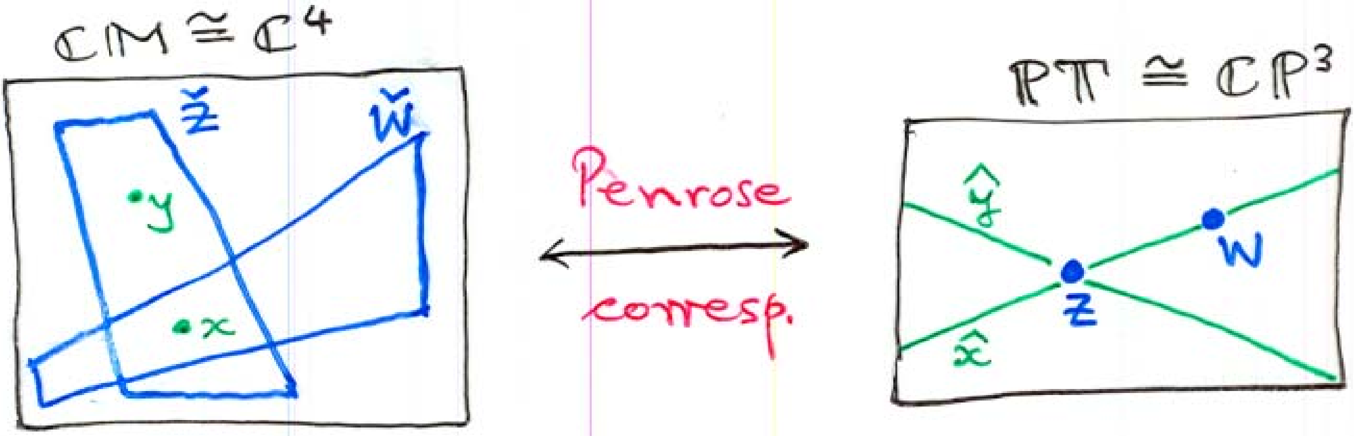
Self-Dual Gauge Fields & Twistors.

- Twistor Description for SDYM
- Reductions & Integrable Systems
- Generalized Nahm Transform



* $\check{Z}$: totally-null anti-self-dual 2-plane
* $\hat{x} \cong \mathbb{CP}^1$ "straight line"

- A gauge field on $\mathbb{CM}$ is <u>self-dual</u> $({*F_{\mu\nu}} = F_{\mu\nu})$ iff $F|_{\check{Z}} = 0$ for each TNASD2P $\check{Z}$.
- Reality: $\mathbb{M}$ could have signature $++++$ or $++--$.

- There is a natural 1-1 corresp between:
  - * SD gauge fields on (a subset of) $\mathbb{C}M$
  - * holomorphic vector bundles E over (the corresponding subset of) $\mathbb{P}\mathbb{T}$, such that $E|_{\hat{x}}$ is trivial for all x.

generic

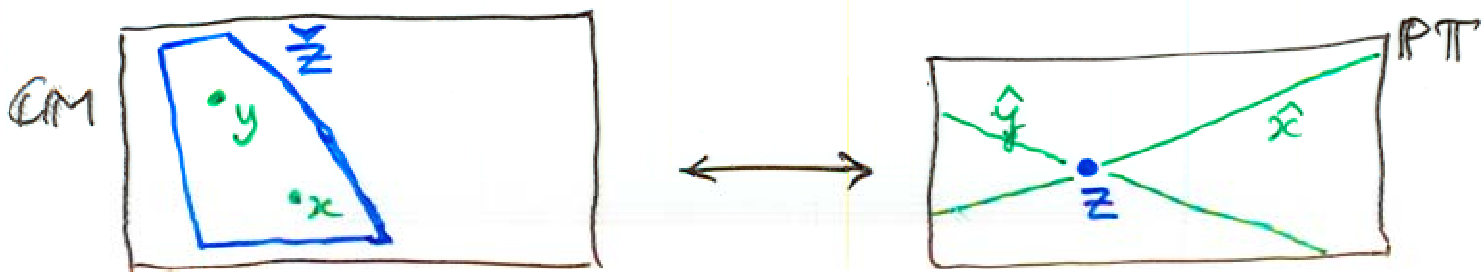
[Rank n, group $GL(n, \mathbb{C})$]

- Given SD gauge field — trivial on plane $\check{Z}$ — space of covariantly-constant isovector fields on $\check{Z}$ is the fibre $E|_Z$.

- Given E, the space of holomorphic sections of $E|_{\hat{x}}$ is $\cong \mathbb{C}^n$

  [use $\hat{x} \cong \mathbb{CP}^1$, Liouville's theorem]

  and this is isovector space at $x \in \mathbb{C}M$. Isovectors at x and y can be identified (namely at $Z = \hat{x} \cap \hat{y}$) $\rightsquigarrow$ SD connection.



- **Higher Dimensions.** Analogous corresp between 4k-dim space-time and $\mathbb{PT} \cong \mathbb{CP}^{2k+1}$, with symmetry

$$[Sp(k) \times Sp(1)]/\mathbb{Z}_2 \subseteq SO(4k) \quad (+\text{conf}).$$

The "SD" and "ASD" equations are (non-equivalent) sets of algebraic relations on $F_{\mu\nu}$.

- **Yang's Equation.** The SDYM eqn can be re-written in terms of $n \times n$ matrix J:

$$\partial^\mu (J^{-1}\partial_\mu J) + \omega^{\mu\nu}(J^{-1}\partial_\mu J)(J^{-1}\partial_\nu J) = 0.$$

const, non-degen, $\omega_{\mu\nu}\omega^{\mu\nu} = -4$

- ## Reductions of SDYM.

  * Choose gauge group & space-time signature.
  * Impose invariance under group of (conformal) motions (dimensional red$^{\underline{n}}$).
  * Impose consistent algebraic constraint on fields, choose gauge, ...

- ## Reduction to 3D gives:

  * Bogomolnyi eqn $D_\mu \Phi = \frac{1}{2} \varepsilon_{\mu\alpha\beta} F^{\alpha\beta}$ **OR**
  * Modified chiral eqn

$$\partial^\mu (J^{-1} \partial_\mu J) + V_\alpha \, \eta^{\alpha\mu\nu} (J^{-1} \partial_\mu J)(J^{-1} \partial_\nu J) = 0$$

$$V_\mu V^\mu = \pm 1$$

on a 3-space of constant curvature ($\mathbb{R}^3$, $\mathbb{R}^{2+1}$, H^3, dS_{2+1}, AdS_{2+1}, S^3).

BPS Monopoles, hyperbolic monopoles, (2+1)-dim soliton systems ...

Examples of 2D Reductions.

* Reduction of (++−−) Yang eqn gives chiral eqn on $\mathbb{R}^{1+1}$, or harmonic-map eqn on $\mathbb{R}^2$.

* Reduction of ++++ SDYM gives Hitchin eqns on $\mathbb{R}^2$ ($\longrightarrow$ Riemann surface)

$$\left.\begin{aligned} \partial_{\bar z}\phi + [A_{\bar z}, \phi] &= 0 \\ F_{z\bar z} &= [\phi, \phi^{+}] \end{aligned}\right\}$$

* Toda Field Theory $\Box \phi_a + \sum_b K_{ab} \exp \phi_b = 0$

(extended) Cartan matrix

Liouville, sin-Gordon, ...

* KdV, NLS, ... from "null" reductions.

* Ernst eqn $\frac{1}{r}\partial_r(rJ^{-1}\partial_r J) + \partial_x(J^{-1}J_x) = 0$

Generalized ADHM-Nahm Transform.

- SDYM in x^μ $\longleftrightarrow$ SDYM in $\tilde{x}^\mu$

$$M(\tilde{x}) = A\tilde{x} + B$$

linear operator on p-vectors

$p \times 2$ matrix

Solve $Mv = 0, \quad v^\dagger v = \mathbb{1}_2$

$$A_\mu(\tilde{x}) = v^\dagger \frac{\partial}{\partial \tilde{x}^\mu} v$$

- Basic system: SDYM on T^4 $\leftrightarrow$ SDYM on $(T^4)^*$

Special cases:
- period of $x^4 \to \infty$
- indep of x^4

- $++++$ case

Calorons

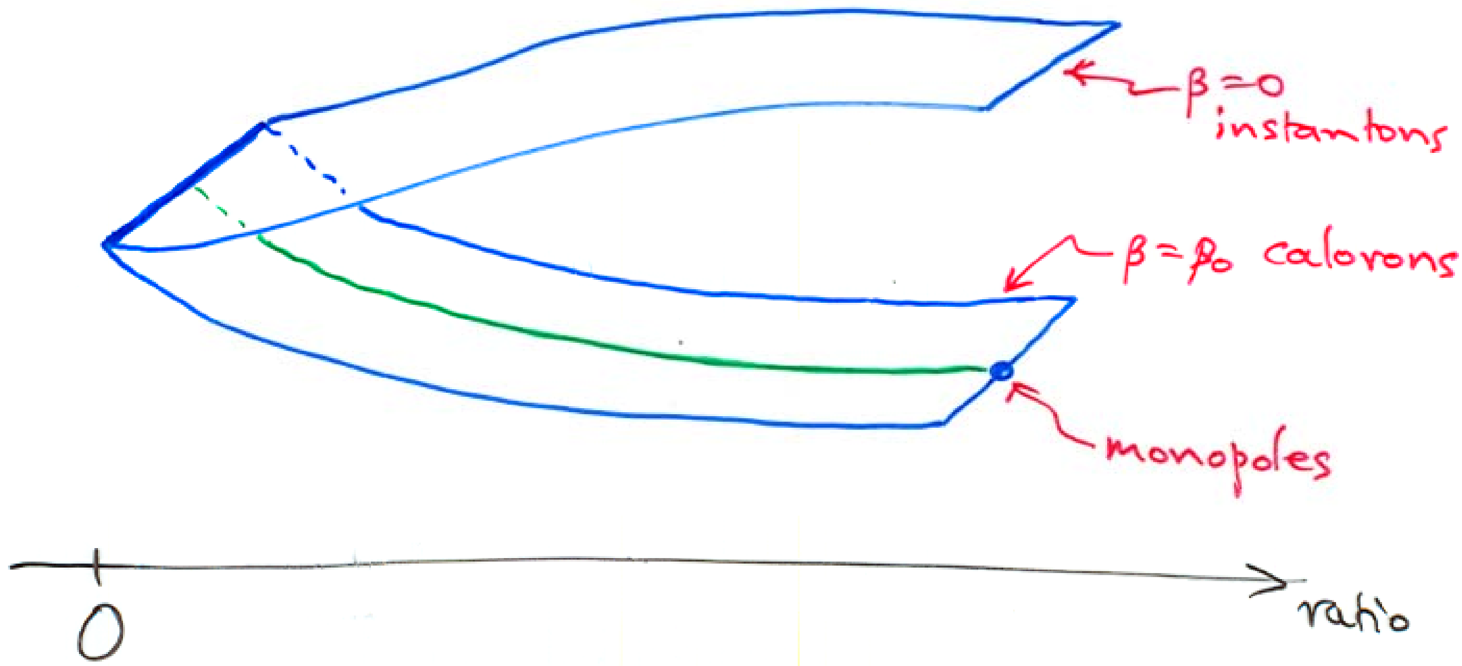
- SD gauge field on $\mathbb{R}^4$, periodic in $t = x^4$ (period β), $\int |F|^2 \, d^4x$ finite.

- Holonomy $\Omega(x^j) = \mathcal{P}\exp\left[-\int_0^\beta A_t \, dt\right]$.

- If $\Omega(\infty) = 1$, then $N = \frac{1}{32\pi^2}\int {*F}\cdot F \, d^4x$ is an integer.

- Dimensionless ratio (size of caloron)/β.

- $\mathcal{M}^{\text{monopole}}_{4N-1} \subset \mathcal{M}^{\text{caloron}}$?



Indep of d coords Periodic in ℓ coords	$\longleftrightarrow$	Indep of $4-d-\ell$ coords Periodic in ℓ coords

- $\ell = 0$: $d=0$ $\mathbb{R}^4$ instanton $\leftrightarrow$ $d=4$ ADHM data

  $d=1$ $\mathbb{R}^3$ monopole $\leftrightarrow$ $d=3$ Nahm eqn

  $d=2$ Hitchin eqn

- $\ell = 1$: $d=0$ Caloron $\leftrightarrow$ $d=3$ periodic Nahm

  $d=1$ Monopole chain $\leftrightarrow$ Hitchin eqn on $\mathbb{R}\times S^1$

- $\ell = 2$: $d=0$ DP instanton $\leftrightarrow$ DP Hitchin

  $d=1$ monopole sheet

- $\ell = 3$: $d=0$ TP instanton $\leftrightarrow$ $d=1$ monopole crystal

- $\ell = 4$: $d=0$ Instanton on T^4.

- Nahm data $T_j(s)$ $(N \times N$, period $2\pi/\beta)$ & W (N-row of quaternions), with

$$\frac{d}{dt}\underline{T} - i\,\underline{T}\times\underline{T} = i\,\mathrm{tr}_2\left(\underline{\sigma}\,W^\dagger W\right)\delta\left(s-\frac{\pi}{\beta}\right)$$

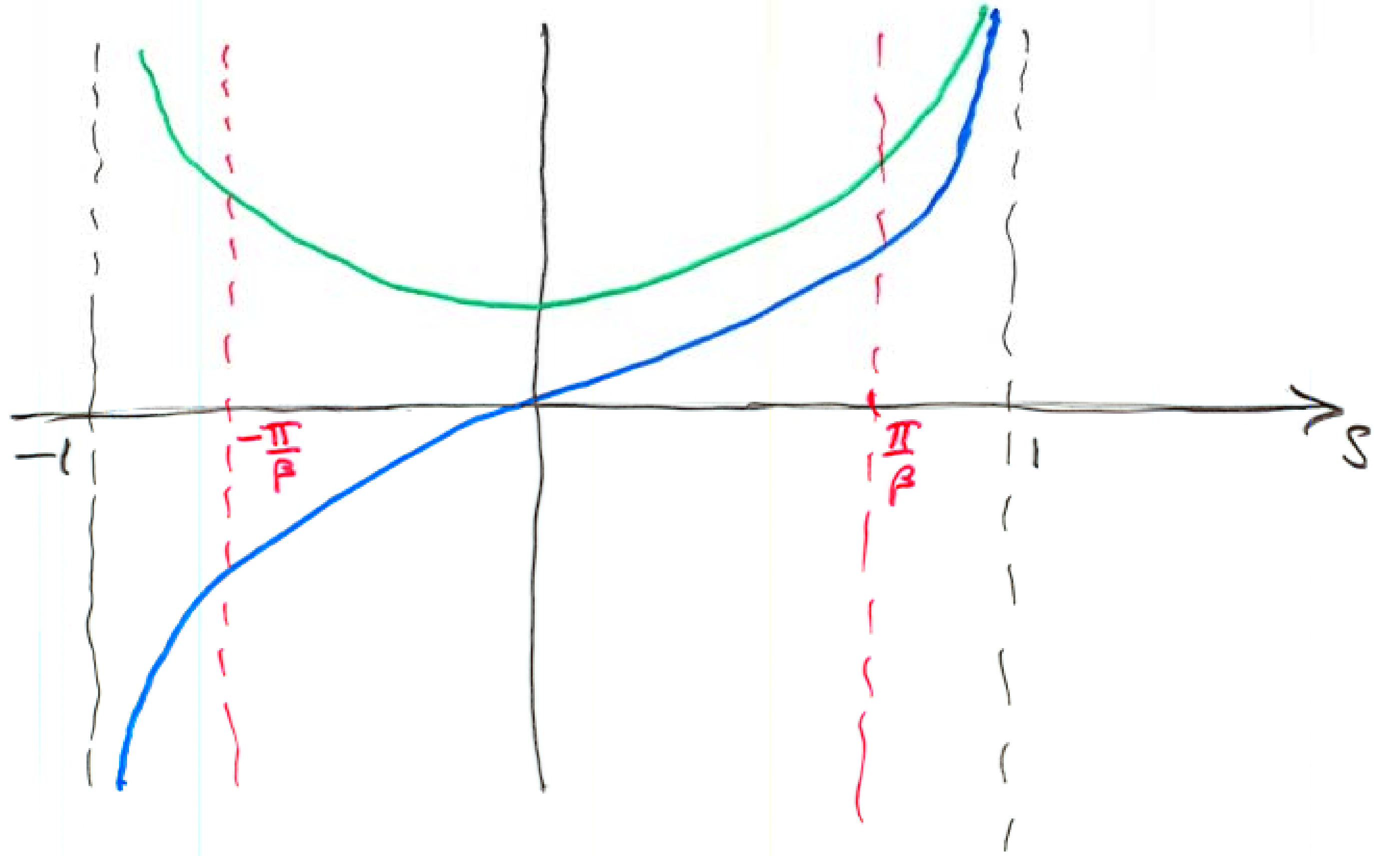
- Examples: explicit Nahm data for monopole, re-cycle to produce caloron, instanton at "other end".

- $N=2$, $SO(2)$ symmetry

  Monopole data $T_1(s) = f_1(s)\,\sigma_1$ etc,

$$f_1 = f_3 = \frac{\pi}{4}\sec\left(\frac{\pi s}{2}\right), \quad f_2 = -\frac{\pi}{4}\tan\left(\frac{\pi s}{2}\right)$$

[Poles at $s = \pm 1$: sets the length-scale.]



$\rightsquigarrow$ periodic solution on $\left[-\frac{\pi}{\beta}, \frac{\pi}{\beta}\right]$ with jump discontinuity. $\beta > \pi$

- Suitable W exists: with $SO(2)$ symmetry, unique: symmetric 2-caloron.

- Limit $\beta \to \pi$ is 2-monopole, limit $\beta \to \infty$ is symmetric 2-instanton (Nahm data $\to$ ADHM data).

- Sim for tetrahedral & cubic monopoles...
  $N=3$ $N=4$