Yang-Mills Amplitudes

from

Twistor String Theory

M. Spradlin
KITP

hep-th/0402016
/0403190
/0412265

with R. Roiban
and
A. Volovich

Outline

Several versions of twistor string theory have been proposed, and two of them (Witten's originally proposed topological B-model on $\mathbb{P}^{3|4}$, and the open string version of Berkovits) have been used to successfully recover tree-level gluon scattering amplitudes in Yang-Mills theory, from twistor string theory.

In fact there are several versions of the calculation, each of which has some poorly understood twists & turns, and each of which reveals different structure. I will discuss in detail the first successful calculation of non-MHV amplitudes, based on the "connected prescription." [Roiban, Volovich, MS]

Setup

The subject of this talk will be (mostly tree-level) gluon scattering amplitudes in Yang-Mills theory (QCD, $\mathcal{N}=4, \ldots$).

Instead of labeling each gluon by its momentum p_i^μ and polarization ϵ_i^μ, which are inefficient variables ($p_i^2=0$, $p_i \cdot \epsilon_i = 0$, $\epsilon_i \sim \epsilon_i + \alpha p_i$), we recall that it is more convenient to use spinor-helicity notation.

We decompose $p_{a\dot{a}} = p^\mu (\sigma_\mu)_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ and use the fact that specifying $\lambda, \tilde{\lambda}$ and a choice of helicity (+ or −) determines the polarization ϵ^μ (up to a gauge transformation).

$$= A_n(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}, \pm_i)$$

The only remaining redundancy in the variables $(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}})$ is that there is a scaling symmetry which leaves $p_\mu = (\sigma_\mu)_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}}$ unchanged: $(\lambda^a, \tilde{\lambda}^{\dot{a}}) \sim (t\lambda^a, t^{-1}\tilde{\lambda}^{\dot{a}})$ for any $t \in \mathbb{C}^*$.

Penrose transform

When "$\frac{1}{2}$-Fourier transformed", the amplitude

$$\tilde{A}(\lambda_i^a, \mu_i^{\dot{a}}, \pm_i) = \int d^{2n}\tilde{\lambda}_i^{\dot{a}} \, \exp\left[i \sum_{i=1}^{n} \mu_i^{\dot{a}} \tilde{\lambda}_{i\dot{a}}\right] A(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}, \pm_i)$$

becomes a function of n points on projective twistor space $\mathbb{P}^3$, with homogeneous coordinates $Z^I = (\lambda^1, \lambda^2, \mu^1, \mu^2)$.

Amplitudes $\Longleftrightarrow$ Curves in $\mathbb{P}^3$

Witten (2003) checked in several cases, and conjectured in general, that an n-gluon amplitude $\tilde{A}_n(\lambda_i^a, \mu_i^{\dot{a}}, \pm_i)$ with

q negative helicity gluons and

$n-q$ positive helicity gluons

is supported on holomorphic curves in $\mathbb{P}^3$ of degree $\boxed{d = q-1.}$

By itself, this remarkable observation is obviously not enough to calculate an amplitude. It implies that

$$\tilde{A}_n = \int d\mathcal{M}_{n,d} \quad (\text{something})$$

moduli space of degree d curves with n marked points

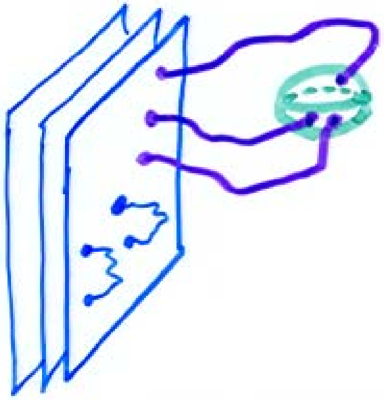
but we still need to know (something).

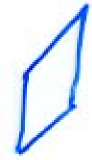
Goal of the Talk

Witten further proposed that the Yang–Mills scattering amplitudes could be calculated from a particular string theory: the open topological B-model on $\mathbb{P}^{3|4}$.

My goal in this talk will be to show (following hep-th/0403190) how to calculate (something) from the B-model. We will thereby obtain a new and interesting formula for any tree-level gluon scattering amplitude in Yang–Mills theory.

<u>How to calculate amplitudes in the B-model</u>

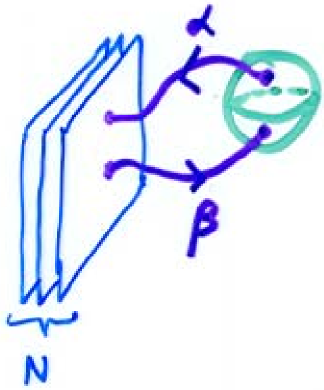


We start with N D5-branes  wrapping the bosonic dimensions of $\mathbb{P}^{3|4}$. The open strings  on the D5-branes will be the gluons (more details on this later...). The effective action for the gluons receives contributions from instantons — D1-branes  wrapping some holomorphic curve (topologically a $\mathbb{P}^1$) inside $\mathbb{P}^{3|4}$. Integrating out the D1-D5 string  degrees of freedom gives the following contribution to the gluon scattering amplitude:

$$A_n \sim \underbrace{\int d\mathcal{M}}_{\substack{②\\ \text{moduli space of curves}}} \underbrace{\int d^n z}_{\boxed{n \text{ points on } \mathbb{P}^1}} \underbrace{\langle J(z_1)\cdots J(z_n)\rangle}_{\substack{①\\ \text{open string}\\ \text{correlation}\\ \text{function}}} \underbrace{\prod_{i=1}^{n} \phi_i\big(\overbrace{Z}^{\text{embedding } \mathbb{P}^1 \to \mathbb{P}^{3|4}}(z_i)\big)}_{\substack{③ \text{ gluon}\\ \text{wavefunctions}}}$$

(1) The open string correlation function

$$A_n \sim \int d\mathcal{M} \int d^n z \; \boxed{\langle J(z_1) \cdots J(z_n) \rangle} \prod_{i=1}^{n} \phi_i(Z(z_i))$$



We combine the D1-D5 strings α and the D5-D1 strings β into $J = \alpha\beta$ which takes values in the Lie algebra of $GL(N,\mathbb{C})$ (since there are N D5-branes).

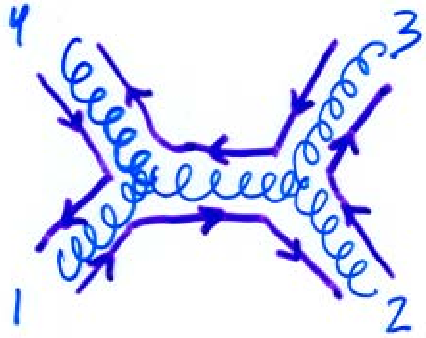
If we choose the wavefunction of the i^{th} gluon to be proportional to T^{a_i} (a generator of $GL(N,\mathbb{C})$) then the correlator is

$$\langle J(z_1) \cdots J(z_n) \rangle = \frac{\mathrm{Tr}\left[T^{a_1} \cdots T^{a_n}\right]}{(z_1 - z_2) \cdots (z_n - z_1)} + \text{permutations}$$

$$+ \left(\frac{\mathrm{Tr}\left[T^{a_1} \cdots T^{a_k}\right]}{(z_1 - z_2) \cdots (z_k - z_1)} \; \frac{\mathrm{Tr}\left[T^{a_{k+1}} \cdots T^{a_n}\right]}{(z_{k+1} - z_{k+2}) \cdots (z_n - z_{k+1})} + \text{perm} \right)$$

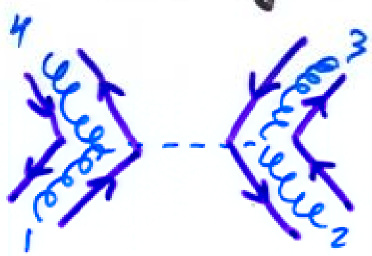
$+$ etc.

Tree-level gluon amplitudes are always single-trace interactions



$\propto \ Tr[T^{a_1} T^{a_2} T^{a_3} T^{a_4}]$

whereas multi-trace interactions indicate the interaction of gauge-singlet particles (scalars or gravitons).



$\propto \ Tr[T^{a_1} T^{a_4}] \ Tr[T^{a_2} T^{a_3}]$

As Witten noted, this already indicates that the B-model has more than just Yang-Mills...

But for our purposes it is sufficient to look at the coefficient of $Tr[T^{a_1} \dots T^{a_n}]$

(2) Integration over the moduli space of curves

At genus 0, a curve in $\mathbb{P}^{3|4}$ is most conveniently parametrized by writing an embedding as an explicit degree d polynomial

homogeneous coordinates on $\mathbb{P}^{3|4}$
$$\begin{cases} Z^I(z) = \sum_{k=0}^{d} z^k a_k^I \\ \Psi^A(z) = \sum_{k=0}^{d} z^k \beta_k^A \end{cases}$$
z = coordinate on $\mathbb{P}^1$

To integrate over all curves means to integrate over all of the parameters a, β.

$$d\mathcal{M} = \left(\prod_{k=0}^{d} d^4 a_k \, d^4 \beta_k \right) \Big/ \underbrace{SL(2)}_{} \times \underbrace{U(1)}_{}$$

[Witten]

reparametrizations of $\mathbb{P}^1$

overall scaling of a, β

(3) Wavefunctions for the gluons

$$A_n \sim \int d\mathcal{M} \int d^n z \; \langle J(z_1) \cdots J(z_n) \rangle \boxed{\prod_{i=1}^{n} \phi_i(Z_I(z_i))}$$

The physical open string states of the B-model correspond to $H^1(\mathbb{P}^{3|4}, \mathcal{O}(-k))$ $k=0,1,2,3,4$.

By the Penrose theorem, these cohomology groups = the space of solutions of the (conformally invariant) free massless wave equation of helicity

$h = 1 - k/2 = 0, \pm\frac{1}{2}, \pm 1$ = field content of $\mathcal{N}=4$ SYM

But which solutions of the free wave equations should we choose? In particle physics we of course usually use plane waves, with definite momentum p (in $\mathbb{R}^4$). $\phi_i(p) = \delta^4(p_i - p)$

What does this look like in twistor variables?

Start with

$$\phi_i(\lambda^a, \mu^{\dot a}) = \int \frac{d\xi}{\xi}\, \delta^2(\lambda_i^a - \xi \lambda^a)\, \delta^2(\mu_i^{\dot a} - \xi \mu^{\dot a})$$

which is a properly normalized δ-function on $\mathbb{P}^3$ (integration over ξ takes care of homogenaeity).

Now Fourier transform $\mu_i \to \tilde\lambda_i$ to get the desired wavefunction

$$\boxed{\phi_i(\lambda^a, \mu^{\dot a}) = \int \frac{d\xi}{\xi}\, \delta^2(\lambda_i^a - \xi \lambda^a)\, \exp\left[i \mu_{\dot a} \tilde\lambda_i^{\dot a}\, \xi\right]}$$

$$\Longleftrightarrow \quad \delta^4(\lambda_i^a \tilde\lambda_i^{\dot a}) = \delta^4(p_i^{a\dot a})$$

Note: we do this for each gluon so we get $\xi_i \quad i = 1, \ldots, n$

The Twistor String Amplitude

Putting together all of the ingredients gives

$$A(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}) = \int \frac{d^{4d+4}a \;\; d^n z}{GL(2)} \; \frac{d^n \xi}{\xi_1 \cdots \xi_n}$$

$$\times \frac{1}{(z_1 - z_2)\cdots(z_n - z_1)} \prod_{i=1}^{n} \delta^2(\lambda_i^a - \xi_i Z^a(z_i))$$

$$\prod_{i=1}^{n} \exp\left[i \tilde{\lambda}_i^{\dot{a}} \xi_i Z^{\dot{a}}(z_i)\right]$$

where the embedding $\mathbb{P}^1 \to \mathbb{P}^3$ is $Z^I(z) = \sum_{k=0}^{d} z^k a_k^I$.

Now half of the a_k^I – those with $I = \dot{a}$ – appear in the exponential and can be integrated out easily to give more δ-functions.

Localization of the Integral

This gives

$$A(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}) = \int \frac{d^{2d+2}a \; d^n z}{GL(2)} \frac{d^n \xi}{\xi_1 \cdots \xi_n} \frac{1}{(z_1 - z_2)\cdots(z_n - z_1)}$$

$$\times \prod_{i=1}^{n} \delta^2\left(\lambda_i^a - \xi_i \mathcal{Z}^a(z_i)\right) \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \xi_i z_i^k \tilde{\lambda}_i^{\dot{a}}\right)$$

There are $(2d+2) + n + n \overset{GL(2)}{-4} = 2n + 2d - 2$ integration variables and $2n + 2d + 2$ δ-functions.

In fact, it is easy to see momentum conservation:

$$\begin{aligned} 0 &= \sum_{k=0}^{d} a_k^a \left(\sum_{i=1}^{n} \xi_i z_i^k \tilde{\lambda}_i^{\dot{a}}\right) \\ &= \sum_{i=1}^{n} \tilde{\lambda}_i^{\dot{a}} \left(\xi_i \sum_{k=0}^{d} a_k^a z_i^k\right) \\ &= \sum_{i=1}^{n} \tilde{\lambda}_i^{\dot{a}} \left(\xi_i \mathcal{Z}^a(z_i)\right) = \sum_{i=1}^{n} \tilde{\lambda}_i^{\dot{a}} \lambda_i^a = \sum_{i=1}^{n} (p_i)^{a\dot{a}} \end{aligned}$$

Accounting for the overall $\delta^4(\Sigma p)$, there are as many δ-functions as integration variables, so the integral localizes onto discrete points.

The "Connected Prescription" For The tree-level S-matrix of Yang-Mills theory

[Roiban, Volovich, MS]

Given $(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}})$, find all solutions to the

$2n+2d+2$ polynomial equations

$$0 = H_r \equiv \begin{cases} \sum_{i=1}^{n} \xi_i z_i^k \tilde{\lambda}_i^{\dot{a}} & \dot{a}=1,2 \quad k=0,\dots,d \\ \lambda_i^a - \xi_i \sum_{k=0}^{d} z_i^k a_k^a & a=1,2 \quad i=1,\dots,n \end{cases}$$

$r=1,\dots,2n+2d+2$

in terms of the $2n+2d+2$ variables $q_r \equiv (z_i, \xi_i, a_k^a)$.

Then the scattering amplitude is NO ABSOLUTE VALUE!

$$\boxed{A(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}) = \delta^4\left(\sum_{i=1}^{n} \lambda_i^a \tilde{\lambda}_i^{\dot{a}}\right) \sum_{q : H(q)=0} \left[J \frac{(\det F)^4}{\det(\partial H_r / \partial q_s)} \right]}$$

where $J = \dfrac{1}{(\xi_1 \cdots \xi_n)} \dfrac{1}{(z_1 - z_2) \cdots (z_n - z_1)} \times GL(2)$ determinant

and $F_{i'k} = \xi_{i'} (z_{i'})^k$ is a $(d+1) \times (d+1)$ matrix.

(i' runs over the negative helicity gluons)

Comment 1: The formula is correct!

For mostly-plus ($d=1$) $(+++\cdots++--)$

and mostly-minus ($d=n-3$) $(---\cdots--++)$

MHV amplitudes, it is straightforward to check that the equations have a unique solution, which reproduces the known expression for these amplitudes. (Reduces to Witten & Nair for MHV)

For other amplitudes, the equations must be solved numerically, to compare with known results.

In general, the formula can be shown to satisfy a number of consistency conditions which suggest that it is correct.

It would be nice to have a direct proof from gauge theory, and/or a more rigorous derivation from the B-model.

Comment 2: Parity Invariance (The Googly Problem)

One important check is parity invariance

$(\lambda \leftrightarrow \tilde{\lambda})$ and $(+ \leftrightarrow -$ helicity$)$

which is completely obscured in twistor space.

Nevertheless, it is elementary to prove that [Roiban, Volovich, MS]

- The roots of $H_r(q_s)=0$, for any n and d, are in 1-1 correspondence with the roots of the parity-flipped equations $\tilde{H}_r(\tilde{q}_s)=0$ (which have $\tilde{d}=n-d-2$). (The proof is constructive.)
- The Jacobian transforms in precisely the right way to account for $+ \leftrightarrow -$.

$\Rightarrow$ Connected prescription is parity symmetric.

(See also a similar proof by Witten.)

Comment 3: The number of roots?

One very interesting question is: what is the number of roots $N_{n,d}$ as a function of n & d? At the moment all we know is

$$N_{n,1} = N_{n,n-3} = 1 \qquad \text{MHV \& googly MHV}$$

$$N_{6,2} = 4$$

$$N_{n,d} = N_{n,n-d-2} \qquad \text{parity invariance}$$

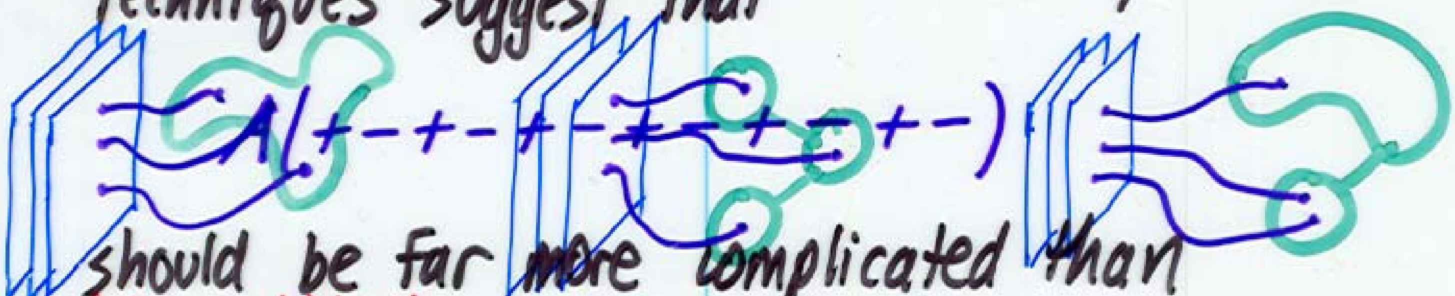
Does $N_{n,d}$ grow polynomially or exponentially?

Of course, this question only makes sense within the context of our particular choice of the manner in which to fix the $GL(2)$ symmetry. Choosing to fix a different set of variables can lead to different equations. Maybe a less obvious $GL(2)$ fixing leads to simpler equations?

Comment 5: Connected or Disconnected?

~~Comment 4: Helicity Permutations~~

We considered only connected instantons,
Almost all of the currently-known calculational
but why not disconnected, or partially connected?
techniques suggest that



should be far more complicated than

= string worldsheet

$A(--++++++)$

Remarkably, we found that connected instantons

... amplitude.

... completely disconnected

... theory, and

... intermediate

... choice of

... equivalence

of ... (det F)

from localization of the integral onto poles which

are shared by the various moduli spaces.

Comment 6: Integration contour

One important but not completely understood issue is the choice of integration contour.

[illegible] space'? is really

[illegible]

[illegible] structure

[illegible] Yang-Mills theory.

[See [illegible] Britto Dixon Gukov ~~Klose~~ Neitzke makes sense in signature [illegible]

Kosower Roiban Svrcek Travaglini Witten, and papers by these collaborators.] But this doesn't work! It can happen

[illegible] complex [illegible] λ & $\tilde\lambda$ are real).

$$A(1^-,2^-,3^-,\ldots,7^+,8^+) = \frac{[1|K_3^{[3]}|5\rangle^3}{t_2^{[4]}\,[23][34][45]\langle 67\rangle\langle 78\rangle\langle 81\rangle\,[2|K_3^{[3]}|6\rangle} - \frac{1}{t_7^{[3]}\,[\text{illegible}]\,\langle 7|K_{[\text{illegible}]}^{[2]}|2]}\left[\frac{\langle 1|K_7^{[2]}K_3^{[4]}[\text{illegible}]|5]^3}{[\text{illegible}]} + \frac{\langle 1|K_7^{[2]}K_5^{[2]}|4\rangle^3}{t_{[\text{illegible}]}^{[3]}\,[23]\langle 45\rangle\langle 56\rangle}\right] + [\text{illegible}]$$

Still one must sum over all roots, whether real or complex, to reproduce Yang-Mills amplitude! [illegible] absolute value!

A Quadratic Recurrence

Taking ① & ② together obviously demonstrates the existence of a new, quadratic recurrence relation governing tree amplitudes! [Roiban, Volovich, MS]

Quadratic: $$A_n^{\text{tree}} \sim \sum_{k=3}^{n-1} A_k^{\text{tree}} \cdot A_{n-k+2}^{\text{tree}}$$

Indeed, a quadratic recurrence of this form was conjectured by Britto, Cachazo, Feng and then proven together with Witten!

Summary

Connected prescription is not an integral!

Twistor string theory has led to many fascinating insights into scattering amplitudes in Yang-Mills theory, revealing rich mathematical structure which is completely obscured in textbook Feynman diagram calculations. In my talk I have explained in detail the connected prescription and some of its properties, and I've tried to indicate, as best as we currently understand, how it can be derived from the B-model.

I've also demonstrated that there must exist a quadratic recurrence for tree amplitudes, which has indeed been realized.