Conformal Supergravity

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Conformal Gravity

The D=4 higher derivative gravity action

$$I_W = \frac{1}{2}\int d^4x\sqrt{-g}\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor

produces upon variation the Bach equation

$$\frac{1}{\sqrt{-g}}\frac{\delta I}{\delta g^{\mu\nu}} = B_{\mu\nu} = 2\nabla_\sigma\nabla_\rho C^\sigma{}_{\mu\nu}{}^\rho + C^\rho{}_{\mu\nu}{}^\sigma R_{\rho\sigma}$$

Another way to write this action is to use the D=4 Gauss-Bonnet identity

$$\sqrt{-g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) = \text{total div}$$

to show that, dropping the topological total derivative, one can also write

$$I_W \sim \int d^4x\sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)$$

Especially for this 2nd version, find for all Einstein spaces

$$R_{\mu\nu} = \lambda g_{\mu\nu} \quad \text{any } \lambda$$

or equivalently

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 0$$

it is clear that Einstein spaces automatically

Weyl symmetry (local)

The action I_W possesses, in addition to general coordinate invariance, another local symmetry: $g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = e^{2\lambda(x)} g_{\mu\nu}(x)$.

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Relation to the rigid conformal group:

Flat space $g_{\mu\nu} = \eta_{\mu\nu}$ in the context of a general coordinate invariant theory has a set of 10 isometries whose transformations close to form the Poincaré group

$$x^\mu \to x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu + c^\mu \qquad \Lambda^\mu{}_\nu \in SO(3,1)$$

($\Lambda^\mu{}_\nu$ and c^μ constant)

Within the context of a general coordinate + Weyl invariant theory, this is expanded to the group of conformal isometries under which $\eta_{\mu\nu} \to \Omega^2(x)\,\eta_{\mu\nu}$ for some $\Omega(x)$.

The extra rigid transformations are

$$x^\mu \to e^\lambda x^\mu \qquad \text{dilatations}$$

$$\frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + \beta^\mu \qquad \text{proper conformal transformations}$$

(λ and β^μ constant)

Conformal algebra

The 15 parameter conformal group has the Lie algebra

$$[M_{\mu\nu}, M^{\rho\sigma}] = 4\, M_{[\mu}{}^{[\sigma}\,\delta_{\nu]}{}^{\rho]} \qquad [M_{\mu\nu}, P_\rho] = 2\, P_{[\mu}\,\eta_{\nu]\rho}$$

$$[M_{\mu\nu}, K_\rho] = 2\, K_{[\mu}\,\eta_{\nu]\rho} \qquad [P_\mu, D] = 0$$

$$[K_\mu, D] = -K_\mu \qquad [P_\mu, K_\nu] = 2(\eta_{\mu\nu} D - M_{\mu\nu})$$

$M_{\mu\nu}$: Lorentz $\qquad$ P_μ : translations

K_μ : proper conformal $\qquad$ D : dilatations

This algebra is isomorphic to that of $SO(4,2)$ or $SU(2,2)$.

Nonlinear realizations

The general formalism for nonlinear realizations of spacetime symmetries developed by Volkov may be applied to a realization Conformal/Poincaré.

A priori, this requires 4+1 Goldstone fields for the nonlinearly realized proper conformal + dilatation transformations: $b_\mu(x)$ (proper conformal) $\quad \sigma(x)$ (dilatation).

This is, however, a case where the "inverse Higgs effect" (I & Ogievetsky) may be employed to eliminate the vector b

Conformal compensator

The nonlinear realization employing a single Goldstone field $\sigma(x)$ extends naturally to local Weyl invariant theories. An example is Einstein's theory promoted to a Weyl-invariant theory using a conformal compensator field $\phi = e^{\sigma}$:

$$I_{EH} = -\frac{1}{2}\int d^4x \sqrt{-g}\left(R\phi^2 - \frac{1}{6}\,\partial_\mu\phi\,\partial_\nu\phi\, g^{\mu\nu}\right)$$

note: "wrong" sign for $(\partial\phi)^2$

Through use of the conformal compensator, where

$$\sigma(x) \to \sigma(x) + \lambda(x)$$

I_{EH} is invariant under local Weyl transformations.

Of course, using $\lambda(x)$ one may set $\phi(x) = 1$ and this returns one to the usual Einstein-Hilbert action.

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However humble, the above construction is at the base of all off-shell supergravity constructions.

Ghosts

The reason for the unpopularity of conformal gravity considered as a physical theory arises from the solutions that are not simply Einstein spaces. These present an unappealing dilema for interpretation: negative energy/or negative state norm in Q.M.

With apologies to the relativists, the easiest way to see this is to consider fluctuations near flat space: $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$

Moreover, it is easiest to see the ghosts in the 2^{nd} + 4^{th} order theory

$$-\int d^4x\sqrt{-g}\left(\gamma R + \alpha\left(R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2\right)\right)$$

where the Einstein γR may be considered here as a regulator. Restoring the compensator $\phi(x)$, this becomes fully Weyl invariant.

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The propagators for this theory have the momentum structure $\dfrac{1}{k^2(\alpha k^2+\gamma)}$, which can be split into partial fractions $\dfrac{1}{\gamma k^2} - \dfrac{1}{(\gamma k^2 + \gamma^2/\alpha)}$, i.e. a <u>difference</u> between a massless propagator and a massive propagator with $m^2 = \gamma/\alpha$.

The N=1 conformal superalgebra

The conformal algebra $su(2,2)$ extends to the superconformal algebra $su(2,2|1)$, which adds to the $so(4,2) \cong su(2,2)/Z_2$ generators two $D=4$ Majorana spinor generators Q and S plus one new bosonic $U(1)$ generator U:

The algebra is the $so(4,2)$ algebra plus

$$\{Q, Q^T\} = \tfrac{1}{2}(\gamma_\mu C^{-1}) P^\mu \qquad \{S, S^T\} = -\tfrac{1}{2}(\gamma_\mu C^{-1}) K^\mu$$

$$[Q, M_{\mu\nu}] = \sigma_{\mu\nu} Q \qquad [S, M_{\mu\nu}] = \sigma_{\mu\nu} S$$

$$[Q, K_\mu] = -\gamma_\mu S \qquad [S, P_\mu] = \gamma_\mu Q$$

$$[Q, D] = \tfrac{1}{2} Q \qquad [S, D] = -\tfrac{1}{2} S$$

$$[Q, U] = -\tfrac{3i}{4}\gamma_5 Q \qquad [S, U] = \tfrac{3i}{4}\gamma_5 S$$

$$\{Q, S^T\} = \tfrac{1}{2} C^{-1} D - \tfrac{1}{2}\sigma^{\mu\nu} C^{-1} M_{\mu\nu} - i\gamma_5 C^{-1} U$$

Gauging

The historical route to the construction of conformal supergravities proceeded by 'gauging' the above algebra. This is analogous to a way of going from the rigid Poincaré algebra to general coordinate symmetry. (In that case, one can make the process rigorous by including affine tangent spaces and making

In the gauging process, a gauge field is initially introduced for each generator, but this rather redundant set is then reduced by choosing a set of constraints on the corresponding curvatures. One of the basic constraints is a standard zero torsion condition $R_{\mu\nu}^{\ a}(P) = \mathcal{D}_\mu e_\nu{}^a - \mathcal{D}_\nu e_\mu{}^a = 0$

(allowing one to solve for the spin connection $\omega_\mu{}^{ab}$)
$e_\mu{}^a$: translational gauge field for P^a, becomes vierbein

For the $N=1$, $D=4$ superconformal theory, one finds the additional constraints

$$\bar{R}_{\mu\nu}(Q)\gamma^\mu = 0$$

$$R_{\mu\nu}^{\ ab}(M)e^\nu{}_b + \frac{1}{2}e^\rho{}_a\left(\bar{R}_{\lambda\rho}(Q)\gamma_\mu\psi^\lambda - i R^*_{\mu\rho}(u)\right) = 0$$

ψ_μ: gauge field for Q supersymmetry

Taken together with the torsion constraint, these allow one to solve algebraically for $\omega_\mu{}^{ab}(e,b,\psi)$ $\phi_\mu(e,b,\psi,A)$ and $f_\mu{}^a(e,b,\psi,A)$ where the generalized gauge field is

$$W_\mu = e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab}M_{ab} + b_\mu D + f_\mu{}^a K_a + \bar{\psi}_\mu Q + \bar{\phi}_\mu S + A_\mu u$$

at this stage, one is left with independent fields

The constraints imposed to solve for some of the gauge fields are not manifestly covariant with respect to all symmetries. This is because one is going to trade in 'gauged' superconformal symmetry for (super) spacetime symmetry. Accordingly, one has to do some opportunistic coefficient fixing in building the action.

Starting from

$$I = \int d^4x \left[a_1 R_{\mu\nu}{}^{ab}(M) R_{\rho\sigma}{}^{cd} \epsilon_{abcd} + a_2 \bar{R}_{\mu\nu}(Q) \gamma_5 R_{\rho\sigma}(S) + a_3 R_{\mu\nu}(\omega) R_{\rho\sigma}(D) \right] \epsilon^{\mu\nu\rho\sigma}$$

one finds invariance for $a_2 = 2 i a_3 = -8 a_1 = -\frac{1}{\alpha^2}$ at which point the D gauge field D_μ mercifully drops out of the action and one is left with

$$I_{CSG} = \frac{1}{\alpha^2} \int d^4x \sqrt{-g} \Big(\tfrac{1}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - \tfrac{3}{4} F_{\mu\nu}(A) F^{\mu\nu}(A)$$
$$- 4 e^{-1} \epsilon^{\mu\nu\rho\sigma} \bar{\phi}_\rho \gamma_5 \gamma_\sigma D_\mu \phi_\nu$$
$$- 2 R_{\mu\nu} \left[\bar{\psi}^\lambda \sigma_\lambda{}^\nu \phi^\mu - \bar{\psi}^\mu \sigma^{\lambda\nu} \phi_\lambda + \bar{\psi}_\lambda \gamma^\nu (D^{[\mu} \psi^{\lambda]} - \gamma^{[\mu} \phi^{\lambda]}) \right.$$
$$+ \tfrac{4}{3} R \bar{\psi}^\lambda \sigma_{\lambda\nu} \phi^\nu - i \bar{\psi}_\mu (\gamma_5 F^{\mu\nu} + F^{*\mu\nu}) \phi_\nu$$
$$- D^\mu (R_{\rho\sigma} - \tfrac{1}{6} g_{\rho\sigma} R) \bar{\psi}^\sigma \gamma^\rho \psi^\mu$$
$$+ \left[R(\bar{\psi}\psi)^2 + F(\bar{\psi}\psi)^2 + (\bar{\psi} D \psi)^2 + \bar{\psi} D \psi (\bar{\psi}\psi)^2 + (\bar{\psi}\psi)^4 \text{ terms} \right]$$

where it should be recalled that ϕ_μ is given by the constraints :

Transformations

After imposing the constraints, the transformations of the remaining fields are (infinitesimal parameters: ξ^λ: gen. coord. ω_{ab}: Lorentz λ: dilatations α: $U(1)$ ϵ: Q-susy η: S-susy)

$$\delta e_\mu{}^a = \xi^\lambda \partial_\lambda e_\mu{}^a + e_\lambda{}^a \partial_\mu \xi^\lambda + \omega^a{}_b e_\mu{}^b - \lambda e_\mu{}^a - \tfrac{1}{2}\bar{\psi}_\mu \gamma^a \epsilon$$

$$\delta \psi_\mu = \xi^\lambda \partial_\lambda \psi_\mu + \psi_\lambda \partial_\mu \xi^\lambda + \tfrac{1}{2}\omega^{ab}\sigma_{ab}\psi_\mu - \tfrac{1}{2}\lambda \psi_\mu$$
$$+ \mathcal{D}_\mu \epsilon - \gamma_\mu \eta + \tfrac{3i}{4}\gamma_5 \alpha \psi_\mu$$

$$\delta A_\mu = \xi^\lambda \partial_\lambda A_\mu + A_\lambda \partial_\mu \xi^\lambda - i\bar{\phi}_\mu \gamma_5 \epsilon + i\bar{\psi}_\mu \gamma_5 \eta + \partial_\mu \alpha$$

Note that these are nonlinear for the Q-susy (ϵ) owing to the nonlinear algebraic solutions to the constraints $\omega_\mu{}^{ab}(e,\psi) \sim e^{-1}\partial e + \bar{\psi}\psi + \ldots$, $\phi_\mu \sim e^{-1}\partial\psi + A\psi + \ldots$

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Off-shell character:

Unlike the case of Poincaré supergravity, where the supersymmetry closes to form a finite set of symmetry transformations only modulo the classical equations of motion ("on-shell" susy), the above conformal supersymmetry transformations already form a closed system.

The difference: the gauge field A_μ is already present in the system. In Poincaré supergravity, this is an auxiliary field, without independent dynamics but needed to close the algebra. Here, it is dynamical: $F_{\mu\nu}(A)F^{\mu\nu}(A)$ term in I_{CSG}.

Superspace

Whenever one has a proper off-shell closing set of supersymmetry transformations, there is an algorithmic procedure called gauge completion which allows the system to be written in manifestly supersymmetric form, i.e. in superspace.

For $N=1$, $D=4$ conformal supergravity, the superspace coordinates $(X^\mu, \Theta_\alpha, \bar{\Theta}_{\dot\alpha})$ are the same as for Poincaré supergravity. There is more local superspace symmetry, however, and correspondingly fewer covariant field strengths:

just $W_{\alpha\beta\gamma}$, a chiral $(\bar{\mathcal{D}}_{\dot\alpha} W_{\alpha\beta\gamma} = 0)$ superfield containing at lowest order in Θ the γ_μ-traceless part of the Rarita-Schwinger field strength $\partial_{[\mu}\psi_{\nu]} + \cdots$ and at the next order in Θ the Weyl tensor $C_{\mu\nu\rho\sigma}$ and the $U(1)$ field strength $F_{\mu\nu}(A)$.

One can now write the action succinctly in superspace:

$$I_{CSG} = \int d^4x\, d^2\Theta\, \mathcal{E}\, W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$$

$\mathcal{E}$: chiral superspace measure for $N=1$ supergravity

Poincaré from conformal supergravity

Given a representation of conformal supergravity, one can construct a representation of Poincaré supergravity using the idea of a compensating supermultiplet. At the least, one needs to supply compensators to counteract the

- dilatations, which affect $\det(e_\mu{}^a)$
- S-susy, which affect $\gamma^\mu \psi_\mu$

– Depending on one's choice, one may or may not counteract the $U(1)$ symmetry.

For $N=1$, $D=4$ supersymmetry, there are basically 3 types of compensator multiplets that have been used:

a) Chiral multiplet (A, B, χ, M, N)

compensate dilatations, $U(1)$, S-susy, remain as auxiliaries

With this choice of compensator, $e_\mu{}^a$ acquires a determinant, ψ_μ a γ-trace, A_μ a longitudinal part, while M and N persist in the theory as auxiliary fields.

The result: $e_\mu{}^a$, ψ_μ, A_μ, M, N "minimal supergravity"

gauges gen. coord. only; gauges Q-susy only; non-gauge auxiliaries

b) Linear multiplet $(\underset{\text{dil}}{A}, \underset{S}{\chi}, \underset{\text{aux}}{V_\mu})$ with $\partial^\mu V_\mu = 0$

gives $e_\mu{}^a$, ψ_μ, $\underset{\text{gauge}}{A_\mu}$, $V_\mu = \epsilon_{\mu\nu\rho\sigma} H^{\nu\rho\sigma}$ "new minimal" $H = dB$

Superconformal ghosts

We can repeat the exercize of combining conformal and ordinary gravity in the context of supergravity.

— as for the nonsupersymmetric case, we get massless normal + massive ghost spin 2

— in addition, now we get a pair of massive spin $3/2$ ghosts as well as normal massless spin $3/2$ from the system $\gamma \bar{\psi} \partial \psi + \alpha \bar{\psi} \partial\partial\partial \psi$

— the $U(1)$ vector's $\alpha F_{\mu\nu} F^{\mu\nu}$ in the conformal theory combines with $\gamma A_\mu A^\mu$ to give a massive vector

upshot: massless supergravity plus

ghost massive max spin 2 multiplet
(1 spin 2, 2 spin $3/2$, 1 spin 1)

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Matter coupling

Conformal supergravity couples straightforwardly to supermatter. If the supermatter is not superconformal, a compensating multiplet must be used to preserve superconformal symmetry. If it is superconformal, the compensator drops out of the coupling.

Resulting current couplings
$$\begin{array}{ccc} \text{stress} & \text{susy} & U(1) \\ (T_{\mu\nu}, & S_\mu{}^\alpha, & J_\mu) \\ T^\alpha{}_\alpha = 0 & \partial^\mu S_\mu = 0 & \partial^\mu J_\mu = 0 \end{array}$$

Extended conformal supergravities

One can construct extended conformal supergravities by several different approaches

- gauging rigid extended conformal superalgebras and choosing appropriate constraints + actions
- deducing the necessary field content + superspace structure from the associated extended massive supermultiplet (always "long", without central charges)

From the structure of the associated massive supermultiplet, one sees that there is a limit to $N \le 4$ extended conformal supergravity.
Larger N gives multiplets with spins > 2.
In the maximal $N=4$ case, the structure of the conformal supergravity multiplet is indicated by the corresponding long massive multiplet
$\leftrightarrow$ $N=4$ conformal stress tensor multiplet:
Bergshoeff, de Roo, de Wit

Spin	SU(4)	SL(2,C)	Dimension		
2	1	$(1,1)$	4	$T_{\mu\nu}$	stress
3/2	$4+4^*$	$(1,\frac{1}{2})+(\frac{1}{2},1)$	7/2	$S_{\mu\alpha}{}^{i}$	susy
1	15	$(\frac{1}{2},\frac{1}{2})$	3	$J_\mu{}^{i}{}_{j}$	SU(4)
	$6+6$	$(1,0)+(0,1)$	3	$A_{\mu\nu}{}^{[ij]}$	
1/2	$4+4^*$	$(\frac{1}{2},0)+(0,\frac{1}{2})$	7/2	$\lambda_\alpha{}^{i}$	
	$20+20^*$	$(\frac{1}{2},0)+(0,\frac{1}{2})$	5/2	$\chi_\alpha{}^{ij}{}_{k}$	auxiliaries
0	$1+1$	$(0,0)$	4	C	
	$10+10^*$	$(0,0)$	3	e_{ij}	
	20	$(0,0)$	2	$d^{ij}{}_{kl}$	

Extended conformal supergravities in superspace

The full off-shell multiplet structure of extended conformal supergravities is known from the corresponding stress tensor multiplets, as we have seen. This has been extended to a full off-shell formulation of the theory for $N=2$ and $N=3$ and this is known at linearized level for $N=4$.

At linearized level, all actions are chiral superspace integrals:

$N=1$ $\int d^4x\, d^2\theta\, W_{\alpha\beta\gamma} W^{\alpha\beta\gamma}$

$N=2$ $\int d^4x\, d^4\theta\, W_{\alpha\beta} W^{\alpha\beta}$

$N=3$ $\int d^4x\, d^6\theta\, W_\alpha W^\alpha$

$N=4$ $\int d^4x\, d^8\theta\, W W$

(should take real parts of all of these)

In each case, the corresponding W superfield contains at lowest θ level the lowest dimensional superconformally covariant object:

$N=1$ Rarita-Schwinger field strength, γ-traceless $W_{\alpha\beta\gamma}$
$N=2$ $U(1)$ field strength $W_{\alpha\beta}$
$N=3$ spinor field W_α
$N=4$ complex scalar W

Renormalizability

Given off-shell formalisms for the various extended theories, one may use the background field method in superspace to derive non-renormalization theorems in superspace.

In all cases, $\ell = 1$ loop is special, owing to complexities with the Feynman-deWit-Faddeev-Popov ghosts.

Result for $D=4$ $N \leq 4$ supergravities at $\ell \geq 2$ loops: lowest allowed divergences in linearized superspace

$$N=1 \quad \int d^4x\, d^2\theta\, d^2\bar{\theta} \left(W_{\alpha\beta\gamma} \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} W^{\alpha\beta\gamma} \bar{W}^{\dot{\alpha}\dot{\beta}\dot{\gamma}} \right)$$

$$N=2 \quad \int d^4x\, d^4\theta\, d^4\bar{\theta} \left(W_{\alpha\beta} \bar{W}_{\dot{\alpha}\dot{\beta}} W^{\alpha\beta} \bar{W}^{\dot{\alpha}\dot{\beta}} \right)$$

$$N=3 \quad \int d^4x\, d^6\theta\, d^6\bar{\theta} \left(W_{\alpha} \bar{W}_{\dot{\beta}} W^{\alpha} \bar{W}^{\dot{\beta}} \right)$$

$$N=4 \quad \int d^4x\, d^8\theta\, d^8\bar{\theta} \left(W \bar{W} W \bar{W} \right)$$

• All of these ruled out by $D=4$ power counting and all would violate Weyl invariance

(However Weyl invariance can be ruined by infinities $\Rightarrow$ anomalies at $\ell = 1$ loop)

The 1-loop cases need to be explicitly calculated. At 1-loop, the starting action is always available

Anomalies

Explicit calculation of the 1-loop divergences shows that all pure conformal supergravities have divergences at this order. These give rise to conformal anomalies plus anomalies in other gauge symmetries of the stress tensor supermultiplet, e.g. $SU(4)$ for the $N=4$ theory.

Accordingly, one may calculate the Weyl multiplet anomalies either from the conformal anomaly (Fradkin + Tseytlin) or from the gauge current anomaly such as the $SU(4)$ anomaly (Romer + Van Nieuwenhuizen).

- The results are the same either way: to obtain cancellation of the Weyl multiplet anomalies, one must couple $N=4$ conformal supergravity to $N=4$ super Yang Mills theory with a gauge group of dimension 4.

Accordingly, the only possibilities for anomaly-free theories are $N=4$ CSG + $N=4$ SYM with gauge group $SU(2)\times U(1)$ or $(U(1))^4$.

References (very incomplete)

Review:

E.S. Fradkin and A.A. Tseytlin
Phys. Rep. 119 (1985) 233.

Early constructions:

Kaku, Townsend + van Nieuwenhuizen

N=2 & N=4:

Bergshoeff, de Roo + de Wit

Linearized superspace formulations:

Howe

N=2 harmonic superspace formulation:

Galperin, Ivanov, Ogievetsky + Sokatchev

Non-renormalization theorems:

Grisaru, Siegel + Rocek
Howe, KS^2 + Townsend
Howe + KS^2

Other approaches:

Ivanov + Niederle
Anderson + Wheeler