Summer Research Project

Campbell Brawley

Supervisor: Prof. Jason Lotay

September 8, 2022

Abstract

In this paper, we discuss some methods for calculating induced maps on homology and cohomology for some Lie groups. We begin with the n-torus and then move on to the special unitary group $SU(2)$. We also describe ways in which our results for $SU(2)$ may be used to calculate pushforward maps on the homology of general $SU(n)$. This motivates the investigation of embeddings of $SU(2)$ into $SU(n)$. We prove two results in this direction, the first that all embeddings of $SU(2)$ into $SU(3)$ which are also group homomorphisms are homotopic, the second that all embeddings of $SU(2)$ into $SU(n)$ which induce isomorphisms on third homology are homotopic. We finish with a brief discussion of Lie algebra cohomology and an example calculation.

1 The n-Torus

The first space we shall consider will be the n-dimensional torus T^n. The n-torus may be thought of as the product space $(S^1)^n = S^1 \times \cdots \times S^1$. Another nice way to think of the torus is as the quotient space of $\mathbb{R}^n$ under the equivalence relation which identifies v, $w \in \mathbb{R}^n$ if $v - w \in \mathbb{Z}^n$. We might think of this as saying that the torus is $\mathbb{R}^n$ modulo the obvious translation action of $\mathbb{Z}^n$ on $\mathbb{R}^n$. For this reason, the n-torus is sometimes denoted by $\mathbb{R}^n/\mathbb{Z}^n$.

Using this identification of the torus, we may easily define a large family of continuous maps from the torus to itself. Consider an $n \times n$ matrix A with entries in $\mathbb{Z}$. There is a corresponding continuous map from $\mathbb{R}^n$ to itself given by pre-multiplication by A.

$$\begin{aligned} f_A &: \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ f_A &: v \mapsto Av \end{aligned} \tag{1}$$

As $v - w \in \mathbb{Z}^n \Rightarrow Av - Aw \in \mathbb{Z}^n$, this descends to give a well-defined continuous map $[v] \mapsto [Av]$ from the torus $\mathbb{R}^n/\mathbb{Z}^n$ to itself.

One natural question to ask about these maps is: *What are the induced maps on homology and cohomology?*. The homology (with coefficients in $\mathbb{Z}$) of the n-torus may easily be calculated using the Künneth formula, which states that there is a natural short exact sequence

$$0 \longrightarrow \bigoplus_i H_i(X) \otimes H_{n-i}(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_i \operatorname{Tor}(H_i(X), H_{n-i-1}(Y)) \longrightarrow 0,$$

where by naturality we mean that continuous maps $X \to X'$ and $Y \to Y'$ induce a commutative diagram of the corresponding short exact sequences [3].

Proposition 1.1. *The homology and cohomology of the n-torus are given by*

$$H_k(T^n) = H^k(T^n) = \mathbb{Z}^{\binom{n}{k}}$$

for $0 \le k \le n$ and the groups are 0 otherwise.

Proof. To compute the homology we will use the description of the n-torus as $S^1 \times \cdots \times S^1$. The homology of S^1 is easily calculated by giving it a Δ-complex structure and computing the simplicial homology, establishing the base case.

Now Künneth's formula gives a short exact sequence

$$0 \longrightarrow \bigoplus_i H_i((S^1)^{n-1}) \otimes H_{k-i}(S^1) \longrightarrow H_k((S^1)^n) \longrightarrow \bigoplus_i \operatorname{Tor}(H_i((S^1)^{n-1}), H_{k-i-1}(S^1)) \longrightarrow 0$$

By induction, the homology groups of the $n-1$-torus are free, so the Tor vanishes and we get an isomorphism

$$\begin{aligned} H_k((S^1)^n) &\cong \left(H_k((S^1)^{n-1}) \otimes H_0(S^1)\right) \oplus \left(H_{k-1}((S^1)^{n-1}) \otimes H_1(S^1)\right) \\ &\cong \mathbb{Z}^{\binom{k-1}{n-1}} \oplus \mathbb{Z}^{\binom{k}{n-1}} \\ &\cong \mathbb{Z}^{\binom{k}{n}} \end{aligned} \tag{2}$$

Because all these groups are free, the cohomology will be the same by the universal coefficient theorem for cohomology.

□

We now turn to calculating the maps induced on (co)homology by the maps (1). Calculating the maps induced on first homology is easy once we remember that the first homology of $(S^1)^n$ is generated by 1-simplices which wind once around one of the circles and remain constant in the other co-ordinates. In our identification of the n-torus with $\mathbb{R}^n/\mathbb{Z}^n$, this corresponds to a 1-simplex starting at the origin and going in a straight line to the point $e_i = (0, \ldots, 1, \ldots, 0)$. The map f_A will take this to a straight line segment from the origin to the point Ae_i. It is not hard to see that this is homologous to $A_{i,1}e_1 + \cdots + A_{i,n}e_n$. We will employ a slight abuse of notation and write this as $(f_A)_*(e_i) = Ae_i$.

Identifiying $H_1(\mathbb{R}^n/\mathbb{Z}^n) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \cong \mathbb{Z}^n$, we get that the pushforward is simply given by pre-multiplication by the matrix A, $(f_A)_*(v) = Av$. It is not immediately clear how to go about calculating the induced maps on the higher homology groups. We could write out generators for the higher homologies and see how they are mapped by f_A, but this would be extremely tedious and inefficient.

The calculation turns out to be easy, however, by considering the cohomology ring. First, we can compute the induced map on cohomology using the Universal Coefficient Theorem. The theorem gives a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_0(T^n), \mathbb{Z}) \longrightarrow H^1(T^n; \mathbb{Z}) \longrightarrow \mathrm{Hom}(H_1(T^n), \mathbb{Z}) \longrightarrow 0.$$

Because $H_0(T^n)$ is free, the Ext vanishes and we get a natural isomorphism $H^1(T^n;\mathbb{Z}) \to \mathrm{Hom}(H_1(T^n), \mathbb{Z})$, where naturality means that the pull-back of a map from T^n to itself will induce a commutative diagam with the isomorphism (see below). Now, as $H_1(T^n)$ is $\mathbb{Z}^n$, there is an obvious identification of $\mathrm{Hom}(H_1(T^n), \mathbb{Z})$ with $\mathbb{Z}^n$ by picking a dual basis such that the dual map $(f_A)'_*$ to $(f_A)_*$ ($(f_A)_*$ being pre-multiplication by A) is pre-multiplication by the transpose A^T. Thus, we get a commutative digram

$$\begin{array}{ccccc} H^1(T^n;\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}(H_1(T^n),\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}^n \\ \big\downarrow (f_A)^* & & \big\downarrow (f_A)'_* & & \big\downarrow \cdot A^T \\ H^1(T^n;\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{Hom}(H_1(T^n),\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}^n \end{array}$$

so we see that the map $(f_A)^*$ induced on first cohomology by f_A is multiplication by A^T.

We now calculate the cohomology ring of T^n. We can do this using a form of the Künneth formula.

Proposition 1.2. *The cohomology ring of the n-torus is the exterior algebra $\Lambda_{\mathbb{Z}}[e'_1, \ldots, e'_n]$ where $|e'_i| = 1$ and each e'_i is the inclusion of a generator for $H^1(S^1)$ into the i-th copy of S^1 in $(S^1)^n$.*

Proof. The Künneth formula (for example, Theorem 3.15 of [3]) gives an isomorphism

$$H^*((S^1)^{n-1}) \otimes_{\mathbb{Z}} H^*(S^1) \longrightarrow H^*((S^1)^n)$$

defined by $a \otimes b \mapsto a \times b$. Thus, by induction, we have $H^*((S^1)^n) \cong \Lambda_{\mathbb{Z}}[e'_1, \ldots, e'_{n-1}] \otimes \Lambda_{\mathbb{Z}}[e'_n] \cong \Lambda_{\mathbb{Z}}[e'_1, \ldots, e'_n]$. The fact that the e_i's arise from generators of each copy of S^1 follows from the definition of the Künneth isomorphism.

□

One important fact about the cup product is that it obeys the following naturality property (see, for example, Proposition 3.10 of [3]): for a map $g : X \to Y$ of topological spaces, its pull-back $g^* : H^*(Y) \to H^*(X)$ satisfies $g^*(\alpha \smile \beta) = g^*(\alpha) \smile g^*(\beta)$. This might be re-phrased as saying that the induced maps on cohomology respect the ring structure given by the cup product.

Using this fact, we may easily calculate the induced maps on the higher cohomologies from the induced map on first cohomology.

Proposition 1.3. *The induced map $(f_A)^*$ on k-th cohomology acts on a general generator by*

$$(f_A)^* : e'_{i_1} \wedge \cdots \wedge e'_{i_k} \mapsto A^T e'_{i_1} \wedge \cdots \wedge A^T e'_{i_k},$$

where we are treating e'_i both as the i-th element of the dual basis to $e_1, \ldots, e_n$ and as the i-th unit column vector.

Proof. This follows immediately from the naturality of the cup product by

$$(f_A)^*(e'_{i_1} \wedge \cdots \wedge e'_{i_k}) = (f_A)^*(e'_{i_1}) \wedge \cdots \wedge (f_A)^*(e'_{i_k}) = A^T e'_{i_1} \wedge \cdots \wedge A^T e'_{i_k}.$$

□

Corollary 1.3.1. *The induced map $(f_A)^*$ on n-th cohomology is multiplication by* $\det(A)$.

Proof. Consider the map $\delta : M_n(\mathbb{Z}) \to \mathbb{Z}$ defined by $(f_A)^*(e'_1 \wedge \cdots \wedge e'_n) = \delta(A) e'_1 \wedge \cdots \wedge e'_n$. In the equation

$$(f_A)^*(e'_1 \wedge \cdots \wedge e'_n) = A^T e'_1 \wedge \cdots \wedge A^T e'_n,$$

each factor in the wedge product represents one of the rows of A, so since the wedge product is bilinear in each factor and $x \wedge x = 0$, the map δ is alternating and bilinear on the rows, and for $A = I$, clearly $(f_I)^*$ is identity so $\delta(I) = 1$. Thus, we must have that $\delta = \det$.

□

Corollary 1.3.2. *The induced map $(f_A)_*$ on k-th homology acts on a generator*

$$(f_A)_*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = A e_{i_1} \wedge \cdots \wedge A e_{i_k}.$$

In particular, the induced map on n-th homology is multiplication by $\det(A)$.

Proof. As before, the Universal Coefficient Theorem gives a commutative diagram

$$\begin{tikzcd}
H^k(T^n;\mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "(f_A)^*"] & \mathrm{Hom}(H_k(T^n),\mathbb{Z}) \arrow[d, "(f_A)'_*"] \\
H^k(T^n;\mathbb{Z}) \arrow[r, "\cong"] & \mathrm{Hom}(H_k(T^n),\mathbb{Z})
\end{tikzcd}$$

so the dual map $(f_A)'_*$ to $(f_A)_*$ is identified with the map $(f_A)^*$. Now writing out the map $(f_A)^*$ with respect to the basis $(e'_{i_1} \wedge \cdots \wedge e'_{i_k})_{1 \le i_1 < \cdots < i_k \le n}$, we have

$$(f_A)^*(e'_{i_1} \wedge \cdots \wedge e'_{i_k}) = A^T e'_1 \wedge \cdots \wedge A^T e'_n = \left(\sum_{j=1}^{n} a_{i_1,j} e'_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} a_{i_k,j} e'_j\right)$$

$$= \sum_{1 \le j_1 < \cdots < j_k \le n} \left(\sum_{\sigma \in \mathrm{Sym}\{j_1,\ldots,j_k\}} a_{i_1,\sigma(j_1)} \ldots a_{i_k,\sigma(j_k)} \mathrm{sign}(\sigma) \right) e'_{j_1} \wedge \cdots \wedge e'_{j_k}.$$

From this, we see that the map $(f_A)_*$ with respect to the basis $(e_{i_1} \wedge \cdots \wedge e_{i_k})_{1 \le i_1 < \cdots < i_k \le n}$ must be given by

$$(f_A)_*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{1 \le j_1 < \cdots < j_k \le n} \left(\sum_{\sigma \in \mathrm{Sym}\{i_1,\ldots,i_k\}} a_{j_1,\sigma(i_1)} \ldots a_{j_k,\sigma(i_k)} \mathrm{sign}(\sigma) \right) e_{j_1} \wedge \cdots \wedge e_{j_k}$$

$$= \sum_{1 \le j_1 < \cdots < j_k \le n} \left(\sum_{\sigma \in \mathrm{Sym}\{j_1,\ldots,j_k\}} a_{\sigma(j_1),i_1} \ldots a_{\sigma(j_k),i_k} \mathrm{sign}(\sigma) \right) e_{j_1} \wedge \cdots \wedge e_{j_k}$$

$$= \left(\sum_{j=1}^{n} a_{j,i_1} e_j\right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} a_{j,i_k} e_j\right) = A e_{i_1} \wedge \cdots \wedge A e_{i_k}$$

□

This completes our computation of the induced maps on homology and cohomology for the torus.

2 $SU(n)$

We will now investigate calculating the induced maps on homology for the Lie group $SU(n)$. We will first focus on the simple case of $SU(2)$. In this case, calculating the induced maps on homology will be relatively simple due to a simple togological characterisation of $SU(2)$.

As a matrix Lie group, $SU(2)$ is defined as the set of invertible 2×2 matrices with entries in $\mathbb{C}$ satisfying $A^* = A^{-1}$ and $\det(A) = 1$. It is not to difficult from this to derive that $SU(2)$ consists precisely of those matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

for $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$. From this there is a clear homeomorphism from $SU(2)$ to the unit sphere in $\mathbb{C}^2$ sending the above matrix to $(a, b) \in \mathbb{C}^2$ so we have that $SU(2)$ and S^3 are homeomorphic as topological spaces.

This gives us that the homology and cohomology of $SU(2)$ are

$$H_n(SU(2)) = H^n(SU(2)) = \begin{cases} \mathbb{Z} & n = 0, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, calculating the induced map on homology for the sphere S^n is relatively easy due to the following standard fact, which may be found, for example, as Proposition 2.30 of [3] or in [8]:

Lemma 2.1. *If $f : S^n \to S^n$ is a continuous map and $x \in S^n$ satisfies that $f^{-1}(x)$ is finite, then the degree of f is given by*

$$\deg f = \sum_{y \in f^{-1}(x)} \deg f|_y$$

where $\deg f|_y$ *is the local degree of f at y.*

We will now use this result to calculate the induced map on cohomology of two classes of maps from $SU(2)$ to itself.

Example 2.2. Define the map $k_A : SU(2) \to SU(2)$ to be evaluation with the commutator at $A \in SU(2)$.

$$k_A : X \mapsto AXA^{-1}X^{-1}.$$

We will show that the induced map on homology is the zero map. We claim that this this map is not surjective, and thus by the above lemma with x taken to be a point not hit by k_A, the induced map on homology is zero.

If $A = I$ then the map sends all elements to I and we are done. Assuming $A \neq I$, observe that that map will not in fact hit A as

$$AXA^{-1}X^{-1} = A \implies XAX^{-1} = I \implies A = I.$$

Example 2.3. Define $p_n : SU(2) \to SU(2)$ to be the map $p_n : X \mapsto X^n$ for $n \in \mathbb{Z}$. We claim that the induced map on homology is multiplication by n. To prove this consider a general element X of $SU(2)$. We first suppose $n \geq 0$. We wish to calculate $p_n^{-1}(X)$; that is, the number of n-th roots of X in $SU(2)$. By Spectral Theorem, X may be diagonalised, so we have

$$X = P \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} P^{-1}$$

with $|\lambda| = 1$. Suppose $Y \in SU(2)$ is a matrix satisfying $Y^n = X$. We also have

$$Y = Q \begin{pmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{pmatrix} Q^{-1},$$

so $Y^n = X$ becomes

$$Q \begin{pmatrix} \mu^n & 0 \\ 0 & \overline{\mu}^n \end{pmatrix} Q^{-1} = P \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} P^{-1}$$

Assuming λ is not 1 or -1, λ and $\overline{\lambda}$ will be distinct and so we will have $\mu^n = \lambda$ and $P = Q$ (or $\mu^n = \overline{\lambda}$ and Q is P times the matrix swapping the two basis vectors), so if $\lambda^{1/n}$ is a square root of λ and ω is a primitive n-th root of unity then X has exactly n n-th roots

$$P \begin{pmatrix} \lambda^{1/n} & 0 \\ 0 & \overline{\lambda^{1/n}} \end{pmatrix} P^{-1}, P \begin{pmatrix} \lambda^{1/n}\omega & 0 \\ 0 & \overline{\lambda^{1/n}\omega} \end{pmatrix} P^{-1}, \ldots, P \begin{pmatrix} \lambda^{1/n}\omega^{n-1} & 0 \\ 0 & \overline{\lambda^{1/n}\omega^{n-1}} \end{pmatrix} P^{-1}.$$

Thus, aside from at I and $-I$, the map p_n is n-to-1 onto all points of $SU(2)$, so for each $X \in SU(2)\backslash\{I,-I\}$, $Y \in p_n^{-1}(X)$, there is a neighbourhood of Y which is mapped homeomorphically by p_n to a neighbourhood of X. By picking some points X and Y and passing to to local co-ordinates, we may find that the derivative of p_n at Y has positive determinant so p_n is orientation preserving. By doing the same for the map p_{-1} we may check that this will be orientation reversing so that $(p_{-n})_* = (p_{-1})_* \circ (p_n)_* = \cdot(-n)$. Thus, the degree of the map p_n is n.

We will now study the Lie group $SU(n)$ for $n \geq 2$. Our first order of business will be to calculate the homology of $SU(n)$. Our method for doing this will be the use of a certain spectral sequence associated to a fibration involving $SU(n)$.

2.1 Homology of $SU(n)$ via the Serre spectral sequence

The group $SU(n)$ can be thought of in some sense as describing the group of 'rotations' of $\mathbb{C}^n$, with $SU(n)$ acting in a natural way on $\mathbb{C}^n$ by pre-multiplication. We may thus let $SU(n)$ act on the unit sphere in $\mathbb{C}^n$ (which is homeomorphic to S^{2n-1}). The stabiliser of any given point will be the set of rotations fixing the axis through that point, and will therefore be a certain subroup of $SU(n)$ consisting of the rotations of some copy of $\mathbb{C}^{n-1}$ in $\mathbb{C}^n$, ie a subgroup $SU(n-1) \subseteq SU(n)$. This gives us a fibration $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$ [9].

Given such a fibration, there is a convenient way to compute the homology using a spectral sequence: [4]

Proposition 2.4. *Given a fibration $F \hookrightarrow X \to B$ such that $\pi_1(B) = 0$, there is a spectral sequence $(E^r_{p,q}, d_r)$ satisfying:*

1. $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ *and* $E^{r+1}_{p,q} = \ker(d_r)/\mathrm{im}(d_r)$,
2. $E^\infty_{p,n-p} \cong F^p_n/F^{p-1}_n$ *for some filtration* $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X)$,
3. $E^2_{p,q} \cong H_p(B; H_q(F))$.

The spectral sequence of the above proposition is known as the *Serre* or *Leray-Serre spectral sequence*. We may use it to calculate the homology of $SU(3)$ as follows.

Consider our fibration $SU(2) \hookrightarrow SU(3) \to S^5$. The E^2 page of our sequence will have terms $E^2_{p,q} \cong H_p(S^5; H_q(SU(2))) \cong H_q(SU(2))$ for $p = 0$ or 5 and 0 otherwise.

We claim that $H_k(SU(3))$ is isomorphic to $\mathbb{Z}$ for $k = 0, 3, 5, 8$ and 0 otherwise. The only columns we need to worry about are $p = 0$ and $p = 5$, so all the maps d_r are 0 except possibly on the $r = 5$ page. However, none of the maps from the $p = 5$ column will hit any of the non-zero entries of the $p = 0$ column (see Figure 1 below), so the sequence collapses on the second page. We see from part 3. of the above proposition that

$$H_k(SU(3)) = \bigoplus_{p=1}^{k} F^p_k/F^{p-1}_k = \begin{cases} \mathbb{Z} & k = 0, 3, 5, 8, \\ 0 & \text{otherwise,} \end{cases}$$

by summing along the diagonals.

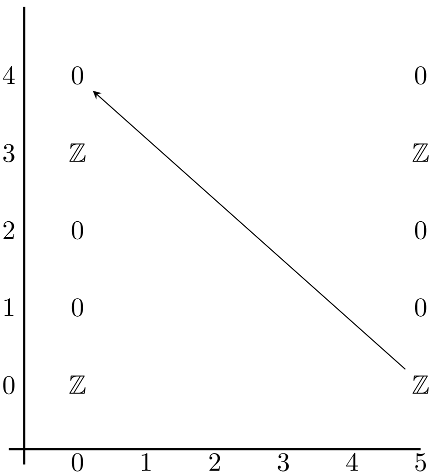

Figure 1: The E^5 page

More generally, there is a similar spectral sequence for the cohomology groups of a fibration satisfying exactly the same propoerties but which converges as an algebra to the cohomology ring $H^*(X)$ of the given space. Using this, we may use essentially the same argument as above and induction using the fibration $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$ to get that the cohomology ring for $SU(n)$ is the exterior algebra $\Lambda[x_3, \ldots, x_{2n-1}]$ with generators satisfying $|x_i| = i$ [9].

2.2 Induced maps on $H_*(SU(n))$

We now wish to calculate the induced maps on homology for some maps from $SU(n)$ to itself. Some of these maps are made easier to calculate due to the calculations we have already made for $SU(2)$. More precisely, consider the following inclusion of $SU(2)$ into $SU(n)$:

$$i : SU(2) \hookrightarrow SU(n);\ A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix}.$$

This inclusion respects raising a matrix to the n-th power in the sense that the following diagram commutes:

$$\begin{tikzcd} SU(2) \arrow[r, "p_n"] \arrow[d, hook, "i"] & SU(2) \arrow[d, hook, "i"] \\ SU(n) \arrow[r, "p_n"] & SU(n) \end{tikzcd}$$

where p_n denotes the power map $\cdot^n$ on both $SU(2)$ and $SU(n)$. Applying the H_3 functor to this diagram gives (we will take as given that the embedding i induces identity on third homology; we may easily justify this using the argument in the proof of Theorem 2.10):

$$\begin{tikzcd} H_3(SU(2)) \cong \mathbb{Z} \arrow[r, "\cdot n"] \arrow[d, "\cong"] & H_3(SU(2)) \cong \mathbb{Z} \arrow[d, "\cong"] \\ H_3(SU(n)) \cong \mathbb{Z} \arrow[r, "(p_n)_*"] & H_3(SU(n)) \cong \mathbb{Z} \end{tikzcd}$$

A simple diagram chase shows that the induced map $(p_n)_*$ on the homology of $SU(2)$ is also multiplication by n.

This argument generalises to prove the following simple fact

Proposition 2.5. *If $i : Y \hookrightarrow X$ is an embedding of Y in X inducing the identity map on k-th homology, $f : X \to X$ is a continuous map and $f|_Y$ its restriction to $Y \cong i(Y) \subseteq X$ satisfying that $f \circ i = i \circ f|_Y$, then f and $f|_Y$ induce the same map on k-th homologies $H_k(Y) \cong H_k(X)$.*

Corollary 2.5.1. *If $f : SU(n) \to SU(n)$ is a continuous map and $f|_{SU(2)}$ its restriction to $SU(2) \cong i(SU(2)) \subseteq SU(n)$ satisfying that $f \circ i = i \circ f|_{SU(2)}$, then f and $f|_{SU(2)}$ induce the same map on third homologies $H_3(SU(2)) \cong H_3(SU(n)) \cong \mathbb{Z}$.*

Example 2.6. The commutator map $k_A(X) = AXA^{-1}X^{-1}$ on $SU(n)$ for $A \in i(SU(2))$ induces the zero map on third homology by Example 2.2 and the above corollary.

We may, however, wish to calculate induced maps on homology where our given map does not commute with an embedding of $SU(2)$. We might however hope that the image of some $SU(2)$ subspace might be able to be 'homotoped back to our original subspace' in some sense so that the induced map may be calculated in a similar fashion. This motivates the study of in which ways $SU(2)$ may be embedded in $SU(n)$.

2.3 Embeddings of $SU(2)$ in $SU(3)$ via representation theory

We will now prove two results regarding embeddings of $SU(2)$ into $SU(n)$ for $n > 2$. The first result is weaker than the second, but is proved using a nice argument utilising the representation theory of $SU(2)$. The second result is much more powerful than the first and uses significantly heavier topological machinery.

Suppose we have an embedding $\rho : SU(2) \hookrightarrow SU(3)$. Assume also both that this embedding is injective and that it is a group homomorphism. I make the following claim.

Theorem 2.7. *The embedding $\rho : SU(2) \hookrightarrow SU(3)$ is homotopic to the embedding i of 2.2. In other words, all embedddings of $SU(2)$ in $SU(3)$ which are also group homomorphisms are homotopic.*

Proof. The key observation is that the map ρ defines a 3-dimensional representation of $SU(2)$. It is a standard result in the basic representation theory of Lie groups that the irreducible representations of $SU(2)$ may be described as follows [2]:

Let V_m be the $m+1$-dimensional space of homogeneous polynomials of degree m in two variables. Define the representation $\Pi_m : SU(2) \to \mathrm{GL}(V_m)$ by

$$(\Pi_m(X)f)(z) = f(U^{-1}z)$$

for $z \in \mathbb{C}^2$. Now, it is a theorem that every finite-dimensional representation of a matrix Lie group is completely reducible[1] (see Theorem 4.28 of [2]), so our representation ρ must be isomorphic to one of the representations $V_0 \oplus V_0 \oplus V_0$, $V_1 \oplus V_0$ or V_2. The first of these is just the trivial representation, which is certainly not faithful and thus does not correspond to an embedding of $SU(2)$. Observe also that in the last of these, $\Pi_2(-I) = \Pi_2(I) = I$, so this representation is also not faithful. We theorefore conclude that all embeddings ρ of $SU(2)$ in $SU(3)$ are isomorphic as representations, and therefore must be isomorphic to the representation i.

We have shown the existence of an isomorphism of representations between ρ and i; that is, a vector space isomorphism $\phi : \mathbb{C}^3 \to \mathbb{C}^3$ satisfying $\phi(\rho(X)v) = i(X)\phi(v)$ for all $X \in SU(2)$, $v \in \mathbb{C}^3$. Now ϕ is given by pre-multiplication by some invertible matrix $A \in \mathrm{GL}(3;\mathbb{C})$, so we have

$$A\rho(X) = i(X)A \implies \rho(X) = A^{-1}i(X)A. \tag{3}$$

We know make use of the following Lemma, the idea for the proof of which I found in [5].

Lemma 2.8. *If $\Pi : G \to \mathrm{GL}(n;\mathbb{C})$, $\Sigma : G \to \mathrm{GL}(n;\mathbb{C})$ are two unitary representations which are isomorphic via an intertwining map A with $\Pi(X)A = A\Sigma(X)$, then Π and Σ are unitarily equivalent.*

Proof. Observe that

$$\Pi(X)A = A\Sigma(X) \implies A^*\Pi(X) = \Sigma(X)A^*$$

since Π and Σ are unitary. We thus have

$$A^*A\Sigma(X) = A^*\Pi(X)A = \Sigma(X)A^*A$$

so that A^*A and $\Sigma(X)$ commute. Now as A^*A is self-adjoint and positive, we may take its positive square root $(A^*A)^{1/2}$. This may be expanded as a power series in A^*A, so it commutes with $\Sigma(X)$ as well. Now $A(A^*A)^{1/2}$ is unitary (see the proof of Theorem 2.17 of [2]) and satisfies

$$\Pi(X)A(A^*A)^{1/2} = A\Sigma(X)(A^*A)^{1/2} = A(A^*A)^{1/2}\Sigma(X),$$

so $A(A^*A)^{1/2}$ defines a unitary equivalence between Π and Σ. □

Now in light of the above lemma, we may assume that A in (3) is unitary. Now as $U(3)$ is a path-connected Lie group (see Proposition 1.13 of [2]), we may pick a path $A(t)$ from I to A lying in $U(3)$. Observe that $A(t)^{-1}\rho(X)A(t) \in SU(3)$ for all $t \in [0,1]$, $X \in SU(2)$, so the map

$$h : SU(2) \times [0,1] \to SU(3)$$
$$h : (X,t) \mapsto A(t)^{-1}\rho(t)A(t)$$

defines a homotopy from ρ to i. □

2.4 Embeddings of $SU(2)$ in $SU(n)$ using the Hurewicz Theorem

An element of the n-th homotopy group of a space X may be characterised as a homotopy class $[f]$ of pointed maps $f : (S^n, y_0) \to (X, x_0)$. Using this characterisation, one can define a homomorphism from the relative homotopy group $\pi'_n(X, A, x_0)$ to the relative singular homology group $H_n(X, A)$ by

$$h' : \pi'_n(X, A, x_0) \to H_n(X, A)$$
$$h' : [f] \mapsto f_*(\alpha)$$

where f_* is the induced map on homology $f_* : H_n(S^n) \to H_n(X, A)$ and α is some fixed generator for the singular homology $H_n(S^n)$. This map is called the *Hurewicz map* [3]. We will make use of the following theorem (Theorem 4.37 of [3]):

[1] We do not actually need this relatively strong result. In our case we have a unitary representation $\rho : SU(2) \to SU(3) \subseteq \mathrm{GL}(3;\mathbb{C})$, and it is also a result that any finite-dimensional unitary representation of a Lie group is completely reducible. See Proposition 4.27 of [2].

Theorem 2.9 (Hurewicz Theorem). *If (X, A) is an $n-1$-connected pair of path-connected spaces with $n \geq 2$ and $A \neq \emptyset$, then the Hurewicz map $h' : \pi'_n(X, A, x_0) \to H_n(X, A)$ is an isomorphism and $H_i(X, A) = 0$ for $i < n$.*

If we let $n = 3$, $X = SU(n)$, $A = \{I\}$, then the Hurewicz map is

$$h' : \pi'_3(SU(n), \{I\}, I) \xrightarrow{\cong} H_3(SU(n), I)$$

$$h' : [f] \mapsto f_*(\alpha)$$

with α a fixed generator of $H_3(S^3)$. Now as $SU(2) \cong S^3$, a map $SU(2) \to SU(n)$ can be thought of as representing an element of the homotopy group $\pi'_3(SU(n), \{I\}, I)$. The Hurewicz theorem tells us that if two maps $SU(2) \to SU(n)$ induce the same map on homology, then they represent the same element of $\pi'_3(SU(n), \{I\}, I)$, in other words they are homotopic as pointed maps.

This is relevant to our discussion of embeddings of $SU(2)$ in $SU(n)$ because it tells us that homologous embeddings of $SU(2)$ will be homotopic. We now prove the following result.

Theorem 2.10. *All embeddings of $SU(2)$ in $SU(n)$ which induce isomorphisms on third homology are homotopic to the obvious inclusion*

$$i : A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix}$$

Proof. In light of the above discussion and the Hurewicz Theorem, we need only prove that the inclusion i incduces an isomorphism on third homotopy. We make use again of the fibration $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$. Now to a fibration $F \hookrightarrow E \to B$ there is an associated long exact sequence of homotopy groups (Theorem 4.41 of [3])

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0.$$

Specialising this to the case of the fibrations $SU(2) \hookrightarrow SU(3) \to S^5$ and $SU(3) \hookrightarrow SU(4) \to S^7$, we get

$$\cdots \to \pi_4(S^5, I) \to \pi_3(SU(2), I) \xrightarrow{(i_1)_*} \pi_3(SU(3), I) \to \pi_3(S^5, I) \to \cdots$$

$$\cdots \to \pi_4(S^7, I) \to \pi_3(SU(3), I) \xrightarrow{(i_2)_*} \pi_3(SU(4), I) \to \pi_3(S^7, I) \to \cdots$$

where the maps i_1, i_2 are the obvious inclusions into the first block $SU(2) \hookrightarrow SU(3)$, $SU(3) \hookrightarrow SU(4)$. Now since $\pi_4(S^5, I) = \pi_3(S^5, I) = 0$, we see that $(i_1)_*$ is an isomorphism by exactness, and as $\pi_4(S^7, I) = \pi_3(S^7, I) = 0$, the same holds for $(i_2)_*$. Thus, $i_* = (i_1)_* \circ (i_2)_*$ is an isomorphism. It is clear how this arguement may be extended by induction to $SU(n)$ for general n. $\square$

3 Lie Algebra Cohomology

In this last section, we will discuss how the cohomology of a compact Lie group may be calculated using its Lie algebra and how the induced maps might also be calculated. A remarkable fact is that the cohomology of a differentiable manifold M may be calculated by calculating the cohomology of the chain complex [7]

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \Omega^2(M) \to \cdots$$

where $\Omega^k(M)$ is the space of differential k-forms on M and the boundary maps are given by the exterior derivative [6]

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

$$d : f dx^1 \wedge \cdots \wedge dx^k \mapsto df \wedge dx^1 \wedge \cdots \wedge dx^k.$$

De Rham's Theorem states that the cohomology calculated in this way will indeed be isomorphic to the singular cohomology of M with coefficients in $\mathbb{R}$ [1].

In the case of G a compact Lie group, the space of differential k-forms may be identified with the dual of the k-th exterior power, $(\Lambda^k \mathfrak{g})'$ of the associated Lie algebra $\mathfrak{g}$. The exterior derivative becomes identified with the map

$$d : (\Lambda^k \mathfrak{g})' \to (\Lambda^{k+1} \mathfrak{g})'$$

$$d(\zeta)(x_1 \wedge \cdots \wedge x_{k+1}) = \frac{1}{k+1} \sum_{i<j} (-1)^{i+j+1} \zeta([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{k+1}).$$

The de Rham cohomology for the Lie group may then be recovered by calculating the cohomology of the associated chain complex. The justification for this identification is described in detail in a paper by Chevalley and Eilenberg [1].

The induced map on de Rham cohomology for a map $f : X \to Y$ is given by the pushforward on differential forms [7]. From this, it should be possible to calculate the corresponding chain map on the associated Lie algebra chain complex and use this to calculate the induced map on cohomology purely algebraically. However, I have found that calculating the pushforward on differential forms and then associating this to the Lie algebra is computationally difficult in practice.

We finish with an example of calculating Lie algebra cohomology.

Example 3.1. We will now calculate the cohomology of $SU(2)$ by calculating the Lie algebra cohomology of the associated Lie algebra $\mathfrak{su}(2)$. The Lie algebra $\mathfrak{su}(2)$ consists of the 2×2 traceless skew-Hermitian matrices with entries in $\mathbb{C}$: [2]

$$\mathfrak{su}(2) = \{X \in M_2(\mathbb{C}) : \mathrm{tr}(X) = 0, X^* = -X\}.$$

A basis for this as a real Lie algebra is given by

$$x = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, y = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, z = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfying the commutation relations

$$[x, y] = z, [y, z] = x, [z, x] = y.$$

Now, as $\mathfrak{su}(2)$ is 3-dimensional, all the k-th exterior powers of $\mathfrak{su}(2)$ for $k > 3$ will be zero, so our chain complex is

$$0 \to \mathbb{R} \to \mathfrak{su}(2)' \to (\Lambda^2\mathfrak{su}(2))' \to (\Lambda^3\mathfrak{su}(2))' \to 0 \to \cdots.$$

We now need only calculate the boundary maps d. The map $d^0 : \mathbb{R} \to \mathfrak{g}$ is always zero. Now, a basis for $\mathfrak{su}(2)'$ is x', y', z' and for $(\Lambda^2\mathfrak{su}(2))'$ is $(x \wedge y)'$, $(y \wedge z)'$, $(z \wedge x)'$. To find what, say, dx' is, we calculate

$$\begin{aligned}(dx')(x \wedge y) &= \frac{1}{2}x'([x, y]) = \frac{1}{2}x'(z) = 0,\\ (dx')(y \wedge z) &= \frac{1}{2}x'([y, z]) = \frac{1}{2}x'(x) = \frac{1}{2},\\ (dx')(z \wedge x) &= \frac{1}{2}x'([z, x]) = \frac{1}{2}x'(y) = 0.\end{aligned}$$

Thus, dx' is the element of $(\Lambda^2\mathfrak{su}(2))' = \mathrm{Hom}(\Lambda^2\mathfrak{su}(2), \mathbb{R})$ sending $x\wedge y$ to 0, $y\wedge z$ to $\frac{1}{2}$ and $z\wedge x$ to 0, so $dx' = \frac{1}{2}(y\wedge z)'$. By performing similar calculations, or by simply noting the symmetry of the commutation relations, we get that $dy' = \frac{1}{2}(z\wedge x)'$ and $dz' = \frac{1}{2}(x\wedge y)'$, so $d^1 : \mathfrak{su}(2)' \to: (\Lambda^2\mathfrak{su}(2))'$ is an isomorphism. Now $\Lambda^3\mathfrak{su}(2)$ is one-dimensional with basis $x\wedge y\wedge z$. Now for any $\zeta \in (\Lambda^2\mathfrak{su}(2))'$,

$$d(\zeta)(x \wedge y \wedge z) = \frac{1}{3}\zeta([x, y] \wedge z) - \frac{1}{3}\zeta([x, z] \wedge y) + \frac{1}{3}\zeta([y, z] \wedge x)$$

$$= \frac{1}{3}\zeta(z \wedge z) - \frac{1}{3}\zeta(y \wedge y) + \frac{1}{3}\zeta(x \wedge x) = \frac{1}{3}\zeta(0) - \frac{1}{3}\zeta(0) + \frac{1}{3}\zeta(0) = 0,$$

so $d^2 = 0$. Thus our chain complex is

$$0 \to \mathbb{R} \xrightarrow{0} \mathfrak{su}(2)' \xrightarrow{\cong} (\Lambda^2\mathfrak{su}(2))' \xrightarrow{0} (\Lambda^3\mathfrak{su}(2))' \to 0 \to \cdots.$$

This gives the Lie algebra cohomology as $H^0_{\mathrm{Lie}}(\mathfrak{su}(2)) \cong H^3_{\mathrm{Lie}}(\mathfrak{su}(2)) \cong \mathbb{R}$ and $H^k_{\mathrm{Lie}}(\mathfrak{su}(2)) = 0$ for $k \neq 0, 3$, which is indeed the same as the de Rham cohomology for $SU(2)$.

4 Supplementary Note and Acknowledgements

The preceding is a summary of a summer project undertaken over the summer of 2022 under the supervision of Prof. Jason Lotay. While the large portion of the project has consisted of background reading and learning about algebraic topology in general, I have decided only to include brief summaries of that reading which I felt necessary to explain and motivate my own arguments and computations.

I make no particular claim to the originality of anything presented here. However, the computations of induced maps on the n-torus and on $SU(2)$, the computation of Example 3.1, the results of Section 2.2, and Theorems 2.7 and 2.10 are my own arguments. I hope the reader finds them as interesting as I have.

I would like to acknowledge the London Mathematical Society for providing a grant to fund this project. Thank you also to the Mathematical Institute at the University of Oxford for providing funding. I'd also like to thank Prof. Jason Lotay for taking the time to supervise this project.

References

[1] CHEVALLEY, Claude and EILENBERG, Samuel, 1948, *Cohomology Theory of Lie Groups and Lie Algebras.* Transactions of the American Mathematical Society 63.1 (1948): 85-124.

[2] HALL, Brian C., 2015, *Lie Groups, Lie Algebras, and Representations, An Elementary Introduction, Second Edition.* Cham. Springer International Publishing.

[3] HATCHER, Allen, 2001, *Algebraic Topology.* Cambridge. Cambridge University Press.

[4] HATCHER, Allen, 2004, *Spectral Sequences in Algebraic Topology, Chapter 1.* `https://pi.math.cornell.edu/~hatcher/SSAT/SSch1.pdf`.

[5] *Inequivalent unitary representations of a compact group must be orthogonal.* Mathematics Stack Exchange. `https://math.stackexchange.com/questions/3261878/inequivalent-unitary-representations-of-a-compact-group-must-be-orthogonal`.

[6] KNAPP, Anthony W., 1988, *Lie Groups, Lie Algebras, and Cohomology.* Princeton. Princeton University Press.

[7] MADSEN, I.H. and TORNEHAVE, Jørgen, 1997, *From Calculus to Cohomology.* Cambridge. Cambridge University Press.

[8] RITTER, Alexander, 2019, *C3.1: Algebraic Topology* (Lecture Notes). Oxford. `https://people.maths.ox.ac.uk/ritter/algebraic-topology/ritter-algebraic-topology-48.pdf`.

[9] TAN, Jane, 2020. *Guide to the Serre Spectral Sequence in 10 easy steps!.* `https://people.maths.ox.ac.uk/tanj/notes/sssguide.pdf`.