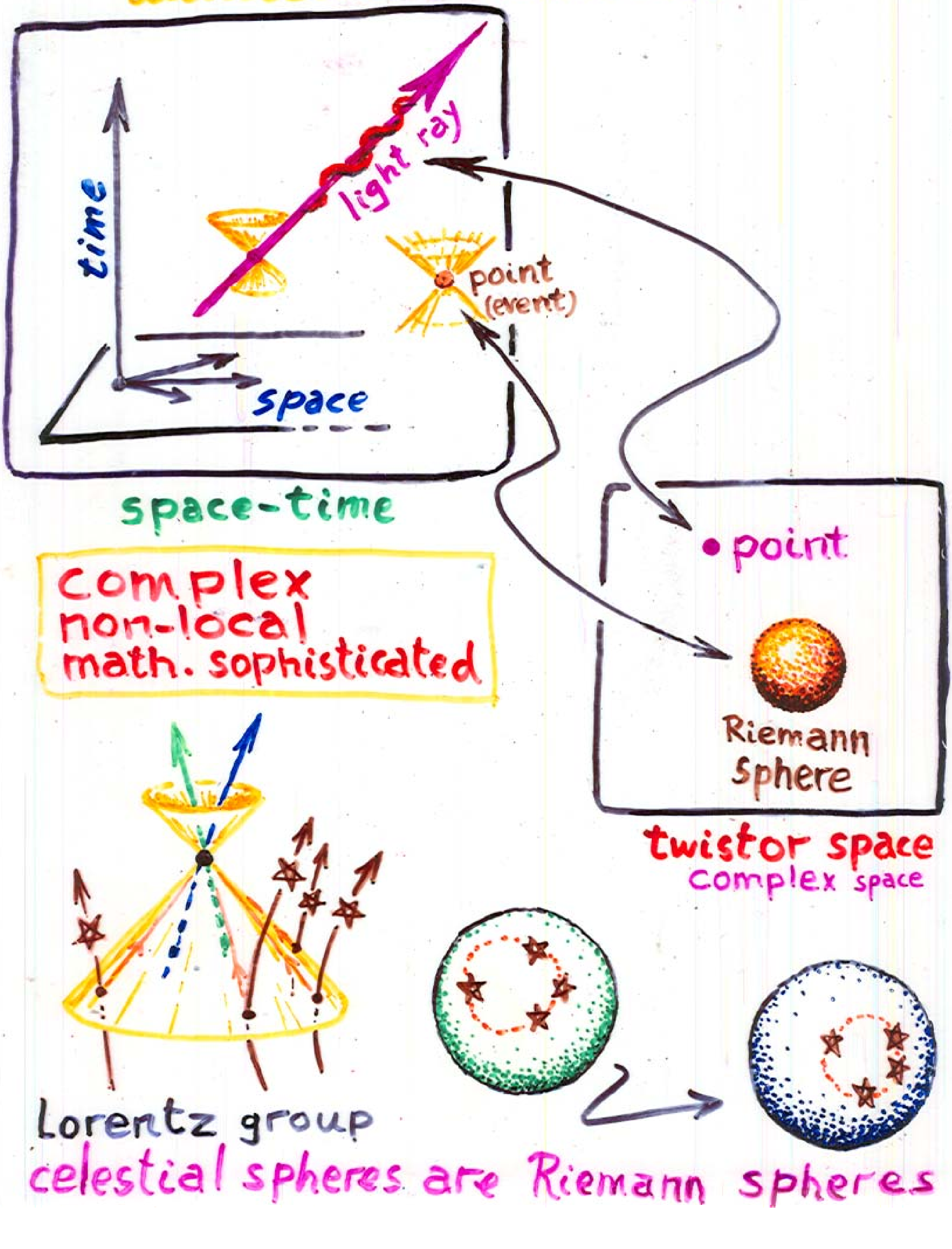
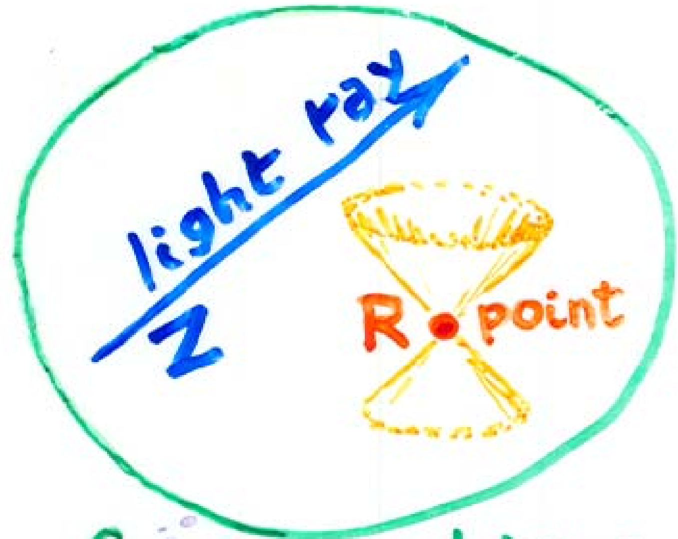
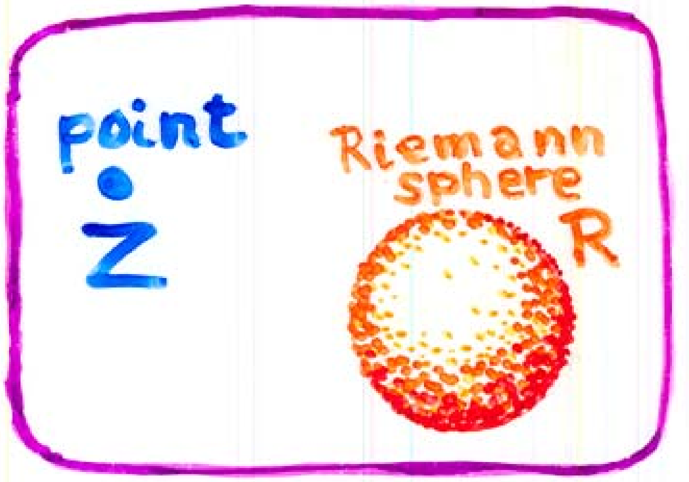
Twistor Theory



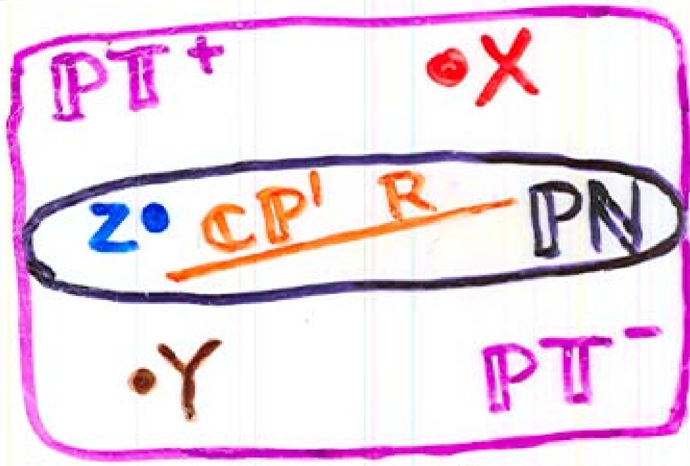


Space-time



Twistor space

Minkowski Space $\mathbb{M}$

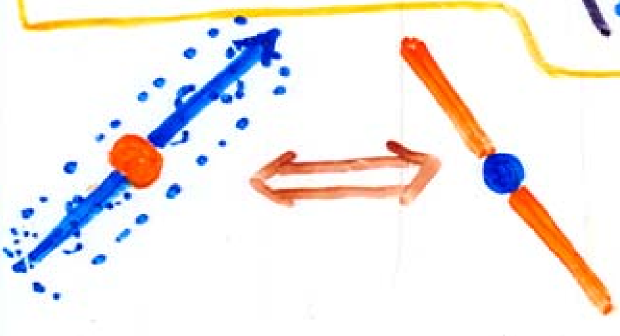


Projective twisto[r] space $\mathbb{PT} = \mathbb{CP}^3$

R has space-time coordinates (r^0, r^1, r^2, r^3)

Z has twistor coordinates (Z^0, Z^1, Z^2, Z^3)

Incidence: $\begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} = \frac{i}{\sqrt{2}} \begin{pmatrix} r^0 + r^3 & r^1 + i r^2 \\ r^1 - i r^2 & r^0 - r^3 \end{pmatrix} \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$



Eqn. of $\mathbb{PN}$:

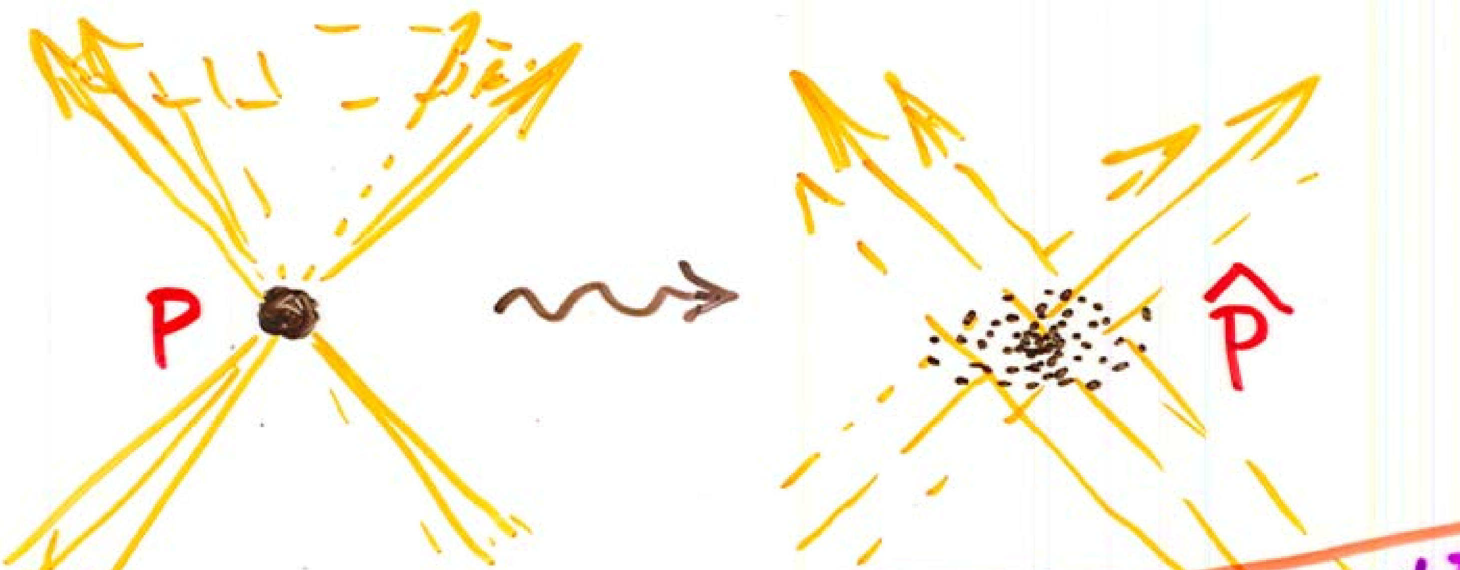
$Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1} =$

Quantum Geometry?

A commonly expressed view:

Quantized metric ⇝ "fuzzy light cone"

Twistor view:

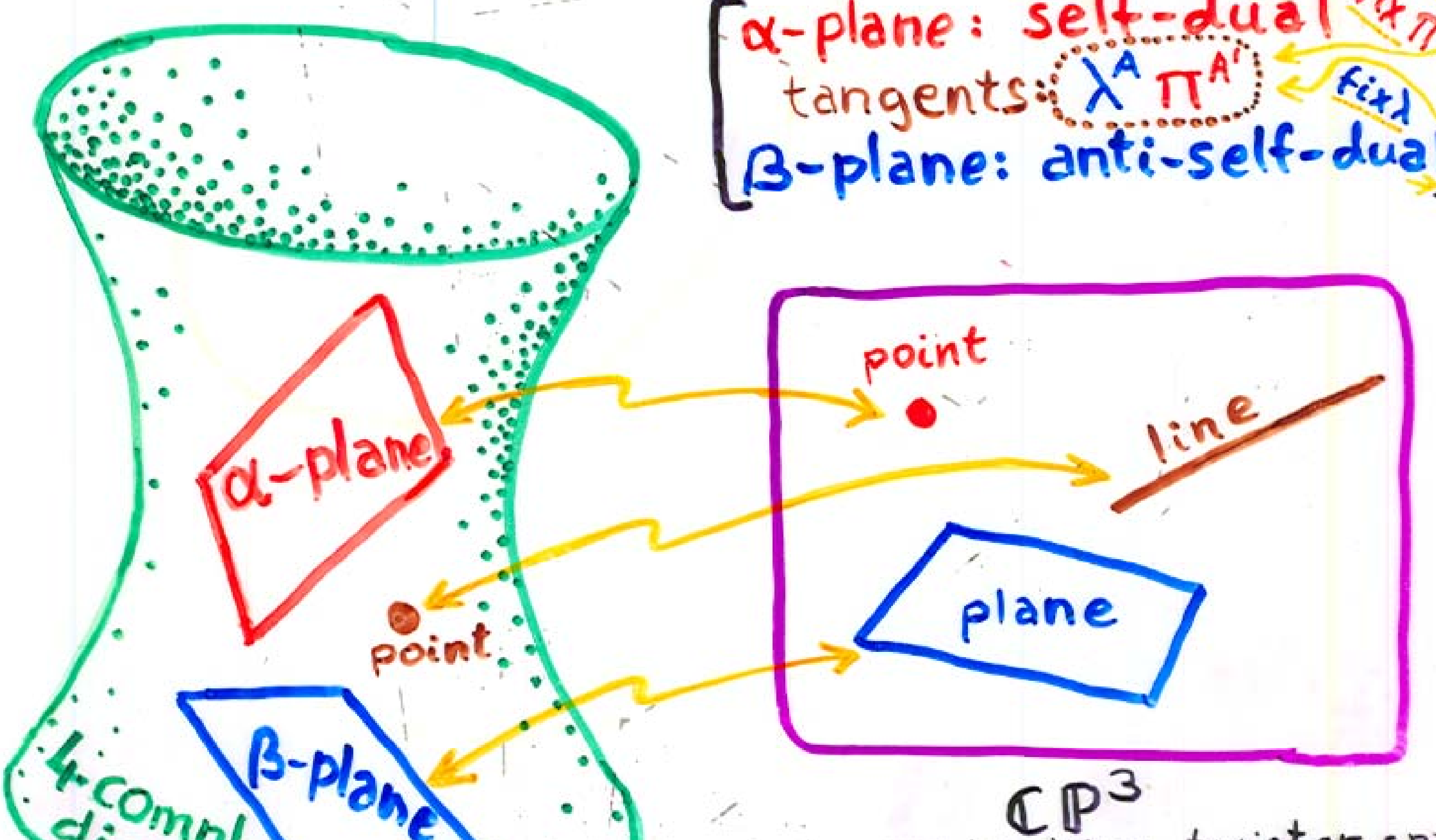


"fuzzy point"

Twistors as spinors for the

Complexified (compactified
Minkowski space-time $\mathbb{CM}$ ($\mathbb{CM}^{\#}$

α-plane: self-dual — fix π
tangents: $\lambda^{A}\pi^{A'}$ — fix λ
β-plane: anti-self-dua



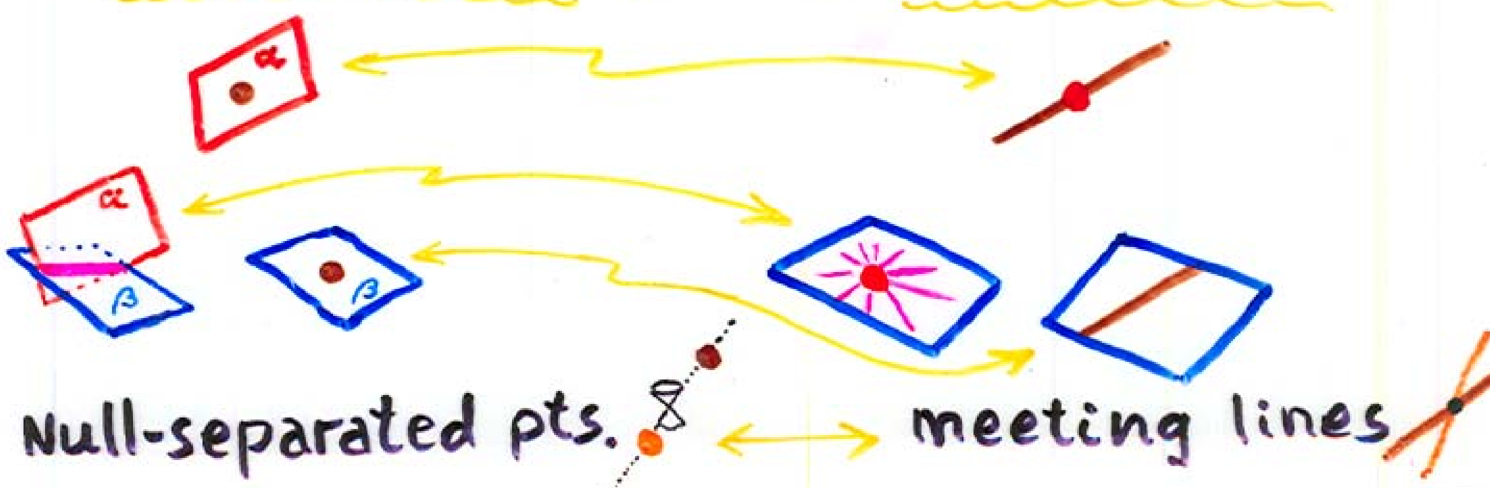
projective twistor sp
$\mathbb{PT}$

Klein quadric in $\mathbb{CP}^5$

Real pts.: signature (2,4)

Twistors ar
(reduced) spinor
for SO(2,4

Incidence ⟷ Incidence



Incidence: $\omega^{A} = i\, r^{AA'} \pi_{A'}$

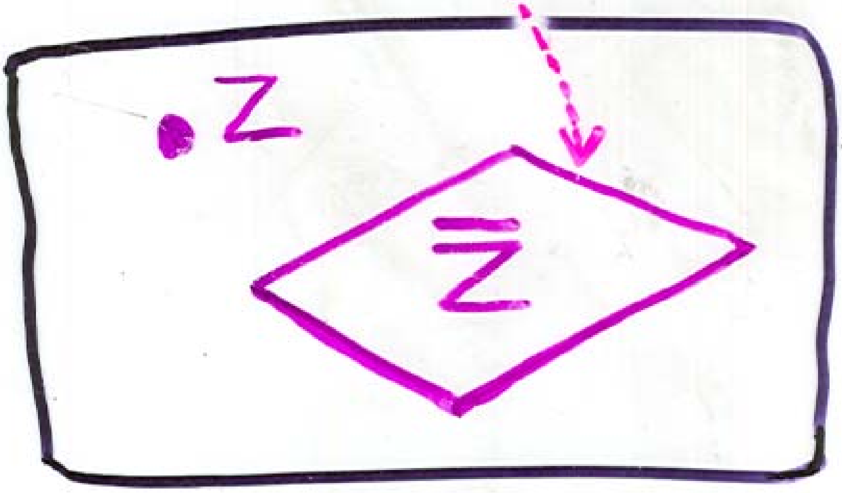
Twistor: $Z^{\alpha} = (\omega^{A}, \pi_{A'})$

Eqn. of $\mathbb{PN}$: $Z^{\alpha}\bar{Z}_{\alpha} = 0$

where $\bar{Z}_{\alpha} = (\bar{\pi}_{A}, \bar{\omega}^{A'})$

is a dual twistor

$\mathbb{PT} = \mathbb{CP}^3$

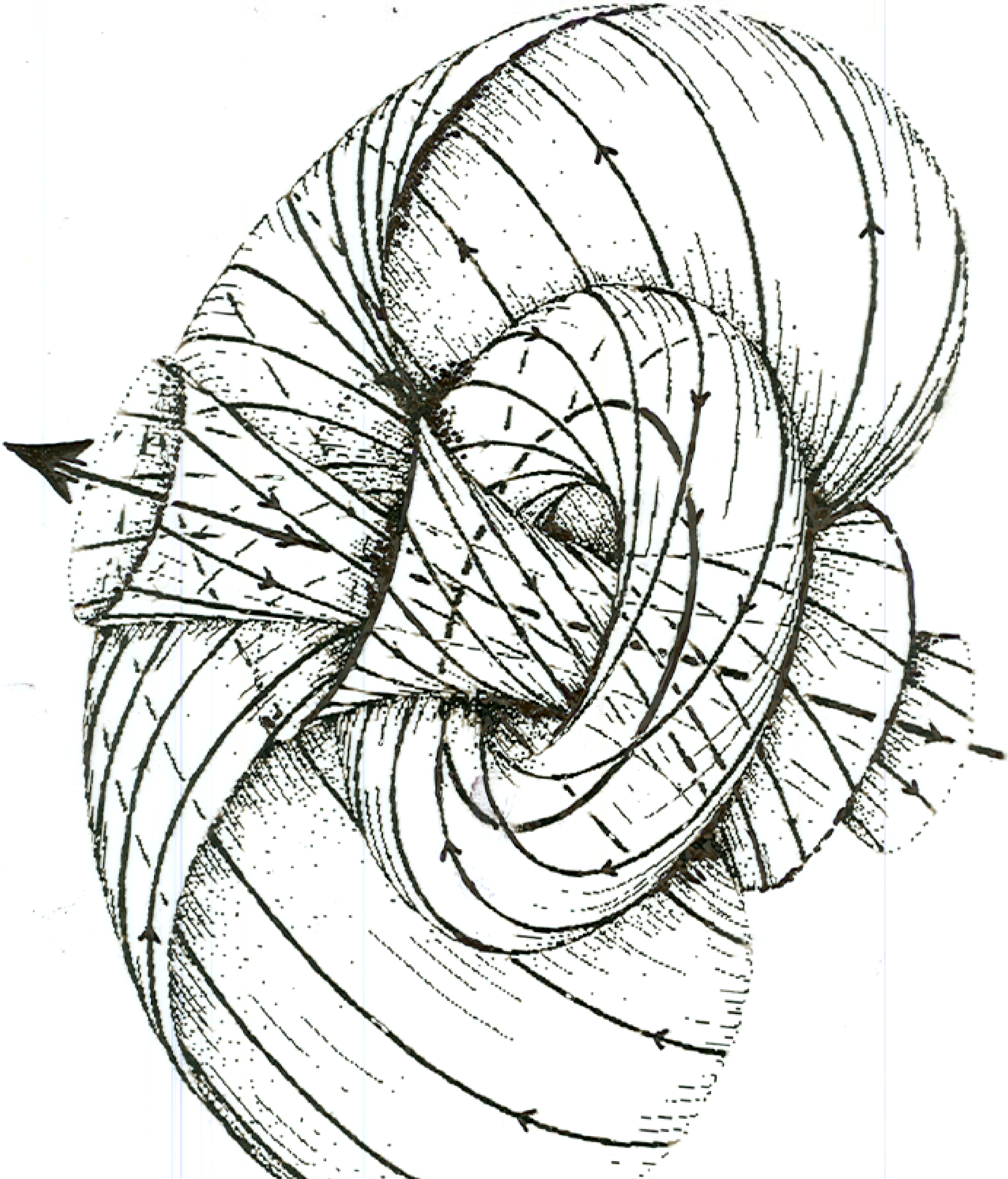


Complexified space-time: $\mathbb{C}M$
allow r^a to be complex.

Fix Z^{α}. Set of points of $\mathbb{C}M$ incident with Z constitute an α-plane. (For a dual twistor: β-plane)

Twistor theory for different spacetime signatures

Space-time signature	Twistor complex conjugation
$++++$ (Atiyah, Hitchin, Singer .....)	$Z^\alpha \mapsto \bar{Z}^\alpha$ $W_\alpha \mapsto \bar{W}_\alpha$ $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3$ $\downarrow$ S^4 no "real" twistors: $Z^\alpha = \bar{Z}^\alpha \Rightarrow Z^\alpha = 0$ quaternionic case
$++--$ (Dujanski, Mason, Witten ....)	$Z^\alpha \mapsto \bar{Z}^\alpha$ $W_\alpha \mapsto \bar{W}_\alpha$ $\mathbb{CP}^3$ $\mathbb{RP}^3$ real twistors $Z^\alpha = \bar{Z}^\alpha$ give real vector 4-space $\mathbb{RP}^3 \subset \mathbb{CP}^3$ real case
Physical $+---$ (or $+++-$)	$Z^\alpha \mapsto \bar{Z}_\alpha$ $W_\alpha \mapsto \bar{W}^\alpha$ $Z^\alpha \bar{Z}_\alpha = 0$ gives light-ray space complex case



Spinor notation

$$r^{AA'} : \begin{pmatrix} r^{00'} & r^{01'} \\ r^{10'} & r^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} r^0 + r^3 & r^1 + ir^2 \\ r^1 - ir^2 & r^0 - r^3 \end{pmatrix}$$

$$\omega^A \rightsquigarrow \begin{pmatrix} \omega^0 \\ \omega^1 \end{pmatrix} = \begin{pmatrix} Z^0 \\ Z^1 \end{pmatrix} ; \quad \pi_{A'} \to \begin{pmatrix} \pi_{0'} \\ \pi_{1'} \end{pmatrix} = \begin{pmatrix} Z^2 \\ Z^3 \end{pmatrix}$$

incidence: $\boxed{\omega = i r \pi}$

$$Z^\alpha = (\omega^A, \pi_{A'}), \quad \bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})$$

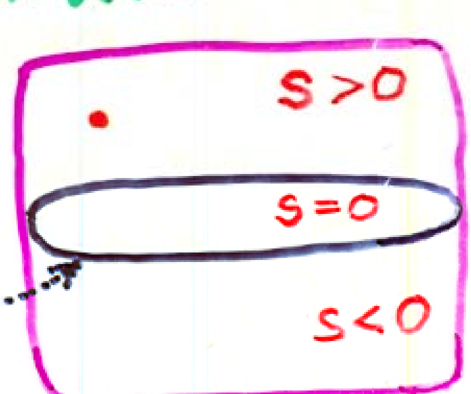
A twistor represents the 4-momentum p_a / 6-angular momentum M^{ab} structure of a massless particle

$$\boxed{\begin{aligned} p_a &\rightsquigarrow p_{AA'} = \bar{\pi}_A \pi_{A'} \\ M^{ab} &\rightsquigarrow M^{AA'BB'} = i\omega^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - i\varepsilon^{AB}\bar{\omega}^{(A'}\pi^{B')} \end{aligned}}$$

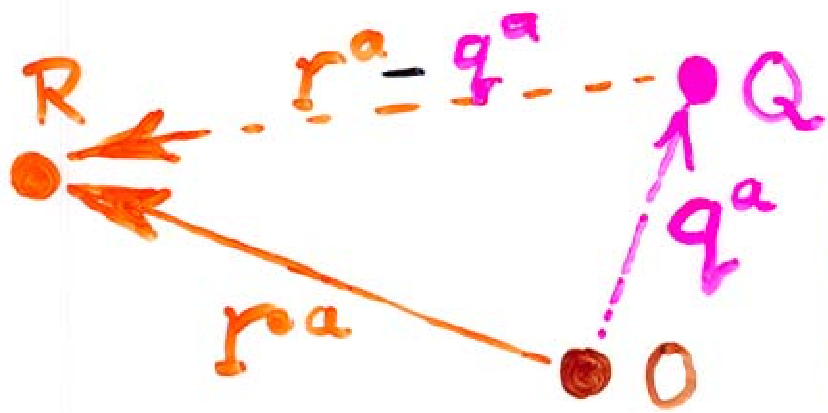
Helicity $\quad s = \frac{1}{2} Z^\alpha \bar{Z}_\alpha = \frac{1}{2}\{\omega^A \bar{\pi}_A + \pi_{A'} \bar{\omega}^{A'}\}$

where $\quad \underbrace{\frac{1}{2}\varepsilon_{abcd}\, p^b M^{cd}}_{\text{Pauli–Lubański spin vector}} = s\, p_a$

This enables us to interpret twistors that are not null as "spinning photons", rather than "light rays".



Shift origin



$$\boxed{\begin{aligned} \overset{Q}{\omega}{}^{A} &= \overset{O}{\omega}{}^{A} - i q^{AA'} \overset{O}{\pi} \\ \overset{Q}{\pi}_{A'} &= \overset{O}{\pi}_{A'} \end{aligned}}$$

Momentum & Angular Momentum for massless particle

$\{p_a, M^{ab}\}$ $\qquad p_a p^a = 0, \quad p_0 > 0$

$M^{ab} = -M^{ba}$ $\qquad S_a = s\, p_a \quad s = \text{helicit}$

where (Pauli-Lubański) $S_a = \frac{1}{2} e_{abcd}\, p^b M^{cd}$

$$p_{AA'} = \bar{\pi}_A \pi_{A'}, \quad M^{AA'BB'} = i\omega^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - i\varepsilon^{AB}\bar{\omega}^{(A'}\pi^{B')}$$

$$s = \tfrac{1}{2} Z^\alpha \bar{Z}_\alpha$$

Quantization: $[Z^\alpha, \bar{Z}_\beta] = \hbar \delta^\alpha_\beta, \ [Z^\alpha, Z^\beta]$

Above unchanged, except $\boxed{s = \frac{1}{4}(Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z}$

In Z^α-repn. twistor wave fn. f is holomorphic in Z^α $\quad (\partial f/\partial \bar{Z}_\alpha =$

$\bar{Z}_\alpha = -\hbar \dfrac{\partial}{\partial Z^\alpha}$ $\qquad s = \dfrac{\hbar}{2}\left(-Z^\alpha \dfrac{\partial}{\partial Z^\alpha} - 2\right)$ $\quad$ [Eule

Massless field equations:

$$\phi_{\underbrace{AB\ldots L}_{n}} = \phi_{(AB\ldots L)}\,,\quad \nabla^{AA'}\phi_{AB\ldots L} = 0$$

helicity $-\frac{n}{2}$

$$\Box\phi = 0$$

helicity 0

$$\phi_{\underbrace{A'B'\ldots L'}_{n}} = \phi_{(A'B'\ldots L')}\,,\quad \nabla^{AA'}\phi_{A'B'\ldots L'} = 0$$

helicity $+\frac{n}{2}$

(assuming these are positive-frequency wave functions)

		Helicity	Twistor function hom. deg.
Scalar wave	$\Box\phi = 0$	0	-2
Dirac-Weyl: neutrino	$\nabla^{AA'}\nu_A = 0$	$-1/2$	-1
Dirac-Weyl: anti-neutrino	$\nabla^{AA'}\tilde{\nu}_{A'} = 0$	$+1/2$	-3
Maxwell photon $F_{ab} \rightsquigarrow \varphi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\tilde{\varphi}_{A'B'}$			
left-handed (anti-s.-d.)	$\nabla^{AA'}\varphi_{AB} = 0$	-1	0
right-handed (self-dual)	$\nabla^{AA'}\tilde{\varphi}_{A'B'} = 0$	$+1$	-4
Linearized Einstein graviton $K_{abcd} \rightsquigarrow \psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\tilde{\psi}_{A'B'C'D'}$			
left-handed (anti-s.-d.)	$\nabla^{AA'}\psi_{ABCD} = 0$	-2	$+2$
right-handed (self-dual)	$\nabla^{AA'}\tilde{\psi}_{A'B'C'D'} = 0$	$+2$	-6

Massless Field Contour Integral

Whittaker 1903, Bateman 1904, 1944, RP 1968, 1969, 1975, Hughston 1973, 1979

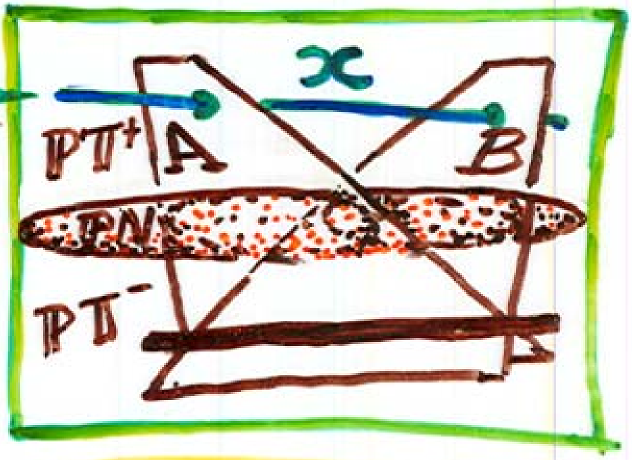
$S=0$: Wave eqn. $\Box\phi=0$ $\quad f(Z)$ hom. deg. $=-2$

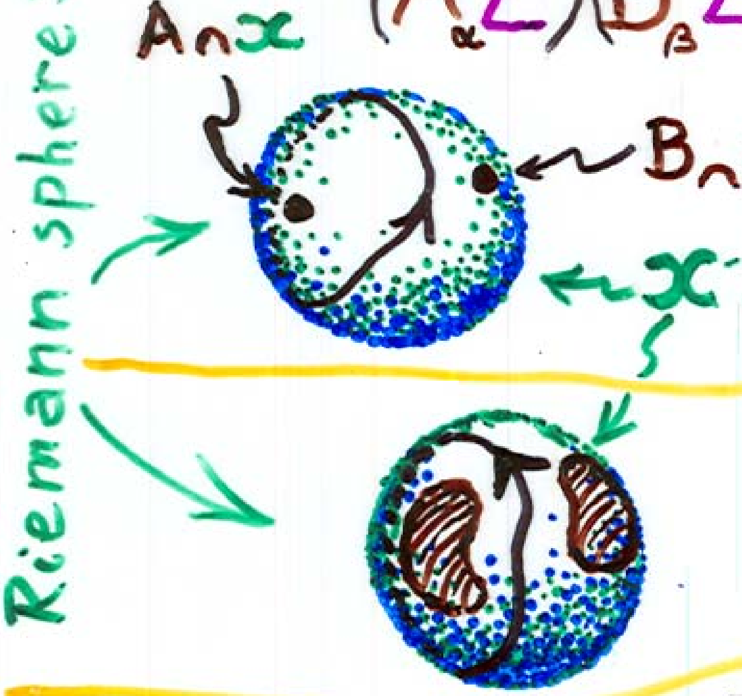
$$\boxed{\phi(x)=\oint_{\omega=ix\pi} f(Z)\,\delta Z}$$

$$\delta Z=\varepsilon^{A'B'}\pi_{A'}\,d\pi_{B'}$$

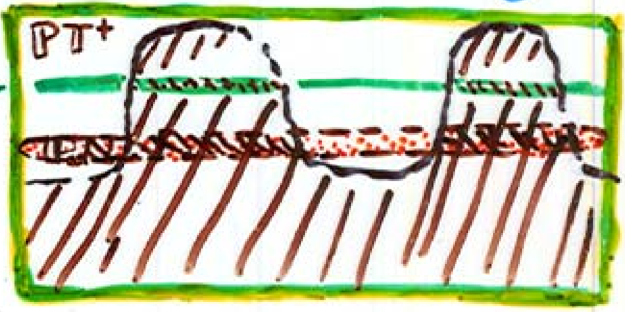
Typical case:

$$f(Z)=\frac{1}{(A_\alpha Z^\alpha)(B_\beta Z^\beta)}$$





Generally:



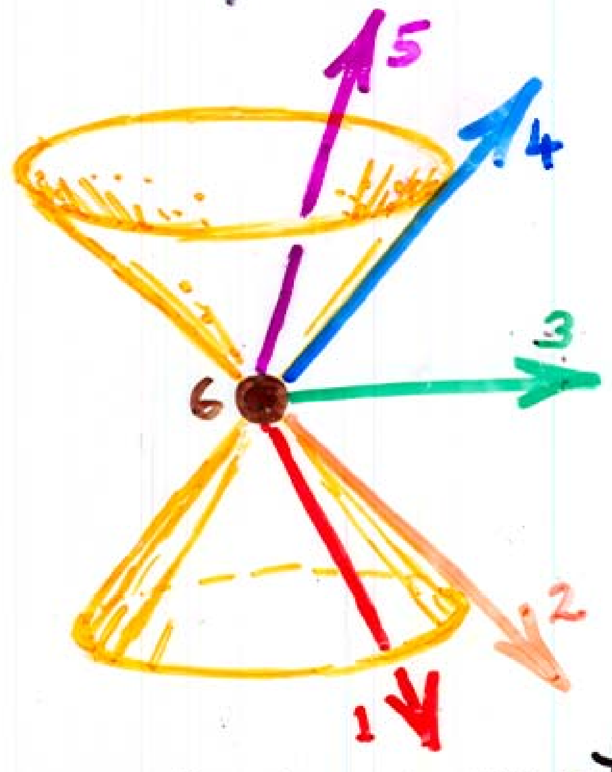
$S>0$ $\quad \nabla^{AA'}\phi_{A'B'C'\dots L'}=0$

$$\boxed{\phi_{A'B'\dots L'}(x)=\oint_{\omega=ix\pi}\pi_{A'}\pi_{B'}\cdots\pi_{L'}\,f(Z)\,\delta Z}$$

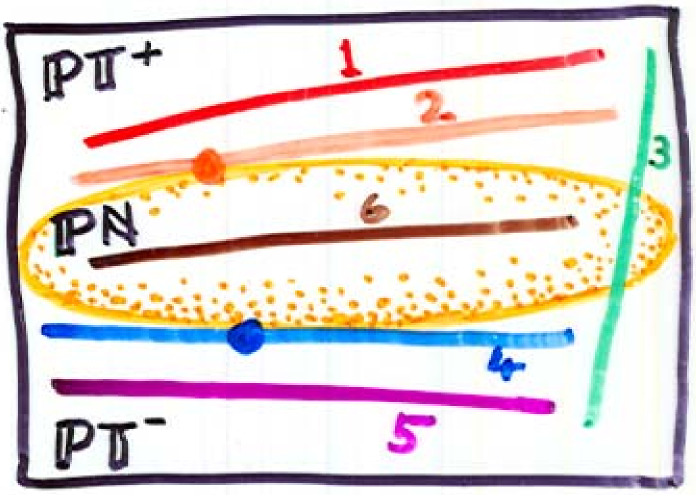
$S<0$ $\quad \nabla^{AA'}\phi_{ABC\dots L}=0$

$$\boxed{\phi_{AB\dots L}(x)=\oint_{\omega=ix\pi}\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B}\cdots\frac{\partial}{\partial\omega^L}f(Z)\,\delta Z}$$

Complex Minkowski Points

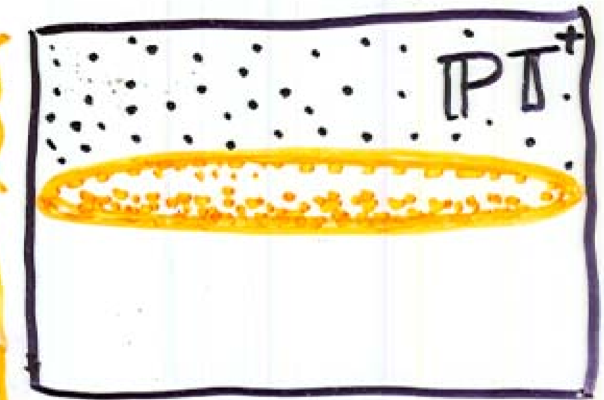


Imaginary part of complex position vector



Projective twistor space $\mathbb{PT}$

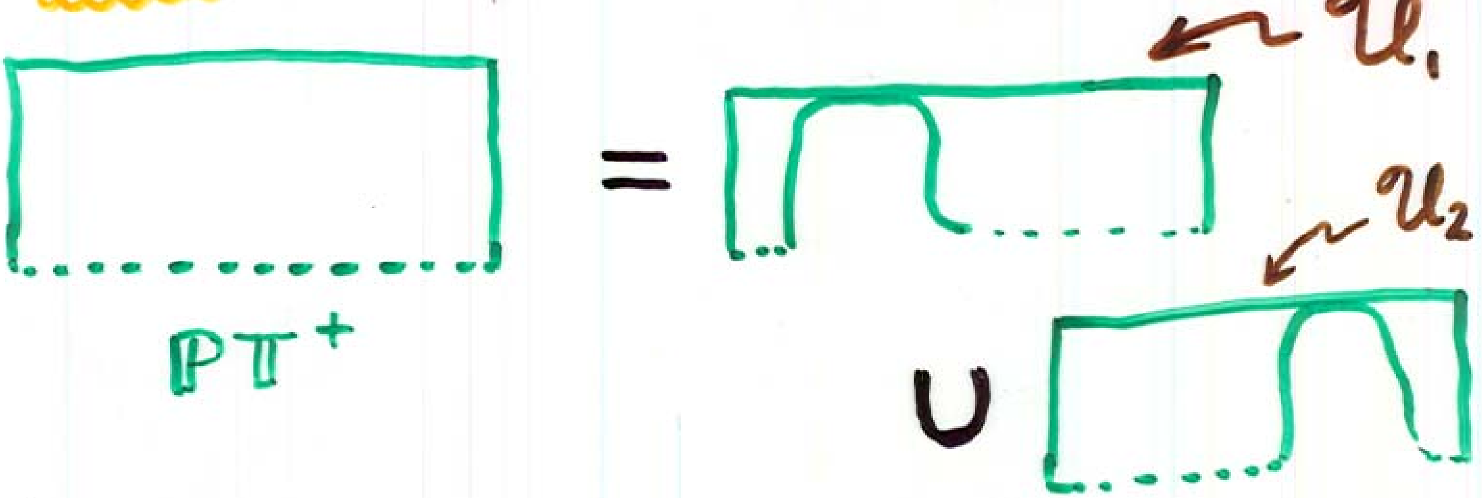
Corresponds to *forward tube* of complex Minkowski space: past-timelike imaginary part



Positive-frequency fields: extend holomorphically to the forward tube

Composed of $e^{P_a x^a / i\hbar}$ with  : tails off in forward tube

Twistor sheaf cohomology



$\mathbb{PT}^+ = \mathcal{U}_1 \cup \mathcal{U}_2$

f defined (holomorphic) on $\mathcal{U}_1 \cap \mathcal{U}_2$

More generally: space $\mathcal{X}$ $\left(\overset{\text{here}}{=} \mathbb{PT}^+\right)$

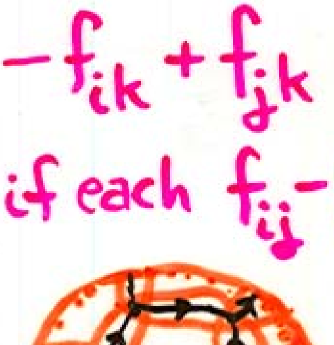
$\mathcal{X} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \ldots \cup \mathcal{U}_n$ ($\{\mathcal{U}_i\}$ open cover)

collection $\{f_{ij}\}$, where $f_{ij}\,(=-f_{ji})$ hol. on $\mathcal{U}_i \cap \mathcal{U}$

we require $f_{ij} - f_{ik} + f_{jk} = 0$ on $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$

and $\{f_{ij}\} \equiv \{g_{ij}\}$ if each $f_{ij} - g_{ij} = h_i - h_j$ with h_k hol. on $\mathcal{U}$

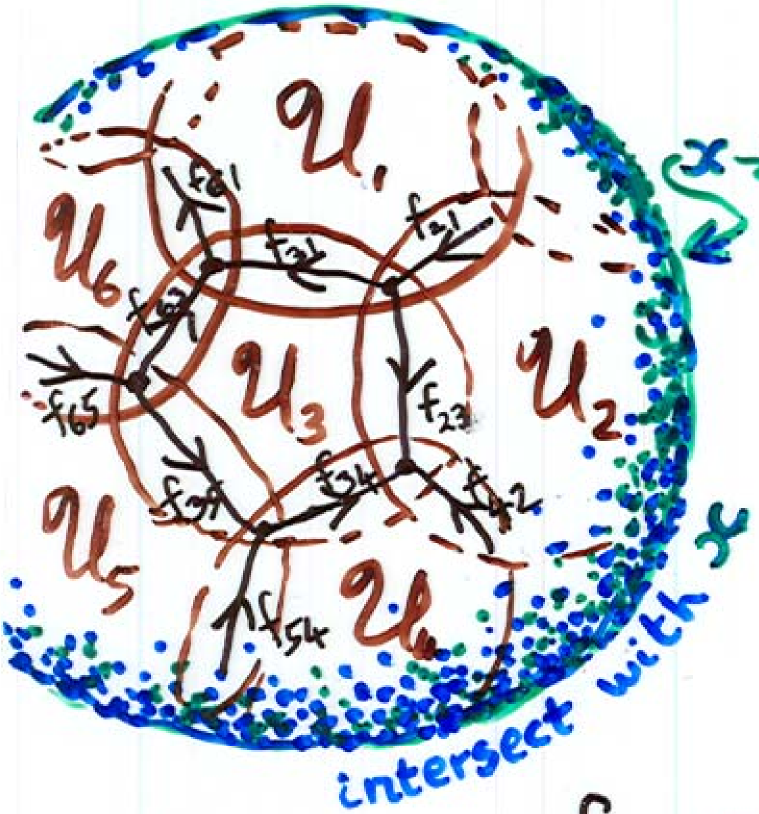
Branched contour integral:



Riemann sphere

Twistor (sheaf) cohomology

Cover (e.g.) $\mathbb{PT}^+$ with a (locally) finite number of open sets $\{\mathcal{U}_i\}$



$$f_{ji} = -f_{ij}$$

$$f_{ij} - f_{ik} + f_{jk} = 0$$ on triple overlaps

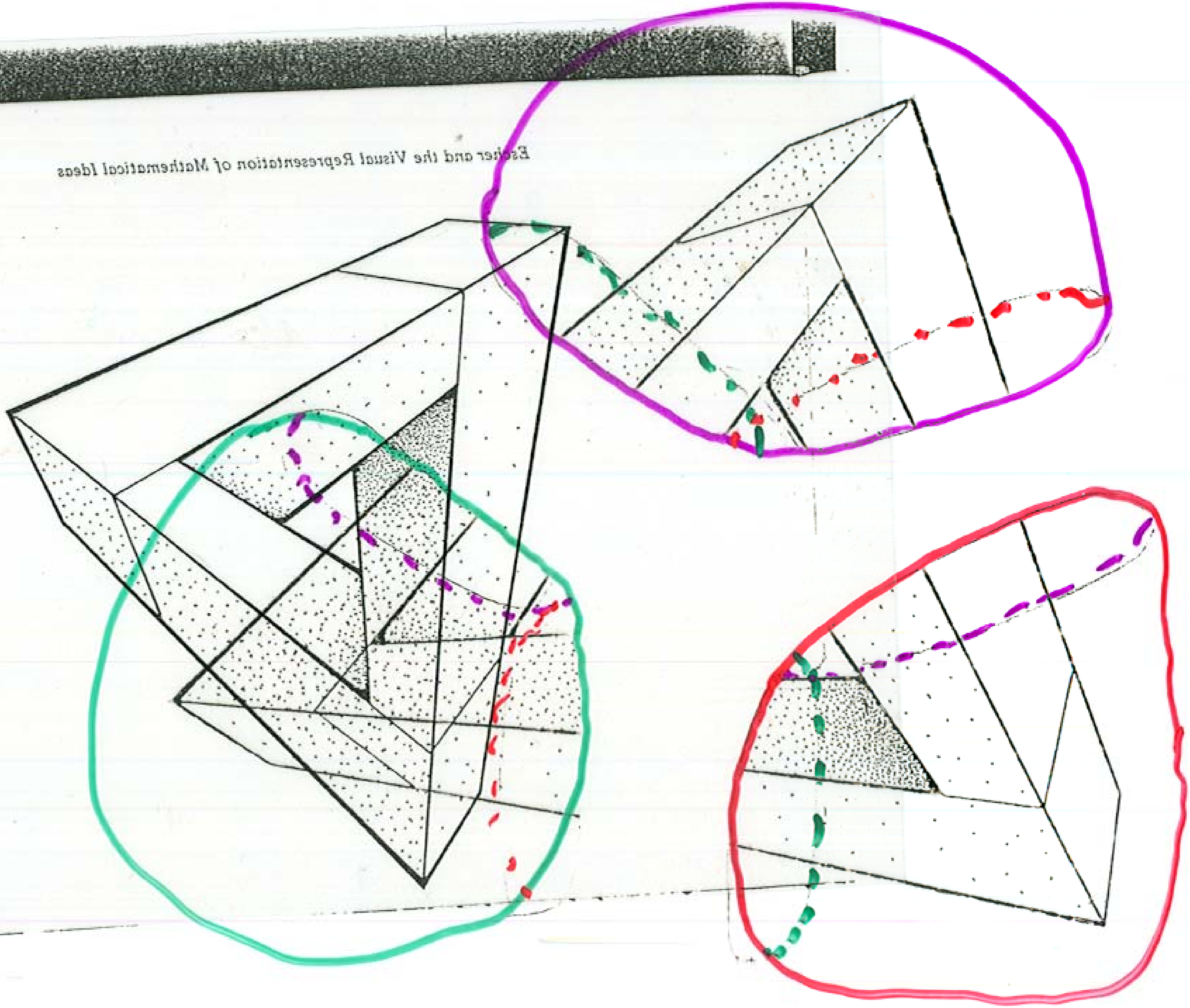
$\{f_{ij}\}$ is Čech cocycle

Branched contour integral evaluates $\{f\}$ cocycle/coboundaries

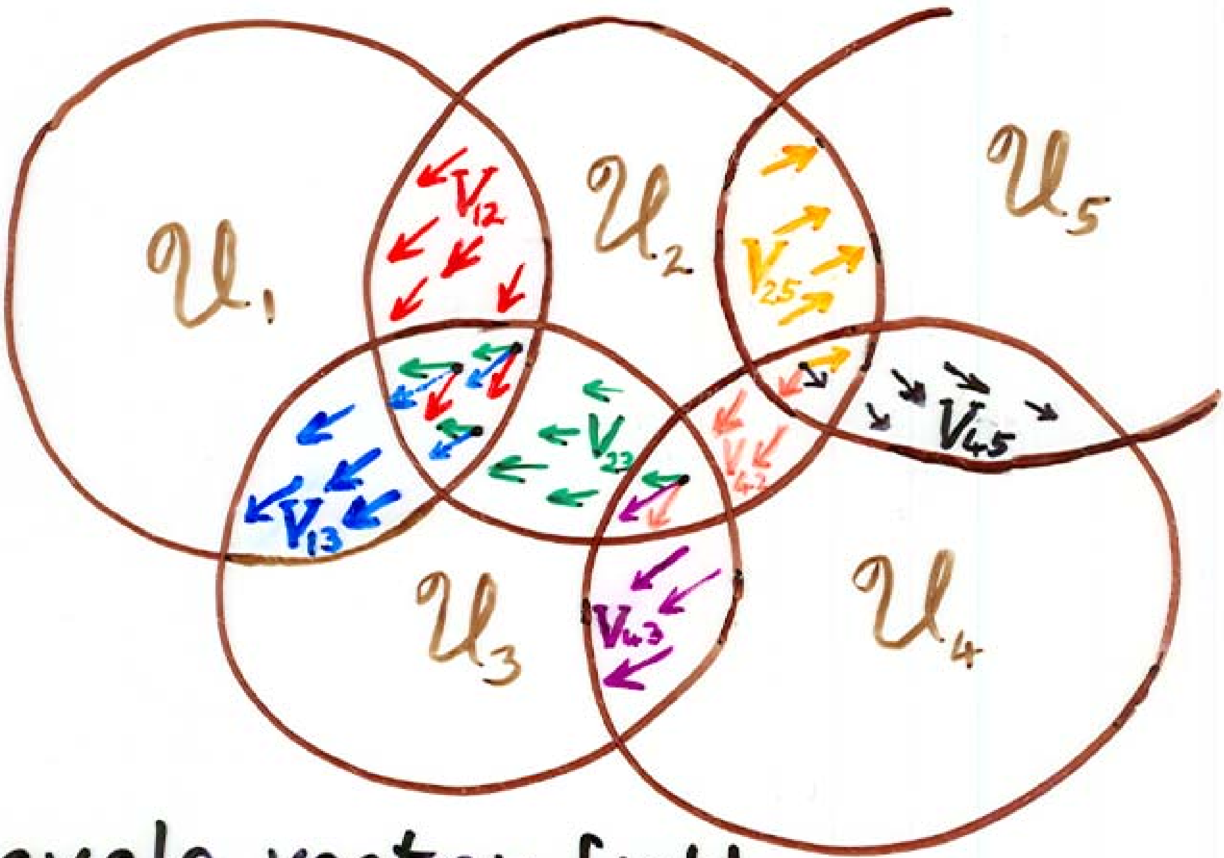
$\{f\} \equiv \{g\}$ if $f_{ij} - g_{ij} = h_i - h_j$ on overlaps

where h_i defined (holomorphic) on $\mathcal{U}_i$

$$\boxed{H^1(\mathbb{PT}^+, \mathcal{O}(-2s-2))}$$ massless free fields helicity s



Finite (Holomorphic) Deformations of Complex Manifolds — "Non-Linear 1st sheaf cohomology"



Cocycle vector field

$\{V_{ij}\}$ is infinitesimal form of

finite deformation — produces genuinely distinct complex manifold, in many cases

Non-linear graviton

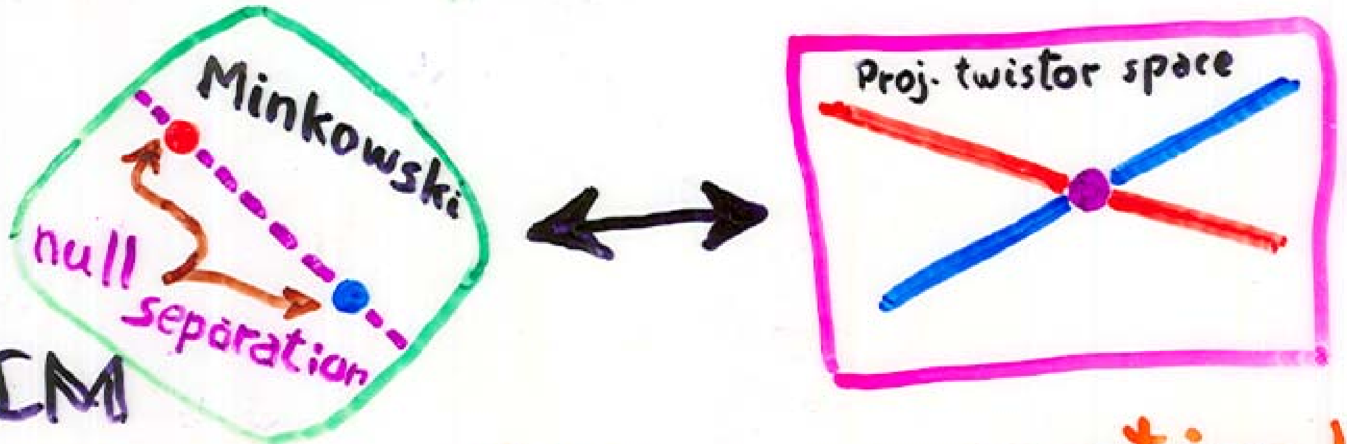
Ward construction of ASD gauge fld

General Relativity

Numerous special applications
(e.g. Woodhouse-Mason: stationary axi-symm.)

As part of general programme:
"non-linear graviton construction" [R.P. '76]

N.B. for flat space:

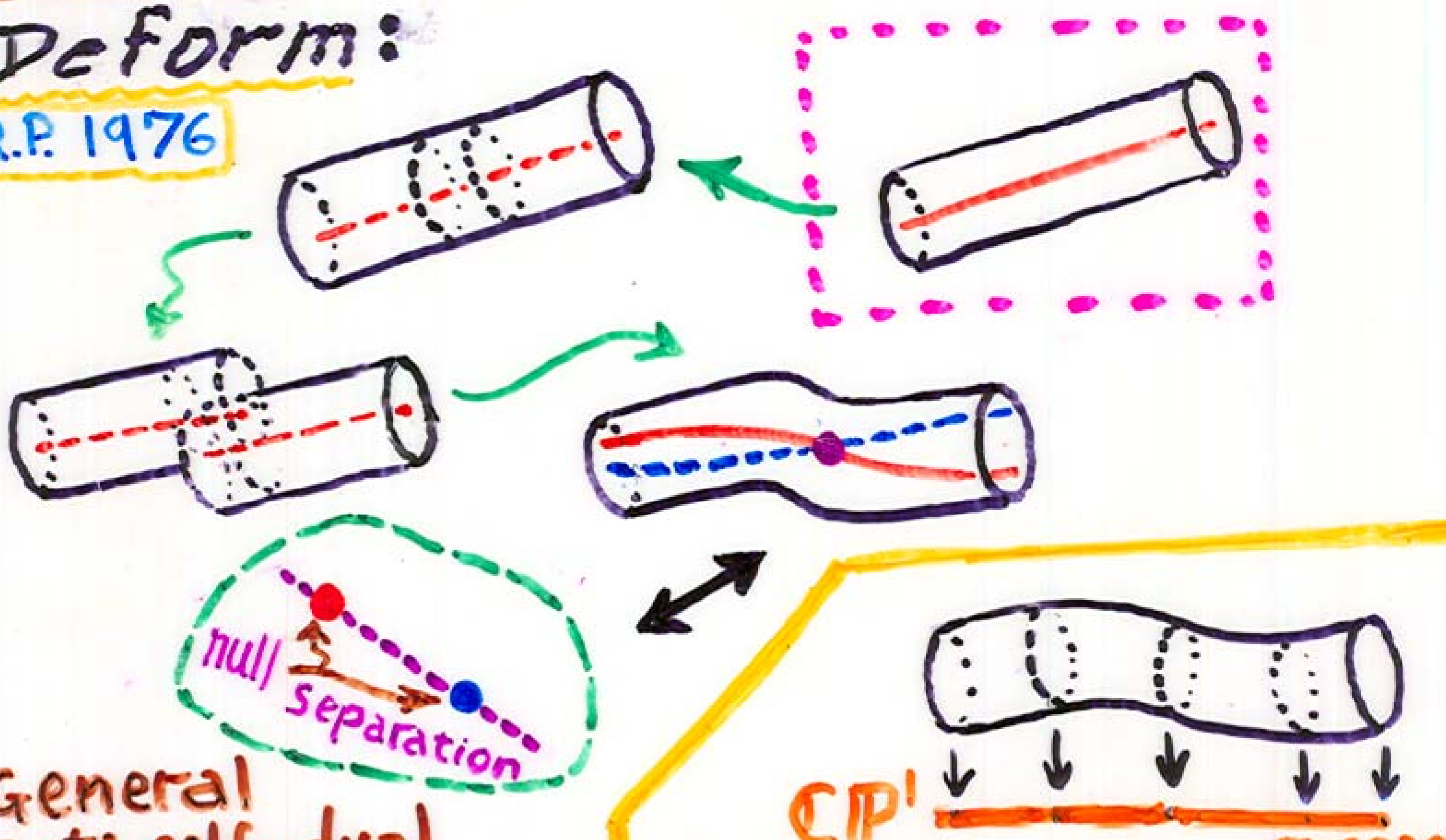


$\mathbb{CM}$ $\mathbb{CP}^3$

null separation ⟷ meeting lines

Deform:

R.P. 1976



$\mathbb{CP}^1$ π-space

General anti-self-dual complex "space-time"

General anti-self-dual Ricci-flat complex "space-time"

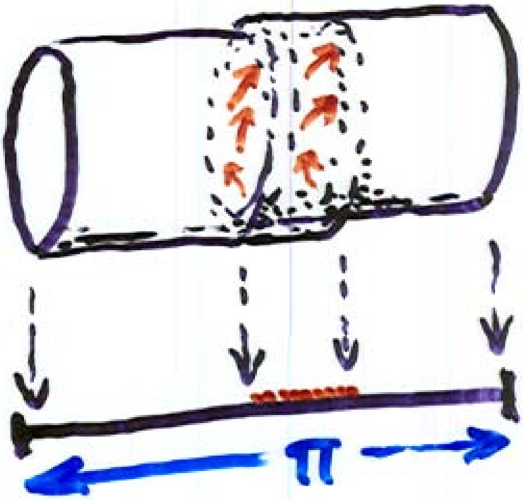
Relation to twistor function $f(Z)$ deg. $= +2$

Infinitesimal shunt:

$$\hat{\omega}^A = \omega^A + \epsilon\, \varepsilon^{AB} \frac{\partial f}{\partial \omega^B}$$

$$\hat{\pi}_{A'} = \pi_{A'}$$

vector field $\varepsilon^{AB} \dfrac{\partial f}{\partial \omega^B} \dfrac{\partial}{\partial \omega^A}$

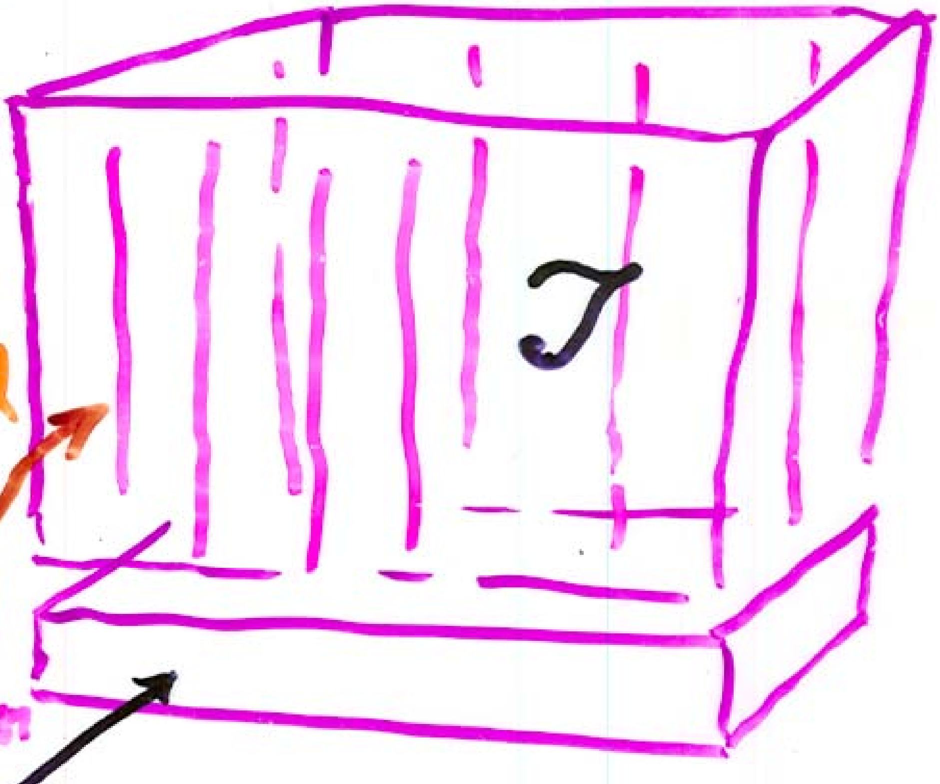


Agrees with

$$\psi_{ABCD} = \oint \frac{\partial}{\partial \omega^A} \cdots \frac{\partial f}{\partial \omega^D}\, \pi \cdot d\pi$$

How do we incorporate $\tilde{f}(Z)$ deg. $= -6$

Information of -6 deg. fn. (right-handed gravity) incorporated in a distortion of Euler fibres



$\{$ 3-dim projective $\mathbb{P}\mathcal{T}$, as before

Why is it important for the programme of twistor theory to have a twistor (not just ambitwistor) formulation of Einstein eqs?

Why twistor space $\mathbb{T}$, not just projective twistor sp. $\mathbb{PT}$?

Mixed helicity states:

$$F_{ab} = \varphi_{AB}\,\varepsilon_{A'B'} + \varepsilon_{AB}\,\tilde{\varphi}_{A'B'}$$

$$= \frac{1}{(2\pi i)^2}\oint\left(-\hbar^2\varepsilon_{A'B'}\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B} + \varepsilon_{AB}\pi_{A'}\pi_{B'}\right)f(Z)\,d^2\pi$$

Not hom.

$$K_{abcd} = \psi_{ABCD}\,\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\,\tilde{\psi}_{A'B'C'D'}$$

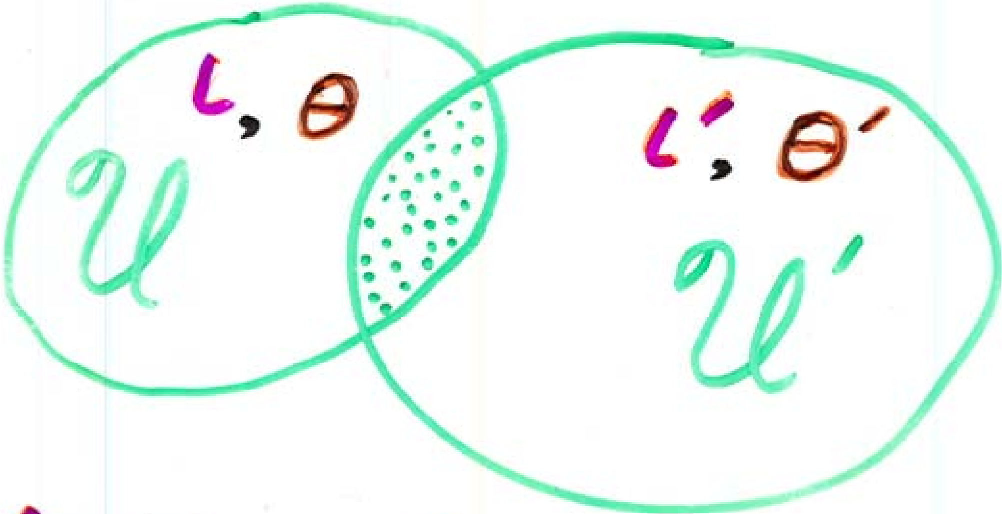
$$= \frac{1}{(2\pi i)^2}\oint\left(\hbar^4\varepsilon_{A'B'}\varepsilon_{C'D'}\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B}\frac{\partial}{\partial\omega^C}\frac{\partial}{\partial\omega^D} + \varepsilon_{AB}\varepsilon_{CD}\pi_{A'}\pi_{B'}\pi_{C'}\pi_{D'}\right)f(Z)\,d^2$$

WAVEFUNCTIONS

TWISTOR PARTICLE THEORY

GENERAL GAUGE FIELDS

Googly scalings



$$\left.\begin{aligned} \iota' &= k\iota \\ \theta' &= k^2\theta \\ d\theta' &= k^{-1}\, d\theta \end{aligned}\right\}$$

preserves Π and Σ

and $\underbrace{d\theta \otimes \iota = -2\theta \otimes d\iota}_{\text{gives}}$

$$\boxed{\Upsilon(k) = 2k^{-2} - 2k}$$

[equiv.: $\Upsilon'(k^{-1}) = 2k^{2} - 2k^{-1}$ since $\Upsilon' = k^3\Upsilon$

recall: $\Upsilon = \theta \div (\tfrac{1}{4} d\theta)$]

Standard param. up Euler curves $\nearrow \Upsilon$

$$\Upsilon(z) = z$$

Find: $k^3 = 1 - F z^{-6}$ where F const. on cur

$$\therefore \boxed{k^3 = 1 - f_{-6}(Z^{\alpha})}$$

Twistor-space forms | $\mathbb{M}$

1-form: $\iota\ (=\delta z)$ $= I_{\alpha\beta} Z^\alpha dZ^\beta = \varepsilon^{A'B'} \pi_{A'} d\pi_{B'}$

2-form: $\tau = \frac{1}{2} d\iota$ $= \frac{1}{2} I_{\alpha\beta} dZ^\alpha \wedge dZ^\beta = d\pi_{0'} \wedge d\pi_{1'}$

3-form: θ

$$= \frac{1}{6} \varepsilon_{\alpha\beta\gamma\delta} Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta$$

$$= Z^0 dZ^1 \wedge dZ^2 \wedge dZ^3 - Z^1 dZ^0 \wedge dZ^2 \wedge dZ^3 + Z^2 dZ^0 \wedge dZ^1 \wedge dZ^3 - Z^3 dZ^0 \wedge dZ^1 \wedge dZ^2$$

4-form: $\phi = \frac{1}{4} d\theta$

$$= \frac{1}{24} \varepsilon_{\alpha\beta\gamma\delta} dZ^\alpha \wedge dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta$$

$$= dZ^0 \wedge dZ^1 \wedge dZ^2 \wedge dZ^3$$

Euler: $\Upsilon = \theta \div \phi = Z^\alpha \partial / \partial Z^\alpha$

$$da \wedge \theta = \Upsilon(a)\, \phi$$

$$\boxed{\iota \wedge \tau = 0, \quad \iota \wedge \theta = 0}$$

Homogeneity degrees

$$\mathcal{L}_{\Upsilon} \begin{bmatrix} \iota \\ \tau \\ \theta \\ \phi \end{bmatrix} = \begin{bmatrix} 2\iota \\ 2\tau \\ 4\theta \\ 4\phi \end{bmatrix} \quad \begin{matrix} \} \Leftrightarrow \theta \otimes \tau = -\phi \otimes \iota \\ \} \text{ automatic } (\theta \otimes \phi \equiv -\phi \otimes \theta) \end{matrix}$$

where

$$\alpha \otimes \beta = r\, \alpha_{[\dots}\, \beta_{\underbrace{.]\dots}_{}}$$

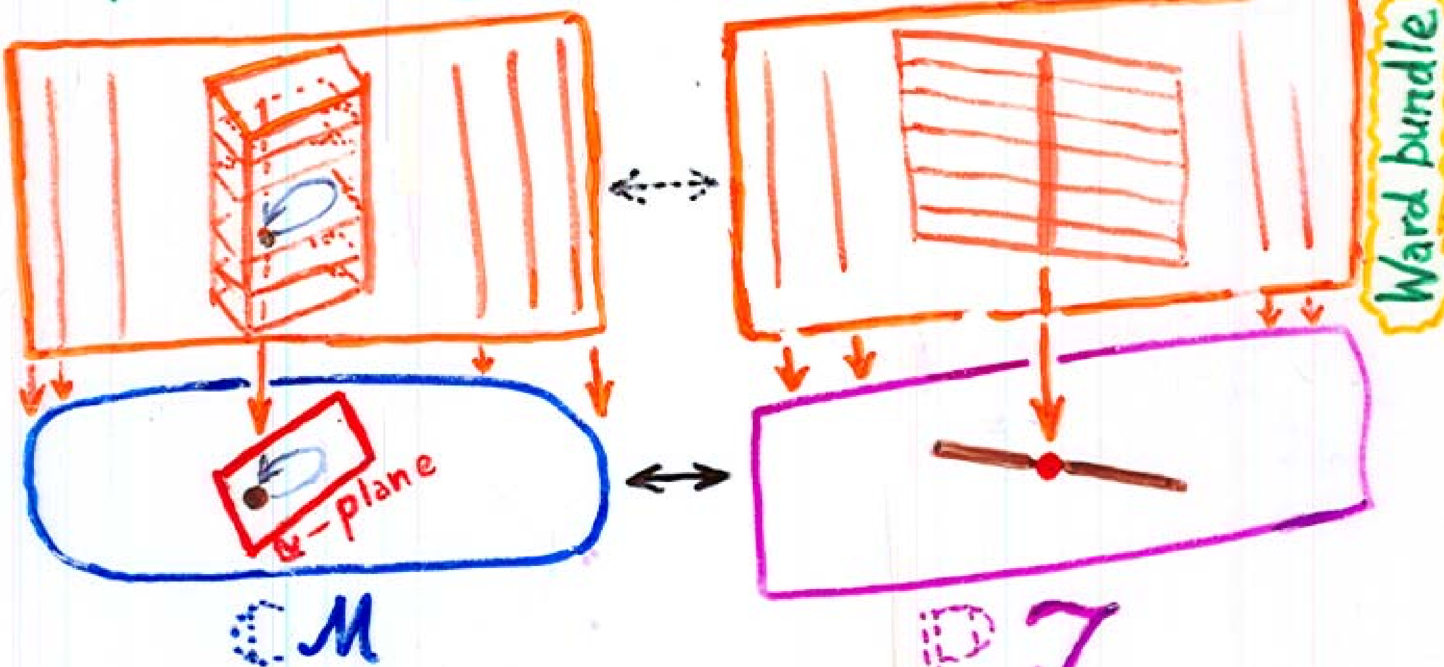
r-form

e.g.

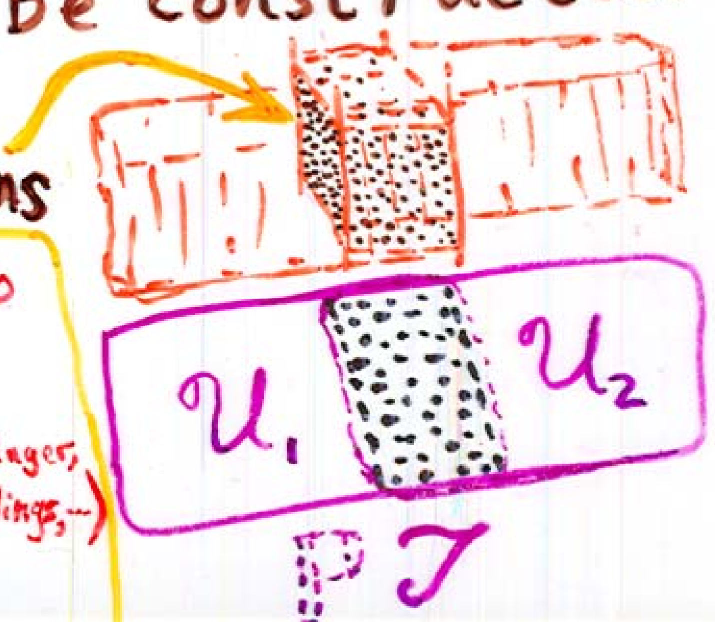
$$\alpha \otimes dp \wedge dq = \alpha \wedge dp \otimes dq - \alpha \wedge dq \otimes dp$$

Ward construction for (anti-)self-dual Yang-Mills fields

Y-M connection on an (analytic) space $\mathcal{M}$, with ASD conformal curvature, is ASD iff it is integrable on α-planes [in its complexification $\mathbb{C}\mathcal{M}$]



Ward bundle can be constructed in terms of **free** transition functions

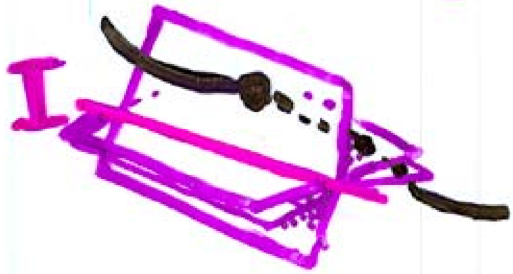


Many applications to integrable systems (KdV, sine-Gordon, non-lin Schrödinger, Toda lattice, Einstein with 2 Killings, --)

Ward, Sparling, Mason, Woodhouse, ...

Questions for String Theorists

- Do we need the extra dims.?
  can $\mathbb{PT}$ take the place
  of Calabi-Yau?
- Do we need the supersymm.?
  Is parameterization invar.
  needed in twistor theory?



(conformal
breaking)

- Do we really want (++−−)
  & a (pseudo-)Wick rot$\underline{\Delta}$?
  Elemental states?

2 "ear
& cohomol

- No helicity for off-shell
  "gluing MHVs"

holomorphic

- Theory?? "Topological" QFT?