MHV Vertices
and
Scattering Amplitudes in Gauge Theory

with F. Cachazo and E. Witten

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In this talk, I will review some recent progress in understanding gauge theory scattering amplitudes and their relation twistor strings.

In particular, I will review MHV rules for computing scattering amplitudes. I will describe their motivation from twistor space and also sketch a derivation of the MHV rules from twistor string theory.

We write onshell momentum as a bispinor

$$P^{a\dot{a}} = \sigma_{\mu}^{a\dot{a}} P^{\mu} = \lambda^{a} \tilde{\lambda}^{\dot{a}},$$

and introduce the skew products

$$\langle \lambda, \lambda' \rangle = \lambda_a \lambda'_b \epsilon^{ab}$$

$$[\tilde{\lambda}, \tilde{\lambda}'] = \tilde{\lambda}_{\dot{a}} \tilde{\lambda}'_{\dot{b}} \epsilon^{\dot{a}\dot{b}}.$$

We take all gluons to be incoming and label them with the spinors $\lambda, \tilde{\lambda}$ and their helicities

$$\epsilon^{-}_{a\dot{a}} = \frac{\lambda_a \rho_{\dot{a}}}{[\tilde{\lambda}, \rho]} \qquad \epsilon^{+}_{a\dot{a}} = \frac{\kappa_a \tilde{\lambda}_{\dot{a}}}{\langle \kappa, \lambda \rangle}$$

Instead of writing $A(\epsilon_i)$ we introduce helicity labels

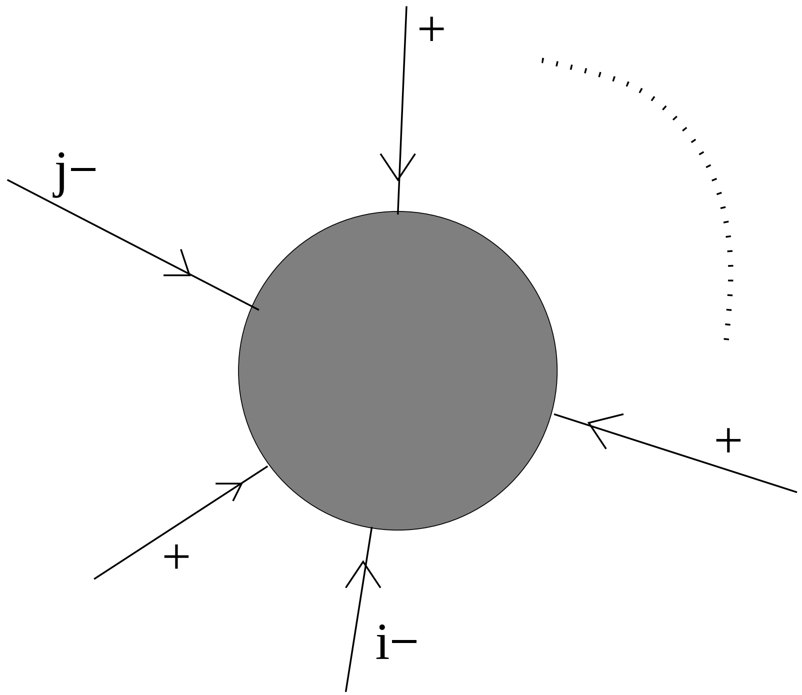
$$h_i = \pm.$$

We factor out delta function of momentum conservation and consider the color ordered amplitudes, that is we consider the ordered amplitude that comes with $\mathrm{Tr}\, T_1 \ldots T_n$.

$$\hat{A} = \delta^4 \left(\sum_i \lambda_i^a \tilde{\lambda}_i^{\dot{a}} \right) \sum_{S_n/Z_n} \mathrm{Tr}\, (T_{\sigma(1)} T_{\sigma(2)} \ldots T_{\sigma(n)}) A(\lambda_{\sigma(i)}, \tilde{\lambda}_{\sigma(i)}, h_{\sigma(i)}).$$

Recall, that the color ordered MHV scattering amplitude with gluons i and j of negative helicity takes the simple form

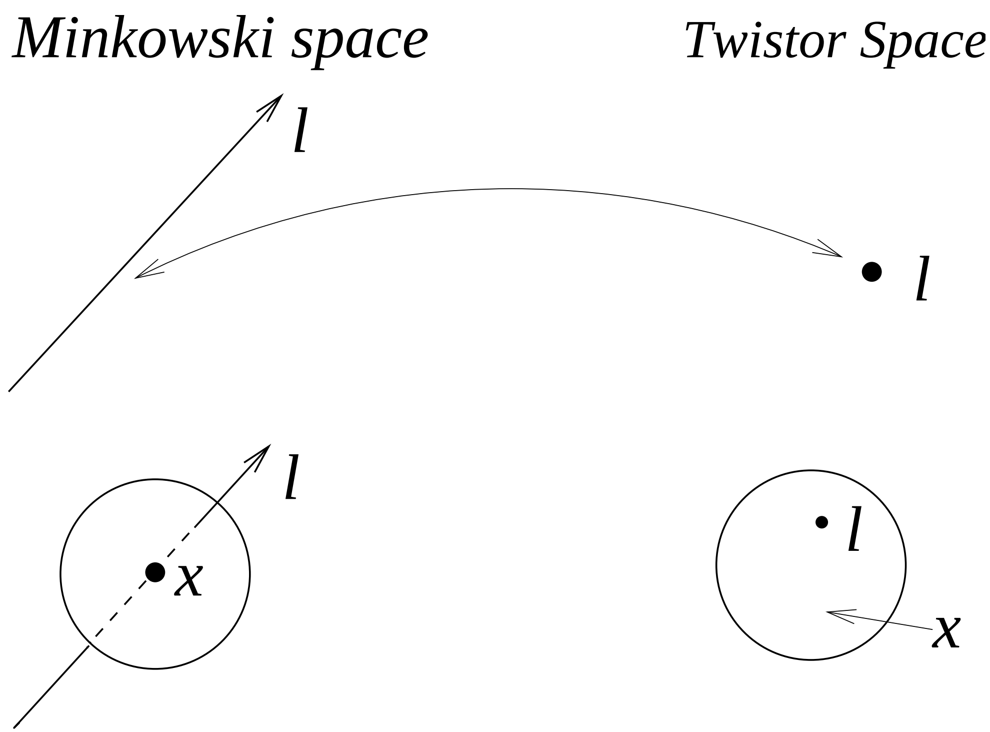
$$A = \frac{\langle i, j \rangle^4}{\prod_{k=1}^{n} \langle k, k+1 \rangle} \delta^4 \left(\sum_k \lambda_k^a \tilde{\lambda}_k^{\dot{a}} \right).$$



We would like to understand why the MHV and other amplitudes have much simpler expressions than what we would expect from textbook recipes.

For that we need to venture into the twistor space.

Penrose's twistor space is the space of light rays in Minkowski space.



A point x in Minskowski space, maps to the sphere of light rays l through x.

Twistor space is the projective space CP^3. Its homogeneous coordinates are

$$(\lambda^a, \mu^{\dot{a}}) \sim (c\lambda^a, c\mu^{\dot{a}}).$$

μ is conjugate to $\tilde{\lambda}$

$$\mu \to -i\frac{\partial}{\partial\tilde{\lambda}.} \qquad \tilde{\lambda} \to i\frac{\partial}{\partial\mu}$$

The conformal group $SU(2,2)$ is the group of motions of twistor space. It acts linearly on (λ, μ).

One gets twistor amplitudes by Fourier transform in $\tilde{\lambda}$ of Minkowski space amplitudes

$$A(\lambda_i, \mu_i) = \int \prod_{i=1}^{n} d^2\tilde{\lambda}_i \, e^{i\tilde{\lambda}_i\mu_i} A(\lambda_j, \tilde{\lambda}_j)\delta^4(\sum_j \lambda_j\tilde{\lambda}_j).$$

MHV amplitudes behave simply in twistor space because they are independent of $\tilde{\lambda}^{\dot{a}}$. They are *holomorphic.*

$$A(\lambda_i, \mu_i) = A(\lambda_i) \int \prod_i d^2\tilde{\lambda}_i \, e^{i\tilde{\lambda}_i\mu_i} \int d^4x \, e^{ix\lambda_i\tilde{\lambda}_i}$$

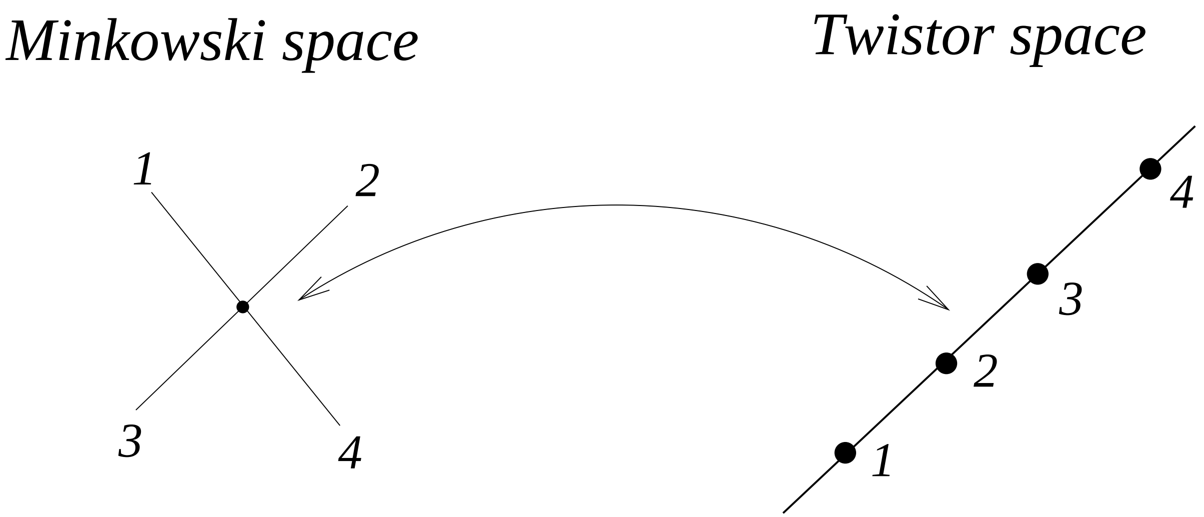
$$= A(\lambda_i) \int d^4x \prod_i \delta^2(\mu_i^{\dot{a}} + x^{a\dot{a}}\lambda_{ia})$$

They vanish unless all gluons lie on a common CP^1.

But a line in twistor space corresponds to a point
in Minkowski space.

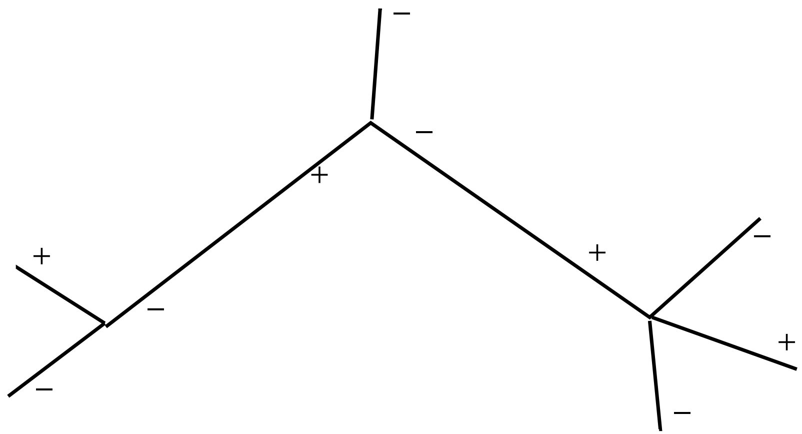
$$\mu_i^{\dot{a}} + x^{a\dot{a}} \lambda_{ia} = 0$$

So MHV amplitudes behave like *local interaction vertices.*



MHV diagrams

We use off-shell continuations of MHV amplitudes as local vertices in Feynman diagrams to compute more complicated amplitudes.



Off-shell continuation of MHV amplitudes

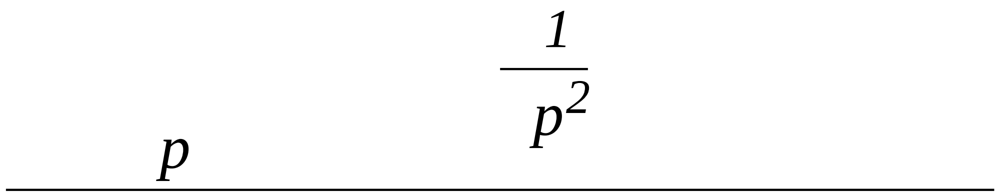
To use MHV amplitudes in Feynman diagrams, we need to define λ^a for an off-shell momentum.

For on-shell momentum $P^{a\dot{a}} = \lambda^a \tilde{\lambda}^{\dot{a}}$, we have

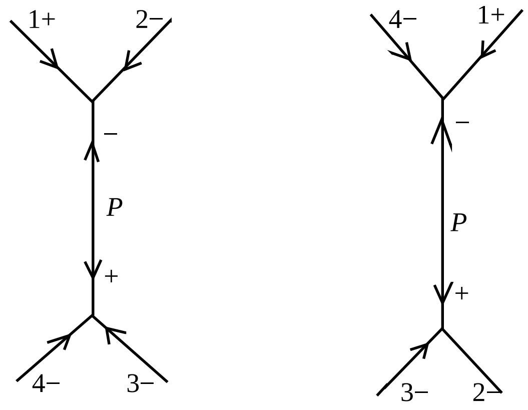
$$\lambda^a = \frac{P^{a\dot{a}} \eta_{\dot{a}}}{[\tilde{\lambda}, \eta]}.$$

We define λ^a for an off-shell line with the same formula. It is well defined because $[\tilde{\lambda}, \eta]$ scales out of Feynman diagrams.

The propagator connecting MHV vertices is the scalar propagator



The simplest example is the $(+,-,-,-)$ helicity amplitude.



The first of the two diagrams gives

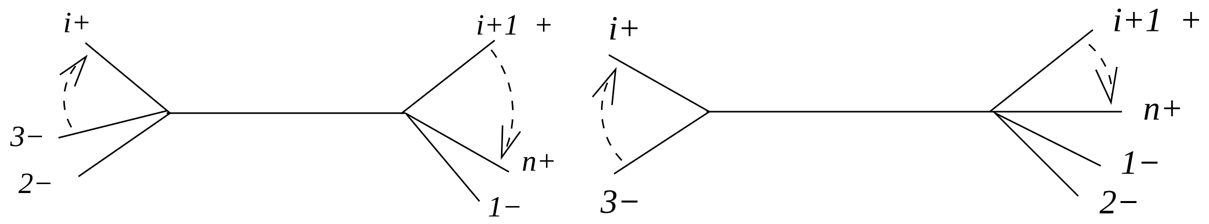
$$\frac{\langle 2,\lambda\rangle^4}{\langle 1,2\rangle\langle 2,\lambda\rangle\langle\lambda,1\rangle}\frac{1}{P^2}\frac{\langle 3,4\rangle^4}{\langle 3,4\rangle\langle 4,\lambda\rangle\langle\lambda,3\rangle},$$

where

$$\lambda^a=(p_1+p_2)^{a\dot{a}}\eta_{\dot{a}}.$$

The second diagram gives equal and opposite contribution.

It is easy to compute more complicated amplitudes. For example, the n gluon $---++\ldots++$ amplitude is a sum of $2n-3$ MHV diagrams.



$$\begin{aligned} A = \quad & \textstyle\sum_{i=3}^{n-1} \frac{\langle 1\lambda_{2,i}\rangle^3}{\langle\lambda_{2,i}i{+}1\rangle\langle i{+}1i{+}2\rangle\ldots\langle n1\rangle}\frac{1}{q_{2i}^2}\frac{\langle 2{,}3\rangle^3}{\langle\lambda_{2,i}2\rangle\langle 34\rangle\ldots\langle i\lambda_{2,i}\rangle} \\ + \quad & \textstyle\sum_{i=4}^{n} \frac{\langle 12\rangle^3}{\langle 2\lambda_{3,i}\rangle\langle\lambda_{3,i}i{+}1\rangle\ldots\langle n1\rangle}\frac{1}{q_{3i}^2}\frac{\langle\lambda_{3,i}3\rangle^3}{\langle 3{,}4\rangle\ldots\langle i{-}1i\rangle\langle i\lambda_{3,i}\rangle}, \end{aligned} \tag{1}$$

MHV diagrams have not yet been derived from gauge theory (but see Khoze's talk. . .)

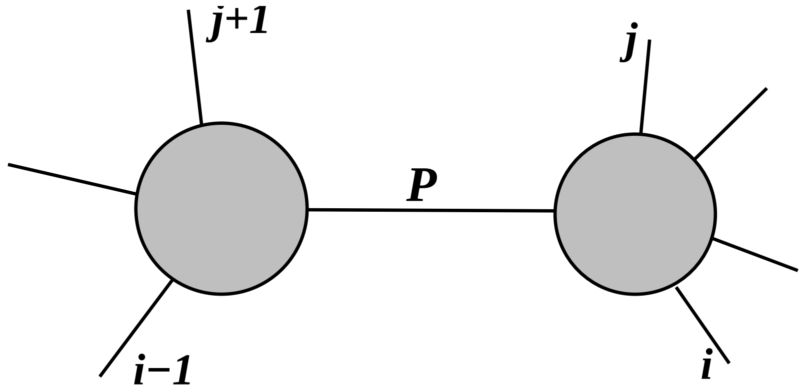
Besides reproducing known amplitudes, MHV diagrams have correct factorization properties:

- multiparticle singularities
- soft and collinear divergences

Multiparticle singularity in the channel $P^2 \to 0$ is $1/P^2$ times the product of the tree amplitudes in the subchannels

$$A \longrightarrow A_L \frac{1}{P^2} A_R.$$

It arises from MHV diagrams with off-shell P line that goes on-shell.



Twistor String Theory

The MHV diagrams have been motivated from twistor space. One may ask whether there is a string theory in twistor space that computes leads to MHV diagrams.

The answer seems to be yes. So far, at least for tree amplitudes in $\mathcal{N} = 4$ gauge theory. Fortunately, tree amplitudes do not depend on supersymmetry.

Recall that from Witten's talk that to define B-model topological strings on twistor space, one has to make it into a Calabi-Yau manifold. This is accomplished by introducing fermionic coordinates

$$\psi^A \qquad A = 1 \dots 4$$

$$(\lambda Z^I, \lambda\psi^A) \sim (Z^I, \psi^A).$$

We get super-twistor space $CP^{3|4}$ with holomorphic $(3|4)$ form

$$\Omega = \epsilon_{IJKL}\epsilon_{ABCD}\, Z^I dZ^J dZ^K dZ^L\, d\psi^A d\psi^B d\psi^C d\psi^D$$

$$\Omega(Z, \psi) = \Omega(cZ, c\psi)$$

The group of motions of super-twistor space is the $\mathcal{N} = 4$ super-conformal group $SU(2, 2|4)$.

The twistor field $\mathcal{A}$ is in $H^{0,1}(CP^{3|4}, \mathcal{O}(0))$. Expanding in ψ^A :

$$\mathcal{A}(Z,\psi) = A + \chi_A \psi^A + \ldots + G(\psi^A)^4$$

So

$$A \in \mathcal{O}(0) \rightarrow h = +1$$

and

$$G \in \mathcal{O}(-4) \rightarrow h = -1$$

are the positive and negative helicity gluons. Similarly,

$$\chi_A \in \mathcal{O}(-1) \rightarrow h = +\frac{1}{2}$$

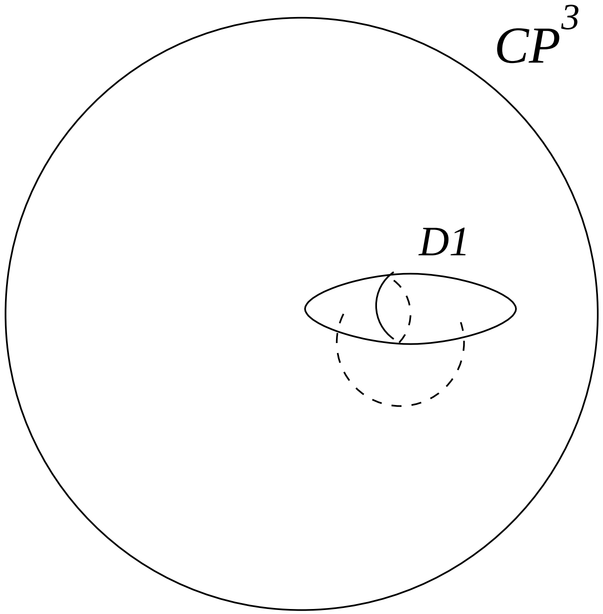
is the positive helicity fermion. We get the entire spectrum of $\mathcal{N} = 4$ SYM!

$$S = \int \Omega \wedge \left(\mathcal{A} \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right)$$

S is an effective action for an open topological string theory, the B-model, in twistor space.

The cubic vertex $\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}$ gives the interactions of the self-dual $\mathcal{N} = 4$ gauge theory. So, we have the correct spectrum, but we do not yet have the interactions. . .

The interactions of the full gauge theory come from *D1-brane instantons* on which the open strings can end.



Current algebra on the world-volume C of a D1-brane:

free fermions

$\alpha \dots$ D1-D5 strings

$\beta \dots$ D5-D1 strings.

$J = \alpha\beta dz \dots$ holomorphic current,

$z \dots$ local coordinate on world-volume

$$S = \int_C \alpha(\bar{\partial} + \mathcal{A})\beta.$$

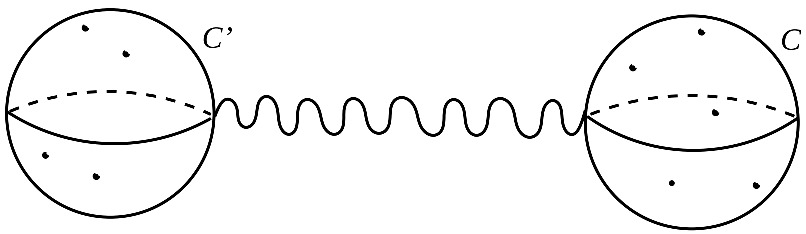
Open string with wavefunction $\psi = \mathcal{A}_I d\bar{z}^I$ couples to D1-instanton via

$$\int_C J\psi = \int_C J_z \mathcal{A}_{\bar{z}}\, dz \wedge d\bar{z}$$

Two disconnected D1-instantons can be connected by an open string with propagator

$$\bar\partial G = \bar\delta^3(Z'^I - Z^I)\delta^4(\psi'^A - \psi^A).$$

$$\bar\delta(z) = d\bar z \delta(z)\delta(\bar z)$$

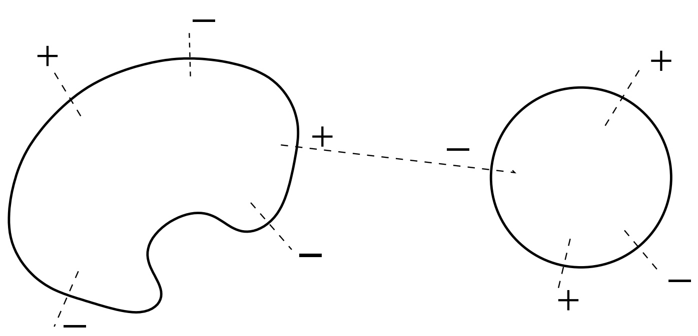


We have computed all tree level amplitudes from twistor string
. . . only genus zero instantons contribute.

A degree d instanton is in the homology class of d times the class of complex line. It's area is quantized.

It contributes to an amplitude with $d + 1$ negative helicity gluons and an arbitrary number of positive helicity gluons.

ie. for $d = 3$:



To compute a scattering amplitude, we evaluate the correlation function

$$\oint d\mathcal{M} \int \prod_i dz_i d\bar{z}_i A^{a_1}_{\bar{z}_1} A^{a_2}_{\bar{z}_2} \dots A^{a_n}_{\bar{z}_n} \langle J^{a_1}_{z_1} J^{a_2}_{z_2} \dots J^{a_n}_{z_n} \rangle.$$

$\psi = A_{\bar{z}} d\bar{z} \dots$ wavefuncion of a scattered particle

$d\mathcal{M} \dots$ holomorphic measure on the moduli space of holomorphic curves of degree d

$$\langle J_1 J_2 \dots J_n \rangle = \frac{1}{(z_1 - z_2)(z_2 - z_3) \dots (z_n - z_1)}$$

is the correlation functions of the currents on D1-instanton.

We would like to compute the scattering amplitude of particles with definite momentum

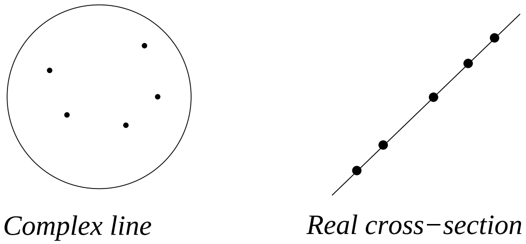
$$\phi(x) = \exp\left(i\, p \cdot x\right) = \exp\left(i\, \pi^a \tilde{\pi}^{\dot{a}} x_{a\dot{a}}\right).$$

Here $p^{a\dot{a}} = \pi^a \tilde{\pi}^{\dot{a}}$ is the on-shell momentum of an external particle. The twistor wavefunctions corresponding to plane waves are

$$\psi(\lambda, \mu) = d\bar{z}\, \delta(\langle \lambda, \pi \rangle) \exp(i[\tilde{\pi}, \mu]).$$

Twistor computation with these wavefunctions gives directly the momentum space scattering amplitudes.

$d = 1$ instantons are complex lines.

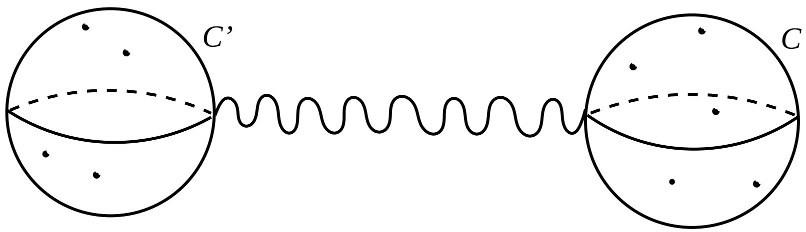


These give MHV amplitudes,

$$A = \frac{\langle i, j\rangle^4}{\prod_{k=1}^{n} \langle k, k+1\rangle} \delta^4\Big(\sum_i \lambda_i \tilde{\lambda}_i\Big).$$

$\delta^4(\sum_i \lambda_i \tilde{\lambda}_i) \ldots$ from integral over moduli $x^{a\dot{a}}$ of lines in $CP^{3|4}$.
$\langle i, j\rangle^4 \ldots$ from integral over fermionic moduli of lines.
$1/\prod_k \langle \lambda_k, \lambda_{k+1}\rangle \ldots$ from the current correlator $\langle J_1 J_2 \ldots J_n\rangle$.

Next is the $d = 2$ amplitude. We evaluate the contribution from two degree one instantons C and C' that are connected by an open string.



The instantons are described by the equations

$$\mu^{\dot{a}} = x_i^{a\dot{a}} \lambda_a \qquad \psi^A = \theta_i^{Aa} \lambda_a$$

$x_i^{a\dot{a}}$ ... bosonic moduli of the i'th instanton.
θ_i^{Aa} ... fermionic moduli.

The current correlators on C and C' and the integral over θ_i^{Aa} give two MHV amplitudes A_L and A_R on D1-instantons just like for $d = 1$. We are left with

$$\oint d^4x d^4x' A_L \frac{1}{(x'-x)^2} A_R \prod_{i \in L} \exp(ix' \cdot p_i) \prod_{j \in R} \exp(ix \cdot p_j).$$

We rewrite the exponential as

$$\exp(iy \cdot P) \prod_{i \in L,R} \exp(ix \cdot p_i).$$

Here, $P = \sum_{i \in L} p_i$ is the off-shell momentum. The integral

$$\oint d^4x \prod_{i \in L,R} \exp(ix \cdot p_i) = \delta^4(\sum_i p_i)$$

gives momentum conservation.

Finally the amplitude becomes

$$A = \oint d^4y \frac{1}{y^2} \exp(iy \cdot P) A_L A_R.$$

The integral has a pole at $y^2 = 0$. This is the condition for C and C' to intersect. We choose a contour that picks the residue at $y^2 = 0$. Parametrizing the surface $y^2 = 0$ as

$$y^{a\dot{a}} = t\lambda^a \tilde{\lambda}^{\dot{a}},$$

the residue is the volume form on the conifold

$$\mathrm{Res}\, \frac{d^4y}{y^2} = t dt \langle \lambda, d\lambda \rangle [\tilde{\lambda}, d\tilde{\lambda}].$$

The amplitude becomes

$$I = \oint t dt \langle \lambda, d\lambda \rangle [\tilde{\lambda}, d\tilde{\lambda}] \exp(itP\lambda\tilde{\lambda}) A_L A_R.$$

Carrying out the t integral

$$\int_0^\infty t dt \exp(it(P\lambda\tilde{\lambda})) = -\frac{1}{(P\lambda\tilde{\lambda})^2}.$$

gives

$$I = \int \langle \lambda, d\lambda \rangle [\tilde{\lambda}, d\tilde{\lambda}] \frac{1}{(P\lambda\tilde{\lambda})^2} g(\lambda; \lambda_i).$$

The integral over $\lambda, \tilde{\lambda}$ can be evaluated using residues

$$I = \sum \mathrm{Res}\left(\frac{g(\lambda; \lambda_i)}{(P\lambda\eta)}\right) \frac{[\tilde{\lambda}, \eta]}{(P\lambda\tilde{\lambda})}.$$

The residues of $1/(P\lambda\eta)$ are at $\lambda = P\eta$. They give

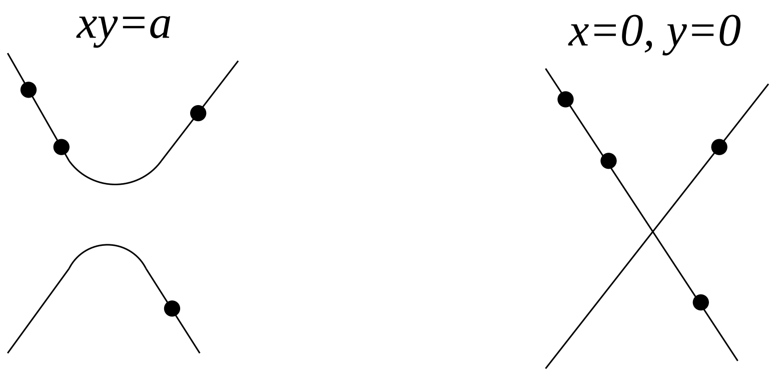
$$A = \sum \frac{1}{P^2} A_L A_R(\lambda = P\eta).$$

But this is exactly the sum over MHV diagrams!

This derivation clearly generalizes to d disconnected degree one instantons. These give MHV diagrams with d MHV vertices.

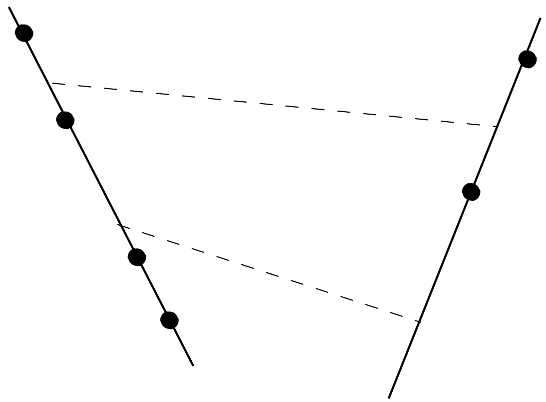
What happened to the higher degree *connected* instantons? They also compute the scattering amplitudes(see Volovich's talk).

The integral over the moduli space of instantons has poles where the instantons degenerate to intersecting lines. Evaluating the residues leads to the MHV diagrams(see Gukov's talk).



$$\oint \frac{da}{a} \oint_{\mathcal{M}'} f(a,b,\ldots) = \oint_{\mathcal{M}'} f(a=0,b,\ldots)$$

It seems that MHV diagrams work for loop amplitudes.
See talks by Britto, Dixon and Travaglini.



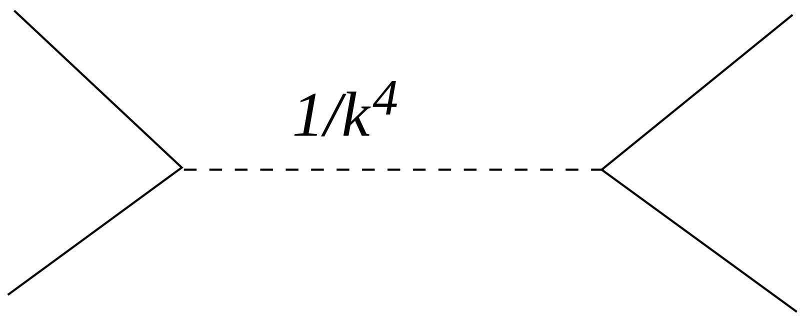
Unfortunately we have not yet learned how to derive them from twistor string theory.

Moreover, we do not even have a twistor string theory that would compute Yang-Mills loop amplitudes!

The topological string theory has closed strings as well. They are related by twistor transform to excitations of $\mathcal{N} = 4$ conformal supergravity.

The conformal supergravity has a fourth order action

$$S = \int d^4x \, W^2 \sim \int d^4x \, (\partial\partial h)^2 .$$



Witten's and Berkovits' twistor string theories describe $\mathcal{N} = 4$ gauge theory coupled to conformal supergravity.

Conclusions

- MHV diagrams give an efficient method to analytically compute YM scattering amplitudes
- They work better than expected: loops in $\mathcal{N} = 1, \ldots, 4$ YM
- Twistor string theory leads to tree level MHV diagrams
- Closed strings mix into loop amplitudes … conformal supergravity
- Need to find the correct twistor string theory…
  MHV vertices for $\mathcal{N} = 0$ gauge theory (QCD)?