right\}$$



DEFINE A NEW SET OF FREE FERMIONS χ, ϕ

$$\langle \phi(1)\, \chi(2) \rangle = \frac{1}{\langle 12 \rangle}$$

$$\mathcal{M} = -\left(\frac{K}{2}\right)^2 \langle 12 \rangle^8 \langle J(1) \cdots J(n) \rangle$$

$$\int_\lambda \langle \phi(\lambda)\, (\chi\phi)_1 \frac{[2P\lambda\rangle}{\langle 2\lambda \rangle} (\chi\phi)_3\, (\chi\phi)_4\, \underbrace{\bar\partial \chi(\lambda)}\rangle$$

$$\rightarrow \bar\partial \frac{1}{\langle 4\lambda \rangle} \sim \delta(\lambda, 4)$$

ALTERNATIVELY

$$\mathcal{M} = -\left(\frac{K}{2}\right)^2 \langle 12 \rangle^8 \langle J(1) \cdots J(n) \rangle$$

$$\times \oint_C \langle \phi(\lambda)\, (\chi\phi)_1 \frac{[2P\lambda\rangle}{\langle 2\lambda \rangle} (\chi\phi)_3\, (\chi\phi)_4\, \chi(\lambda) \rangle$$

$\uparrow$

PENROSE CONTOUR INTEGRAL

C ENCLOSES POLE AT 4.

GENERALIZATION

$$\mathcal{M} = -\left(\frac{K}{2}\right)^{n-2} \langle 1\,2\rangle^8 \left\{ \frac{1}{C(n)} \oint \langle \phi(\lambda)\, (\chi\phi)_1 \frac{[2 P \lambda\rangle}{\langle 2\lambda\rangle} \frac{[3 P\lambda\rangle}{\langle 3\lambda\rangle} \cdots \right.$$

FROM $\delta^{16}_{(\theta)}$ From J's

$$\left. \cdots \frac{[n-2\; P\, \lambda\rangle}{\langle n-2\,\lambda\rangle} (\chi\phi)_{n-1} (\chi\phi)_n\, \chi\rangle \right\}$$

+ perm's. $(2, \ldots, n-2)$

$$\mathcal{A} = -\left(\frac{K}{2}\right)^{n-2} \int d^4x\, d^{16}\theta \oint_C \langle \phi(\lambda)\; \underline{J(1) V(1)}\; J(2)\, \mathcal{E}(2)\; J(3)\, \mathcal{E}(3) \cdots$$

$$\cdots J(n-2)\, \mathcal{E}(n-2)\; \underline{J(n-1) V(n-1)}\; \underline{J(n) V(n)}$$

$$\times\, \chi(\lambda)\rangle$$

$$= -\left(\frac{K}{2}\right)^{n-2} \int \oint \langle J(1) \cdots J(n)\rangle$$

$$\langle \phi\;\; V(1)\, \mathcal{E}(2) \cdots \mathcal{E}(n-2)\, V(n-1)\, V(n)\, \chi\rangle$$

VERTEX OPERATORS

1) $J = \alpha\, \beta$

2) $V = \chi\,\phi\;\; e^{i p\cdot x}\; e^{i \pi^A \eta_\alpha \theta^\alpha_A}$

3) $\mathcal{E}(2) = e^{i p_2 \cdot x}\; e^{i \pi^A_2 \eta_\alpha \theta^\alpha_A} \left\{ \frac{[2\, P\, \lambda\rangle}{\langle 2\lambda\rangle} \right\} = \frac{\bar{\pi}^{\dot A}_2 \lambda^A (-i \nabla_{\dot A A})}{\langle 2 \lambda\rangle}$

YANG-MILLS:

$$\mathcal{L} = \beta \bar{\mathcal{D}} \alpha$$

$$\bar{\mathcal{D}} = \bar{\partial} - \mathcal{A}$$

HELICITY 1 PROJECTION OF $A_{\dot{A}A}$

FERMIONS COUPLED TO GRAVITY

$$\gamma^a D_a = \gamma^a \left[\underset{\text{TETRAD}}{e_a^\mu} \partial_\mu - \omega_a^{bc} \underset{\text{SPIN MATRIX}}{S^{bc}} - A_a^\alpha T^\alpha \right]$$

T^α: LIE ALGEBRA ~~ELEMENT~~ GENERATORS GAUGE GROUP

$$e_a^\mu \approx \delta_a^\mu - h_a^\mu$$

$$\gamma \cdot D \approx \gamma^a \Big[\partial_\mu - \underbrace{\left(h_a^\mu \partial_\mu + \omega_a^{bc} S^{bc} \right)}_{\mathcal{A}} - A_a^\alpha T^\alpha \Big]$$

$\sim$ field x generator

PROJECT OUT HELICITY 1.

$$\partial_\mu \rightarrow \frac{\bar{\pi} \nabla \lambda}{\langle \pi \lambda \rangle} \sim \mathcal{E}$$

∂_μ: ANALOG OF T^α (CHARGE CORRESPONDING TO MOMENTUM)

$$\omega S \rightarrow \frac{\bar\pi\ \omega_a^{bc} S^{bc} \lambda}{\langle \pi \lambda \rangle} \sim \bar\Psi^{A\hat{A}}_{\hat{B}}\ \psi_{A\hat{A}C} \frac{\pi^{\hat{B}} \lambda^{C}}{\langle \pi \lambda \rangle}$$

GRAVITINO

$$\underbrace{\psi_{A\hat{A}C} \frac{\pi^{\hat{B}} \lambda^{C}}{\langle \pi \lambda \rangle}}_{V\ ?}$$

(CHARGE CORRESPONDING TO SPIN ANGULAR MOM.?)

$$\mathcal{M} = -\left(\frac{K}{2}\right)^{n-2} \int d^4x\, d^{16}\theta \oint \langle J(1) \cdots J(n) \rangle$$

$$\underbrace{\langle \phi\ \mathcal{A}(1)\ \mathcal{A}(2) \cdots \mathcal{A}(n)\ \chi \rangle}_{\text{ANALOG OF CHAN-PATON}}$$

1. NON VANISHING TERMS HAVE A MINIMUM OF 3 V's. WHAT ARE TERMS WITH >3 V's ?

2. A "TWISTOR STRING" (BRANE?) VERSION MAY BE POSSIBLE FOR $\mathcal{N}=8$ SUPERGRAVITY IF ONE CAN INCORPORATE ~~THESE~~ $\langle \phi\ \mathcal{A}(1) \cdots \rangle$ AS "CHAN-PATON FACTORS"

AN APPROACH TO MHV IN $\mathcal{N}=4$

(WITH ABE, PARK)

SPINOR DERIVATIVES

$$D_{A\alpha} = \frac{\partial}{\partial \theta^{A\alpha}} + i\bar{\theta}^{\dot{A}}{}_{\alpha} \nabla_{\dot{A}A}$$

$$\bar{D}_{\dot{A}}{}^{\alpha} = -\frac{\partial}{\partial \bar{\theta}^{\dot{A}}_{\alpha}} - i\,\theta^{A\alpha} \nabla_{A\dot{A}}$$

CONSTRAINTS FOR $\mathcal{N}=4$

$$\{\mathcal{D}_{A\alpha}, \mathcal{D}_{B\beta}\} = 0$$

$$\{\bar{\mathcal{D}}_{\dot{A}}{}^{\alpha}, \bar{\mathcal{D}}_{\dot{B}}{}^{\beta}\} = 0$$

$$\{\mathcal{D}_{A\alpha}, \bar{\mathcal{D}}_{\dot{A}}{}^{\beta}\} = 2i\, \mathcal{D}_{A\dot{A}}\, \delta_{\alpha}^{\beta}$$

FOR $\mathcal{N}=4$ $\underbrace{\text{CONSTRAINTS}}$ $\Rightarrow$ EQNS OF MOTION

SOLVE THESE $\Rightarrow$ CLASSICAL AMPLITUDES

INTRODUCE SPINOR VARIABLE u^A, $\bar{w}^A = K^{A\dot{A}} \bar{u}_{\dot{A}}$

REFERENCE MOMENTUM

$$D^+_\alpha = u^A D_{A\alpha}$$

$$D^-_\alpha = -\bar{w}^A D_{A\alpha}$$

FURTHER

$$D^{++} = u^A \frac{\partial}{\partial \bar{w}^A} \qquad D^{--} = -\bar{w}^A \frac{\partial}{\partial u^A}$$

$$D^0 = \left(u^A \frac{\partial}{\partial u^A} - \bar{w}^A \frac{\partial}{\partial \bar{w}^A} \right)$$

$(x, \theta, u) \rightarrow$ HARMONIC SUPERSPACE (CLOSELY RELATED TO TWISTORS)

EXPRESS CONSTRAINTS USING $D^\pm_\alpha$, $D^{\pm\pm}$, ETC.

$\mathcal{N}=4$ SYM HAS $A^{\pm\pm} = 0$

$\rightarrow$ DO A "GAUGE TRANSFORMATION" $g(x, \theta, u)$

SUCH THAT $A^+_\alpha = 0$

NOW $A^{++} \neq 0$

NOW CONSTRAINTS ARE

$$\mathcal{D}^+_\alpha A^{++} = 0 \qquad \text{("HOLOMORPHICITY" CONDITION)}$$

$$A^-_\alpha = -\mathcal{D}^+_\alpha A^{--}$$

$$D^{++} A^{--} - D^{--} A^{++} + [A^{++}, A^{--}] = 0$$

$$\mathcal{D}^+_\alpha A^-_\beta + \mathcal{D}^+_\beta A^-_\alpha = 0$$

$$\cdots$$

$$\cdot\ \cdot$$

IMPOSE ONE EXTRA CONDITION

$$\mathcal{D}^\alpha_{\dot{A}} A^{++} = 0$$

THIS IS WHERE RESTRICTION TO MHV ARISES

$\rightarrow$ ALL CONSTRAINTS EASILY SOLVED

A^{++} DEPENDS ON $y = x - i\,\bar{\theta}\,\theta$

$$\mathcal{D}^+_\alpha A^{++} = 0 \quad \Rightarrow$$

$$A^{++} = t^a(a^a_- + \xi^\alpha a^a_\alpha + \cdots \qquad)\, e^{ip\cdot y}$$

$$\xi^\alpha = \theta^{A\alpha} u_A$$

ALSO $\quad p^2 = 0, \quad p_{\dot{A}A} = \bar{\pi}_{\dot{A}} \pi_A$

$$\Rightarrow \quad u_A = \pi_A$$

FINAL CONDITION

$$D^{++} A^{--} - D^{--} A^{++} + [A^{++}, A^{--}] = 0$$

$$\sim \quad \partial_{\bar{z}} A_z - \partial_z A_{\bar{z}} + [A_{\bar{z}}, A_z] = 0$$

ACTION FOR THIS: GAUGED WZW ACTION

$$S = -\int d^4x \, d^8\theta \; S_{WZW}(U) + \frac{1}{\pi} \int Tr \left(A_{\bar{z}} \, \partial_z U U^{-1} \right)$$

$\Longrightarrow$ ALL MHV AMPLITUDES

RELAX THE EXTRA CONDITION $D^{\alpha}_{\dot{A}} A^{++} = 0$.

CAN ONE GET NON-MHV AMPLITUDES?