Nonperturbative Topological Strings

With Applications to the Twistor-String (?)

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Preface

Witten's version of the twistor-string is a B model on $\mathbb{CP}^{3|4}$. More precisely it is a nonperturbative version of the B model: one has to include solitonic objects (D1-branes) and integrate over their moduli. This raises the question: shouldn't what (little) we know about the nonperturbative B model be relevant here?

So far, what we have learned about the nonperturbative topological string has been closely tied to its interpretation as computing quantities in the physical string theory. That interpretation seems to make sense only when the target space is a Calabi-Yau threefold.

Nevertheless, the lessons can ultimately be phrased purely in terms of the topological string theory itself, and they involve structures which are still present in the twistor-string.

Outline

Nonperturbative topological strings from the physical superstring

- Vector multiplet couplings
- Counting black holes

A reformulation of the B model

- Kodaira-Spencer gravity
- Hitchin's functional

S-duality in topological strings

- An intuitive picture of the role of S-duality
- Matching of observables in the compact case
- Application to the twistor-string?

Until further notice

For a while we just consider the topological string on a Calabi-Yau threefold X: the so-called critical case. We'll comment on $\mathbb{CP}^{3|4}$ later.

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Vector multiplet couplings

Consider Type IIA/B theory on $X \times \mathbb{R}^{3,1}$. This gives an effective $\mathcal{N} = 2$ supergravity theory on $\mathbb{R}^{3,1}$, which has the gravitational multiplet plus vector and hyper multiplets. In IIA, vector multiplets correspond to the Kähler moduli of X; in IIB, they correspond to the complex moduli.

The dilaton (string coupling) is always in a hyper multiplet. $\mathcal{N} = 2$ supersymmetry implies that F-terms involving vector and hyper multiplets are decoupled in the supergravity theory; therefore F-terms involving vector multiplets are protected from string loop corrections $\implies$ they are good things to compute!

Vector multiplet couplings

Some of these F-terms are computed by the topological string.

As Candelas described, there are two versions of the topological string, called "A model" and "B model". In perturbation theory, they depend on different moduli: the A model partition function Z_A depends on the Kähler moduli of X, while the B model partition function Z_B depends on the complex moduli.

So in Type IIA, Z_A depends on the vector multiplet moduli, while in Type IIB, Z_B depends on the vector multiplet moduli. (Lucky coincidence?)

Vector multiplet couplings

Writing $Z = \exp(\mathcal{F}/g_s^2)$, with $\mathcal{F} = \mathcal{F}_0 + g_s^2 \mathcal{F}_1 + g_s^4 \mathcal{F}_2 + \cdots$, the topological string computes terms in the 4-dimensional action of the form

$$\int \mathrm{d}^4 x \int \mathrm{d}^4 \theta \, \mathcal{F}_g(X^I)(\mathcal{W})^{2g}.$$

Here X^I are the vector multiplet chiral superfields, and $\mathcal{W}$ is the Weyl superfield, built from the gravity multiplet — expanding it gives e.g.

$$\int \mathrm{d}^4 x \, \mathcal{F}_g(X^I)(R_+^2 F_+^{2g-2}).$$

Geometric engineering

What are these couplings good for?

In some cleverly engineered geometries, the low energy supergravity gets enhanced by a nonabelian $\mathcal{N} = 2$ gauge theory. In that case $\mathcal{F}_0$ turns out to be useful — it allows one to solve the IR dynamics of the gauge theory (as done by Seiberg and Witten).

To actually compute $\mathcal{F}_0$ one uses mirror symmetry, and the Seiberg-Witten curve appears directly in the mirror geometry.

[Katz-Klemm-Vafa]

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Counting black holes in $d = 5$

The higher $\mathcal{F}_g$ are also useful: one can use them for counting BPS states of black holes in M-theory on $X \times \mathbb{R}^{4,1}$. These black holes are obtained by wrapping M2-branes on homology cycles $Q \in H_2(X, \mathbb{Z})$. They make a contribution to the term $R_+^2 F_+^{2g-2}$ which can be evaluated: one finds that the A model partition function can be expanded [Gopakumar-Vafa]

$$\mathcal{F}(t, g_s) = \sum_{j \geq 0} \sum_{Q \in H_2(X,\mathbb{Z})} \mathcal{N}_{j,Q} \left(\sum_{n \geq 0} \left(2 \sinh \frac{n g_s}{2} \right)^{2j-2} e^{-n \langle Q, t \rangle} \right).$$

This recodes $\mathcal{F}(t, g_s)$ in terms of the integer invariants $\mathcal{N}_{j,Q}$ which count states of black holes of charge Q and spin j.

Counting black holes in $d = 4$

Recently topological strings have also been applied to black holes in four dimensions. This is much deeper! [Ooguri-Strominger-Vafa]

Take Type IIB on $X \times \mathbb{R}^{3,1}$. Then can obtain black holes by wrapping D3-branes on a homology cycle $C \in H_3(X, \mathbb{Z})$. This C includes both electric and magnetic charges, $C = Q + P$: splitting into electric and magnetic corresponds to a symplectic marking of $H_3(X, \mathbb{Z})$,

$$A^i \cap A^j = 0, \quad B_i \cap B_j = 0, \quad A^i \cap B_j = \delta^i_j.$$

B model topological string computes the partition function of a mixed ensemble where we fix P and an electric potential Φ conjugate to Q:

$$|Z(P + i\Phi)|^2 = \sum_Q \Omega_{P,Q} e^{-\Phi Q}$$

Counting black holes in $d = 4$

$$|Z(P + i\Phi)|^2 = \sum_Q \Omega_{P,Q} e^{-\Phi Q}$$

Here $Z(P + i\Phi)$ is the B model topological string partition function, written as a function of the A cycle periods ("electric periods.") Note that this depends on the choice of marking (basis of A and B cycles) — there is a different Z for every such choice.

The various Z are related by Fourier transforms. This is an aspect of the holomorphic anomaly of the B model: quantization of $H^3(X, \mathbb{R})$. [Witten]

Passing to $\Omega_{P,Q}$ gets rid of the holomorphic anomaly — the number of black holes with charge $P + Q$ is a canonically defined object which is independent of the choice of basis. ("Wigner function")

Counting black holes in $d = 4$

$$|Z(P + i\Phi)|^2 = \sum_Q \Omega_{P,Q} e^{-\Phi Q}$$

On the left side g_s does not appear explicitly; it has been traded for the overall scaling of P and Φ. So the genus expansion of the topological string on the left side corresponds to an expansion around large P and Φ on the right side.

The perturbative topological string only gives the asymptotics of $\Omega_{P,Q}$! The integers $\Omega_{P,Q}$ should correspond to a hypothetical nonperturbative completion of the topological string.

Maybe one should define the nonperturbative topological string by these numbers.

The case of 2-d Yang-Mills

One example where this can be worked out explicitly: Type IIA on a non-compact Calabi-Yau threefold $\mathcal{L}_1 \oplus \mathcal{L}_2 \to T^2$. In this case black hole charges are the even homology $H_0 \oplus H_2 \oplus H_4 \oplus H_6$ (wrap D0, D2, D4, D6 branes). Consider N D4 branes and 0 D6 branes, and sum over D0 and D2 branes: this is equivalent to computing the partition function of $U(N)$ Yang-Mills theory on T^2.

This partition function was known to be factorized, $Z_{YM} = |Z|^2$, but *only* perturbatively in $1/N$. Can identify Z with the A model partition function, and $1/N$ with g_s. The point is that this Z seems to make sense only perturbatively: nonperturbatively only $|Z|^2$ exists!

[Gross-Taylor, Vafa, Aganagic-Saulina-Ooguri-Vafa]

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The target space

The A and B models also have target space descriptions, both in their open and closed string versions. [Witten, Bershadsky-Cecotti-Ooguri-Vafa, Bershadsky-Sadov]

Open A model: Chern-Simons theory

Closed A model: Kähler gravity

Open B model: Holomorphic Chern-Simons theory

Closed B model: Kodaira-Spencer gravity

The target space

So far in the twistor-string, the target space description of the open B model has been the most useful one. With that in mind, it might be worthwhile to describe a recent reformulation of the classical target space description of the closed B model.

[Dijkgraaf-Gukov-AN-Vafa, Gerasimov-Shatashvili]

This reformulation incorporates the idea that $|Z|^2$ is the natural object to consider.

Kodaira-Spencer gravity

The Kodaira-Spencer gravity theory (closed B model) on a Calabi-Yau threefold X has classical solutions given by complex structures on X: conveniently captured by specifying the holomorphic 3-form Ω.

This Ω has to obey two properties: it should be of type $(3,0)$ in some underlying complex structure on X, and $\mathrm{d}\Omega = 0$.

Such Ω could be obtained from "Kodaira-Spencer gravity," where the fundamental field is a $(2,1)$-form giving the variation of Ω; but that gravity theory has some unwanted features, such as holomorphic anomaly, also known as background dependence (the partition function depends on a choice of background complex structure, even classically.)

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Real 3-forms and complex structures

Recently there appeared a new action functional for which the classical solutions are the desired Ω. The construction begins with the crucial observation that a "sufficiently generic" real 3-form ρ on a real 6-manifold X is enough to determine an almost complex structure on X! [Hitchin]

A warmup

Before discussing the case of a real 3-form, recall something simpler. Consider a nondegenerate real 2-form in dimension $2n$: it can always locally be written

$$\omega = e_1 \wedge f_1 + \cdots + e_n \wedge f_n,$$

for some choice of basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ for T^*X, varying over X ("vielbein"). If $d\omega = 0$, then there exist local coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ such that

$$\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n.$$

In this case ω defines a symplectic structure.

The 3-form case

Now consider a real 3-form ρ in dimension 6. If ρ is suitably generic ("stable") it can be written in the form

$$\rho = \frac{1}{2}\left(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 + \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3\right)$$

where $\zeta_1 = e_1 + ie_2$, $\zeta_2 = e_3 + ie_4$, $\zeta_3 = e_5 + ie_6$, and the e_i are a basis for T^*X, varying over X ("sechsbein"). The ζ_i determine an almost complex structure. If we are lucky, there exist complex coordinates (z_1, z_2, z_3) such that $\zeta_i = dz_i$; in that case we say the almost complex structure is integrable, i.e. it is an honest complex structure.

This only works in $d = 6$! This "exceptional" fact is crucially related to the fact that the topological string naturally lives in $d = 6$.

Hitchin's functional

We can rephrase this condition as $\mathrm{d}\Omega = 0$, where

$$\Omega = \zeta_1 \wedge \zeta_2 \wedge \zeta_3.$$

This Ω is completely determined by ρ; and $\rho = \mathrm{Re}\ \Omega$.

The integrability condition can be obtained from the action

$$S(\rho) = \frac{1}{2i} \int_X \Omega \wedge \bar{\Omega},$$

where we consider ρ as the field strength of a $U(1)$ gauge 2-form β (so we require $\mathrm{d}\rho = 0$ and fix the cohomology class $[\rho] \in H^3(X, \mathbb{R})$.)

Wild speculation

This action $S(\rho)$ can be thought of as a "non-chiral" version of the closed B model. At least classically, the partition function constructed from it agrees with $|Z|^2$.

Could something similar exist in the open B model — a "non-chiral" version of holomorphic Chern-Simons? It would be related to moduli of holomorphic vector bundles rather than threefolds with complex structures. The relevant integrability condition is $F^{(0,2)} = 0$. (In $d = 4$ a candidate would be self-dual Yang-Mills, but we want $d = 6$.)

If it did exist, it might help us to understand some features of the nonperturbative open B model, and perhaps the twistor-string.

(Witten's talk also could be interpreted as suggesting the need for a non-chiral ingredient in the twistor-string... want to integrate over moduli $\mathrm{d}\beta\mathrm{d}\overline{\beta}$ instead of just $\mathrm{d}\beta$?)

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The reformulation of the closed B model in terms of Hitchin's functional has a parallel in the closed A model. These two feature prominently in the conjectural "topological M-theory," which will be discussed (a bit) in Sergei Gukov's talk. [Dijkgraaf-Gukov-AN-Vafa, Nekrasov]

In topological M-theory one also sees hints of another structure: namely, the A model and B model both appear, but their degrees of freedom appear as canonical conjugates in the quantization of a 7-dimensional theory on $X \times \mathbb{R}$. Could A and B model be somehow dual to one another?

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Role of S-duality?

The T-duality symmetry of the physical string survives into the topological string (mirror symmetry).

What about the S-duality of Type IIB?

Does it relate the integrable sectors we already know about to one another? Or does it imply the existence of new integrable sectors of the superstring?

To make this question precise we would need a nonperturbative definition of the topological string. We don't understand this in general, but can still study an example.

An intuitive picture

The A model partition function gets contributions only from worldsheet instantons — BPS configurations of the fundamental string (aka holomorphic maps). S-duality relates the fundamental string to the D-string. So we might expect that the A model partition function could also be computed by summing over BPS D-strings (aka holomorphic curves).

D-strings do exist in the topological string, but they are B model objects, not A model; and they should be in some sense nonperturbative in the B model. So the perturbative A model could be related to the nonperturbative B model. [AN-Vafa]

To make this more precise, we need to be able to define the "sum over D-brane charges."

Summing over B model D-brane charges

In the B model there is a candidate way to sum over D1 brane charges, which seems consistent with the conjecture. [Nekrasov-Ooguri-Vafa]

Namely, wrap a B model D5 brane on X. It supports a $U(1)$ gauge theory, holomorphic Chern-Simons on X. Classical solutions are then holomorphic line bundles. If we extend the moduli space of such bundles to include singular configurations ("semistable sheaves"), then configurations of the gauge field can carry "induced" D1 (and D-1) brane charges. So the path integral is summing over these charges.

So we have a candidate definition of the nonperturbative B model in terms of holomorphic Chern-Simons — at least if we are willing to fix the D5 brane charge to be 1 (rather than the arguably more "natural" value 0).

Summing over B model D-brane charges

It was recently argued that this $U(1)$ gauge theory does indeed compute the A model partition function. This was proven in the special case where X is a toric variety. In that case one can compute the partition function of the gauge theory and check it agrees with the A model partition function. The configurations which contribute get interpreted in the A model as fluctuations of the Kähler geometry, or "quantum foam." [Okounkov-Reshetikhin-Vafa, Iqbal-Nekrasov-Okounkov-Vafa, Maulik-Nekrasov-Okounkov-Pandharipande]

Caveat: some uncertainty exists about whether the $U(1)$ gauge theory that appears here (mathematically formalized as "Donaldson-Thomas theory," which computes integrals over moduli of semistable sheaves, and in particular in the $U(1)$ case computes integrals over Hilbert schemes of curves) is truly equivalent to holomorphic Chern-Simons. If it is, then this is an example where the nonperturbative B model is indeed equivalent to the perturbative A model.

Embedding in the physical superstring

The role of the S-duality of Type IIB here, as well as the origin in the physical string of the single D5-brane, can be made more precise. [Kapustin]

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Matching of observables in the compact case

Might expect that S-duality of Type IIB will imply that nonperturbatively there is only a single topological string — strong coupling in the A model would be related to weak coupling in the B model.

At first this seems inconsistent with the statement that the A model and B model depend perturbatively on different moduli.

Matching of observables in the compact case

But in the nonperturbative A model, one might expect a sum over sectors labeled by different D-brane charges. It would then be natural to weigh each sector by the action of the D-brane. D-branes in the A model are Lagrangian 3-cycles L, naturally calibrated by holomorphic 3-form: weigh them by $\exp \int_L \Omega$.

This means the nonperturbative A model should depend on Ω.

Similarly, nonperturbative B model should depend on the Kähler form k, which calibrates the B model D-branes.

So at least there is no obvious inconsistency. Still, a proof, or even an exact statement of what the conjecture should mean, is so far missing.

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Application to the twistor-string?

One could try to apply S-duality to the twistor-string, but there is an immediate problem.

The B model on $\mathbb{T}' \subset \mathbb{CP}^{3|4}$ has an infinite-dimensional space of observables, given by $H^1_{\bar\partial}(\mathbb{T}', \mathcal{O}(k))$. These are related to scattering states of $\mathcal{N} = 4$ super Yang-Mills by the Penrose transform.

In the A model one sees no trace of these observables. Putting A model branes on $\mathbb{RP}^{3|4} \subset \mathbb{CP}^{3|4}$ does not seem to help: one would expect to get de Rham cohomology, which looks like it would be finite-dimensional.

Mirror symmetry

If we have strong faith in mirror symmetry, then it cannot be true that the A model always has a finite-dimensional space of observables while the B model can have an infinite-dimensional one, since the two are equivalent. In that case the observables may be hiding somewhere. But we need strong faith! The mirror of $\mathbb{CP}^{3|4}$ has been computed by linear sigma model methods, but a priori this might not apply to $\mathbb{T}'$. [Aganagic-Vafa, Kumar-Policastro]

The mirror was found to be the quadric hypersurface in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$ — possibly related to an ambitwistor version of $\mathcal{N} = 4$ super Yang-Mills? [Witten, Sinkovics-Verlinde]

Summary

- Nonperturbative topological strings are still mysterious.
- There is evidence that one should try coupling the topological to the anti-topological sectors: not Z but $|Z|^2$ is the well defined object. This is also natural in a recent reformulation of the target space dynamics of the closed string sector.
- There is tentative evidence that there is an S-duality between A and B models in the topological string on a Calabi-Yau threefold, induced from the S-duality of the Type IIB superstring.