ONE-LOOP YANG-MILLS AMPLITUDES FROM MHV VERTICES

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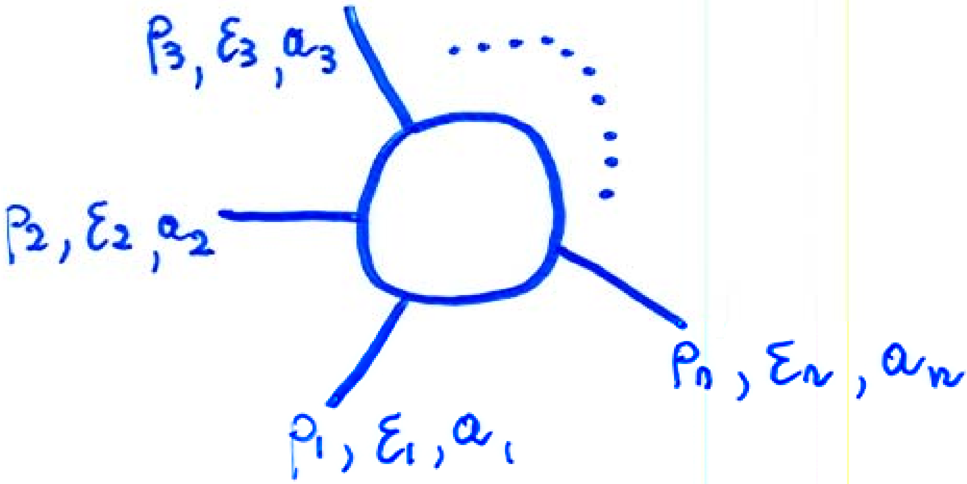
PLAN

1. MOTIVATION AND GOALS

2. SCATTERING AMPLITUDES IN YANG-MILLS
   - COLOUR DECOMPOSITION
   - SPINOR HELICITY FORMALISM
   - MHV DIAGRAMS

3. LOOP AMPLITUDES FROM MHV VERTICES
   - $N=4$ SUPER YANG-MILLS
   - $N=1$ SUPER YANG-MILLS
   - PURE YANG-MILLS
     ("CUT-CONSTRUCTIBLE" PART)

A SCATTERING PROCESS :



GLUON SCATTERING

TO DESCRIBE A SCATTERING PROCESS, WE NEED, FOR EACH GLUON:

1) MOMENTUM, p_i^μ
2) POLARISATION VECTOR, ε_i^μ
3) COLOUR, a_i

- $A = A(\{p_i^\mu, \varepsilon_i^\mu, a_i\})$ IS, IN GENERAL, A VERY COMPLICATED EXPRESSION

- USE OF CONVENTIONAL FEYNMAN RULES HIDES THE SIMPLICITY OF SCATTERING AMPLITUDES

$gg \to n\,g$	$n=7$	$n=8$	$n=9$
# DIAGRAMS	559,405	10,525,900	224,449,225

FACTORIAL GROWTH !

a. COLOUR DECOMPOSITION

COLOUR-STRIPPED PARTIAL AMPLITUDES

b. SPINOR HELICITY FORMALISM

MARCH 2004

c. MHV DIAGRAMS

[Cachazo, Svrček, Witten]

- OUR GOAL :

  USE MHV VERTICES TO BUILD LOOP DIAGRAMS

- CHECK AGAINST KNOWN RESULTS

  - ONE-LOOP MHV AMPLITUDES IN $N=4$ AND $N=1$ SUPER YANG-MILLS

    (Bern, Dixon, Dunbar, Kosower 1994)

- DERIVE NEW RESULTS

  - CUT-CONSTRUCTIBLE PART OF THE ONE-LOOP MHV AMPLITUDE IN PURE YM

- WHY ?

  PERTURBATIVE GAUGE THEORY AMPLITUDES RELEVANT FOR LHC BACKGROUND

COLOUR DECOMPOSITION

SEPARATE COLOUR STRUCTURE

- AT TREE LEVEL, YANG-MILLS INTERACTIONS ARE PLANAR

$$A_n^{\text{tree}}(\{p_i, \varepsilon_i, a_i\}) = \sum_{\sigma \in S_n/\mathbb{Z}^n} \mathrm{Tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}\right) \cdot A(\sigma(p_1, \varepsilon_1), \ldots, \sigma(p_n, \varepsilon_n))$$

COLOUR-ORDERED PARTIAL AMPLITUDE

- INCLUDE ONLY DIAGRAMS WITH A FIXED CYCLIC ORDERING OF GLUONS
- ANALYTIC STRUCTURE IS SIMPLER !

- AT LOOP LEVEL, MULTI-TRACE STRUCTURES APPEAR
  - SUBLEADING IN 1/N

SPINOR HELICITY FORMALISM

(Xu, Zhang, Chang; Berends, Kleiss, De Causmaecker; ....)

- p^μ, 4-VECTOR, $p^\mu = (p_0, \vec{p})$

① DEFINE $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$ 2x2 MATRIX

$\sigma^0 = \mathbb{1}_{2\times 2}$, σ^i = PAULI MATRICES

② MASSLESS PARTICLES $\Rightarrow p_\mu p^\mu = 0$

$\Rightarrow \det p_{a\dot{a}} = 0 \Rightarrow$

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

$\left.\begin{array}{l}\text{* MOMENTA} \\ \text{* WAVEFUNCTIONS}\end{array}\right\}$ FUNCTIONS of $\lambda, \tilde{\lambda}$

↳ e.g. $h = \frac{1}{2}$ $\quad \psi_a \sim \lambda_a e^{i \lambda_a \tilde{\lambda}_{\dot{a}} x^{\dot{a}a}}$

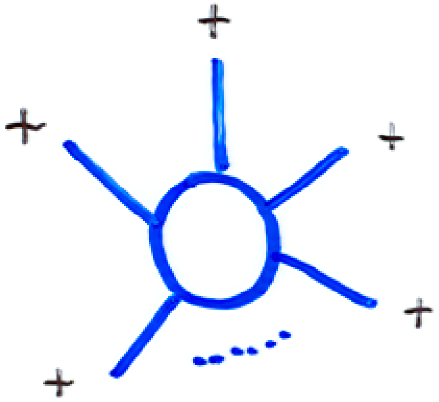
- $\lambda, \tilde{\lambda}$ RELATED TO TWISTOR COORDINATES

- $A = A(\{\lambda_i, \tilde{\lambda}_i; h_i\})$

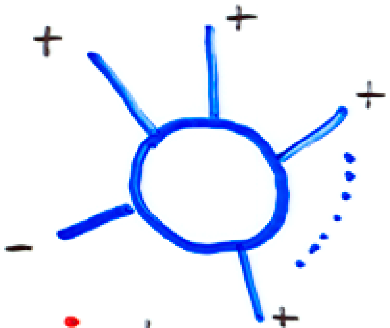
WHY? DRASTIC SIMPLIFICATION !

n-GLUON TREE AMPLITUDES

[CONTRIBUTION PROPORTIONAL TO $\mathrm{Tr}(T_1 \cdots T_n)$]



$= 0$



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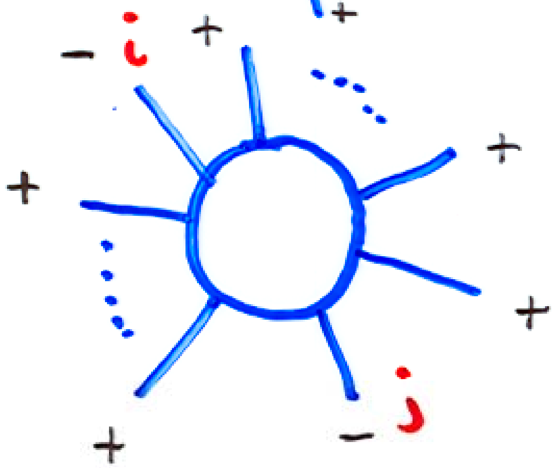


$= A^{(n)}_{MHV}$ PARKE-TAYLOR

- $$A^{(n)}_{MHV} = i\, g^{n-2} (2\pi)^4 \delta^{(4)}\Big(\sum_{i=1}^{n} p_i\Big) \cdot \frac{\langle \lambda_i \lambda_j \rangle^4}{\prod_{k=1}^{n} \langle \lambda_k \lambda_{k+1} \rangle}$$
- $\langle \lambda_i \lambda_j \rangle = \varepsilon_{ab} \lambda_i^a \lambda_j^b$
- "HOLOMORPHIC"

Amplitude for n-Gluon Scattering

Stephen J. Parke and T. R. Taylor
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

PACS numbers: 12.38.Bx

Computations of the scattering amplitudes for the vector gauge bosons of non-Abelian gauge theories, besides being interesting from a purely quantum-field-theoretical point of view (determination of the S matrix), have a wide range of important applications. In particular, within the framework of quantum chromodynamics (QCD), the scattering of vector gauge bosons (gluons) gives rise to experimentally observable multijet production at high-energy hadron colliders. The knowledge of cross sections for the gluon scattering is crucial for any reliable phenomenology of jet physics, which holds great promise for testing QCD as well as for the discovery of new physics at present (CERN $Sp\bar{p}S$ and Fermilab Tevatron) and future (Superconducting Super Collider) hadron colliders.[1]

In this short Letter, we give a nontrivial squared helicity amplitude for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors. To our knowledge this is the first time in a non-Abelian gauge theory that a nontrivial, on-mass-shell, squared Green's function has been written down for an arbitrary number of external points. Our result can be used to improve the existing numerical programs for the QCD jet production, and in particular for the studies of the four-jet production for which no analytic results have been available so far. Before presenting the helicity amplitude, let us make it clear that this result is an educated guess which we have compared to the existing computations and verified by a series of highly nontrivial and nonlinear consistency checks.

For the n-gluon scattering amplitude, there are $(n+2)/2$ independent helicity amplitudes. At the tree level, the two helicity amplitudes which most violate the conservation of helicity are zero. This is easily seen by the embedding of the Yang-Mills theory in a supersymmetric theory.[2,3] Here we give an expression for the next helicity amplitude, also at tree level, to leading order in the number of colors in SU(N) Yang-Mills theory.

If the helicity amplitude for gluons $1, \ldots, n$, of momenta $p_1, \ldots, p_n$ and helicities $\lambda_1, \ldots, \lambda_n$, is $\mathcal{M}_n(\lambda_1, \ldots, \lambda_n)$, where the momenta and helicities are labeled as though all particles are outgoing, then the three helicity amplitudes of interest, squared and summed over color, are

$$|\mathcal{M}_n(+++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{1}$$

$$|\mathcal{M}_n(-++++\cdots)|^2 = c_n(g,N)[0+O(g^4)], \tag{2}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 = c_n(g,N)[(p_1\cdot p_2)^4$$

MHV AMPLITUDE

$$\times \textstyle\sum_P [(p_1\cdot p_2)(p_2\cdot p_3)(p_3\cdot p_4)\cdots(p_n\cdot p_1)]^{-1} + O(N^{-2}) + O(g^2)], \tag{3}$$

where $c_n(g,N) = g^{2n-4}N^{n-2}(N^2-1)/2^{n-4}n$. The sum is over all permutations P of $1, \ldots, n$.

Equation (3) has the correct dimensions and symmetry properties for this n-particle scattering amplitude squared. Also it agrees with the known results[4,5] for $n=4$, 5, and 6. The agreement for $n=6$ is numerical.[5,6] More importantly, this set of amplitudes is consistent with the Altarelli and Parisi[7] relationship for all n, when two of the gluons are made parallel. This is trivial for the first two helicity amplitudes but is a highly nontrivial statement for the last amplitude, as shown here:

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{1\parallel 2}{\longrightarrow} 0, \tag{4}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{2\parallel 3}{\longrightarrow} 2g^2N\frac{z^4}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{5}$$

$$|\mathcal{M}_n(--+++\cdots)|^2 \underset{3\parallel 4}{\longrightarrow} 2g^2N\frac{1}{z(1-z)}\frac{1}{s}|\mathcal{M}_{n-1}(--++\cdots)|^2, \tag{6}$$

where s is the corresponding pole and z is the momentum fraction. The result for particles 2 and 3 nearly parallel, Eq. (5), is only simple because $\mathcal{M}_{n-1}(-+++\cdots)$ is zero to this order in g so that there is no interference term and therefore azimuthal averaging is not required.

The surprise about this result is that all denominators are simple dot products of two external momenta. The Feynman diagrams for n-gluon ($n>5$) scattering contain propagators $(p_i+p_j+p_k)^2$, $(p_i+p_j+p_k+p_m)^2$, ... These propagators must cancel for Eq. (3) to be correct; this occurs for $n=6$. Of course, Altarelli and Parisi have taught us that many cancellations are expected.

We do not expect such a simple expression for the other helicity amplitudes. Also, we challenge the string theorists to prove more rigorously that Eq. (3) is correct.

Fermilab is operated by the Universities Research Association Inc. under contract with the United States Department of Energy.

[1] E. Eichten, I. Hinchliffe, K. Lane, and C. Quigg, Rev. Mod. Phys. **56**, 579 (1984).

[2] M. T. Grisaru, H. N. Pendleton, and P. van Nieuwenhuizen, Phys. Rev. D **15**, 997 (1977); M. T. Grisaru and H. N. Pendleton, Nucl. Phys. **B124**, 81 (1977).

[3] S. J. Parke and T. R. Taylor, Phys. Lett. **157B**, 81 (1985).

[4] T. Gottschalk and D. Sivers, Phys. Rev. D **21**, 102 (1980); F. A. Berends, R. Kleiss, P. de Causmacker, R. Gastmans, and T. T. Wu, Phys. Lett. **103B**, 124 (1981).

[5] S. J. Parke and T. R. Taylor, Fermilab Report No. Pub-85/118-T, 1985 (to be published); Z. Kunszt, CERN Report No. TH-4319, 1985 (to be published).

[6] Another numerical fact worth mentioning is that to leading order in g but to all orders in N, the amplitude $|\mathcal{M}_{n=6}(--++++)|^2$ is permutation symmetric apart from the factor $(p_1\cdot p_2)^4$. This allows all permutations of this amplitude to be trivially calculated from one such permutation.

[7] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).

KEY OBSERVATION OF

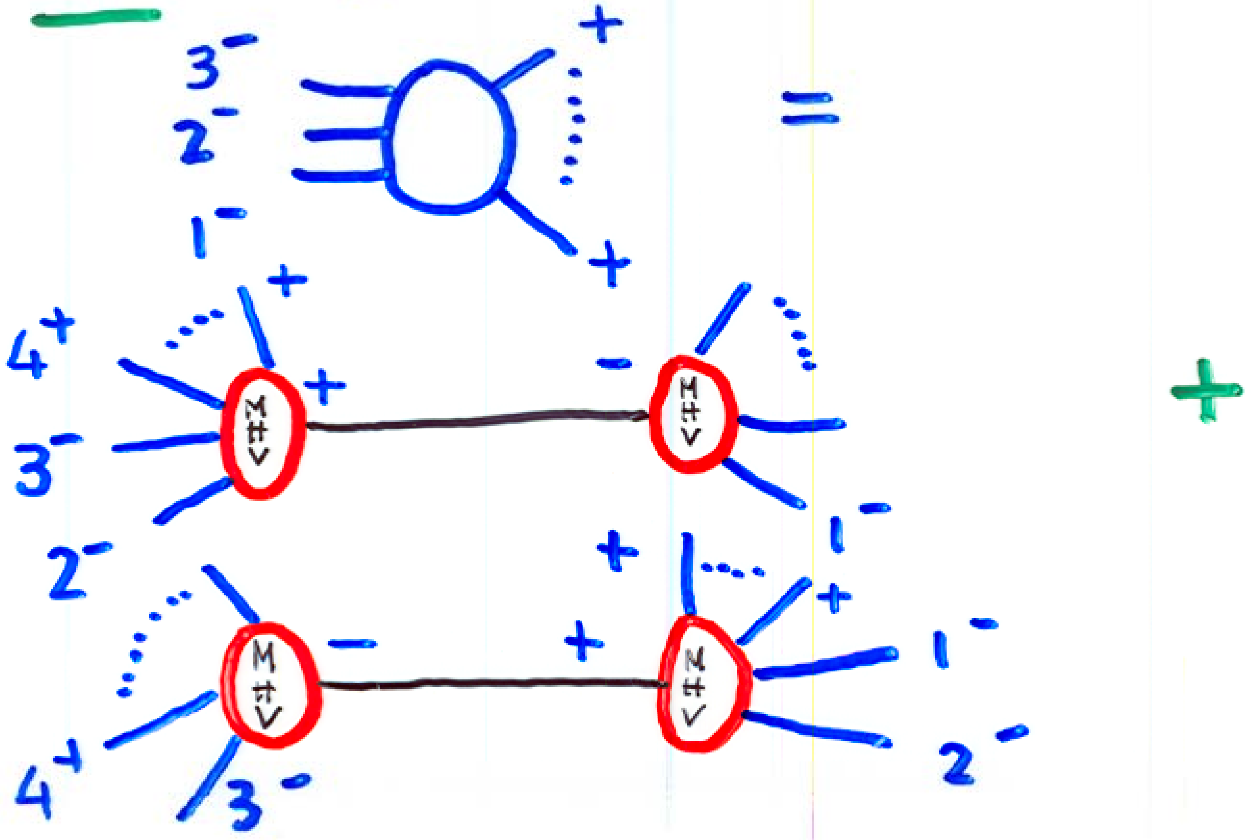
CACHAZO, SVRČEK, WITTEN :

LIFT MHV SCATTERING AMPLITUDES TO EFFECTIVE VERTICES

- REQUIRES AN OFF-SHELL PRESCRIPTION
- SCALAR PROPAGATORS

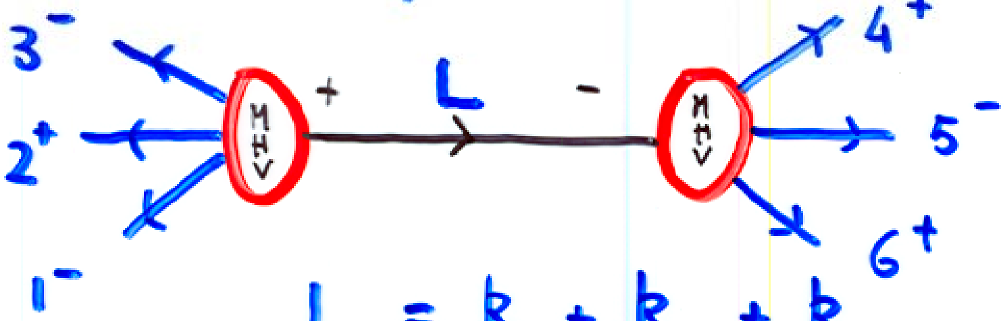
- OBTAIN PREVIOUSLY KNOWN AND UNKNOWN SCATTERING AMPLITUDES WITH DRAMATIC SIMPLIFICATIONS

E.G. "NEXT-TO-MHV":



OFF-SHELL PRESCRIPTION

CONSIDER, FOR EXAMPLE $\quad 1^-2^+3^-4^+5^-6^+$



$$L = k_4 + k_5 + k_6 \qquad L^2 \neq 0$$

- WHAT ARE $\langle 1 L \rangle, \langle 2 L \rangle, \ldots$?

- $$L_{a\dot{a}} = \ell_a \tilde{\ell}_{\dot{a}} + z \underbrace{\eta_a \tilde{\eta}_{\dot{a}}}_{\text{REFERENCE NULL VECTOR}}$$

$$\ell_a = \frac{L_{a\dot{a}} \tilde{\eta}^{\dot{a}}}{[\tilde{\ell}\,\tilde{\eta}]}$$

- $\langle 1 L \rangle \longrightarrow k_{1a} L^{\dot{a}a} \tilde{\eta}_{\dot{a}}$

- $$z = \frac{L^2}{2 L \cdot \eta}$$

- WITH ORDINARY METHODS:
  - TOO MANY DIAGRAMS!
  - NO APPARENT RECURSIVE STRUCTURE

- CSW METHOD SUCCESSFULLY APPLIED TO MANY TREE-LEVEL PROCESSES

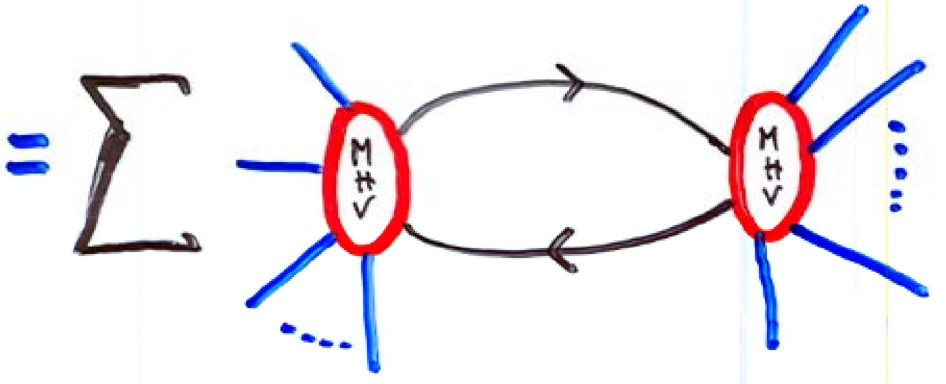
  [ZHU, GEORGIOU + GLOVER + KHOZE, BENA BERN KOSOWER ...]

- NEXT STEP: LOOP DIAGRAMS

  [BRANDHUBER, SPENCE, GT]

FOR EXAMPLE:

MHV, 1 LOOP N=4 SYM



[SCHEMATICALLY]

MHV DIAGRAMS AT LOOP LEVEL

- PROBLEMS WITH (KNOWN) TWISTOR STRING THEORIES
  - SUGRA STATES RUN IN LOOPS
  - DUALITY WITH N=4 SYM IS SPOILED

    [BERKOVITS, WITTEN]

- TRY SIMPLEST APPROACH!
  - SEW V MHV VERTICES

    $V = q - 1 + \ell$ q = # NEGATIVE HELICITIES

    ℓ = # LOOPS

  - SCALAR PROPAGATORS
  - CSW OFF SHELL PRESCRIPTION
  - SUM OVER DIAGRAMS WITH FIXED CYCLIC ORDERING OF EXTERNAL GLUONS

    (DIFFERENT FROM CUT-CONSTRUCTIBILITY APPROACH OF BDDK)

- SIMPLEST EXAMPLE :

ONE-LOOP MHV AMPLITUDE IN $N=4$ SUPER YANG-MILLS

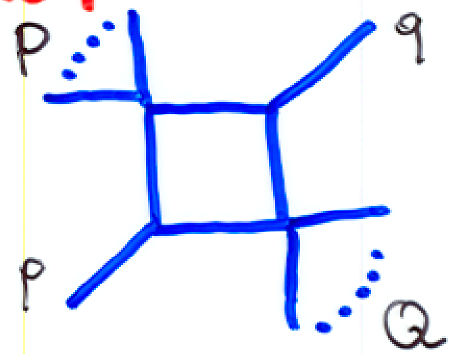
- COMPUTED BY

  BERN, DIXON, DUNBAR, KOSOWER (1994)

  USING CUT-CONSTRUCTIBILITY

- EXPRESSED IN TERMS OF "2-MASS EASY" BOX FUNCTIONS

$*$ $F(s,t,P^2,Q^2) =$



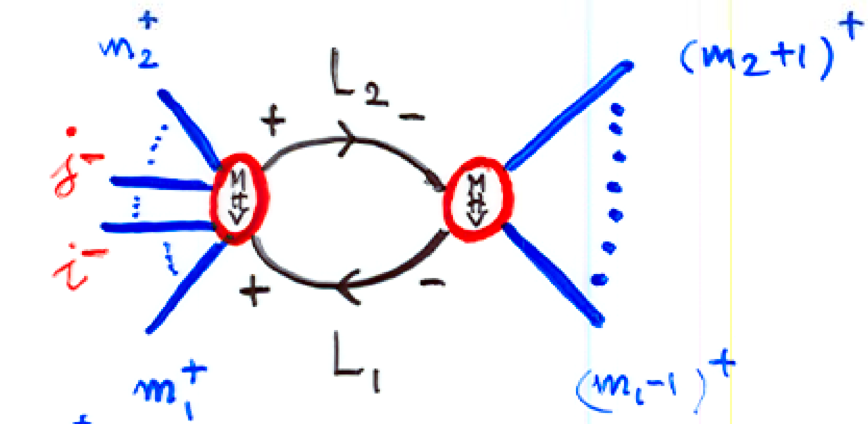
$s = (P+p)^2$

$t = (P+q)^2$

$*$ $$A^{1\text{loop}}_{MHV} = \sum_{p,q} F(s,t,P^2,Q^2)$$

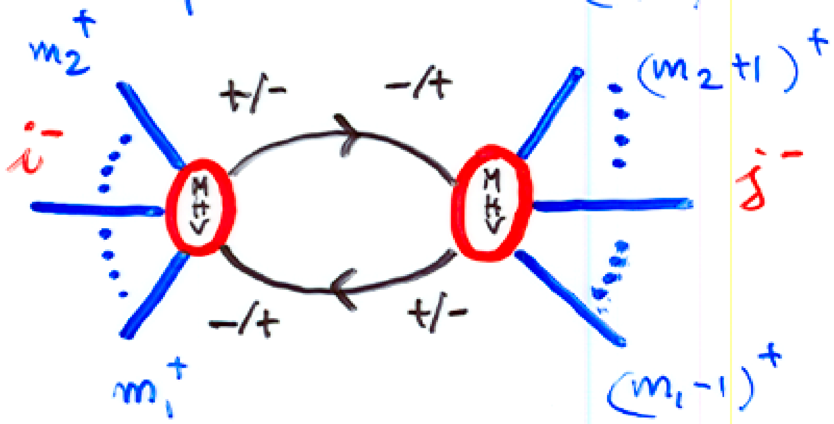
2 CASES:

Ⓐ



THESE ARE TYPICAL MHV DIAGRAMS

Ⓑ



WE HAVE TO SUM OVER ALL POSSIBLE SUCH DIAGRAMS

Ⓐ ONLY **GLUONS** RUNNING

Ⓑ **GLUONS, FERMIONS, SCALARS**

$$\boxed{\begin{aligned} A(\cdots g_i^- \cdots f_r^- \cdots f_s^+ \cdots) &= \frac{\langle is\rangle}{\langle ir\rangle} A^{\text{gluons}} \\ A(\cdots g_i^- \cdots s_r^- \cdots s_s^+ \cdots) &= \left(\frac{\langle is\rangle}{\langle ir\rangle}\right)^2 A^{\text{gluons}} \end{aligned}}$$

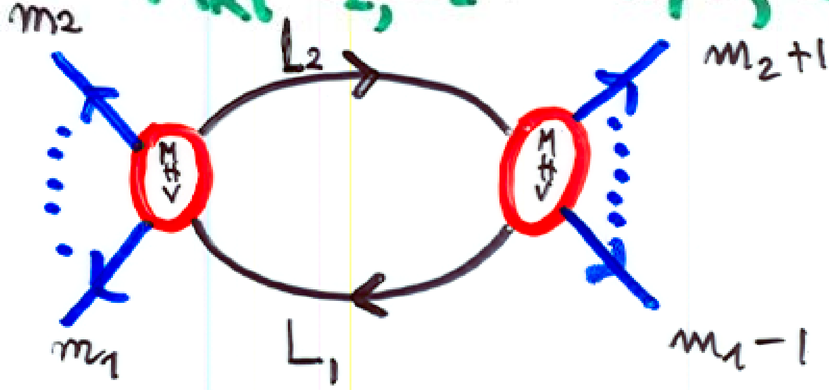
- Ⓐ AND Ⓑ TURN OUT TO BE IDENTICAL

- ALTERNATIVELY THIS CAN ELEGANTLY BE SEEN BY USING THE **NAIR SUPERVERTEX**

SCHEMATICALLY:

$$A = \sum_{m_1, m_2} d\mathcal{M} \; A_L(\ell_2, -\ell_1, m_1 \ldots m_2) \; A_R(-\ell_2, m_2+1 \ldots m_1-1, \ell_1)$$

$$= \sum_{m_1, m_2}$$



- A_L, A_R = MHV VERTICES, WITH CSW OFF-SHELL PRESCRIPTION

$$L_1 = \ell_1 + z_1 \eta \; , \; \ell_{1a\dot{a}} = \ell_{1a} \tilde{\ell}_{1\dot{a}}$$
$$L_2 = \ell_2 + z_2 \eta \; , \; \ell_{2a\dot{a}} = \ell_{2a} \tilde{\ell}_{2\dot{a}}$$

- $d\mathcal{M}$ INTEGRATION MEASURE

$$d\mathcal{M} = (2\pi)^4 \delta^{(4)}(L_2 - L_1 + P_L) \; \frac{d^4 L_1}{L_1^2} \; \frac{d^4 L_2}{L_2^2}$$

$$P_L = \sum_{m_1}^{m_2} k_i$$

NEXT, ELABORATE ON $d\mathcal{M}$

THE INTEGRATION MEASURE

- $$d\mathcal{M} = (2\pi)^4 \delta^{(4)}(L_2 - L_1 + P_L) \frac{d^4 L_1}{L_1^2} \frac{d^4 L_2}{L_2^2}$$

- $$L_{a\dot{a}} = \ell_a \tilde{\ell}_{\dot{a}} + z\, \eta_a \tilde{\eta}_{\dot{a}}$$

- $$\frac{d^4 L}{L^2} = \underbrace{\frac{dz}{z}}_{\text{DISPERSIVE MEASURE}} \; \underbrace{dN(\ell, \tilde{\ell})}_{\text{NAIR MEASURE}}$$

- $$dN \equiv \frac{1}{4i}\left[\langle \ell d\ell \rangle d^2\tilde{\ell} - [\tilde{\ell} d\tilde{\ell}] d^2\ell \right]$$

$$= d^4\ell \;\; \delta^{(+)}(\ell^2)$$

= LORENTZ-INVARIANT PHASE SPACE (LIPS) MEASURE [NAIR]

- $$L_2 - L_1 + P_L =$$

$$= \ell_2 - \ell_1 + \underbrace{P_L + (z_2 - z_1)\,\eta}_{\text{SHIFTED MOMENTUM}}$$

- $d\mathcal{M} \rightarrow \frac{dz_1}{z_1} \frac{dz_2}{z_2} \cdot$

$$d^4\ell_1 \delta^{(+)}(\ell_1^2) d^4\ell_2 \delta^{(+)}(\ell_2^2) \delta^{(4)}(\ell_2 - \ell_1 + P_{L;z})$$

$$z = z_1 - z_2 \qquad z' = z_1 + z_2$$

$$\longrightarrow \quad P_{L;z} = P_L - z\eta \quad \longleftarrow$$

- INTEGRATE OUT z'

- $d^4\ell_1 \delta^{(+)}(\ell_1^2) d^4\ell_2 \delta^{(+)}(\ell_2^2) \, \delta^{(4)}(\ell_2 - \ell_1 + P_{L;z})$

$$\equiv d\text{LIPS}(\ell_2, -\ell_1; P_{L;z})$$

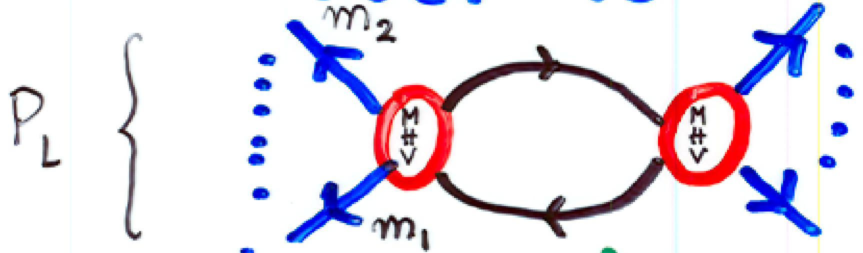
2-PARTICLE LIPS MEASURE

- DIMENSIONAL REGULARISATION (INFRARED DIVERGENCES)

• $d\mathcal{M} \sim \frac{dz}{z}\, dLIPS(\ell_2, -\ell_1; P_{L;z})$

$\longrightarrow \quad P_{L;z} \equiv P_L - z\eta \quad \longleftarrow$

• RESULT IS OF THE FORM:



$$= \int \frac{dz}{z}\, dLIPS(\ell_2, -\ell_1; P_{L;z}) \left[R(m_1, m_2) + 3 \text{ terms} \right]$$

• $\frac{dz}{z} = \frac{d P^2_{L;z}}{P^2_{L;z} - P^2_L}$

[FROM $P^2_{L;z} = P^2_L - 2z(\eta \cdot P_L)$]

• $d\mathcal{M} \sim \frac{d P^2_{L;z}}{P^2_{L;z} - P^2_L}\, d^{4-2\varepsilon}LIPS(\ell_2, -\ell_1; P_{L;z})$

DISPERSIVE INTEGRAL

CUT INTEGRAL

$P_{L;z}$ FLOWS IN THE CUT

COMMENTS:

a) **LIPS** COMPUTES THE DISCONTINUITY OF THE DIAGRAM ACROSS THE BRANCH CUT! WHICH DISCONTINUITY DEPENDS ON $P_{L;z}$

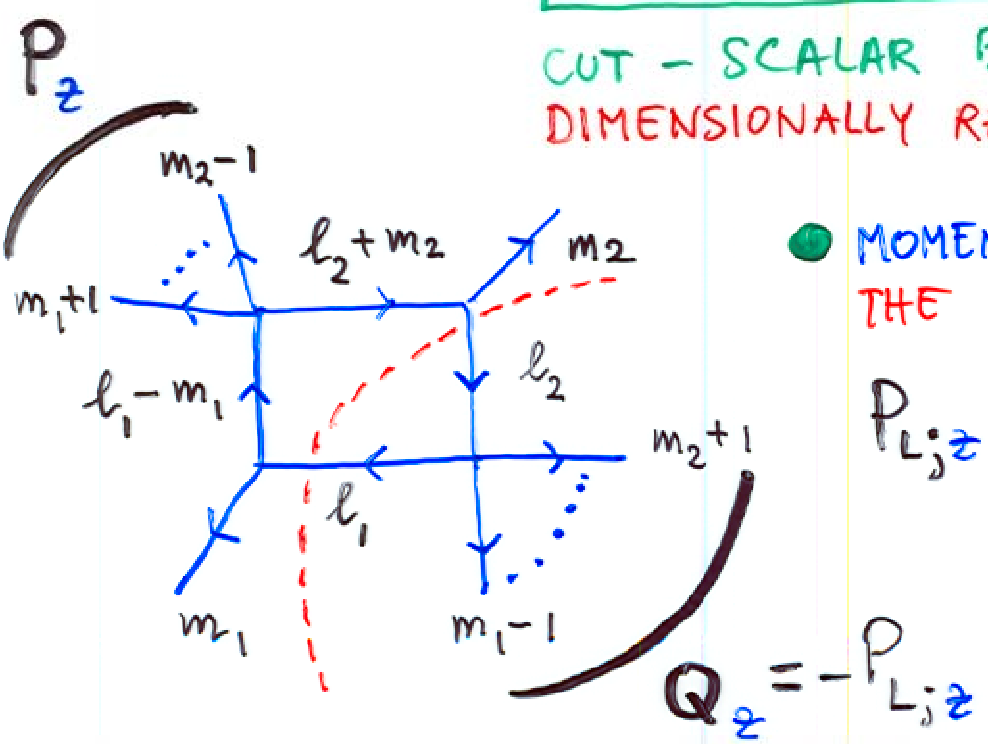
b) **DISPERSIVE INTEGRAL** RECONSTRUCTS THE AMPLITUDE FROM ITS DISCONTINUITIES

- AFTER SPINOR MANIPULATIONS, WE GET:

$$A^{1\text{-loop}}_{MHV} = A^{tree}_{MHV} \cdot \sum_{m_1, m_2} \mathcal{L}_{m_1, m_2}$$

- $\mathcal{L}_{m_1 m_2}$ CONTAINS 4 TERMS. ONE OF THEM IS:

$$\int \frac{dz}{z} \underbrace{\int dLIPS(\ell_2, -\ell_1, P_{L;z}) \frac{N(P_{L;z})}{(\ell_1 - m_1)^2 (\ell_2 + m_2)^2}}_{\text{CUT - SCALAR BOX INTEGRAL, DIMENSIONALLY REGULARISED}}$$



- MOMENTUM FLOWING IN THE CUT IS

$$P_{L;z} = P_L - z\eta$$

$$P_L = \sum_{m_1}^{m_2} k_i$$

INCLUDE ALL MHV DIAGRAMS

- EACH SCALAR BOX FUNCTION APPEARS EXACTLY ONCE IN EACH OF ITS (FOUR) CUTS

- COLLECT THE 4 CUTS CORRESPONDING TO THE CHANNELS

  * $P_z^2 = (P - z\eta)^2$
  * $Q_z^2 = (Q + z\eta)^2$
  * $S_z = (P_z + p)^2$
  * $t_z = (P_z + q)^2$

  AND DO DISPERSION INTEGRALS

  - η-DEPENDENCE DISAPPEARS AFTER SUMMING THE 4 INTEGRATED CUTS

- SUM ALL THE BOXES

DISPERSION INTEGRALS

- CONSIDER ONE BOX FUNCTION AT A TIME
- FOR EACH CHANNEL
  $P^2_{L;z} \quad [= s_z, t_z, P^2_z, Q^2_z]$
  REWRITE $\dfrac{dz}{z} = \dfrac{d\,P^2_{L;z}}{P^2_{L;z} - P^2_L}$
- SUM THE 4 DISPERSION INTEGRALS OF A BOX FUNCTION:

$$\mathcal{F}(s,t,P^2,Q^2) =$$

$$= \int ds' \frac{\Delta_{s'} F}{s'-s} + \int dt' \frac{\Delta_{t'} F}{t'-t} + \int dP^{2\prime} \frac{\Delta_{P^{2\prime}} F}{P^{2\prime}-P^2} + \int dQ^{2\prime} \frac{\Delta_{Q^{2\prime}} F}{Q^{2\prime}-Q^2}$$

- $\mathcal{F}$ HAS PRECISELY THE SAME DISCONTINUITIES OF THE DESIRED F

$$\mathcal{F} = F \quad ?$$

  - RATIONAL TERMS
  - IR DIVERGENCES
- DO THE CALCULATION !

EXPLICIT CALCULATION GIVES

$$A^{MHV}_{1-loop} = A^{MHV}_{tree} \sum_{p,q} \mathcal{F}(s,t,P^2,Q^2)$$

$$\mathcal{F}(s,t,P^2,Q^2) = -\frac{1}{\varepsilon^2}\left[(-s)^{-\varepsilon} + (-t)^{-\varepsilon} - (-P^2)^{-\varepsilon} - (-Q^2)^{-\varepsilon}\right] + B(s,t,P^2,Q^2)$$

- $$B(s,t,P^2,Q^2) = \mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at)$$

$$a = \frac{u}{P^2Q^2 - st} \qquad s+t+u = P^2+Q^2$$

- $$B'(s,t,P^2,Q^2) = \mathrm{Li}_2\left(1-\frac{P^2}{s}\right) + \mathrm{Li}_2\left(1-\frac{P^2}{t}\right) + \mathrm{Li}_2\left(1-\frac{Q^2}{s}\right) + \mathrm{Li}_2\left(1-\frac{Q^2}{t}\right) - \mathrm{Li}_2\left(1-\frac{P^2Q^2}{st}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right)$$

- IF $B = B'$, THEN AGREEMENT WITH BDDK (1994)

- $B - B' \ni$ 9 DILOGARITHMS

$\Rightarrow$ MANTEL'S IDENTITY (1898)

$$\mathrm{Li}_2\left(\frac{vw}{xy}\right) = \mathrm{Li}_2\left(\frac{v}{x}\right) + \mathrm{Li}_2\left(\frac{w}{x}\right) + \mathrm{Li}_2\left(\frac{v}{y}\right) + \mathrm{Li}_2\left(\frac{w}{y}\right) +$$
$$+ \mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2(v) - \mathrm{Li}_2(w) +$$
$$+ \frac{1}{2}\log^2\left(-\frac{x}{y}\right)$$

$$x, y, v, w \in (0,1)$$
$$(1-v)(1-w) = (1-x)(1-y)$$

$x = as$ $\qquad$ $y = at$

$v = aP^2$ $\qquad$ $w = aQ^2$

- $P^2 = 0,\ Q^2 \neq 0$ OR $P^2 \neq 0,\ Q^2 = 0$
(5-POINT CASE): HILL'S IDENTITY

- $P^2 = Q^2 = 0$ (4-POINT CASE):
EULER'S IDENTITY

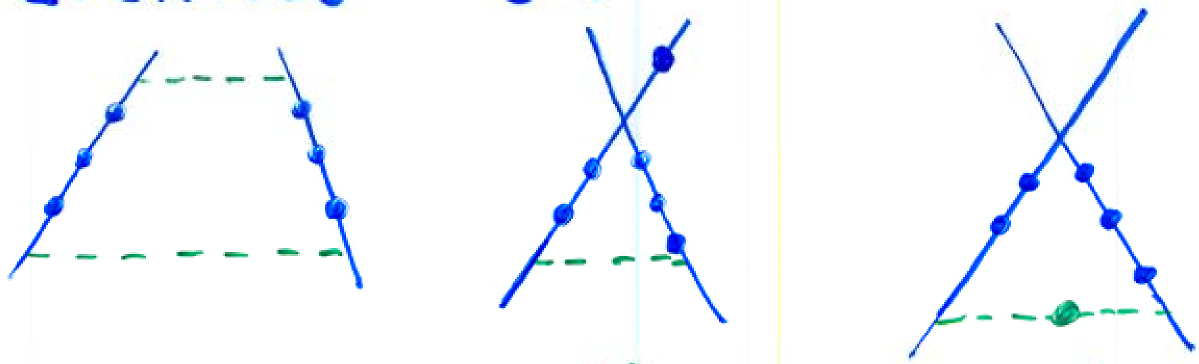
OUR RESULT :

- INCORPORATES LARGE NUMBERS OF CONVENTIONAL DIAGRAMS
- IS (SLIGHTLY) SIMPLER
  - NEW EXPRESSION FOR THE 2-MASS EASY BOX FUNCTION, CONTAINING 4 DILOGS
- CONTACT WITH BEAUTIFUL RESULTS OF BERN, DIXON, DUNBAR, KOSOWER

COMMENT ON TWISTOR SPACE PICTURE

- CACHAZO, SVRČEK and WITTEN HAVE STUDIED LOCALISATION PROPERTIES IN TWISTOR SPACE OF ONE-LOOP AMPLITUDES, USING DIFFERENTIAL OPERATORS IN SPINOR SPACE.

- **CSW** **FIND THAT 1-LOOP AMPLITUDES LOCALISE ON**



a) b) c)

- a) AND b) CONSISTENT WITH OUR RESULT

- c) GENERATED BY A **HOLOMORPHIC ANOMALY** [CACHAZO, SVRČEK, WITTEN]

- **VERY USEFUL !**

GENERALISATIONS

- OUR APPROACH CAN BE APPLIED TO
  - OTHER (NON-MHV) AMPLITUDES
  - THEORIES WITH LESS SUPERSYMMETRY

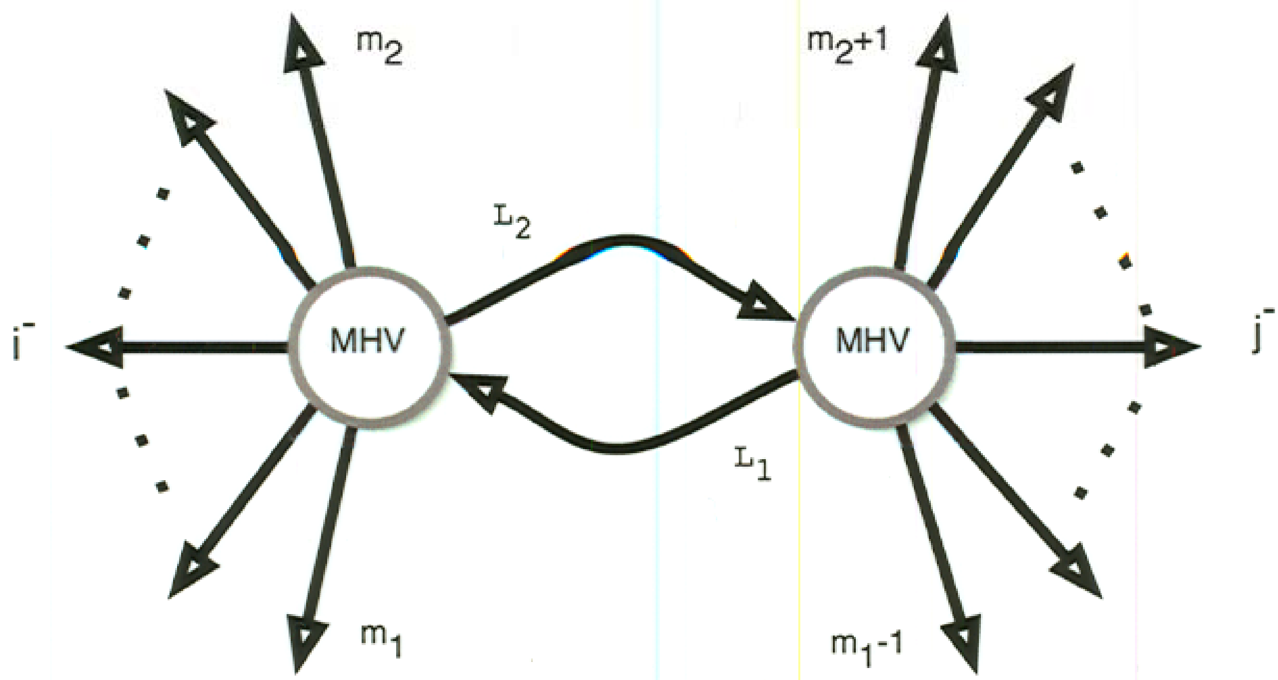
1-LOOP MHV AMPLITUDES IN $\mathcal{N}=1$ SYM

[BEDFORD, BRANDHUBER, SPENCE, GT ; QUIGLEY, ROZALI]

- SUPERSYMMETRIC DECOMPOSITION:

$$A^{\mathcal{N}=1,\,\text{VECTOR}}_{1-\text{loop}} = \underbrace{A^{\mathcal{N}=4}_{1-\text{loop}}}_{\text{KNOWN !}} - 3 \cdot \underbrace{A^{\mathcal{N}=1,\,\text{CHIRAL}}_{1-\text{loop}}}_{\text{TO BE COMPUTED, CHECK WITH BDDK (1994)}}$$

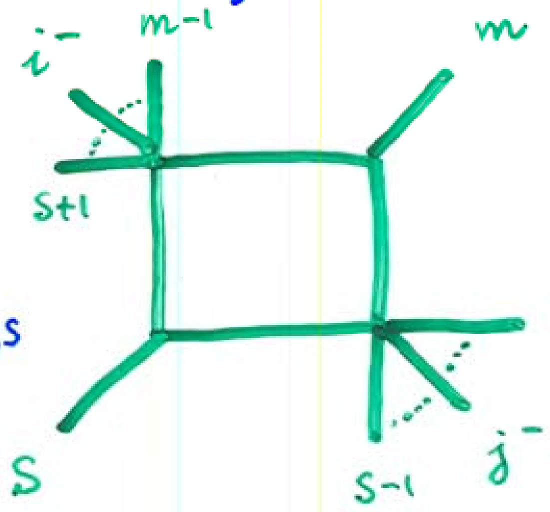
- FEATURES of the RESULT:
  - $A^{\mathcal{N}=1,\,\text{CHIRAL}}_{1-\text{loop}} = A_{\text{tree}} \cdot \mathcal{J}$
  - $\mathcal{J}$ CONTAINS SCALAR BOX FUNCTIONS, TRIANGLE FUNCTIONS
  - $\mathcal{J}$ DEPENDS ON i^-, j^-



An $\mathcal{N} = 1$ one-loop MHV diagram, using two MHV vertices. The two external gluons with negative helicity are labelled by i and j.

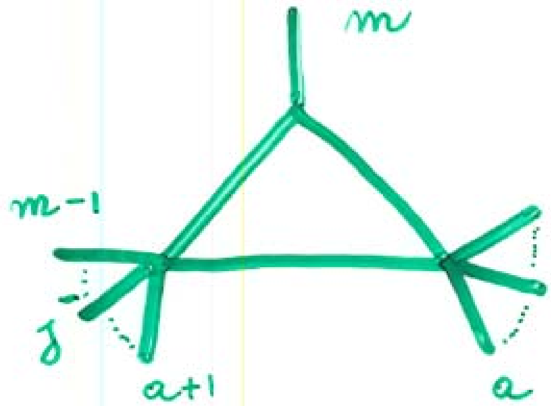
- SCHEMATICALLY, THE RESULT IS

$$\mathcal{J} = \sum_{m,s} b^{i,j}_{m,s}$$



$+$

$$\sum_{m,a} c^{i,j}_{ma}$$



$+$

$$\sum_{m,a} c^{i,j}_{m,a} \quad [\, i^- \longleftrightarrow j^- \,]$$

- AGREEMENT WITH

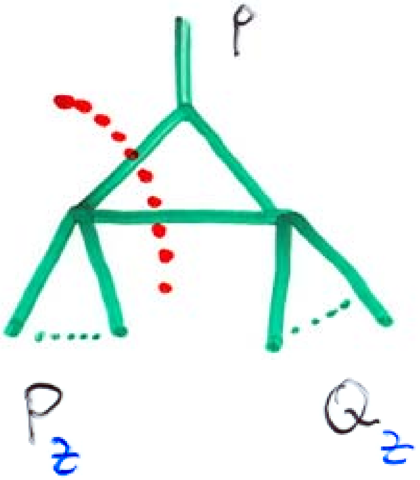
BDDK (1994)

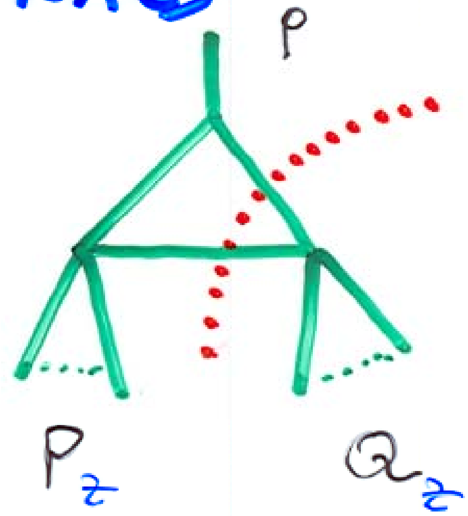
• TRIANGLE FUNCTION

$$T(p,P,Q) = \frac{\log Q^2/P^2}{Q^2 - P^2}$$

$p + P + Q = 0$

RECONSTRUCTED VIA DISPERSION INTEGRALS

$\int \frac{dz}{z}$ 

$+ \int \frac{dz}{z}$ 

$P_z = P - z\eta \qquad Q = Q + z\eta$

• EXPLICITLY

η- INDEPENDENT

MORE IN DETAIL:

$$\mathcal{J} = \sum_{m,s} b^{ij}_{m,s} \, B\,(s,t,P^2,Q^2) \, +$$

$$+ \frac{1}{1-2\varepsilon}\left[\sum_{m,a} c^{ij}_{m,a} \, T_\varepsilon(p_m, P^2, Q^2) + (i \leftrightarrow j) \right]$$

- B IS THE FINITE PART OF THE BOX FUNCTION
- $T_\varepsilon(p, P, Q) = \frac{1}{\varepsilon} \frac{(-P^2)^{-\varepsilon} - (-Q^2)^{-\varepsilon}}{Q^2 - P^2}$
- **THE COEFFICIENTS ARE**

$$b^{ij}_{m,s} = -2 \, \frac{\mathrm{Tr}_+(\hat{k}_i \hat{k}_j \hat{k}_m \hat{k}_s)\, \mathrm{Tr}_+(\hat{k}_i \hat{k}_j \hat{k}_s \hat{k}_m)}{[(k_i+k_j)^2]^2 \, [(k_a+k_m)^2]^2}$$

$$c^{ij}_{m,a} = \left[\frac{\mathrm{Tr}_+(\hat{k}_m \hat{k}_{a+1} \hat{k}_j \hat{k}_i)}{(k_{a+1}+k_m)^2} - \frac{\mathrm{Tr}_+(\hat{k}_m \hat{k}_a \hat{k}_j \hat{k}_i)}{(k_a+k_m)^2} \right] \times$$

$$\frac{\mathrm{Tr}_+(\hat{k}_i \hat{k}_j \hat{k}_m \hat{q}_{m,a}) - \mathrm{Tr}_+(\hat{k}_i \hat{k}_j \hat{k}_m \hat{q}_{m,a+1})}{(k_i+k_j)^2}$$

$$q_{m,a} = k_m + \cdots + k_a$$

MHV AMPLITUDES IN PURE YANG-MILLS

[BEDFORD, BRANDHUBER, SPENCE, GT]

- CONTAINS A CUT-CONSTRUCTIBLE PART, BUT ALSO RATIONAL TERMS
- FROM MHV VERTICES WE OBTAIN THE CUT-CONSTRUCTIBLE TERM
- SUPERSYMMETRIC DECOMPOSITION

$$A^g = \underbrace{(A^g + 4A^f + 3A^s)}_{N=4} - 4\underbrace{(A^f + A^s)}_{N=1} + \underbrace{A^s}_{\text{TO BE COMPUTED}}$$

- RESULT IS EXPRESSED IN TERMS OF $B(s,t,P^2,Q^2)$ AND

$$T^{(r)}(p,P,Q) = \frac{\log Q^2/P^2}{(Q^2-P^2)^r}$$

- COEFFICIENT OF B IS $\left[b^{ij}_{m_1 m_2}\right]^2$

– AGREES WITH 5-GLUON CASE [BDK]

– ADJACENT NEGATIVE HELICITY GLUONS [BDDK]

– NEW RESULT FOR NEGATIVE HELICITY GLUONS IN ARBITRARY POSITIONS