Universal Moduli of Sheaves over Curves and Moduli of Flags of Varieties via Geometric Invariant Theory

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In memory of Thomas Gater (1996 – 2017)
and Dr Vicky Neale (1984 – 2023).

Declaration

Sections 6.2 to 6.6 of Chapter 6 are based upon the preprint [37]. Sections 7.2 to 7.4 of Chapter 7 are based upon the preprint [38]. Certain preliminary results from these aforementioned pieces of work appear in Chapters 2 through 5, the preliminary chapters of this thesis. The results of Chapter 8 will form part of the upcoming work [36]; these results have yet to appear in any preprint or publication. All three of these pieces of work are original to myself as the sole author.

To the best of my knowledge, all other results in this thesis which are not the original work of myself have been duly cited.

No part of this thesis has been submitted previously for any degree of the University of Oxford or elsewhere.

George Edward Cooper
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Abstract

This thesis is comprised of three main results.

The first main result is the existence of GIT constructions for good moduli spaces of compactified universal Jacobians, and their higher rank analogues, over stacks of marked stable curves and stable maps. The construction is carried out in a way which allows for changes in stability conditions to be governed by variation of GIT. We also prove that the singularities of these good moduli spaces (when working over $\overline{\mathcal{M}}_{g,n}$) are canonical when $g \geq 4$, extending work of Casalaina-Martin–Kass–Viviani.

The second main result concerns the construction of quasi-projective coarse moduli spaces parametrising complete flags of subschemes of a fixed projective space $\mathbb{P}(V)$ up to projective automorphisms; these flags are obtained by intersecting non-degenerate non-singular subvarieties of $\mathbb{P}(V)$ of dimension n by flags of linear subspaces of $\mathbb{P}(V)$ of length n, with each positive dimension component of the flags being required to be non-singular and non-degenerate, and with the dimension 0 component being required to satisfy a Chow stability condition. These moduli spaces are constructed using non-reductive Geometric Invariant Theory, making use of a non-reductive analogue of quotienting-in-stages developed by Hoskins and Jackson.

The final main result is the existence of quasi-projective fine moduli spaces of Gieseker unstable torsion-free coherent sheaves of uniform rank 1 on a reducible projective curve with Gorenstein singularities. These are the first examples of projective moduli spaces of unstable sheaves on projective schemes which admit fully modular descriptions, without any restriction on the Harder–Narasimhan length. These moduli spaces are constructed by taking iterated universal extension bundles over fine compactified Jacobians of the subcurves.

Contents

Chapter 1

Introduction

In this thesis, we present the following three main results:

1. The existence of Geometric Invariant Theory (GIT) constructions for good moduli spaces of compactified universal Jacobians, and their higher rank analogues, over stacks of marked stable curves and stable maps, and its implications for the geometries of these moduli spaces (Chapter 6).

2. The construction of quasi-projective coarse moduli spaces parametrising complete flags of subschemes of a projective space $\mathbb{P}(V)$ which we call *hyperplanar admissible flags*, using non-reductive GIT (Chapter 7).

3. The existence of quasi-projective fine moduli spaces of Gieseker unstable torsion-free coherent sheaves of uniform rank 1 on a reducible projective curve with Gorenstein singularities (Chapter 8).

We refer the reader to the introductions of Chapters 6, 7 and 8 for more information about these results, including the significance of these results and how these results relate to other work in the literature.

The original motivation for the work presented in this thesis involved seeking new, GIT based approaches to constructing moduli spaces of projective varieties. A fruitful approach when studying a moduli problem in algebraic geometry using GIT is to add additional structure/decoration into the moduli problem and formulate stability conditions for the enlarged setup, depending on the additional data, in a way which allows for the original setup to be recovered by restricting to certain stability conditions in the enlarged setup. Having a larger space of stability conditions allows for more flexibility to work with well-behaved stability conditions, in the sense that there are no strictly semistable objects (so that the resulting GIT quotient is geometric) and that the points of the resulting quotient can be described explicitly. As the stability condition is allowed to vary, the resulting GIT quotients are related by variation of GIT, yielding information about the original setup.

In addition to considering as usual ample invertible sheaves on the variety, the envisioned additional structure comes from considering morphisms from (marked) curves into the variety, making use of the fact that moduli spaces of stable curves and stable maps have well-understood geometries, especially when compared with moduli spaces of higher-dimensional varieties. Including morphisms from curves into the varieties being parametrised can be thought of as capturing information about dimension 1 subschemes of the varieties, in contrast to the invertible sheaves corresponding to divisors (i.e. codimension 1 subschemes). It is hoped that the results of this thesis serve as useful assistance in providing constructions of moduli spaces of such decorated varieties.

Chapter Summaries

Chapters 2 through 5 constitute the preliminary chapters of this thesis. We begin by reviewing the results from GIT that are required in this thesis, before moving on to review results concerning moduli spaces and stacks of coherent sheaves on projective schemes.

In Chapter 2 we focus on the classical situation in GIT, involving the construction of categorical quotients for the action of a reductive linear algebraic group on a quasi-projective scheme. We recap the connection between GIT and moduli theory, in particular concerning the existence of corepresentations, coarse and good moduli spaces for moduli functors and algebraic quotient stacks. We also review a selection of additional results relating to reductive GIT quotients, including the Hesselink–Kempf–Kirwan–Ness instability stratification of the unstable locus.

Chapter 3 provides an overview of the results we require from non-reductive GIT, with a focus on the existence of categorical quotients of projective schemes by actions of linear algebraic groups with an internally graded unipotent radical. This includes an overview of the quotienting-in-stages construction of Hoskins and Jackson, which applies when forming non-reductive quotients of actions by parabolic subgroups of the special linear group. We also review how the problem of forming quotients of unstable HKKN strata relates to non-reductive GIT.

We review in Chapter 4 the notion of Gieseker (semi)stability of coherent sheaves on a projective scheme, and how this yields projective schematic moduli spaces. We also recap the existence of Harder–Narasimhan filtrations of unstable coherent sheaves, and of the existence of schematic Harder–Narasimhan stratifications arising from flat families of coherent sheaves. We also review how the problem of constructing moduli spaces of unstable sheaves of a fixed Harder–Narasimhan type is related to the existence of quotients of certain unstable HKKN strata. We finish by reviewing the notion of Gieseker stability with respect to multiple ample invertible sheaves, as formulated by Greb–Ross–Toma.

In Chapter 5, the final preliminary chapter of this thesis, we review the aspects of the theory of compactified Jacobians of a projective curve required in this thesis. In particular, we explain how these moduli spaces relate to moduli spaces of Gieseker semistable sheaves. We also prove some technical results relating to torsion-free sheaves on reducible curves, in advance of Chapter 8.

The final three chapters of this thesis, Chapters 6, 7 and 8, are where we present the main results of this thesis. Summaries about the contents of these chapters can be found in their respective introductions.

Notation and Conventions

- Throughout, we work over an algebraically closed field $\mathbb{C}$ of characteristic zero. All schemes are assumed to be schemes over $\mathbb{C}$. In particular, the notions of reductivity, linear reductivity and geometric reductivity for algebraic groups coincide.

- The end of an example is denoted by a black square $\blacksquare$.

- If X is a scheme, we occasionally use $|X|$ to denote the underlying topological space of X.

- A point of a scheme X is always understood to be a closed point of X, unless specified otherwise. We use the notation $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ to denote the residue field of $x \in X$.

- A *variety* is an integral separated scheme of finite type over $\mathbb{C}$.

- We often use the abbreviation 1PS for *one-parameter subgroup* or the plural *one-parameter subgroups.*

- If V is a vector space, then $\mathbb{P}(V)$ is always understood to be the projective space of lines in V. More generally, if E is a locally free coherent sheaf on a scheme X, then $\mathbb{P}_X(E) := \mathbf{Proj}_X(\mathrm{Sym}^\bullet E^\vee)$ is the fine moduli space parametrising invertible quotients $E^\vee \to L \to 0$ of $E^\vee$.

- If G is an algebraic group, and if H is an algebraic subgroup of G which acts on a quasi-projective scheme Z, then $G \times^H Z$ denotes the geometric quotient of H acting on $G \times Z$ by $h \cdot (g, y) = (gh^{-1}, hy)$; by [109, Theorem 4.19] $G \times^H Z$ exists and is a scheme.

- Let C be a reduced projective curve.

  (i) C is said to be a *nodal curve* if each singular point $p \in C$ is a node, meaning that there is an isomorphism of complete local $\mathbb{C}$-algebras $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x, y]]/(xy)$.

  (ii) C is said to have *Gorenstein singularities* if each local ring $\mathcal{O}_{C,p}$ of C is Gorenstein, equivalently if C admits an invertible dualising sheaf ω_C.

  (iii) C is said to have *locally planar singularities* if at each singular point $p \in C$, there exists an isomorphism of complete local rings $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x, y]]/(f)$, where f is a reduced power series. In particular, nodal curves have locally planar singularities. Any curve with locally planar singularities is Gorenstein.

Chapter 2

Reductive Geometric Invariant Theory

In this chapter, we present a review of the key results of reductive GIT and how it can be used to form categorical quotients of schemes by actions of linear algebraic groups. We also include various technical results which will be called upon in later parts of this thesis, including the existence of the HKKN instability stratification of the GIT unstable locus. We indicate how reductive GIT relates to moduli theory, as understood both classically (i.e. in terms of existence of corepresentations of moduli functors) as well as the contemporary perspective, in terms of the existence of good moduli spaces of algebraic stacks.

The standard reference for reductive GIT remains the seminal book [101], originally written by Mumford and with later additions by Fogarty and Kirwan.[1] Other references include the books [103] and [118].

2.1 Schematic Quotients in Algebraic Geometry

Let G be a linear algebraic group acting on a scheme X.

Definition 2.1.1. *A* categorical quotient *of the action $G \circlearrowright X$ is a scheme $X /\!\!/ G$ together with a G-invariant morphism $X \to X /\!\!/ G$ which is universal for all G-invariant morphisms $X \to Y$ where Y is a scheme. A* universal categorical quotient *is a categorical quotient $X \to X /\!\!/ G$ such that the base change $X \times_{X /\!\!/ G} T \to T$ is a categorical quotient for all morphisms $T \to X /\!\!/ G$ where T is a scheme.*

[1] A word of caution: the now accepted notion of GIT stability in the literature is what Mumford referred to as *proper stability* in [101].

From the universal property, if a categorical quotient exists then it is unique up to unique isomorphism. Within the context of GIT, a special class of quotients are given by good quotients.

Definition 2.1.2. *A morphism of schemes $q : X \to Y$ is said to be a* good quotient *if:*

1. *q is G-invariant;*
2. *q is surjective;*
3. *q is an affine morphism;*
4. *the natural morphism $\mathcal{O}_Y \to q_*\mathcal{O}_X$ induces an isomorphism of $\mathcal{O}_Y$-algebras $\mathcal{O}_Y \xrightarrow{\cong} (q_*\mathcal{O}_X)^G$;*
5. *if $Z \subset X$ is a G-invariant closed subset of X then the image $q(Z)$ is closed in Y; and*
6. *q separates disjoint G-invariant closed subsets of X.*

A geometric quotient *is a good quotient $q : X \to Y$ with the additional property that for each closed point $y \in Y$, $q^{-1}(y)$ is a single G-orbit. A* universal good quotient *(resp.* universal geometric quotient*) is a good (resp. geometric) quotient $q : X \to Y$ such that the base change $X \times_Y T \to T$ is a good (resp. geometric) quotient for all morphisms $T \to Y$, where Y is a scheme.*

Remark. A good quotient of an action of a linear algebraic group is always a categorical quotient. If $q : X \to Y$ is a good quotient then for any point $x \in X$, the fibre $q^{-1}(q(x))$ coincides with the orbit closure $\overline{G \cdot x}$. In particular, if each point of x has a closed orbit then q is a geometric quotient.

Remark. The key example of a good quotient is given as follows. Let A be an algebra of finite type acted on by a reductive group G. By a theorem of Nagata [102], the invariants A^G are of finite type. The morphism $\operatorname{Spec} A \to \operatorname{Spec} A /\!/ G := \operatorname{Spec} A^G$ is then a good quotient. Conversely, any good G'-quotient $q : X \to Y$ (whether G' is reductive or not) is locally given by taking G'-invariants of rings of sections of G'-invariant open affine subschemes of X.

We give a few elementary results needed in this thesis concerning geometric quotients. These results are very likely standard, however we provide proofs where necessary when satisfactory references are lacking.

Lemma 2.1.3 ([118], Exercise 1.5.3.3)**.** *Let X be a scheme acted on by $G \times H$, where G and H are linear algebraic groups.*

1. *If there exists a good quotient $X \to X /\!\!/ G \times H$, then there are good quotients $X \to X /\!\!/ G$ and $X /\!\!/ G \to (X /\!\!/ G) /\!\!/ H$, and there is an isomorphism $(X /\!\!/ G) /\!\!/ H \cong X /\!\!/ G \times H$.*

2. *Conversely, if there are good quotients $X \to X /\!\!/ G$ and $X /\!\!/ G \to (X /\!\!/ G) /\!\!/ H$, then there exists a good quotient $X \to X /\!\!/ G \times H$, and there is an isomorphism $(X /\!\!/ G) /\!\!/ H \cong X /\!\!/ G \times H$.*

Lemma 2.1.4. *Let $q : X \to X/G$ be a geometric quotient for the action of a reductive linear algebraic group G on a scheme X of finite type. Let $Z \subset X$ be a G-invariant locally closed subscheme of X. Then q restricts to give a geometric G-quotient $q|_Z : Z \to q(Z) \subset X/G$, where $q(Z)$ is a locally closed subscheme of X/G.*

Proof. Let $\overline{Z}$ be the closure of Z in X; as Z is locally closed in X then Z is open in $\overline{Z}$. Since q is a good quotient then $q(\overline{Z})$ is closed in X/G; moreover $q(\overline{Z})$ inherits a canonical scheme structure from $\overline{Z}$ (if $\overline{Z} \subset X$ is locally $\operatorname{Spec} A/I \subset \operatorname{Spec} A$ then $q(\overline{Z}) \subset X/G$ is locally $\operatorname{Spec}(A/I)^G = \operatorname{Spec} A^G/I^G \subset \operatorname{Spec} A^G$). The restriction $q|_{\overline{Z}} : \overline{Z} \to q(\overline{Z})$ is then a geometric G-quotient.

As q is surjective, there is an equality $q(Z) = q(\overline{Z}) \setminus q(\overline{Z} \setminus Z)$, and so $q(Z)$ is open in $q(\overline{Z})$. Since the property of a morphism being a geometric quotient is local on the base, q restricts to a geometric quotient $Z \to q(Z)$. □

Lemma 2.1.5. *Let $q : X \to X/H$ be a Zariski-locally trivial quotient for the action of a linear algebraic group H (not assumed reductive) on a scheme X of finite type. Let $Z \subset X$ be an H-invariant locally closed subscheme of X. Then there exists a unique scheme structure on the set-theoretic image $q(Z)$ and a locally closed immersion $q(Z) \to X/H$ such that $Z \cong q(Z) \times_{X/H} X$ as schemes and such that $q|_Z : Z \to q(Z)$ is a locally trivial quotient.*

Proof. Once again consider the closure $\overline{Z}$. The image $q(\overline{Z})$ is closed in X/H; we endow $q(\overline{Z})$ with the scheme-theoretic image scheme structure. Over an affine open $\operatorname{Spec} A \subset X/H$ where q is trivial, we have $q^{-1}(\operatorname{Spec} A) = \operatorname{Spec} A \times H$, where H acts trivially on the first factor and by multiplication on the second. Writing $q(\overline{Z}) \cap \operatorname{Spec} A = \operatorname{Spec} A/I$ for $I \subset A$ an ideal, as $\overline{Z}$ is H-invariant we have $\overline{Z} = q^{-1}(q(\overline{Z})) = \operatorname{Spec} A/I \times H$, and $q|_{\overline{Z}}$ is the projection $\operatorname{Spec}(A/I) \times H \to \operatorname{Spec}(A/I)$. It follows that $q|_{\overline{Z}} : \overline{Z} \to q(\overline{Z})$ is a locally trivial H-quotient and that $\overline{Z} \cong q(\overline{Z}) \times_{X/H} X$ as schemes. By restricting to the open subscheme $Z \subset \overline{Z}$, the same statements hold for Z in place of $\overline{Z}$, since $q(Z)$ is open in $q(\overline{Z})$.

The uniqueness of the scheme structure follows from the fact that locally trivial quotients are categorical quotients and categorical quotients are unique up to unique isomorphism. □

Lemma 2.1.6. *Let G be a linear algebraic group acting on quasi-projective schemes X and Y. Let $H = \operatorname{Stab}_G(y_0) \subset G$ be the stabiliser of a point $y_0 \in Y$, and let $Z = X \times \{y_0\} \subset X \times Y$. Then the map $\sigma_Z : G \times Z \to GZ$, $(g, z) \mapsto gz$ descends to an isomorphism $G \times^H Z \cong GZ$.*

Proof. The orbit map $\sigma_{y_0} : G = G \times \{y_0\} \to G \cdot y_0$ is faithfully flat (cf. [95, Proposition 7.4]) and hence is a principal H-bundle (with respect to the fppf topology). As the diagram

$$\begin{array}{ccc} G \times Z & \xrightarrow{\mathrm{id}_G \times \mathrm{pr}_Y} & G \times \{y_0\} \\ \downarrow{\scriptstyle \sigma_Z} & & \downarrow{\scriptstyle \sigma_{y_0}} \\ GZ & \xrightarrow[\mathrm{pr}_Y]{} & G \cdot y_0 \end{array}$$

is Cartesian, σ_Z is also a principal H-bundle, and in particular a geometric H-quotient. The lemma then follows from the uniqueness of geometric quotients. □

2.2 GIT for Reductive Linear Algebraic Group Actions

2.2.1 Definition of the GIT Quotient

Throughout, let X be a quasi-projective scheme acted on by a reductive linear algebraic group G. In this situation, GIT is used to construct quotients of appropriate G-invariant open subschemes of X which are themselves quasi-projective, satisfying many other desirable properties. In order to construct GIT quotients of X by G, an extra piece of data is required.

Definition 2.2.1. *A* linearisation L *of the action* $G \circlearrowright X$ *is an invertible sheaf* L *on the quotient stack* $[X/G]$. *It is said to be* (very) ample *if the pullback of* L *along* $X \to [X/G]$ *is (very) ample.*[2]

Equivalently, a linearisation is a choice of an invertible sheaf L on X together with a lifting of the action $G \circlearrowright X$ to a bundle action of G on the underlying line bundle of L. Given a choice of linearisation L, for each integer n the invertible sheaf L^n is also a linearisation, and there is an induced action of G on the space of sections $H^0(X, L^n)$. Setting $R(X, L) := \bigoplus_{n\geq 0} H^0(X, L^n)$, we may consider the subalgebra of invariants $R(X, L)^G = \bigoplus_{n\geq 0} H^0(X, L^n)^G$. By a theorem of Nagata [102], the ring of invariants $R(X, L)^G$ for the reductive group G is finitely generated over $\mathbb{C}$.

Definition 2.2.2. *The* GIT quotient *of* X *by* G *(with respect to* L*) is the scheme* $X /\!\!/_L G = \operatorname{Proj} R(X, L)^G$.

The inclusion $R(X, L)^G \subset R(X, L)$ defines a map $X \dashrightarrow X /\!\!/_L G$; the domain of definition is the locus of semistable points.

Definition 2.2.3. *Let* L *be a linearisation for the action* $G \circlearrowright X$.

1. *A point* $x \in X$ *is said to be* semistable (with respect to L) *if there exists* $n \geq 1$ *and* $f \in H^0(X, L^n)^G$ *such that* $f(x) \neq 0$ *and* X_f *is affine, where* $X_f = \{x' \in X : f(x') \neq 0\}$.

2. *A point* $x \in X$ *is said to be* stable (with respect to L) *if* $\dim G \cdot x = \dim G$ *and if there exists* $n \geq 1$ *and* $f \in H^0(X, L^n)^G$ *such that* $f(x) \neq 0$, X_f *is affine and the action of* G *on* X_f *is closed.*

3. *A point* $x \in X$ *is said to be* polystable (with respect to L) *if there exists* $n \geq 1$ *and* $f \in H^0(X, L^n)^G$ *such that* $f(x) \neq 0$, X_f *is affine and the orbit of* x *in* X_f *is closed.*

[2] By an abuse of notation, we will consider linearisations as being invertible sheaves on X; that is, we identify a linearisation L of the action $G \circlearrowright X$ with its pullback along $X \to [X/G]$.

The locus of semistable points, denoted $X^{ss}(L)$, is naturally a G-invariant open subscheme of X. $X^{ss}(L)$ contains $X^s(L)$, the locus of stable points, as an open subscheme, and $X^{ps}(L)$, the locus of polystable points, as a subset. All stable points are polystable, and in turn all polystable points are semistable.

Definition 2.2.4. *The complement $X^{us}(L) = X \setminus X^{ss}(L)$ of the semistable locus is known as the* unstable locus. *A point $x \in X$ is said to be* unstable (with respect to L) *if $x \in X^{us}(L)$.*

Definition 2.2.5. *Let $x, y \in X^{ss}(L)$ be two points. x and y are said to be* S-equivalent[3] *if their orbit closures in $X^{ss}(L)$ intersect.*

The S-equivalence class of a polystable point x is the orbit $G \cdot x$. Every semistable point x is S-equivalent to a unique orbit of polystable points; this orbit is characterised as being the unique orbit of minimal dimension in the semistable orbit closure of x.

2.2.2 Main Theorem of Reductive GIT

The main properties of the GIT quotient $X /\!\!/_L G$ are given by the following theorem.

Theorem 2.2.6 (Mumford, [101])**.** *Let X be a quasi-projective scheme acted on by a reductive linear algebraic group G with linearisation L.*

1. *The restriction $q : X^{ss}(L) \to X /\!\!/_L G$ of $X \dashrightarrow X /\!\!/_L G$ to $X^{ss}(L)$ is a universal good quotient and a universal categorical quotient for the action of G on $X^{ss}(L)$. In particular, q is surjective.*

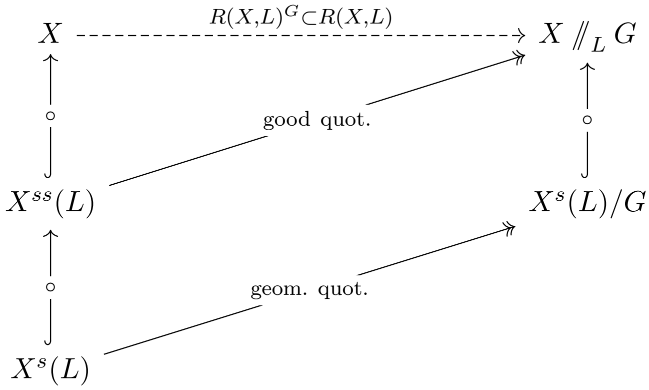
2. *The image $q(X^s(L))$ is open in $X /\!\!/_L G$, and the restriction $q|_{X^s(L)} : X^s(L) \to q(X^s(L)) =: X^s(L)/G$ is a universal geometric quotient.*

3. *The GIT quotient $X /\!\!/_L G$ admits an ample invertible sheaf M such that $q^*(M) \cong L^N|_{X^{ss}(L)}$ for some $N > 0$; in particular, $X /\!\!/_L G$ is quasi-projective. If L is ample and if X is projective, then $X /\!\!/_L G$ is projective.*

4. *If $x, y \in X^{ss}(L)$, then $q(x) = q(y)$ if and only if x and y are S-equivalent. In particular, there is a bijection of sets $X^{ps}(L)/G \to X /\!\!/_L G$.*

[3]The 'S' stands for Seshadri.

This theorem is summarised by the following diagram:



2.2.3 The Hilbert–Mumford Criterion

At first glance, the definitions of GIT stability and semistability seem intractable to work with in practice. However, in many situations the GIT stable and semistable loci can be characterised in a more computationally-friendly manner, in terms of one-parameter subgroups (1PS) of the group.

Let X be a projective scheme acted on by a reductive linear algebraic group G, with an *ample* linearisation L. Let $\lambda : \mathbb{G}_m \to G$ be a 1PS. Given a point $x \in X$, let

$$x_0 := \lim_{t \to 0} \lambda(t) \cdot x \in X^{\lambda(\mathbb{G}_m)}.$$

The $\lambda(\mathbb{G}_m)$-action on the restriction of L to the fixed point x_0 is given by a character of $\mathbb{G}_m$ of the form $t \mapsto t^r$.

Definition 2.2.7. *The* Hilbert–Mumford weight *of λ on x is given by $\mu^L(x, \lambda) := -r$.*

Suppose L is very ample[4]. The Kodaira embedding of L yields a G-equivariant closed embedding

$$X \overset{|L|}{\hookrightarrow} \mathbb{P}^N = \mathbb{P}(H^0(X, L)^\vee),$$

with L the pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$. Given $x \in X$ and a 1PS λ, write $x = [v] \in \mathbb{P}^N$, where $v = x_0 e_0 + \cdots + x_N e_N$, with $e_0, \ldots, e_N$ a basis of $\mathbb{C}^{N+1}$ chosen such that $\lambda(t) \cdot e_i = t^{r_i} e_i$ for integers $r_0, \ldots, r_N$. One may then express the Hilbert–Mumford weight of λ on x as

$$\mu^L(x, \lambda) = \min\{r_i : x_i \neq 0\}.$$

[4] This is not a genuine restriction, since replacing L by L^n for any positive integer n does not alter the semistable loci nor the GIT quotient.

Theorem 2.2.8 (Hilbert–Mumford Criterion: Projective Case, [101]). *Let X be a projective scheme acted on by a reductive linear algebraic group G, with an* ample *linearisation L. Let $x \in X$ be a point. Then:*

1. *$x \in X^{ss}(L)$ if and only if $\mu^L(x, \lambda) \geq 0$ for all non-trivial 1PS of G; and*

2. *$x \in X^s(L)$ if and only if $\mu^L(x, \lambda) > 0$ for all non-trivial 1PS of G.*

We have the following equivalent formulation of the Hilbert–Mumford criterion; let T be a maximal torus of G. Since any 1PS of G is conjugate to a 1PS of T, we have equalities

$$X^{G-(s)s}(L) = \bigcap_{g \in G} g \cdot X^{T-(s)s}(L).$$

The condition that $\mu^L(x, \lambda) > 0$ (resp. $\mu^L(x, \lambda) \geq 0$) for all non-trivial 1PS λ of T can be expressed in terms of the T-weights of the coordinates of the point x.

Theorem 2.2.9 (Hilbert–Mumford Criterion for Tori). *Let X be a projective scheme acted on by a reductive linear algebraic group G. Let T be a maximal torus of G. Assume X is endowed with a very ample G-linearisation L, with $X \subset \mathbb{P}^N = \mathbb{P}(H^0(X, L)^\vee)$. Choose coordinates on $\mathbb{C}^{N+1}$ which diagonalise the action of T. For each $x \in X$, let* $\mathrm{conv}(x) \subset \mathrm{Lie}(T)^\vee$ *be the convex hull of the T-weights corresponding to the non-zero coordinates of x. Then:*

1. *$x \in X^{T-ss}(L)$ if and only if $0 \in \mathrm{conv}(x)$; and*

2. *$x \in X^{T-s}(L)$ if and only if $0 \in \mathrm{int}(\mathrm{conv}(x))$.*

Another version of the Hilbert–Mumford criterion exists in the situation when a reductive linear algebraic group G acts on an *affine space* V with respect to a character $\chi : G \to \mathbb{G}_m$.[5] In other words, we endow V with the G-linearisation $\mathcal{O}_V(\chi)$ whose underlying line bundle is the trivial line bundle, with action

$$g \cdot (v, z) = (g \cdot v, \chi(g)z), \quad g \in G,\ v \in V,\ z \in \mathbb{C}.$$

The ring of G-invariants $R(V, \mathcal{O}_V(\chi))^G = \bigoplus_{n \geq 0} H^0(V, \mathcal{O}_V(\chi^n))^G$ coincides with the ring of χ-*semi-invariants* of the underlying coordinate ring of the affine space V.

Let $\langle -, - \rangle$ denote the natural pairing between characters and 1PS of G.

[5]This situation is sometimes referred to in the literature as "twisted affine GIT".

Theorem 2.2.10 (Hilbert–Mumford Criterion: Twisted Affine Case, [75], [58])**.** *Suppose G is a reductive linear algebraic group acting on an affine space V linearised by $\mathcal{O}_V(\chi)$. Let $v \in V$ be a point.*

1. *$v \in V^{ss}(\mathcal{O}_V(\chi))$ if and only if $\langle \chi, \lambda \rangle \geq 0$ for all non-trivial 1PS λ of G for which $\lim_{t\to 0} \lambda(t) \cdot v$ exists in V.*

2. *$v \in V^{s}(\mathcal{O}_V(\chi))$ if and only if $\langle \chi, \lambda \rangle > 0$ for all non-trivial 1PS λ of G for which $\lim_{t\to 0} \lambda(t) \cdot v$ exists in V.*

2.2.4 Quotients by Finite Groups

Since finite groups are reductive linear algebraic groups, one immediate consequence of the Hilbert–Mumford criterion is that if L is an ample G-linearisation on a projective scheme X for the action of a finite group G, then $X = X^s(L)$, whence there exists a projective geometric quotient X/G.[6] In a similar vein, an action of a finite group on a quasi-projective scheme always yields a quasi-projective geometric quotient.

Lemma 2.2.11. *Let X be a quasi-projective scheme acted on by a finite group G. Let L be an ample G-linearisation on X. Then there exists a quasi-projective geometric G-quotient $X \to X/G$.*

Proof. Since L is an ample linearisation, for $r > 0$ sufficiently large there exists a finite-dimensional subspace $W \subset H^0(X, L^r)^\vee$ and a locally closed immersion $\iota : X \to \mathbb{P}(W)$, such that $\iota^*\mathcal{O}_{\mathbb{P}(W)}(1) = L^r$. By averaging W over the finite group G if necessary we can always ensure that W is a G-invariant subspace, and that ι is G-equivariant. By Lemma 2.1.4, the restriction of the geometric quotient $\mathbb{P}(W) \to \mathbb{P}(W)/G$ to X yields a quasi-projective geometric quotient $X \to X/G$. □

2.3 GIT and Moduli Theory

2.3.1 Existence of Corepresentations

As highlighted numerous times in the seminal text [101], the primary *raison d'être* of GIT concerns the existence and construction of corepresentations and (coarse) moduli

[6] It is possible to give a direct proof that $X = X^s(L)$; see for instance [118, Paragraph after Remark 2.1.1.1].

spaces of algebro-geometric moduli functors. Here we indicate how GIT can be used to construct corepresentations of a given moduli functor $\mathcal{F}$.

Let S be a scheme. Fix a functor

$$\mathcal{F} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}.$$

For any S-scheme T, let $h_T = \mathrm{Hom}_{\mathbf{Sch}_S}(-, T)$ denote the functor of points of T.

Definition 2.3.1. *A* corepresentation *of $\mathcal{F}$ is a pair (M, Φ), where M is an S-scheme and $\Phi : \mathcal{F} \to \mathrm{Hom}_{\mathbf{Sch}_S}(-, M)$ is a natural transformation, such that for any S-scheme T, any natural transformation $\mathcal{F} \to \mathrm{Hom}_{\mathbf{Sch}_S}(-, T)$ factors uniquely through Φ; if (M, Φ) is a corepresentation of $\mathcal{F}$, we also say that M* corepresents *$\mathcal{F}$. A corepresentation (M, Φ) is said to be a* coarse moduli space *for $\mathcal{F}$ if for each morphism $\mathrm{Spec}\,\mathbb{C} \to S$, the induced map $\mathcal{F}(\mathrm{Spec}\,\mathbb{C}) \to M(\mathrm{Spec}\,\mathbb{C})$ is a bijection.*

A universal corepresentation (resp. universal coarse moduli space) *of $\mathcal{F}$ is a corepresentation (M, Φ) such that for all morphisms $T \to M$, T corepresents (resp. is a coarse moduli space for) the contravariant functor*

$$\mathcal{F}_T : T' \mapsto \mathrm{Hom}_{\mathbf{Sch}_S}(T', T) \times_{\mathrm{Hom}_{\mathbf{Sch}_S}(T', M)} \mathcal{F}(T').$$

GIT can be used to construct corepresentations of moduli functors, provided that the functor $\mathcal{F}$ admits an object with the so-called *local universal property*, admitting appropriate symmetries by an algebraic group.

Definition 2.3.2. *If T is an S-scheme and if $W \in \mathcal{F}(T)$, we say that W* has the local universal property for *$\mathcal{F}$ if for all S-schemes T', for all $W' \in \mathcal{F}(T')$ and for all points $t \in T'$, there exists an open neighbourhood U of T' and a morphism of S-schemes $f : U \to T$ such that $W'|_U = f^*(W) \in \mathcal{F}(U)$, where $W'|_U$ denotes the pullback of W' along the open immersion $U \to T'$.*

Proposition 2.3.3 (cf. [103], Proposition 2.13)**.** *Let $\mathcal{F} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ be a functor. Suppose there exists $W \in \mathcal{F}(T)$ with the local universal property for $\mathcal{F}$. Suppose in addition that there is a linear algebraic group G acting on T (and acting trivially on S) with the property that for any two geometric points $t_1, t_2 \in T$ lying over a common point of S,*

$$W_{t_1} = W_{t_2} \iff G \cdot t_1 = G \cdot t_2, \tag{2.3.1}$$

where W_t is the pullback of W along $\mathrm{Spec}\,\mathbb{C} \xrightarrow{t} T$. Then:

1. *an S-scheme M corepresents $\mathcal{F}$ if and only if M is a categorical quotient of T by G (in the category of S-schemes); and*

2. *a categorical quotient M of T by G is a coarse moduli space for $\mathcal{F}$ if and only if the preimages of closed points of M under the quotient $T \to M$ are single G-orbits.*

Proof. This result is likely known to experts, though we give a proof (largely following the proof of [103, Proposition 2.13]) for lack of an appropriate reference. To prove the first assertion, we claim that for any S-scheme M, there exists a natural one-to-one correspondence between natural transformations $\Phi : \mathcal{F} \to \mathrm{Hom}_{\mathbf{Sch}_S}(-, T)$ and G-invariant morphisms of S-schemes $\phi : T \to M$. Given the natural transformation Φ we set $\phi = \Phi_T(W)$. On the other hand, given the morphism ϕ we define Φ as follows. Suppose T' is an S-scheme and $W' \in \mathcal{F}(T')$. By the local universal property of W, we may cover T' by open subschemes U_i such that there are morphisms $\psi_i : U_i \to T$ with $W'|_{U_i} = \psi_i^* W$. The G-invariance of ψ along with Property (2.3.1) implies that the morphisms $\phi \circ \psi_i$ glue to give a morphism $T' \to M$ which is independent of the choice of trivialising cover U_i and the morphisms ψ_i; we set $\Phi_{T'}(W')$ to be this morphism. This defines a natural transformation Φ, and the two correspondences are mutually inverse; this establishes the claim.

The first assertion of Proposition 2.3.3 now follows from the observation that $T \to M$ is a categorical quotient of T by G if and only if the corresponding natural transformation $\mathcal{F} \to \mathrm{Hom}_{\mathbf{Sch}_S}(-, T)$ corepresents $\mathcal{F}$. Moreover, the preimages of closed points of M under the quotient $T \to M$ are single G-orbits if and only if the map $\mathcal{F}(\operatorname{Spec}\mathbb{C}) \to M(\operatorname{Spec}\mathbb{C})$ is a bijection, which yields the second assertion. □

2.3.2 Good Moduli Spaces and GIT

Another frequent application of GIT, closely related to the existence of corepresentations of moduli functors, concerns the existence of good moduli spaces for certain (quotient) algebraic stacks.

Definition 2.3.4. *A morphism $\pi : \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a* coarse moduli space *if*

1. *the induced map $\mathcal{X}(\mathbb{C})/\sim \to X(\mathbb{C})$ is a bijection, where $\mathcal{X}(\mathbb{C})/\sim$ is the set of isomorphism classes of objects of $\mathcal{X}$ over $\mathbb{C}$; and*

2. *π is universal for morphisms from $\mathcal{X}$ to algebraic spaces.*

Definition 2.3.5 (Alper, [6]). *A morphism $\pi : \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a* good moduli space *if*

1. *π is quasi-compact;*

2. *$\pi_* : \mathbf{QCoh}(\mathcal{X}) \to \mathbf{QCoh}(X)$ is exact; and*

3. *the induced morphism $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.*

If in addition the induced map $\mathcal{X}(\mathbb{C})/\sim \to X(\mathbb{C})$ is a bijection, we say that $\pi : \mathcal{X} \to X$ is a tame moduli space.

If $\pi : \mathcal{X} \to X$ is a good moduli space, then π is surjective and universally closed, and in particular X has the quotient topology inherited from $\mathcal{X}$. The property of being a good moduli space is stable under base changes $X' \to X$, where X' is an algebraic space. If $\mathcal{X}$ is locally Noetherian, then π is universal for morphisms from $\mathcal{X}$ to algebraic spaces. In addition, every closed point of $\mathcal{X}$ has a reductive stabiliser (cf. Proposition 12.14 of *loc. cit.*). In particular, a tame moduli space whose source is locally Noetherian is a coarse moduli space.

By the Keel–Mori theorem [73], if $\mathcal{X}$ is a Deligne–Mumford stack which is separated and of finite type over an algebraic space S, then there exists a coarse moduli space $\pi : \mathcal{X} \to X$ with the additional properties that $\pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$, that X is separated and of finite type over S, and that any flat base change of π is also a coarse moduli space. In the case where $\mathcal{X}$ is an algebraic stack of finite presentation, whose automorphism groups are affine and whose diagonal is separated over a Noetherian algebraic space, a result of Alper–Halpern-Leistner–Heinloch [7] characterises the existence of a good moduli space of $\mathcal{X}$ in a manner which is intrinsic to $\mathcal{X}$.[7] If the algebraic stack $\mathcal{X}$ is known to be isomorphic to a quotient stack, GIT can also be used to prove the existence of a good moduli space.

Proposition 2.3.6 (Alper, [6]). *The following statements hold:*

[7]Namely, $\mathcal{X}$ admits a good moduli space if and only if $\mathcal{X}$ is S-complete and Θ-complete/Θ-reductive; the notions of S-completeness and Θ-completeness are defined using valuative criteria.

1. *Let G be a linear algebraic group acting on a scheme X. If $X \to X /\!\!/ G$ is a good (resp. geometric) quotient, then the morphism $[X/G] \to X /\!\!/ G$ is a good (resp. tame) moduli space. Conversely, if $[X/G]$ admits a schematic good (resp. tame) moduli space Y, then the composition $X \to [X/G] \to Y$ is a good (resp. geometric) quotient.*

2. *In particular, let G be a reductive linear algebraic group acting on a quasi-projective scheme X. Let L be a linearisation for the action $G \circlearrowright X$. Then the natural morphism $[X^{ss}(L)/G] \to X /\!\!/_L G$ is a good moduli space, and the natural morphism $[X^s(L)/G] \to X^s(L)/G$ is a tame moduli space.*

2.4 Reductive GIT Relative to a Base

In this section, we indicate how the formation of reductive GIT quotients can be extended to the relative situation, where one works over a quasi-compact base scheme S.

Definition 2.4.1 ([6], Definition 11.1). *Let S be a quasi-compact scheme, let G be a reductive linear algebraic group scheme over S, let $p : X \to S$ be a quasi-compact morphism and let L be a G-linearisation on X. A point $x \in X$ is said to be* relatively semistable *(with respect to L) if there exists an open neighbourhood $U \subset S$ of $p(x)$, a positive integer n and an invariant section $t \in H^0(p^{-1}(U), L^n)^G$ for which $t(x) \neq 0$ and for which the locus $\{y \in p^{-1}(U) : t(y) \neq 0\}$ is affine. The locus of points which are relatively semistable with respect to L is denoted $X^{ss}(L/S)$.*

Remark. By Remark 11.2 of *loc. cit.*, if S is affine and if p is quasi-projective, then the relative semistable locus coincides with the semistable locus as defined in Definition 2.2.3: $X^{ss}(L/S) = X^{ss}(L)$.

Proposition 2.4.2 ([6], Theorem 13.6). *Let S be a quasi-compact scheme, let G be a reductive linear algebraic group scheme over S, let $p : X \to S$ be a quasi-compact morphism and let L be a G-linearisation on X.*

1. *There exists an open subscheme $Y \subset \mathbf{Proj}_S \bigoplus_{k=0}^{\infty} (p_* L^k)^G$ and a good moduli space $[X^{ss}(L/S)/G] \to Y$. In other words, there exists a good quotient $q : X^{ss}(L/S) \to Y = X^{ss}(L/S) /\!\!/ G$.*

2. *If in addition $X^{ss}(L/S)$ is quasi-compact (e.g. if X is Noetherian), there exists an S-ample invertible sheaf*[8] *M on Y such that $q^*(M) \cong L^N|_{X^{ss}(L/S)}$ for some $N > 0$.*

2.4.1 Relative GIT for Projective Morphisms

Suppose $p : X \to S$ is a *projective* morphism of finite type to a finite type base scheme S. Suppose in addition that X is acted on by a reductive linear algebraic group G in such a way that the morphism p is G-invariant, so that G acts on each geometric fibre of f. Equivalently, we may consider this action as one of the S-group scheme $G \times S$ on the S-scheme X. Let L be a relatively ample linearisation on X (i.e. a linearisation on X for the action $G \circlearrowright X$ whose underlying invertible sheaf is relatively ample). Since $X^{ss}(L/S) /\!\!/ G$ is a categorical quotient, the morphism $p : X^{ss}(L/S) \to S$ factors through $X^{ss}(L/S) /\!\!/ G$.

Proposition 2.4.3 ([6], Theorem 13.6)**.** *The good quotient $X^{ss}(L/S) /\!\!/ G$ is projective over S, and there is an equality of S-schemes*

$$X^{ss}(L/S) /\!\!/ G = \mathbf{Proj}_S \bigoplus_{n \geq 0} (p_* L^n)^G.$$

Proposition 2.4.4 ([124], Lemma 1.13)**.** *Let $t \in S$ be a geometric point. Then there are equalities $(X^{(s)s}(L/S))_t = X_t^{(s)s}(L_t)$. In particular, there is an equality*

$$(X^{ss}(L/S) /\!\!/ G)_t = X_t /\!\!/_{L_t} G.$$

2.4.2 Relative GIT for Affine Morphisms

We require in Chapter 6 versions of Propositions 2.4.3 and 2.4.4 which hold for relative twisted affine GIT, where the morphism $X \to S$ is no-longer projective.

Proposition 2.4.5. *Let S be a finite type base scheme, let $p : X \to S$ be an affine morphism of finite type, let G be a reductive linear algebraic group (over $\mathbb{C}$), and let χ be a character of G. Suppose G acts on X (and acts trivially on S) in such a way that the morphism p is invariant. Let $L = \mathcal{O}_X(\chi)$ be the linearisation given by twisting*

[8] Here, M can be taken to be the restriction of $\mathcal{O}(N)$ for some $N > 0$, where $\mathcal{O}(1)$ denotes the twisting sheaf.

the structure sheaf $\mathcal{O}_X$ by the character χ, and let $X^{ss}(L/S)$ denote the resulting relatively semistable locus. Assume that for each geometric point $s \in S$, the (non-relative) semistable locus $(X_s)^{ss}(L_s)$ for the induced action $G \circlearrowright X_s$ is non-empty.

Then the good moduli space of $[X^{ss}(L/S)/G]$ is given by

$$Y = \mathbf{Proj}_S \bigoplus_{k=0}^{\infty} (p_* L^k)^G = \mathbf{Proj}_S \bigoplus_{k=0}^{\infty} (p_* \mathcal{O}_X(\chi^k))^G,$$

and for each geometric point $s \in S$ there is an equality of schemes

$$(X^{ss}(L/S))_s = (X_s)^{ss}(L_s). \tag{2.4.1}$$

Proof. There is a map $X \dashrightarrow \mathbf{Proj}_S \bigoplus_{k=0}^{\infty} (p_* L^k)^G$ arising from the inclusions of G-invariants $(p_* L^k)^G \subset p_* L^k$; the domain of definition of this map is the open subscheme $X^{ss}(L/S)$, and the image coincides with the good moduli space Y.

Take a geometric point $s \in S$. We have an inclusion $(X^{ss}(L/S))_s \subset (X_s)^{ss}(L_s)$. To show that the reverse inclusion holds, without loss of generality we may assume that $S = \operatorname{Spec} A$ is affine, so that $X = \operatorname{Spec} B$ is also affine. Let $\mathfrak{m} \subset A$ be the maximal ideal which corresponds to $s \in S$, so $X_s = \operatorname{Spec} B/\mathfrak{m}B$. If $x \in (X_s)^{ss}(L_s)$ is a point, there exists a positive integer n and a χ^n-semi-invariant $\overline{f} \in B/\mathfrak{m}B$ which does not vanish at x. The ring of χ-semi-invariants of $B/\mathfrak{m}B$ can be identified with a ring of G-invariants of the graded ring $(B/\mathfrak{m}B)[t]$ (cf. [99, Page 193]). As G is reductive then taking G-invariants of graded rings is exact; consequently there exists a χ^n-semi-invariant $f \in B$ whose image in $B/\mathfrak{m}B$ is $\overline{f}$. In particular, f does not vanish at x, and so $x \in (X^{ss}(L/S))_s$; this gives the equality (2.4.1).

It remains to show $Y = \mathbf{Proj}_S \bigoplus_{k=0}^{\infty} (p_* L^k)^G$. However, the restriction of the morphism $X^{ss}(L/S) \to \mathbf{Proj}_S \bigoplus_{k=0}^{\infty} (p_* L^k)^G$ over a geometric point $s \in S$ coincides with the surjective good quotient

$$X_s^{ss}(L_s) \to \left(\mathbf{Proj}_S \bigoplus_{k=0}^{\infty} (p_* L^k)^G \right) \times_S \operatorname{Spec} \mathbb{C} = \operatorname{Proj} \bigoplus_{k=0}^{\infty} H^0(X_s, \mathcal{O}_{X_s}(\chi^k))^G,$$

and so $X^{ss}(L/S) \to \mathbf{Proj}_S \bigoplus_{k=0}^{\infty} (p_* L^k)^G$ must be surjective. $\square$

2.4.3 A Miscellaneous Relative GIT Result

We require one further relative GIT result, which will also be used in Chapter 6. Let G be a reductive linear algebraic group, let X, Y be quasi-projective schemes, and let $q : X \to Y$ be a projective morphism. Assume X and Y admit G-actions with respect to which the morphism q is equivariant. Let N be a G-linearised relatively ample invertible sheaf on X. Assume there exists an equivariant open immersion $\iota : Y \hookrightarrow \overline{Y}$ into a projective scheme $\overline{Y}$ acted on by G with ample linearisation L. Assume further that

$$\overline{Y}^{ss}(L) = \overline{Y}^{s}(L) = Y.$$

Proposition 2.4.6. *Let $q : X \to Y$ be as above. Then there are projective geometric quotients $X /\!\!/ G$ and $Y /\!\!/ G$ of X and Y respectively, and the morphism q induces a projective morphism $\hat{q} : X /\!\!/ G \to Y /\!\!/ G$. If $y \in Y$ is a closed point, the fibre of $\hat{q}$ over the orbit $G \cdot y$ is isomorphic to $X_y/\mathrm{Stab}_G(y)$, the geometric quotient of X_y by the group $\mathrm{Stab}_G(y)$.*

Proof. In the situation of Proposition 2.4.6, there are representations $G \to GL(V_i)$, $i = 1, 2$, a commutative diagram

$$\begin{array}{ccccc} X & & \xhookrightarrow{\quad\kappa\quad} & & \mathbb{P}(V_1) \times \mathbb{P}(V_2) \\ {\scriptstyle q}\downarrow & & & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ Y & \xhookrightarrow{\iota} & \overline{Y} & \xhookrightarrow{\eta} & \mathbb{P}(V_2) \end{array}$$

whose rows are given by G-equivariant locally closed immersions, and positive integers a and b with

$$N^a \cong \kappa^*(\mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(1,0)) \text{ and } L^b \cong \eta^*(\mathcal{O}_{\mathbb{P}(V_2)}(1)).$$

Let $\overline{X}$ denote the closure of the image of X in $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$. The morphism $\mathrm{pr}_2 : \overline{X} \to \mathbb{P}(V_2)$ factors through $\overline{Y}$, giving a projective morphism $\overline{q} : \overline{X} \to \overline{Y}$ making the following diagram commute:

$$\begin{array}{ccccc} X & \xhookrightarrow{\kappa} & \overline{X} & \xhookrightarrow{\subset} & \mathbb{P}(V_1) \times \mathbb{P}(V_2) \\ {\scriptstyle q}\downarrow & & {\scriptstyle \overline{q}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ Y & \xhookrightarrow{\iota} & \overline{Y} & \xhookrightarrow{\eta} & \mathbb{P}(V_2) \end{array}$$

By [68, Theorem 3.11][9] for all $d \gg 0$ there are equalities of semistable loci (over $\operatorname{Spec}\mathbb{C}$)

$$\begin{aligned}\overline{X}^{ss}(\mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(1,d)|_{\overline{X}}) = \overline{X}^{s}(\mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(1,d)|_{\overline{X}}) &= \overline{q}^{-1}(\overline{Y}^{ss}(\eta^*\mathcal{O}_{\mathbb{P}(V_2)}(1))) \\ &= \overline{q}^{-1}(\overline{Y}^{ss}(L^b)) \\ &= \overline{q}^{-1}(Y) = X.\end{aligned}$$

Applying Theorem 3.13 of *loc. cit.* to $\bar{q}$ yields the result. □

2.5 Additional Reductive GIT Results

We conclude this chapter by collecting some additional results concerning reductive GIT quotients which are required for this thesis, including the existence of the Hesselink–Kempf–Kirwan–Ness (HKKN) instability stratification of the unstable locus.

2.5.1 Matsushima's Criterion

An important result of Matsushima [86] states that GIT polystable points always have reductive stabiliser groups.

Proposition 2.5.1 (Matsushima)**.** *Let X be an affine scheme acted on by a reductive linear algebraic group G. Suppose $x \in X$ is polystable, i.e. has closed orbit. Then the group $\operatorname{Stab}_G(x)$ is reductive.*

2.5.2 Existence of Étale Slices

The étale-local behaviour of a reductive GIT quotient is described via the slice theorem of Luna [85]. Let $X = \operatorname{Spec} A$ be an affine scheme acted on by a reductive linear algebraic group G, with good quotient $\pi : X \to X /\!\!/ G = \operatorname{Spec} A^G$. Let $x \in X$ be a closed point, with stabiliser $H = \operatorname{Stab}_G(x) \subset G$.

Definition 2.5.2. *A* slice *of the G-action on X through x is an H-invariant locally closed subscheme $Y \subset X$ containing x such that:*

1. *the morphism $G \times^H Y \to X$ arising from the action map $G \times Y \to X$ is étale, and its image is an open affine $U \subset X$ which is π-saturated, i.e. $\pi^{-1}(\pi(u)) \subset U$ for all $u \in U$;*

[9] As indicated by Schmitt in [119], Hu's result is erroneously stated and only applies in certain circumstances; the case where both the domain and target are projective is one such circumstance.

2. there are categorical quotients $U /\!\!/ G$ and $U /\!\!/ H$;

3. the induced morphism $(G \times^H Y)/G \to U /\!\!/ G$ is étale; and

4. the induced morphism $G \times^H Y \to U \times_{U /\!\!/ G} U /\!\!/ H$ is an isomorphism.

Theorem 2.5.3 (Luna)**.** *Assume that $x \in X$ is polystable. Then there exists a slice of the G-action on X through x.*

2.5.3 Variation of Reductive GIT

The GIT quotient $X /\!\!/_L G$ depends on the choice of linearisation; different choices of linearisations in general yield different semistable loci and hence different GIT quotients. *Variation of GIT (VGIT)* concerns the study of the dependence of $X /\!\!/_L G$ on L, and was first studied in independent work by Thaddeus [126] and Dolgachev–Hu [41]. We give a brief overview of VGIT, with a particular emphasis on the notion of a Thaddeus flip.

Let X is a projective scheme acted on by a reductive linear algebraic group G. Let $\mathrm{Pic}^G(X) = \mathrm{Pic}([X/G])$ be the set of isomorphism classes of G-linearised invertible sheaves on X; $\mathrm{Pic}^G(X)$ is an abelian group under the tensor product.

Definition 2.5.4. *Two linearisations $L_1, L_2 \in \mathrm{Pic}^G(X)$ are said to be G-*algebraically equivalent *if there exists a connected scheme S, a linearisation $\mathcal{L}$ on $X \times S$ (where G acts trivially on S) and points $s_1, s_2 \in S$ such that $\mathcal{L}|_{X \times s_i} \cong L_i$ as G-linearisations, for $i = 1, 2$.*

This defines an equivalence relation on $\mathrm{Pic}^G(X)$; the set of equivalence classes is denoted $\mathrm{NS}^G(X)$.

The set $\mathrm{NS}^G(X)$ also inherits an abelian group structure coming from the tensor product. Since L and L^n (for any $n > 0$) define the same (semi)stable loci and the same GIT quotient, one can pass to the space of *rational linearisations* $\mathrm{NS}^G(X)_{\mathbb{Q}} = \mathrm{NS}^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}$; it is shown in [126] that the semistable locus $X^{ss}(L)$ (and hence the GIT quotient $X /\!\!/_L G$) depends only on the span of $[L]$ in $\mathrm{NS}^G(X)_{\mathbb{Q}}$, and, at least when X is a normal variety, that $\mathrm{NS}^G(X)_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space.

Inside $\mathrm{NS}^G(X)_{\mathbb{Q}}$ are two cones $C^G \subset A^G$:

1. A^G consists of all rational linearisation classes for which the corresponding G-algebraic equivalence class is represented by an ample linearisation; and

2. C^G consists of those classes in A^G for which the semistable locus is non-empty.

An important result of VGIT states that in many cases (for instance, when X is a projective normal variety), the cone C^G carries a finite wall-and-chamber decomposition, where two linearisations in the same chamber yield the same semistable locus, and where linearisations on adjacent sides of a wall are related through a transformation known as a Thaddeus flip:

Definition 2.5.5. *Let Y_+, Y_- and Y_0 be schemes. We say that Y_+ and Y_- are* related by a Thaddeus flip through Y_0 *if there exists a Noetherian scheme X over a quasi-compact scheme S, acted on by a reductive linear algebraic group G over S, with G-linearisations L_+, L_-, L_0 on X such that:*

1. *there are equalities $Y_\pm = X^{ss}(L_\pm/S) /\!\!/ G$ and $Y_0 = X^{ss}(L_0/S) /\!\!/ G$; and*
2. *there exists a diagram*

$$\begin{array}{ccc} Y_+ & \dashleftarrow\!\dashrightarrow & Y_- \\ & {\scriptstyle\psi_+}\searrow \quad \swarrow{\scriptstyle\psi_-} & \\ & Y_0 & \end{array}$$

   where $\psi_\pm$ is the morphism arising from an inclusion of relative semistable loci $X^{ss}(L_\pm/S) \subset X^{ss}(L_0/S)$.

Thaddeus flips enjoy rich geometric properties, and are studied by Thaddeus in [126]. As we do not explicitly rely on these properties in this thesis, we state here only a couple of these,[10] referring the reader to *loc. cit.* for the full list.

Proposition 2.5.6 ([126], Theorem 3.3)**.** *In the situation of Definition 2.5.5, assume X is integral. Then the morphisms $\psi_\pm$ are proper and birational. If they are both small, meaning that their exceptional loci are of codimension at least 2, then the rational morphism $g : Y_- \dashrightarrow Y_+$ is a flip, in the following sense: if $M_\pm$ are the invertible sheaves on $Y_\pm$ arising from the twisting sheaves on $\mathbf{Proj}_S \bigoplus_{k=0}^\infty (p_* L_\pm^k)^G$, then these correspond to $\mathbb{Q}$-Cartier divisors $D_\pm$ and $g_*(-D_-)$ is equivalent to D_+.*

Proposition 2.5.7 ([126], Theorem 3.5)**.** *In the situation of Definition 2.5.5, there exist ideal sheaves $\mathcal{J}_\pm$ on $Y_\pm$ and $\mathcal{J}_0$ on Y_0, defined by taking G-invariants of ideal sheaves generated by sections of $L_\pm$ and L_0 respectively, such that:*

[10] Thaddeus works in the absolute setting $S = \operatorname{Spec}\mathbb{C}$, however the above stated results concern properties which are affine-local over $Y_0 = X^{ss}(L_0/S) /\!\!/ G$, and so carry over to the relative setting.

1. *the pullback of* $\mathcal{J}_0$ *under* $\psi_\pm$ *is equal to* $\mathcal{J}_\pm$*; and*

2. *if* X *is irreducible, the blow-ups* $\mathrm{Bl}_{\mathcal{J}_\pm} Y_\pm$ *and* $\mathrm{Bl}_{\mathcal{J}_0} Y_0$ *are all isomorphic to the irreducible component of the fibre product* $Y_+ \times_{Y_0} Y_-$ *dominating* Y_0.

2.5.4 Actions of $\mathbb{G}_m$ on Algebraic Spaces

We take a slight detour to state the following result of Drinfeld [44] concerning actions of the group $\mathbb{G}_m$ on algebraic spaces; this result generalises the classical *Białynicki-Birula theorem* [21].

Let X be an algebraic space of finite type over $\mathbb{C}$ acted on by $\mathbb{G}_m$.

Theorem/Definition 2.5.8 (Drinfeld)**.** *Let* $\mathbb{G}_m$ *act on* $\mathbb{A}^1$ *in the usual way. The functors*

$$\mathbf{Mor}^{\mathbb{G}_m}(-, X), \quad \mathbf{Mor}^{\mathbb{G}_m}(\mathbb{A}^1 \times -, X)$$

are represented by algebraic spaces of finite type, respectively denoted $X^0 = X^{\mathbb{G}_m}$*, the* fixed point locus *of* X*, and* X^+*, the* attractor *of* X*. Moreover:*

1. *The natural morphism* $X^0 \to X$ *is a closed embedding.*

2. *There is a natural surjective morphism* $p : X^+ \to X^0$*, which on points coincides with the map* $x \mapsto \lim_{t\to 0} t \cdot x$*, and the morphism* p *is affine.*

3. *The natural morphism* $X^+ \to X$ *is unramified, and the fibre over any geometric point of* X^0 *has a single point.* $X^+ \to X$ *is a monomorphism if* X *is separated. If* X *is proper then each geometric fibre of* $X^+ \to X$ *is reduced and has exactly one point.*

4. $X^+ \to X$ *is an isomorphism if and only if the* $\mathbb{G}_m$*-action on* X *can be extended to an action of the monoid* $\mathbb{A}^1$*, in which case the extension is unique.*

5. *If* X *is smooth, then so are* X^0 *and* X^+*, and the morphism* $p : X^+ \to X^0$ *is also smooth.*

2.5.5 The HKKN Stratification

In addition to furnishing a schematic quotient of the semistable locus, reductive GIT also yields a finite locally closed stratification of the unstable locus by "instability type", known as the *Hesselink–Kempf–Kirwan–Ness (HKKN) stratification* or as the *instability stratification.* Here we describe this stratification.

Let X be a projective scheme acted on by a reductive linear algebraic group G with respect to a very ample linearisation L; we identify X with its image in $\mathbb{P}^N = \mathbb{P}(H^0(X, L)^\vee)$ under the (G-equivariant) Kodaira embedding of L. Fix a maximal torus $T \subset G$, fix a positive definite Weyl-invariant integral bilinear form on the Lie algebra of T, and use this to define an invariant norm on the Lie algebra of G. The norm gives an identification of the character lattice of T with the co-character lattice of T, as well as an identification of the Lie algebra of T with the dual Lie algebra.

Definition 2.5.9. *Let $x \in X$ be a point.*

1. *If λ is a non-trivial 1PS of G, the* normalised Hilbert–Mumford weight *of λ on x is given by $M^L(x, \lambda) = \mu^L(x, \lambda)/||\lambda||$, where $|| - ||$ is the induced norm on 1PS of G.*
2. *A non-trivial 1PS λ of G is said to be a* Kempf 1PS *or a* maximally destabilising 1PS *for x if $M^L(x, \lambda) \leq M^L(x, \lambda')$ for all non-trivial 1PS λ' of G.*

Proposition 2.5.10 (Kempf, [74])**.** *Suppose $x \in X^{us}(L)$ is an unstable point.*

1. *The set of Kempf 1PS for x is non-empty.*
2. *There exists a parabolic subgroup P_x of G such that for any Kempf 1PS λ for x,*
$$P_x = P(\lambda) := \left\{ g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G \right\}.$$
Any P_x-conjugate of a Kempf 1PS for x is a Kempf 1PS for x.
3. *A Kempf 1PS λ of x is unique up to conjugation by P_x and replacing $\lambda \mapsto \lambda^n$, where n is a positive integer.*

The HKKN stratification is indexed by conjugacy classes β of rational 1PS of G. The semistable locus is a stratum S_0, indexed by the conjugacy class of the trivial 1PS. The unstable strata S_β are described as follows. Pick a representative λ_β of the conjugacy class β (if one fixes a positive Weyl chamber $\mathfrak{t}_+$ of Lie(T), one can always choose the representatives to come from $\mathfrak{t}_+$), and let $P_\beta = P(\lambda_\beta)$ be the corresponding parabolic subgroup of G. If $U_\beta \subset P_\beta$ is the unipotent radical of P_β, given by the subgroup of elements $g \in G$ for which $\lim_{t\to 0} \lambda_\beta(t) g \lambda_\beta(t)^{-1}$ is the identity, and $R_\beta \subset P_\beta$ is the Levi subgroup given by elements of the form $\lim_{t\to 0} \lambda_\beta(t) g \lambda_\beta(t)^{-1}$ with $g \in P_\beta$, we have a Levi decomposition $P_\beta = U_\beta \rtimes R_\beta$.

Definition 2.5.11. *Let Z_β be the union of the components of the fixed locus $X^{\lambda_\beta(\mathbb{G}_m)}$ on which the Hilbert–Mumford weight $\mu^L(-, \lambda_\beta)$ takes the value $-||\lambda_\beta||^2$. Let X_β be the locally closed subscheme of X consisting of all points $x \in X$ with $p_\beta(x) := \lim_{t\to 0} \lambda_\beta(t) \cdot x \in Z_\beta$.*

Remark. By Theorem/Definition 2.5.8, the schemes Z_β and X_β inherit canonical scheme structures from X, making them locally closed subschemes of X. The retraction $p_\beta : X_\beta \to Z_\beta$ is a morphism of schemes.

The scheme X_β is invariant under the action of P_β, the scheme Z_β is invariant under the action of R_β and the retraction $p_\beta : X_\beta \to Z_\beta$ is equivariant with respect to the quotient map $q_\beta : P_\beta \to R_\beta$. Let $\chi_{-\beta} : R_\beta \to \mathbb{G}_m$ denote the character corresponding to the 1PS $\lambda_\beta^{-1} : \mathbb{G}_m \to Z(R_\beta)$. Let L_β denote the R_β-linearisation on Z_β given by twisting the linearisation L by the character $\chi_{-\beta}$.

Definition 2.5.12. *We define the following loci:*

1. $Z_\beta^{ss} := Z_\beta^{R_\beta - ss}(L_\beta)$;
2. $X_\beta^{ss} := p_\beta^{-1}(Z_\beta^{ss})$; *and*
3. $S_\beta := G \cdot X_\beta^{ss}$.

We may now state the results of [76] concerning the existence and properties of the HKKN stratification.

Theorem 2.5.13 (Hesselink–Kempf–Kirwan–Ness Stratification)**.** *Let X be a projective scheme acted on by a reductive linear algebraic group G with respect to a very ample linearisation L.*

1. *There is a stratification*

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

of X *into disjoint locally closed* G*-invariant subschemes, with* $S_0 = X^{ss}(L)$*. The indexing set* β *is finite.*

2. *The closure* $\overline{S_\beta}$ *of a stratum satisfies*

$$\overline{S_\beta} \subset S_\beta \sqcup \bigsqcup_{||\gamma||>||\beta||} S_\gamma.$$

3. *For every* $\beta \in \mathcal{B}$*, there is a* G*-equivariant isomorphism* $S_\beta \times^{P_\beta} G \cong X_\beta^{ss}$*.*

4. *For* $\beta \in \mathcal{B} \setminus \{0\}$*, a point* $x \in X$ *is contained in* X_β^{ss} *if and only if* λ_β *is a Kempf 1PS for* x*.*

Chapter 3

Non-Reductive Geometric Invariant Theory

The theory of reductive GIT is a well-established toolkit for constructing and studying quotients of schemes in algebraic geometry by reductive groups. Much more recently, considerable progress has been made in extending this theory to cover, in certain well-behaved cases, quotients by non-reductive groups. In addition to giving a statement of the $\widehat{U}$*-Theorem* of Bérczi–Doran–Hawes–Kirwan, which will play an important role in this thesis, we also give an overview of the non-reductive quotienting-in-stages construction of Hoskins–Jackson, which concerns the formation of quotients by actions of parabolic subgroups of $SL(V)$; this construction will be utilised in Chapter 7. We conclude this chapter by stating how non-reductive GIT (NRGIT) can be used to form quotients of unstable HKKN strata.

3.1 Groups with a Graded Unipotent Radical

Let H be a linear algebraic group. If U is the unipotent radical of U and R is a choice of Levi subgroup of H,[1] we have a semi-direct product decomposition $H = U \rtimes R$, known as a *Levi decomposition* of H. Fix such a Levi decomposition.

Definition 3.1.1. *An* internal grading *of* H *is a 1PS* $\lambda : \mathbb{G}_m \to Z(R)$ *such that* $\lambda(\mathbb{G}_m)$ *acts by conjugation on the Lie algebra* $\mathrm{Lie}(U)$ *with strictly positive weights; if an internal grading exists, we also say that* H has an internally graded unipotent radical*. Given a*

[1] A Levi subgroup $R \subset H$ always exists and is unique up to conjugation.

choice of internal grading λ, we denote

$$\widehat{U} = \widehat{U}_\lambda := U \rtimes \lambda(\mathbb{G}_m),$$

and denote $\overline{R} := R/\lambda(\mathbb{G}_m)$.

Remark. In this thesis, we we will be concerned with quotients by groups with an internally graded unipotent radical. However, there is the slightly more general notion of an *external grading* of the unipotent radical of a linear algebraic group; the formation of quotients by groups with an externally graded unipotent radical is considered in [19].

Remark. When forming quotients by internally graded groups, one can always replace λ by λ^n for a positive integer n without affecting the resulting quotients. In particular, this allows us to work with *rational* internal gradings λ.

Example 3.1.2. Our main examples of internally graded groups come from parabolic subgroups of reductive groups G. Let $\lambda : \mathbb{G}_m \to G$ be a (rational) 1PS, with associated parabolic subgroup

$$P = P(\lambda) = \left\{ g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G \right\} = U \rtimes R.$$

This has unipotent radical $U = \left\{ g \in G : \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = e \right\}$ and Levi factor $R = \left\{ \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} : g \in P \right\}$. We have an inclusion $\lambda(\mathbb{G}_m) \subset Z(R)$. That we have $\lim_{t\to 0} \lambda(t) u \lambda(t)^{-1} = e$ for each $u \in U$ implies that the conjugation action of $\lambda(\mathbb{G}_m)$ on $\mathrm{Lie}(U)$ has strictly positive weights. In other words, λ internally grades the unipotent radical of P. ■

3.2 Quotients by Internally Graded Groups

3.2.1 Setup for the $\widehat{U}$-Theorem

Let X be a projective scheme acted on by a linear algebraic group $H = U \rtimes R$, such that there exists a very[2] ample H-linearisation L on X. Assume in addition that H has an internally graded unipotent radical; fix an internal grading $\lambda : \mathbb{G}_m \to Z(R)$. Let $W := H^0(X, L)^\vee$, and view $X \subset \mathbb{P}(W)$. Let $\omega_{\min} = \omega_0 < \omega_1 < \cdots < \omega_{\max}$ be the weights by which $\lambda(\mathbb{G}_m)$ acts on the fibres of $L^\vee$ over points of the connected components of the

[2]The distinction between working with ample and very ample H-linearisations is not crucial, since one is free to replace an ample H-linearisation on X with any positive integer power.

fixed locus $X^{\lambda(\mathbb{G}_m)}$ for the induced action of $\lambda(\mathbb{G}_m)$ on X. Without loss of generality we assume that there are at least two distinct weights, since otherwise the unipotent radical U acts trivially on X,[3] and so forming a quotient by H is equivalent to forming a quotient by the reductive group R. Let $W_{\min} \subset W$ be the weight space where λ acts with weight $\omega_{\min}$.

Remark. For the purposes of simplicity of exposition,[4] we will always assume that there exists a dense open subscheme $X^\circ \subset X$ such that for all points $x \in X^\circ$, $\lambda(\mathbb{G}_m)$ acts on the fibre of $L^\vee$ over $\lim_{t\to 0} \lambda(t) \cdot x$ with a common weight; this common weight is necessarily the minimal weight $\omega_{\min}$. This assumption always holds if X is irreducible, and this assumption holds for all graded unipotent quotients considered in this thesis.

Definition 3.2.1. *We introduce the following loci:*

1. *the* λ-minimal weight space $Z_{\min} = Z(X,\lambda)_{\min}$ *is the closed subscheme* $Z_{\min} = X \cap \mathbb{P}(W_{\min})$*;*
2. *the* λ-attracting open subscheme $X^0_{\min} = X(\lambda)^0_{\min}$ *is the open subscheme of* X *whose set of points is* $X^0_{\min} = \{x \in X : \lim_{t\to 0} \lambda(t) \cdot x \in Z_{\min}\}$*;*
3. *the* λ-retraction *is the morphism* $p : X^0_{\min} \to Z_{\min}$ *given on points by* $p(x) = \lim_{t\to 0} \lambda(t) \cdot x$*; and*
4. *the* (semi)stable minimal weight space *is* $Z^{(s)s}_{\min} = Z^{\overline{R}-(s)s}_{\min}$*, and we set* $X^{0,(s)s}_{\min} = p^{-1}(Z^{(s)s}_{\min})$*.*

Remark. It follows from Theorem/Definition 2.5.8 that all of the loci appearing in Definition 3.2.1 admit canonical scheme structures, making them locally closed subschemes of X.

Since $\lambda(\mathbb{G}_m) \subset Z(R)$, the λ-minimal weight space $Z_{\min}$ is invariant under the action of R, and the group $\lambda(\mathbb{G}_m)$ acts trivially on $Z_{\min}$; as such $Z_{\min}$ inherits a residual $\overline{R}$-action. The property $\lambda(\mathbb{G}_m) \subset Z(R)$ also implies that the λ-retraction p is R-equivariant, and the property that $\lambda(\mathbb{G}_m)$ acts on $\mathrm{Lie}(U)$ with strictly positive weights implies that p is U-invariant. In particular, the λ-attracting open subscheme $X^0_{\min}$ is invariant under the action of H, and p is equivariant with respect to the quotient $H \to H/U = R$.

[3]For in this case $X^0_{\min} = Z_{\min}$ and p is the identity on $X^0_{\min}$, but p is U-invariant.

[4]A priori, different irreducible components could have different minimal $\lambda(\mathbb{G}_m)$-weights; without this simplifying assumption, each of these minimal weights would need to be kept track of.

Definition 3.2.2. *The very ample linearisation L is said to be:*

1. borderline *if* $\omega_{\min} = 0 < \omega_1 < \cdots < \omega_{\max}$*; and*

2. adapted *if* $\omega_{\min} < 0 < \omega_1 < \cdots < \omega_{\max}$.

Given a property $\mathcal{P}$ of very ample H-linearisations of X, we say that property $\mathcal{P}$ holds for L being a well adapted *linearisation if there exists $\epsilon > 0$ such that if $-\epsilon < \omega_{\min} < 0 < \omega_1$, then property $\mathcal{P}$ applies to L.*

Remark. The only properties $\mathcal{P}$ for which the distinction between an adapted and a well adapted linearisation is necessary are properties of finite generation of invariants; for forming quotients, the distinction does not matter.

3.2.2 Quotients with Borderline Linearisations

In the case where L is borderline, the formation of quotients for the induced action $H \circlearrowright X^{0,ss}_{\min}$ can be handled using reductive GIT.

Proposition 3.2.3 ([66], Proposition 2.17)**.** *Let $H = U \rtimes R$ be a linear algebraic group with internal grading $\lambda : \mathbb{G}_m \to Z(R)$, and let X be a projective scheme acted on by H with a very ample borderline linearisation L. Then there is an isomorphism of graded rings*

$$\bigoplus_{n \geq 0} H^0(X, L^n)^H \cong \bigoplus_{n \geq 0} H^0(Z_{\min}, L^n|_{Z_{\min}})^R,$$

and the following actions all admit a categorical quotient given by the reductive GIT quotient $Z_{\min} /\!\!/_L R$:

1. *the action of R on $Z^{ss}_{\min}$;*

2. *the action of R on $X^{0,ss}_{\min}$; and*

3. *the action of H on $X^{0,ss}_{\min}$.*

Moreover, the categorical quotient $X^{0,ss}_{\min} \to X^{0,ss}_{\min} /\!\!/ H = Z_{\min} /\!\!/_L R$ factors through the λ-retraction $p : X^{0,ss}_{\min} \to Z^{ss}_{\min}$.

Unfortunately, since $X^{0,ss}_{\min} \to X^{0,ss}_{\min} /\!\!/ H$ factors through the λ-retraction, the categorical quotient $X^{0,ss}_{\min} /\!\!/ H$ is almost never a geometric H-quotient, as too many orbits get identified with each other.

3.2.3 The $\widehat{U}$-Theorem

The approach taken with the $\widehat{U}$-Theorem is to instead work solely with linearisations which are adapted, which has the effect of destabilising the locus $UZ_{\min}$, so that points $x \in X^0_{\min}$ can no-longer be identified with their limits under the λ-attraction. This comes at the cost of having to first prove the existence of a geometric U-quotient of $X^0_{\min}$ or of $X^{0,ss}_{\min}$, which imposes constraints on the U-stabilisers of points $x \in X^0_{\min}$.

Definition 3.2.4. *We define the following stabiliser conditions concerning the action* $H \circlearrowright X$*:*

1. *We say* semistability coincides with stability for $\widehat{U}$ *if*

$$\dim \mathrm{Stab}_U(z) = 0 \textit{ for all } z \in Z_{\min}. \qquad [\widehat{U}]_0$$

2. *We say* semistability coincides with stability for $\widehat{U}$ on the reductive semistable locus *if*

$$\dim \mathrm{Stab}_U(z) = 0 \textit{ for all } z \in Z^{ss}_{\min}. \qquad [\widehat{U}; \overline{R} - ss]_0$$

3. *If U is abelian, we say that* semistability coincides with Mumford stability for $\widehat{U}$ *if*

$$\dim \mathrm{Stab}_U(-) \textit{ is constant on } X^0_{\min}. \qquad [\widehat{U}]$$

4. *If U is abelian, we say that* semistability coincides with Mumford stability for $\widehat{U}$ on the reductive semistable locus *if*

$$\dim \mathrm{Stab}_U(-) \textit{ is constant on } X^{0,ss}_{\min}. \qquad [\widehat{U}; \overline{R} - ss]$$

5. *We say that* semistability coincides with stability for $\overline{R}$ *if*

$$\dim \mathrm{Stab}_{\overline{R}}(z) = 0 \textit{ for all } z \in Z^{ss}_{\min}. \qquad [\overline{R}]_0$$

In the case where U is not abelian, statements (3) and (4) should be modified to require that there exists a series of subgroups $\{e\} = U^0 \subset U^1 \subset \cdots \subset U^\ell = U$ which is normalised by H and whose successive quotients are abelian, such that (3) (resp. (4)) holds for each subgroup U^i.

We can now state the $\widehat{U}$-Theorem of Bérczi–Doran–Hawes–Kirwan [18] [19] [20].

Theorem 3.2.5 (Bérczi–Doran–Hawes–Kirwan)**.** *Let $H = U \rtimes R$ be a linear algebraic group with internal grading $\lambda : \mathbb{G}_m \to Z(R)$, and let X be a projective scheme acted on by H with a very ample adapted linearisation L. Assume in addition that condition ($[\widehat{U}]$) holds. Then the following statements hold:*

1. *The $\widehat{U}$-stable locus $X_{\min}^{\widehat{U}-s} := X_{\min}^0 \setminus UZ_{\min}$ is an open, H-invariant subscheme of $X_{\min}^0$ which admits a projective geometric $\widehat{U}$-quotient*
$$q_{\widehat{U}} : X_{\min}^{\widehat{U}-s} \to X /\!\!/_L \widehat{U}.$$
The quotient $X /\!\!/_L \widehat{U}$ is endowed with an ample $\overline{R}$-linearisation $\overline{L}$ inherited from L, for the residual action of $\overline{R}$. Moreover, if L is well adapted, there exists a sufficiently divisible positive integer c with the properties that the ring of $\widehat{U}$-invariants $\bigoplus_{n\geq 0} H^0(X, L^{cn})^{\widehat{U}}$ is finitely generated and $X /\!\!/_L \widehat{U} = \operatorname{Proj}\left(\bigoplus_{n\geq 0} H^0(X, L^{cn})^{\widehat{U}}\right)$.

2. *Let $q : X_{\min}^{\widehat{U}-s} \dashrightarrow X /\!\!/_L H := (X /\!\!/_L \widehat{U}) /\!\!/_{\overline{L}} \overline{R}$ be the composition of $q_{\widehat{U}}$ with the reductive GIT quotient $q_{\overline{R}} : (X /\!\!/_L \widehat{U})^{\overline{R}-ss}(\overline{L}) \to (X /\!\!/_L \widehat{U}) /\!\!/_{\overline{L}} \overline{R}$. Then q is defined on an open subscheme $\operatorname{dom}(q) \subset X_{\min}^{\widehat{U}-s}$, and $q : \operatorname{dom}(q) \to X /\!\!/_L H$ is a projective good H-quotient. Moreover, if L is well adapted, there exists a sufficiently divisible positive integer c' such that the ring of H-invariants $\bigoplus_{n\geq 0} H^0(X, L^{c'n})^H$ is finitely generated and $X /\!\!/_L H = \operatorname{Proj}\left(\bigoplus_{n\geq 0} H^0(X, L^{c'n})^H\right)$.*

The following result of Hoskins–Jackson can be used in certain cases to determine the domain of the quotient maps appearing in Theorem 3.2.5.

Theorem 3.2.6 ([66], Theorem 2.28)**.** *In the situation of Theorem 3.2.5, denote by $X_{\min}^{H-s} := X_{\min}^{0,s} \setminus UZ_{\min}^s$ the H-stable locus. Then:*

1. *If $x \in X_{\min}^{H-s}$ then $q_{\widehat{U}}(x) \in (X /\!\!/_L \widehat{U})^{\overline{R}-s}(\overline{L})$.*

2. *Fix a maximal torus T of R which contains $\lambda(\mathbb{G}_m)$. Then there are open inclusions*
$$X_{\min}^{H-s} \subset \operatorname{dom}(q) \subset \bigcap_{h\in H} hX^{T-ss}(L) = \bigcap_{u\in U} uX^{R-ss}(L) \subset X_{\min}^{0,ss} \setminus UZ_{\min}^{ss}.$$
If ($[\overline{R}]_0$) holds, then all of these inclusions are equalities.

3. *The restriction of q to the open subscheme $X^{H-s}_{\min} \subset X^0_{\min}$ yields a quasi-projective geometric H-quotient $X^{H-s}_{\min} \to q(X^{H-s}_{\min}) = X^{H-s}_{\min}/H$ with projective completion $X /\!\!/_L H$. In particular, if $([\overline{R}]_0)$ holds, then $X^{H-s}_{\min}/H = X /\!\!/_L H$ is a projective geometric H-quotient of $X^{H-s}_{\min}$.*

It is possible to weaken the condition $([\widehat{U}])$ to $([\widehat{U}; \overline{R} - ss])$ to still obtain a projective good H-quotient of an open subscheme of $X^{0,ss}_{\min} \setminus UZ^{ss}_{\min}$; the difference here is that the $\widehat{U}$-quotient will only be quasi-projective.

Theorem 3.2.7 (cf. [66], Theorem 2.29). *Let $H = U \rtimes R$ be a linear algebraic group with internal grading $\lambda : \mathbb{G}_m \to Z(R)$, and let X be a projective scheme acted on by H with a very ample adapted linearisation L. Assume in addition that condition $([\widehat{U}; \overline{R} - ss])$ holds. Then the following statements hold:*

1. *(Bérczi–Doran–Hawes–Kirwan) There exists a quasi-projective geometric $\widehat{U}$-quotient $q_{\widehat{U}} : X^{0,ss}_{\min} \setminus UZ^{ss}_{\min} \to (X^{0,ss}_{\min} \setminus UZ^{ss}_{\min})/\widehat{U}$. By composing $q_{\widehat{U}}$ with a reductive GIT quotient map for the residual $\overline{R}$-action, there is a map*

$$q : X^{0,ss}_{\min} \setminus UZ^{ss}_{\min} \xrightarrow{q_{\widehat{U}}} (X^{0,ss}_{\min} \setminus UZ^{ss}_{\min})/\widehat{U} \overset{q_{\overline{R}}}{\dashrightarrow} X /\!\!/_L H$$

which restricts to a projective good H-quotient $q : \mathrm{dom}(q) \to X /\!\!/_L H$ of the domain of q. Moreover, if L is well adapted, there exists a sufficiently divisible positive integer c such that the ring of H-invariants $\bigoplus_{n\geq 0} H^0(X, L^{cn})^H$ is finitely generated and such that there is an equality $X /\!\!/_L H = \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(X, L^{cn})^H\right)$.

2. *(Hoskins–Jackson) There is an inclusion $X^{H-s}_{\min} := X^{0,s}_{\min} \setminus UZ^s_{\min} \subset \mathrm{dom}(q)$, and the restriction of q to the open subscheme $X^{H-s}_{\min} \subset X^0_{\min}$ yields a quasi-projective geometric quotient $X^{H-s}_{\min} \to q(X^{H-s}_{\min}) = X^{H-s}_{\min}/H$ with projective completion $X /\!\!/_L H$. If in addition $([\overline{R}]_0)$ holds, then $\mathrm{dom}(q) = X^{H-s}_{\min}$, and $X^{H-s}_{\min}/H = X /\!\!/_L H$ is a projective geometric H-quotient of $X^{H-s}_{\min}$.*

In order to prove the results of Chapter 7, we require a slightly modified version of the $\widehat{U}$-Theorem, which is able to handle the situation where, on a non-empty open of $X^0_{\min}$, points have trivial unipotent stabilisers, with the same holding for the $\lambda(\mathbb{G}_m)$-limits for these points, whilst allowing for the possibility of some points of $Z^s_{\min}$ to have positive unipotent stabiliser dimensions. The cost is that the resulting non-reductive quotients are only guaranteed to be quasi-projective. In order to state the modified version of the $\widehat{U}$-Theorem, we introduce the following terminology.

Definition 3.2.8. *Let $H = U \rtimes R$ be a linear algebraic group with internal grading $\lambda : \mathbb{G}_m \to Z(R)$, and let X be a projective scheme acted on by H with a very ample adapted linearisation L.*

1. *The U-*very stable locus *is the open subscheme of X whose set of points is*

$$X_{\min}^{U-vs} = \{x \in X_{\min}^0 : \dim \mathrm{Stab}_U(p(x)) = 0\}.$$

2. *The H-*very stable locus *is the open subscheme of X whose set of points is*

$$X_{\min}^{H-vs} = \{x \in X_{\min}^{U-vs} : p(x) \in Z_{\min}^s\} \setminus UZ_{\min}.$$

Remark. That $X_{\min}^{U-vs}$ is open in $X_{\min}^0$ follows from the semicontinuity of dimensions of U-stabiliser dimensions. In turn, the same argument given in [20, Proof of Theorem 4.21, Page 21] shows that the U-sweep of $\{z \in Z_{\min} : \dim \mathrm{Stab}_U(z) = 0\}$ is closed in $X_{\min}^{U-vs}$, whence $X_{\min}^{H-vs}$ is an open subscheme of $X_{\min}^0$. $X_{\min}^{U-vs}$ is invariant under the action of U, and $X_{\min}^{H-vs}$ is invariant under the action of H.

Theorem 3.2.9. *Let $H = U \rtimes R$ be a linear algebraic group with internal grading $\lambda : \mathbb{G}_m \to Z(R)$, and let X be a projective scheme acted on by H with a very ample adapted linearisation L. Assume in addition that $X_{\min}^{U-vs}$ is non-empty.*

1. *There is a locally trivial U-quotient $q_U : X_{\min}^{U-vs} \to X_{\min}^{U-vs}/U$. Moreover, there exists a positive integer $m > 0$ such that $X_{\min}^{U-vs}/U$ admits a locally closed immersion into $\mathbb{P}((H^0(X, L^m)^U)^\vee)$ in such a way that q_U is induced by the linear projection $\mathbb{P}(H^0(X, L^m)^\vee) \dashrightarrow \mathbb{P}((H^0(X, L^m)^U)^\vee)$.*

2. *Assume further that $X_{\min}^{H-vs}$ is non-empty. Fixing m as above, let Q denote the closure of $X_{\min}^{U-vs}/U$ in $\mathbb{P}((H^0(X, L^m)^U)^\vee)$. Endow Q with the R-linearisation obtained by restricting the $\mathcal{O}(1)$ of $\mathbb{P}((H^0(X, L^m)^U)^\vee)$, and let $Q /\!\!/ R$ denote the resulting reductive GIT quotient. Then there exists a map*

$$q : X_{\min}^{U-vs} \xrightarrow{q_U} X_{\min}^{U-vs}/U \overset{q_R}{\dashrightarrow} Q /\!\!/ R$$

with the following properties:

   (i) $X_{\min}^{H-vs}$ is contained in the domain of q, and the restriction $q : X_{\min}^{H-vs} \to q(X_{\min}^{H-vs})$ is a geometric H-quotient.

(ii) There exists an integer $M \geq m$ and a locally closed immersion of $q(X_{\min}^{H-vs})$ into $\mathbb{P}((H^0(X, L^M)^H)^\vee)$ with the property that q is induced by the linear projection $\mathbb{P}(H^0(X, L^M)^\vee) \dashrightarrow \mathbb{P}((H^0(X, L^M)^H)^\vee)$.

Proof. It is possible to prove Theorem 3.2.9 using the results of [110], however we provide a simplified proof, by following the proofs of Theorems 3.2.5 and 3.2.6.

As before, set $W := H^0(X, L)^\vee$. By [20, Proposition 4.26], there exists a locally trivial U-quotient $\mathbb{P}(W)_{\min}^{U-vs} \to \mathbb{P}(W)_{\min}^{U-vs}/U$, given by the restriction of the enveloping quotient (cf. [43, Section 4]). Moreover, from the proof of [18, Proposition 3.1.19] there exists a positive integer m such that for all positive multiples m' of m, $\mathbb{P}(W)_{\min}^{U-vs}/U$ admits a locally closed immersion into $\mathbb{P}((H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m'))^U)^\vee)$, with the composition $\mathbb{P}(W)_{\min}^{U-vs} \to \mathbb{P}(W)_{\min}^{U-vs}/U \to \mathbb{P}((H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m'))^U)^\vee)$ coinciding with the morphism obtained by restricting the rational map between projective spaces corresponding to the inclusion $H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m'))^U \subset H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m'))$.

By Lemma 2.1.5, this quotient restricts to a locally trivial quotient $q_U : X_{\min}^{U-vs} \to X_{\min}^{U-vs}/U$, with $X_{\min}^{U-vs}/U$ a closed subscheme of $\mathbb{P}(W)_{\min}^{U-vs}/U$. By replacing m with a larger multiple if necessary, so that the natural map $\mathrm{Sym}^m H^0(X, L) \to H^0(X, L^m)$ is onto, the closed immersion $X \to \mathbb{P}(H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m))^\vee)$ factors through the projective space $\mathbb{P}(H^0(X, L^m)^\vee)$, and in addition the composite

$$H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m))^U \xrightarrow{\subset} H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m)) \longrightarrow H^0(X, L^m)$$

factors through $H^0(X, L^m)^U$. It follows that the composite

$$X_{\min}^{U-vs}/U \to \mathbb{P}(W)_{\min}^{U-vs}/U \to \mathbb{P}((H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(m))^U)^\vee)$$

factors through $\mathbb{P}((H^0(X, L^m)^U)^\vee)$, and so q_U fits into a diagram

$$\begin{array}{ccc} X_{\min}^{U-vs} & \longrightarrow & \mathbb{P}(H^0(X, L^m)^\vee) \\ \downarrow q_U & & \downarrow H^0(L^m)^U \subset H^0(L^m) \\ X_{\min}^{U-vs}/U & \longrightarrow & \mathbb{P}((H^0(X, L^m)^U)^\vee) \end{array}$$

whose horizontal arrows are given by locally closed immersions. This completes the proof of the first part of the theorem.

The proof of the second part of the theorem boils down to reductive GIT for the residual action of R. Let Q be the closure of $X_{\min}^{U-vs}/U$ in $\mathbb{P}((H^0(X, L^m)^U)^\vee)$, and endow Q with the very ample linearisation L_Q obtained as the restriction of the $\mathcal{O}(1)$ of $\mathbb{P}((H^0(X, L^m)^U)^\vee)$. Choosing $s > 0$ sufficiently large such that $\bigoplus_k H^0(Q, L_Q^{sk})^R$ is generated in degree 1 and such that the natural morphism $\operatorname{Sym}^r H^0(X, L^m)^U \to H^0(Q, L_Q^r)$ is a surjection for all $r \geq s$, by Theorem 2.2.6 the map

$$q_R : Q \dashrightarrow Q /\!\!/ R = \operatorname{Proj} \bigoplus_k H^0(Q, L_Q^{sk})^R \subset \mathbb{P}((H^0(Q, L_Q^s)^R)^\vee)$$

induced by the inclusion $\bigoplus_k H^0(Q, L_Q^{sk})^R \subset \bigoplus_k H^0(Q, L_Q^{sk})$ restricts to give a geometric R-quotient of the open stable locus $Q^{R-s}(L_Q)$. Increasing s if necessary, the above morphisms yield a surjection $H^0(X, L^{sm})^U \to H^0(Q, L_Q^s)$; taking R-invariants (which is exact, as R is reductive) then yields a surjection $(H^0(X, L^{sm})^U)^R = H^0(X, L^{sm})^H \to H^0(Q, L_Q^s)^R$. This yields a closed immersion $\mathbb{P}((H^0(Q, L_Q^s)^R)^\vee) \to \mathbb{P}((H^0(X, L^{sm})^H)^\vee)$, such that the composite $Q /\!\!/ R \to \mathbb{P}((H^0(X, L^{sm})^H)^\vee)$ fits into the following diagram:

$$\begin{array}{ccc}
X_{\min}^{U-vs} & \longrightarrow & \mathbb{P}(H^0(X, L^{sm})^\vee) \\
\downarrow q_U & & \\
X_{\min}^{U-vs}/U & & \Big\downarrow{\scriptstyle H^0(L^{sm})^H \subset H^0(L^{sm})} \\
\downarrow q_R & & \\
Q /\!\!/ R & \longrightarrow & \mathbb{P}((H^0(X, L^{sm})^H)^\vee)
\end{array}$$

In light of Lemmas 2.1.4 and 2.1.5, it remains to show that the locally closed subscheme $q_U(X_{\min}^{H-vs})$ of Q is contained in the GIT stable locus $Q^{R-s}(L_Q)$. But this is a consequence of the first assertion of [66, Theorem 2.28]. □

Remark. Unlike in reductive GIT, where the resulting quotients are always projective, we remark that in the setting of the $\widehat{U}$-Theorem, without the stabiliser conditions $([\widehat{U}]_0)$ or $([\widehat{U}; \overline{R} - ss]_0)$ holding one only obtains a *quasi-projective* good H-quotient of an invariant open subscheme of $X_{\min}^0$. If one seeks projective quotients, one must first modify the H-scheme X so that either $([\widehat{U}]_0)$ or $([\widehat{U}; \overline{R} - ss]_0)$ hold. In favourable situations, this can be achieved (albeit at the cost of not having a modular completion) by blowing-up X along a suitable H-invariant subscheme, supported along the U-sweep of the locus of points in $Z_{\min}$ whose U-stabiliser dimension is strictly larger than the generic dimension [20, Section 4.5] [110]; this should be thought of as being analogous to the procedure of Kirwan desingularisation [77] in reductive GIT, as well as its stacky analogue [47].

3.2.4 Application to Quotients of Unstable HKKN Strata

Let G be a reductive linear algebraic group acting on a projective scheme with respect to a very ample linearisation L. Recall from Section 2.5.5 the HKKN stratification of X:

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta, \quad S_\beta = G \cdot X_\beta^{ss} \cong G \times^{P_\beta} X_\beta^{ss}.$$

As in Example 3.1.2, P_β is internally graded by the 1PS λ_β. From the isomorphism $S_\beta \cong G \times^{P_\beta} X_\beta^{ss}$, the existence of a categorical quotient $S_\beta /\!\!/ G$ is equivalent to the existence of a categorical quotient $X_\beta^{ss} /\!\!/ P_\beta$.

Let $Y_\beta = \overline{X_\beta}$ denote the closure of X_β in X. For $\epsilon \in \mathbb{Q}^{\geq 0}$, let $L_{(1+\epsilon)\beta}$ be the (rational) P_β-linearisation on Y_β obtained by twisting L with respect to the character $\chi_{-(1+\epsilon)\beta}$ corresponding under the Weyl-invariant norm $|| - ||$ on Lie(G) to the rational 1PS $\lambda_{(1+\epsilon)\beta}^{-1} : \mathbb{G}_m \to Z(R_\beta)$.

Proposition 3.2.10 ([66], Proposition 3.5)**.** *Let $\beta \in \mathcal{B} \setminus \{0\}$ be a non-zero index of an HKKN stratum of X. Endow P_β with the grading 1PS λ_β. Then, with respect to the linearisation $L_{(1+\epsilon)\beta}$:*

1. *there is an equality $Z_{\beta,\min} := Z(Y_\beta, \lambda_\beta) = Z_\beta$;*
2. *there is an equality $Y_{\beta,\min} := Y_\beta(\lambda_\beta)^0_{\min} = X_\beta$.*
3. *the λ_β-retraction is given by the morphism $p_\beta : X_\beta \to Z_\beta$; and*
4. *if $\epsilon > 0$ is sufficiently small, the linearisation $L_{(1+\epsilon)\beta}$ is well adapted for the graded unipotent group $\widehat{U}_{\lambda_\beta}$.*

Note that in order to apply Theorems 3.2.5 and 3.2.6 to this situation, there is still the requirement to check that the various stabiliser assumptions hold. For many examples of unstable strata (see for instance [66] and [70]), the conditions ([$\widehat{U}$]) and ([$\widehat{U}; \overline{R} - ss$]) can both fail, and even if these conditions hold then the condition ([$\overline{R}$]$_0$) can also fail.

For example, in the setting of moduli of unstable sheaves of a fixed Harder–Narasimhan type of length ℓ (cf. [66] and Section 4.3.2), there is an $(\ell-1)$-dimensional torus $T \subset R_\beta$ which acts trivially on Z_β (but not on X_β), meaning that unless $\ell = 2$ then $([\overline{R}]_0)$ automatically fails.

3.3 Parabolic Groups of $SL(V)$ and the Quotienting-in-Stages Construction

In order to rectify the issue of $([\overline{R}]_0)$ failing when dealing with grading 1PS with $\ell > 2$ distinct weights, the approach of Hoskins–Jackson [66] (in the case of forming quotients of parabolic subgroup $P \subset SL(V)$ for some finite dimensional vector space V) is to break the process of quotienting by P into stages, according to the row filtration of P, and at each stage form a quotient of a group graded by a 1PS with only two distinct weights. The condition $([\overline{R}]_0)$ has a better chance of holding stage-by-stage, at least over a non-empty open subscheme of $Z_{\min}$.

3.3.1 Notation for Parabolic Subgroups of $SL(V)$

Let $\lambda : \mathbb{G}_m \to SL(V)$ be a 1PS. Fix a basis $\{v_1, \dots, v_N\}$ for V diagonalising the action of $\lambda(\mathbb{G}_m)$, with weights $r_1 \geq r_2 \geq \cdots \geq r_N$. Let $\ell = \ell(\lambda)$ be the number of distinct weights of λ, let $\beta_1 > \cdots > \beta_\ell$ be these distinct weights, and suppose the weight β_i occurs m_i-times.

Let

$$P = P(\lambda) = \left\{ g \in SL(V) : \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } SL(V) \right\} = U \rtimes R$$

be the parabolic subgroup of $SL(V)$ associated to λ, with unipotent radical $U = U(\lambda)$ and Levi factor $R = R(\lambda)$. The groups P, U and R can be explicitly described in terms of block upper-triangular matrices (defined with respect to the basis $\{v_1, \dots, v_N\}$) as follows:

$$\text{(i)} \;\; P = \left\{ A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,\ell-1} & A_{1,\ell} \\ 0 & A_{2,2} & \cdots & A_{2,\ell-1} & A_{2,\ell} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{\ell-1,\ell-1} & A_{\ell-1,\ell} \\ 0 & 0 & \cdots & 0 & A_{\ell,\ell} \end{pmatrix} \in SL(V) : A_{i,j} \in \mathbb{C}^{m_i \times m_j} \right\}.$$

(ii) $R = \{A \in P : A_{i,j} = 0 \text{ for all } 1 \leq i < j \leq \ell\}$ consists of all block diagonal matrices in P.

(iii) The centre $T = Z(R)$ of R is given by

$$T = \left\{ \operatorname{diag}(t_1 I_{m_1}, \ldots, t_\ell I_{m_\ell}) : t_i \in \mathbb{G}_m, \ \prod_{i=1}^{\ell} t_i = 1 \right\} \cong \mathbb{G}_m^{\ell-1}.$$

(iv) The semisimple part of R is given by

$$R' = \{\operatorname{diag}(A_{1,1}, \ldots, A_{\ell,\ell}) \in R : \text{each } A_{ii} \in SL(m_i, \mathbb{C})\} \cong \prod_{i=1}^{\ell} SL(m_i, \mathbb{C}).$$

(v) $U = \{A \in P : A_{i,i} = I_{m_i} \text{ for all } 1 \leq i \leq \ell\}$.

The 1PS λ grades the unipotent radical U. Let $H := U \rtimes R' \subset P$ and $\widehat{H} := H \rtimes T = U \rtimes (R' \times T)$. There are surjections $R' \to R/T$ and $H \to P/T$, both with finite kernels. Forming quotients by the actions of these finite kernels can always be done, by Lemma 2.2.11. As such, the problem of forming a categorical (resp. good, geometric) quotient of an action by P is equivalent to that of forming a categorical (resp. good, geometric) quotient by an action of $\widehat{H}$.

Following [66, Definition 4.6], we introduce the following notation.

(vi) For each $1 \leq i < \ell$, set $m_{>i} = \sum_{j>i} m_j$, $m_{\leq i} = \sum_{j \leq i} m_j$,

$$\beta_{>i} = \frac{\sum_{j>i} \beta_j m_j}{m_{>i}} \quad \text{and} \quad \beta_{\leq i} = \frac{\sum_{j \leq i} \beta_j m_j}{m_{\leq i}}.$$

We define 1PS $\lambda^{(i)}$ and $\lambda^{[i]}$ of $T = Z(R)$ by setting

$$\lambda^{(i)}(t) := \operatorname{diag}(t^{\beta_1} I_{m_1}, \ldots, t^{\beta_i} I_{m_i}, t^{\beta_{>i}} I_{m_{>i}}), \quad \lambda^{[i]}(t) = \operatorname{diag}(t^{\beta_{\leq i}} I_{m_{\leq i}}, t^{\beta_{>i}} I_{m_{>i}}).$$

(vii) Let $U^{[i]}$ be the unipotent radical of the parabolic $P^{[i]} := P(\lambda^{[i]}) \supset P$:

$$U^{[i]} = \{A \in U : \text{for all } p < q, A_{p,q} = 0 \text{ if } q \leq i \text{ or } p > i\}.$$

This is graded by the 1PS $\lambda^{[i]}$ and is normal in P. For $i \geq 2$, we also set $U^{[i-1,i]} = U^{[i-1]} \cap U^{[i]}$.

(viii) Let $U^{(i)}$ be the unipotent radical of the parabolic $P(\lambda^{(i)}) \supset P$; this is graded by the 1PS $\lambda^{(i)}$ and is normal in P.

(ix) Let $P^{(i)} := U^{(i)} \rtimes R^{(i)} \subset P$, where

$$R^{(i)} = \begin{cases} \{A \in L : A_{j,j} = I_{m_j} \text{ if } j > i\} & \text{if } i < \ell - 1, \\ R & \text{if } i = \ell - 1. \end{cases}$$

We denote the successive quotients by $L_i := L^{(i)}/L^{(i-1)}$, $U_i := U^{(i)}/U^{(i-1)}$ and $P_i := P^{(i)}/P^{(i-1)}$.

(x) Let $H^{(i)} := U^{(i)} \rtimes R'^{(i)} \subset P^{(i)}$, where

$$R'^{(i)} = \begin{cases} \{A \in R' : A_{j,j} = I_{m_j} \text{ if } j > i\} & \text{if } i < \ell - 1, \\ R' & \text{if } i = \ell - 1. \end{cases}$$

Let $R'_i := R'^{(i)}/R'^{(i-1)}$ and $H_i := H^{(i)}/H^{(i-1)} = U_i \rtimes R'_i$.

(xi) Let $T^{(i)} = \prod_{j \leq i} \lambda^{(j)}(\mathbb{G}_m) = \prod_{j \leq i} \lambda^{[j]}(\mathbb{G}_m) \subset T$, and set $\widehat{H}^{(i)} := H^{(i)} \rtimes T^{(i)}$.

(xii) For each $j \leq i$ let $\lambda_j^{[i]}$ be the 1PS of length 2 given by the composition

$$\lambda_j^{[i]} : \mathbb{G}_m \xrightarrow{\lambda^{[i]}} P \longrightarrow P/P^{(j-1)},$$

Note that $\lambda^{[i]} = \lambda_1^{[i]}$ for all i.

(xiii) As a special case of the above, set $\lambda_i = \lambda_i^{[i]}$ for each i. The associated parabolic $P(\lambda_i)$ has unipotent radical isomorphic to U_i, and λ_i grades U_i. Let $\widehat{H}_i := \widehat{H}^{(i)}/\widehat{H}^{(i-1)} \cong H_i \rtimes \lambda_i(\mathbb{G}_m) = U_i \rtimes (R'_i \times \lambda_i(\mathbb{G}_m))$.

Suppose X_i is a projective scheme acted on by $\widehat{H}_i$ with a very ample linearisation L_i. The group $\widehat{H}_i$ has a one-dimensional character group, which is generated by a character χ_i dual to the 1PS $\lambda_i = \lambda_i^{[i]}$ grading the unipotent radical U_i of $\widehat{H}_i$. Twisting L_i by some multiple $\epsilon_i\chi_i$ of χ_i corresponds to shifting the $\lambda_i(\mathbb{G}_m)$-weights each by ϵ_i. As such, by twisting L_i by $\epsilon_i\chi_i$ for suitable ϵ_i, we can always ensure that L_i is (well) adapted; twisting by such a character does not alter the loci $X_i(\lambda_i)^0_{\min}$ nor $Z(X_i, \lambda_i)_{\min}$.

3.3.2 Non-Reductive Quotienting-in-Stages

We now present the quotienting-in-stages construction of Hoskins–Jackson, concerning the formation of non-reductive quotients of projective schemes X acted on by the parabolic subgroup $P \subset SL(V)$ with respect to very ample P-linearisations. We quotient out the groups $\widehat{H}_i$ turn-by-turn by applying the $\widehat{U}$-Theorem at each stage, arriving at a quotient of an open subscheme of X by the whole parabolic group P at the end. The presented construction closely follows [66, Construction 4.20]. However, unlike in *loc. cit.*, which utilises Theorems 3.2.5 and 3.2.6, the version of the $\widehat{U}$-Theorem we apply at each stage is Theorem 3.2.9. Construction 3.3.1 is the version of the quotienting-in-stages construction that will be required in Chapter 7.

Construction 3.3.1 ([66], Construction 4.20)**.** *Base step:* Let $X = X_1$ be a projective scheme acted on by P with respect to a very ample P-linearisation $L = L_1$. Viewing L as a $\widehat{H}_1$-linearisation, by twisting L_1 by a suitable multiple $\epsilon_1\chi_1$ of χ_i if necessary, we may ensure that L_1 is adapted.

Assuming we can apply Theorem 3.2.9 to the action $\widehat{H}_1 \circlearrowright X_1$, in other words assuming that $(X_1)^{\widehat{H}_1-vs}_{\min}$ is non-empty, there exists a geometric $\widehat{H}_1$-quotient $q_1 : (X_1)^{\widehat{H}_1-vs}_{\min} \to (X_1)^{\widehat{H}_1-vs}_{\min}/\widehat{H}_1$, together with a locally closed immersion of schemes $(X_1)^{\widehat{H}_1-vs}_{\min}/\widehat{H}_1 \subset \mathbb{P}((H^0(X_1, L_1^{s_1})^{\widehat{H}_1})^\vee)$ for some integer $s_1 > 0$, with the quotient being induced by the projection $\mathbb{P}(H^0(X_1, L_1^{s_1})^\vee) \dashrightarrow \mathbb{P}((H^0(X_1, L_1^{s_1})^{\widehat{H}_1})^\vee)$. We set X_2 to be the closure of $(X_1)^{\widehat{H}_1-vs}_{\min}/\widehat{H}_1$ in $\mathbb{P}((H^0(X_1, L_1^{s_1})^{\widehat{H}_1})^\vee)$, and set L_2 to be the very ample linearisation for the residual action of $P/\widehat{H}^{(1)} = P/\widehat{H}_1$ obtained by restricting the $\mathcal{O}(1)$ of the projective space $\mathbb{P}((H^0(X_1, L_1^{s_1})^{\widehat{H}_1})^\vee)$.

Induction step: Suppose $i < \ell - 1$, and suppose we have constructed successive quotients q_j of the form

$$\begin{array}{ccc}
(X_j)^{\widehat{H}_j-vs}_{\min} & \xrightarrow{\quad q_j \quad} & (X_j)^{\widehat{H}_j-vs}_{\min}/\widehat{H}_j \\
\downarrow & & \downarrow \\
X_j & \overset{q_j}{\dashrightarrow} & X_{j+1} \\
\downarrow & & \downarrow \\
\mathbb{P}(H^0(X_j, L_j^{s_j})^\vee) & \underset{H^0(X_j, L_j^{s_j})^{\widehat{H}_j} \subset H^0(X_j, L_j^{s_j})}{\dashrightarrow} & \mathbb{P}((H^0(X_j, L_j^{s_j})^{\widehat{H}_j})^\vee)
\end{array}$$

for each $j = 1, \ldots, i$, where the top horizontal arrow is a geometric $\widehat{H}_j$-quotient, where X_{j+1} is the closure of the locally closed subscheme

$$(X_j)^{\widehat{H}_j-vs}_{\min}/\widehat{H}_j \subset \mathbb{P}((H^0(X_j, L_j^{s_j})^{\widehat{H}_j})^\vee),$$

and where each X_j is endowed with the very ample linearisation L_j obtained by restricting the $\mathcal{O}(1)$ of $\mathbb{P}((H^0(X_{j-1}, L_{j-1}^{s_{j-1}})^{\widehat{H}_{j-1}})^\vee)$ to X_j, twisted by a suitable character.

The very ample invertible sheaf L_{i+1} on the projective scheme X_{i+1} carries a linearisation for a residual action of $P/\widehat{H}^{(i)}$, and in particular for $\widehat{H}_{i+1}$. By twisting L_{i+1} by a character of $\widehat{H}_{i+1}$ of the form $\epsilon_{i+1}\chi_{i+1}$, we can ensure that this $\widehat{H}_{i+1}$-linearisation is adapted. Assuming once again that we can apply Theorem 3.2.9, we obtain, for some integer $s_{i+1} > 0$, a diagram of the form

$$\begin{array}{ccc}
(X_{i+1})_{\min}^{\widehat{H}_{i+1}-vs} & \xrightarrow{\quad q_{i+1} \quad} & (X_{i+1})_{\min}^{\widehat{H}_{i+1}-vs}/\widehat{H}_{i+1} \\
\downarrow \circ & & \downarrow \circ \\
X_{i+1} & \overset{q_{i+1}}{\dashrightarrow} & X_{i+2} \\
\downarrow & & \downarrow \\
\mathbb{P}(H^0(X_{i+1}, L_{i+1}^{s_{i+1}})^\vee) & \underset{H^0(X_{i+1},L_{i+1}^{s_{i+1}})^{\widehat{H}_{i+1}} \subset H^0(X_{i+1},L_{i+1}^{s_{i+1}})}{\dashrightarrow} & \mathbb{P}((H^0(X_{i+1}, L_{i+1}^{s_{i+1}})^{\widehat{H}_{i+1}})^\vee)
\end{array}$$

where the top horizontal arrow is a geometric $\widehat{H}_{i+1}$-quotient and where X_{i+2} is the closure of the locally closed subscheme $(X_{i+1})_{\min}^{\widehat{H}_{i+1}-vs}/\widehat{H}_{i+1} \subset \mathbb{P}((H^0(X_{i+1}, L_{i+1}^{s_{i+1}})^{\widehat{H}_{i+1}})^\vee)$. If $i < \ell - 2$, we define L_{i+2} to be the restriction of the $\mathcal{O}(1)$ of $\mathbb{P}((H^0(X_{i+1}, L_{i+1}^{s_{i+1}})^{\widehat{H}_{i+1}})^\vee)$ to X_{i+2}. The induction step of Construction 3.3.1 is complete. ∎

Assuming that Theorem 3.2.9 can always be applied at each stage, that is assuming that $(X_i)_{\min}^{\widehat{H}_i-vs} \neq \emptyset$ for all $i = 1, \ldots, \ell-1$, after $\ell-1$ stages Construction 3.3.1 terminates, yielding a diagram of the form

$$\begin{array}{ccccccccc}
X_1 & \overset{q_1}{\dashrightarrow} & X_2 & \overset{q_2}{\dashrightarrow} & \cdots & \overset{q_{\ell-2}}{\dashrightarrow} & X_{\ell-1} & \overset{q_{\ell-1}}{\dashrightarrow} & X_\ell \\
\uparrow \circ & \nearrow q_1 & \uparrow \circ & \nearrow q_2 & & & \uparrow \circ & \nearrow q_{\ell-1} & \\
(X_1)_{\min}^{\widehat{H}_1-vs} & & (X_2)_{\min}^{\widehat{H}_2-vs} & & \cdots & & (X_{\ell-1})_{\min}^{\widehat{H}_{\ell-1}-vs} & &
\end{array}$$

where for each i the restriction $q_i : (X_i)_{\min}^{\widehat{H}_i-vs} \to q_i((X_i)_{\min}^{\widehat{H}_i-vs}) = (X_i)_{\min}^{\widehat{H}_i-vs}/\widehat{H}_i$ is a geometric $\widehat{H}_i$-quotient. For each $i = 1, \ldots, \ell - 1$, set

$$q_{(i)} := q_i \circ q_{i-1} \circ \cdots \circ q_1 : X_1 \dashrightarrow X_{i+1}.$$

We inductively define open subschemes $X_{(i)} \subset X = X_1$ for $i = 1, \ldots, \ell - 1$ by setting $X_{(1)} := (X_1)_{\min}^{\widehat{H}_1 - vs}$ and setting for $i > 1$

$$X_{(i)} := X_{(i-1)} \cap q_{(i-1)}^{-1}((X_i)_{\min}^{\widehat{H}_i - vs}).$$

Then, for each i, the morphism $q_{(i)} : X_{(i)} \to q_{(i)}(X_{(i)}) = X_{(i)}/\widehat{H}^{(i)} \subset X_{i+1}$ is a well-defined geometric $\widehat{H}^{(i)}$-quotient. In particular, $q_{(\ell-1)} : X_{(\ell-1)} \to q_{(\ell-1)}(X_{(\ell-1)}) = X_{(\ell-1)}/\widehat{H} \subset X_\ell$ is a quasi-projective geometric quotient for the action of the group $\widehat{H}$.

Proposition 3.3.2. *Assume that Theorem 3.2.9 can be carried out at each stage of Construction 3.3.1. Then Construction 3.3.1 yields a non-empty invariant open subscheme $X_{(\ell-1)}$ of X which admits a quasi-projective geometric quotient by P.*

Proof. Recall that we have surjections $R' \to R/T$ and $H \to P/T$, both with finite kernels. Since Lemma 2.2.11 can be used to form quotients by the residual actions of these finite kernels, the existence of a quasi-projective geometric quotient of $X_{(\ell-1)}$ by the action of $\widehat{H}$ implies the existence of a quasi-projective geometric quotient of $X_{(\ell-1)}$ by the action of P. □

Remark. By [66, Proposition 4.27] there exists a (rational) character χ of P with the property that, if L_χ is the twist of the very ample P-linearisation L by χ, then L_χ is adapted, and at each stage of Construction 3.3.1 the linearisations $(L_\chi)_i$ are adapted without further twisting.

Remark. If the so-called *quotienting-in-stages assumption* (Assumption 4.29 of *loc. cit.*) holds, it is possible to write down an explicit invariant open subscheme X^{P-qs} (cf. Definition 4.11 of *loc. cit.*) of X for which non-reductive quotienting-in-stages produces a quasi-projective geometric quotient. Without this assumption, the open subscheme X^{P-ps} is no-longer guaranteed to be contained in the domain of the quotient morphism resulting from implementing Construction 3.3.1 or Construction 4.20 of *loc. cit.*

Chapter 4

Moduli Spaces of Coherent Sheaves

In this chapter we review results relating to coherent sheaves and their moduli which are required in this thesis. We first recall the notion of Gieseker stability for coherent sheaves on a polarised projective scheme $(X, \mathcal{O}_X(1))$, and then outline the GIT construction of the projective moduli space of Gieseker semistable sheaves on $(X, \mathcal{O}_X(1))$, due to Simpson. Next, we recall the notion of Harder–Narasimhan filtrations of unstable sheaves and of Harder–Narasimhan stratifications, and state results of Hoskins–Kirwan and Hoskins concerning how such stratifications relate to HKKN stratifications coming from Simpson's construction. We also review the notion of multi-Gieseker stability, due to Greb–Ross–Toma, in advance of Chapter 6.

A standard reference for the theory of Gieseker semistable sheaves and their moduli is the book [69] of Huybrechts and Lehn.

4.1 Gieseker Stability for Coherent Sheaves

Let X be a projective scheme, and let $\mathcal{O}_X(1)$ be an ample invertible sheaf on X. For any coherent sheaf F on X with $d = \dim \operatorname{supp} F$, the Hilbert polynomial of F with respect to X is of the form

$$P(F,t) = P_{\mathcal{O}_X(1)}(F,t) := \chi(X, F(t)) = \sum_{i=0}^{d} \alpha_i(F) \cdot \frac{t^i}{i!}$$

for integers $\alpha_i(F)$.

Definition 4.1.1. *The* reduced Hilbert polynomial *of F with respect to $\mathcal{O}_X(1)$ is the polynomial*

$$p(F,t) = p_{\mathcal{O}_X(1)}(F,t) := \frac{P(F,t)}{\alpha_d(F)} \in \mathbb{Q}[t].$$

Let $\prec$ be the total ordering on monic polynomials in $\mathbb{Q}[t]$ defined by $f \prec g$ if and only if $\deg f > \deg g$,[1] or $\deg f = \deg g$ and $f(m) < g(m)$ for all $m \gg 0$.

Definition 4.1.2. *A coherent sheaf F on X is said to be* Gieseker semistable (with respect to $\mathcal{O}_X(1)$) *if for all non-trivial subsheaves $G \subset F$, $p(G,t) \preceq p(F,t)$. F is said to be* Gieseker stable *if F is Gieseker semistable and for all proper non-trivial subsheaves $G \subset F$, $p(G,t) \prec p(F,t)$. F is said to be* Gieseker polystable *if F is Gieseker semistable and is isomorphic to a direct sum of Gieseker stable sheaves.*

Remark. If F is a Gieseker semistable sheaf on X, it follows as a straightforward consequence of Definition 4.1.2 that F is necessarily *pure*, i.e. for all all non-zero subsheaves $G \subset F$, $\dim \operatorname{supp} G = \dim \operatorname{supp} F$.

Remark. Let F be a coherent sheaf of pure dimension d. By [69, Proposition 1.2.6], in order to check for the Gieseker (semi)stability of F, it is enough to consider *saturated subsheaves*, i.e. subsheaves $G \subset F$ for which the quotient F/G is 0 or pure of dimension d. The Gieseker (semi)stability of F can also be characterised in terms of proper quotients $F \to G \to 0$ with G of pure dimension d; F is Gieseker (semi)stable if and only if for all such quotients, one has $p(F,t) \prec (\preceq)\ p(G,t)$.

Remark. The condition of being Gieseker stable (resp. semistable) is open in flat families (cf. Proposition 2.3.1 of *loc. cit.*); that is, if F is an B-flat, B-finitely presented coherent sheaf over a projective B-scheme X endowed with a relatively ample invertible sheaf $\mathcal{O}_X(1)$, then the locus of geometric points $b \in B$ such that F_b is Gieseker stable (resp. semistable) with respect to $\mathcal{O}_{X_b}(1)$ is an open subscheme of B.

Example 4.1.3. Suppose F is a locally free sheaf on a smooth projective curve C of rank r and degree d. Then for any choice of ample invertible sheaf on C, one has

$$p(F,t) = \mu(F) + t, \quad \mu(F) = \frac{d}{r}.$$

As such, F is Gieseker (semi)stable if and only if F is *(Mumford) slope (semi)stable*, that is $\mu(G) < (\leq)\ \mu(F)$ for all proper non-trivial (locally free) subsheaves $G \subset F$. ■

[1] Note the change of direction in the inequality!

Every semistable sheaf on X admits a canonical filtration; this filtration plays a role in determining which sheaves correspond to a given point in the moduli space of all Gieseker semistable sheaves on X.

Proposition/Definition 4.1.4 ([69], Proposition 1.5.2)**.** *Let F be a Gieseker semistable sheaf on X with respect to a given ample invertible sheaf $\mathcal{O}_X(1)$. Then there exists a filtration*

$$0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$$

of F, known as a Jordan–Hölder filtration *of F, with the property that each subquotient $F_i := F^i/F^{i-1}$ is Gieseker stable and has the same reduced Hilbert polynomial as F. Up to isomorphism, the* Jordan–Hölder associated graded sheaf *of F, namely the sheaf*

$$\mathrm{gr}^{\mathrm{JH}}(F) := \bigoplus_i F_i,$$

does not depend on the choice of Jordan–Hölder filtration.

Definition 4.1.5. *Let F, G be semistable coherent sheaves on X. F and G are said to be* S-equivalent *if $\mathrm{gr}^{\mathrm{JH}}(F) \cong \mathrm{gr}^{\mathrm{JH}}(G)$.*

Remark. The Jordan–Hölder filtration of a stable sheaf F is the trivial filtration $0 \subset F$; as such, up to isomorphism the only coherent sheaf which is S-equivalent to F is F itself. More generally, two polystable sheaves are S-equivalent if and only if they are isomorphic.

One further property of Gieseker stable sheaves is that they are so-called simple coherent sheaves.

Definition 4.1.6. *A coherent sheaf F on a projective scheme X is said to be* simple *if $\mathrm{End}_X(F) = \mathbb{C}$.*

Remark. As with Gieseker (semi)stability, the condition of being simple is open in flat families (cf. Proposition 2.3.1 of *loc. cit.*).

Proposition 4.1.7 ([69], Corollary 1.2.8)**.** *Let F be a Gieseker stable sheaf on a projective scheme X. Then F is simple.*

4.2 Moduli of Gieseker Semistable Sheaves on a Projective Scheme

4.2.1 Families of Semistable Sheaves

Let $f : X \to B$ be a projective morphism (of finite type), where B is a fixed base scheme. Let $\mathcal{O}_X(1)$ be a relatively ample invertible sheaf on X.

Definition 4.2.1. *Let T be a B-scheme. A* family of Gieseker (semi)stable sheaves parametrised by T *is a T-flat, T-finitely presented coherent sheaf F over $X_T = X \times_B T$ such that for each geometric point $t \in T$, the sheaf $F_t \in \mathbf{Coh}(X_t)$ is Gieseker (semi)stable with respect to $\mathcal{O}_{X_t}(1) = \mathcal{O}_X(1)|_{X_t}$.*

Definition 4.2.2. *Fix a polynomial $P(t) \in \mathbb{Q}[t]$ of degree d. Define moduli functors*

$$\mathcal{M}^{(s)s} = \mathcal{M}^{(s)s}_{X/B}(\mathcal{O}_X(1), P) : \mathbf{Sch}_B^{\mathrm{op}} \to \mathbf{Set}$$

by associating to a B-scheme T the set of all isomorphism classes of families of Gieseker (semi)stable sheaves parametrised by T with Hilbert polynomial P.

The existence of moduli schemes corepresenting the functors $\mathcal{M}^{(s)s}$ was first proved by Simpson [124], making use of reductive GIT.

Theorem 4.2.3 (Simpson)**.** *Assume that the base scheme B is of finite type. The moduli functor $\mathcal{M}^{ss}$ is universally corepresented by a scheme $M^{ss} = M^{ss}_{X/B}(\mathcal{O}_X(1), P)$ which is projective over B. There exists an open subscheme $M^s = M^s_{X/B}(\mathcal{O}_X(1), P)$ of M^{ss} which universally corepresents the moduli functor $\mathcal{M}^s$. Moreover:*

1. *The points of M^{ss} over a geometric point $b \in B$ are in bijection with the set of S-equivalence classes of Gieseker semistable sheaves on X_b with Hilbert polynomial P, and the points of M^s over a geometric point $b \in B$ are in bijection with the set of isomorphism classes of Gieseker stable sheaves on X_b with Hilbert polynomial P.*

2. *Locally in the étale topology on M^s, the fibre product $X \times_B M^s$ admits a universal sheaf.*

4.2.2 An Outline of Simpson's Construction

Here we give an outline of how M^{ss} is constructed using GIT, as we will make use of this construction both in Chapter 6 and later on in this chapter, when discussing moduli of unstable sheaves. Proofs for the results stated in this subsection can be found in *loc. cit.*, as well as in Huybrechts–Lehn [69].

Let $f : X \to B$ be a projective morphism, where B is a finite type base scheme, and fix a choice of a relatively ample invertible sheaf $\mathcal{O}_X(1)$ on X. Fix a polynomial $P \in \mathbb{Q}[t]$. A key result required for Simpson's construction is that the collection of all Gieseker semistable coherent sheaves F on X_b with Hilbert polynomial P, taken across all geometric fibres X_b of f, forms a bounded family, in the sense that there exists a scheme B' of finite type over B and a coherent sheaf E over $X_{B'}$, such that each such sheaf F appears as a fibre of E over a geometric fibre of $f_{B'}$.

For a natural number N, let $V_N = \mathbb{C}^{P(N)}$, and let $Q_N = \mathrm{Quot}_{X/B}(V_N \otimes \mathcal{O}_X(-N), P)$ be the relative Grothendieck Quot scheme parametrising quotients of $V_N \otimes \mathcal{O}_X(-N)$ with Hilbert polynomial P. Let

$$V_N \otimes \mathcal{O}_{X_{Q_N}}(-N) \to \mathcal{U}_N \to 0$$

be the universal quotient over $X_{Q_N} = X \times_B Q_N$.

The group $GL(V_N)$ acts naturally on Q_N via composition:[2] for $g \in GL(V_N)$ and $[V_N \otimes \mathcal{O}_{X_b}(-N) \stackrel{q}{\to} F \to 0] \in Q_N$,

$$g \cdot [V_N \otimes \mathcal{O}_{X_b}(-N) \stackrel{q}{\to} F \to 0] := [V_N \otimes \mathcal{O}_{X_b}(-N) \stackrel{q\circ(g^{-1}\otimes \mathrm{id})}{\to} F \to 0].$$

This action factors through the quotient $PGL(V_N)$ of $GL(V_N)$. In addition, the structure morphism $Q_N \to B$ is $GL(V_N)$-invariant.

Denote by R_N the open subscheme of Q_N consisting of all quotients $[V_N \otimes \mathcal{O}_{X_b}(-N) \to F \to 0]$ where F is Gieseker semistable and where the map $V_N \to H^0(X_b, F(N))$, obtained by twisting the given morphism $V_N \otimes \mathcal{O}_{X_b}(-N) \to F$ by $\mathcal{O}_{X_b}(N)$ and taking global sections, is an isomorphism; let R_N^s be the open subscheme of R_N where F is Gieseker stable. The subschemes $R_N^s \subset R_N$ are invariant with respect to the action of the subgroup $G_N := SL(V_N) \subset GL(V_N)$ on Q_N. Let $\overline{R}_N$ denote the closure of R_N in Q_N. Fixing N, for all sufficiently large positive integers $M \gg N$, we have that

[2] Observe that this action factors through the quotient $PGL(V_N)$ of G_N.

$L_{N,M} := \det((f_{Q_N})_*(\mathcal{U}_N(M)))$ is a relatively very ample invertible sheaf on Q_N (and hence on the closed subscheme $\overline{R}_N$), which is naturally linearised with respect to the G_N-action on Q_N (resp. $\overline{R}_N$).

By combining the boundedness of the collection of all semistable sheaves over all geometric fibres X_b with Grothendieck's results on cohomology and base change [57], there exists a positive integer N_0 such that for all $N \geq N_0$, for all B-schemes T and for all T-flat, T-finitely presented coherent sheaves F on X_T whose fibres over geometric points $t \in T$ are Gieseker semistable coherent sheaves with Hilbert polynomial P, the following statements hold:

(i) for each $t \in T$, the sheaf F_t is N-regular: $H^i(X_t, F_t(N-i)) = 0$ for all $i > 0$;

(ii) the higher derived pushforwards $R^i(f_T)_*(F(N))$ for $i > 0$ are all zero;

(iii) the sheaf $(f_T)_*(F(N))$ is locally free of rank $P(N)$ and is compatible with base change; and

(iv) the sheaf $F(N)$ is generated by its global sections, i.e. the natural morphism

$$(f_T)^*(f_T)_*(F(N)) \to F(N) \to 0$$

is surjective.

Taking $T = \operatorname{Spec}\mathbb{C} \to B$ to be a geometric point $b \in B$, we have $(f_T)_*(F(N)) = H^0(X_b, F(N))$; twisting the surjection $(f_T)^*(f_T)_*(F(N)) \to F(N) \to 0$ by $\mathcal{O}_{X_b}(-N)$ then yields a surjection

$$H^0(X_b, F(N)) \otimes \mathcal{O}_{X_b}(-N) \to F \to 0.$$

By *choosing* an isomorphism $V_N = \mathbb{C}^{P(N)} \cong H^0(X_b, F(N))$, we obtain a G_N-orbit of points in $R_N \subset Q_N = \mathrm{Quot}_{X/B}(V_N \otimes \mathcal{O}_X(-N), P)$ which corresponds to the sheaf F. Two sheaves F, F' are isomorphic if and only if they lie in the same G_N-orbit of R_N. Moreover, the scheme R_N, together with the restriction of the universal quotient to the open subscheme $X_{R_N} \subset X_{Q_N}$, has the local universal property for the functor $\mathcal{M}^{ss}$. Invoking Proposition 2.3.3, it follows that a categorical quotient of R_N by G_N corepresents the moduli functor $\mathcal{M}^{ss}$.

Proposition 4.2.4 (cf. [69], Lemma 4.3.1). *If $R_N \to M^{ss}$ (resp. $R_N^s \to M^s$) is a universal categorical quotient for the action of G_N on R_N (resp. R_N^s) then M^{ss} (resp. M^s) universally corepresents the moduli functor $\mathcal{M}^{ss}$ (resp. $\mathcal{M}^s$).* □

The existence of categorical quotients of $R_N^s \subset R_N$ is given by reductive GIT.

Theorem 4.2.5 (Simpson, [124]). *Fix a positive integer $N \geq N_0$, and in turn fix a sufficiently large positive integer M. Then for the action of G_N on $\overline{R}_N$, one has equalities*

$$\overline{R}_N^{ss}(L_{N,M}/B) = R_N, \quad \overline{R}_N^{s}(L_{N,M}/B) = R_N^s.$$

Moreover:

1. *The good quotient $\overline{R}_N /\!\!/_{L_{N,M}} G_N = R_N /\!\!/ G_N$ universally corepresents $\mathcal{M}^{ss}$, and the open subscheme $R_N^s/G_N \subset \overline{R}_N /\!\!/_{L_{N,M}} G_N$ given by the geometric quotient of the (relative) GIT stable locus universally corepresents $\mathcal{M}^s$; in particular, $M^{ss} = \overline{R}_N /\!\!/_{L_{N,M}} G_N$ and $M^s = R_N^s/G_N$.*

2. *The closures of the orbits of points $[V_N \otimes \mathcal{O}_{X_b}(-N) \to F \to 0]$ and $[V_N \otimes \mathcal{O}_{X_{b'}}(-N) \to G \to 0]$ in R_N coincide if and only if $b = b'$ and if F, G are S-equivalent coherent sheaves over X_b.*

3. *The orbit of a point $[V_N \otimes \mathcal{O}_{X_b}(-N) \to F \to 0]$ is GIT polystable if and only if F is a polystable sheaf.*

4.2.3 Moduli Stacks of Semistable Sheaves

Continue to assume that the base scheme B is of finite type. Let $\mathcal{C}oh_{X/B}$ be the algebraic stack parametrising all flat, finitely presented families of coherent sheaves on X/B (cf. [127, Tag 08KA]). There is an open and closed substack $\mathcal{C}oh_{X/B}(\mathcal{O}_X(1), P) \subset \mathcal{C}oh_{X/B}$ parametrising those subsheaves which have Hilbert polynomial P with respect to the choice of relatively ample invertible sheaf $\mathcal{O}_X(1)$. In turn, by the openness of Gieseker (semi)stability in flat families there are open substacks $\mathcal{C}oh^s_{X/B}(\mathcal{O}_X(1), P) \subset \mathcal{C}oh^{ss}_{X/B}(\mathcal{O}_X(1), P) \subset \mathcal{C}oh_{X/B}(\mathcal{O}_X(1), P)$ parametrising those sheaves which are (fibrewise) Gieseker (semi)stable with respect to $\mathcal{O}_X(1)$.

There are natural morphisms $R_N \to \mathcal{C}oh^{ss}_{X/B}(\mathcal{O}_X(1), P)$ and $R_N^s \to \mathcal{C}oh^s_{X/B}(\mathcal{O}_X(1), P)$ obtained by associating to a quotient in R_N the underlying coherent sheaf.

Proposition 4.2.6. *Let R_N and $M^{(s)s}$ be as in the statement of Theorem 4.2.5.*

1. *There are isomorphisms of algebraic stacks*

$$\mathcal{C}oh^{ss}_{X/B}(\mathcal{O}_X(1), P) \cong [R_N/GL(V_N)], \quad \mathcal{C}oh^{s}_{X/B}(\mathcal{O}_X(1), P) \cong [R^s_N/GL(V_N)].$$

2. *The moduli space M^{ss} is a good moduli space for the stack $\mathcal{C}oh^{ss}_{X/B}(\mathcal{O}_X(1), P)$, and M^s is a tame moduli space for $\mathcal{C}oh^{s}_{X/B}(\mathcal{O}_X(1), P)$.*

Proof. By the same arguments given in the proof of [81, Théorème 4.6.2.1], the morphism $R_N \to \mathcal{C}oh^{ss}_{X/B}(\mathcal{O}_X(1), P)$ is a surjective smooth morphism which is a $GL(V_N)$-torsor; this yields an isomorphism $\mathcal{C}oh^{ss}_{X/B}(\mathcal{O}_X(1), P) \cong [R_N/GL(V_N)]$. This isomorphism restricts to give an isomorphism $\mathcal{C}oh^{s}_{X/B}(\mathcal{O}_X(1), P) \cong [R^s_N/GL(V_N)]$.

Since $GL(V_N)$ acts on $\overline{R}_N$ through the quotient $PGL(V_N)$, there are equalities of good quotients

$$R_N /\!\!/ GL(V_N) = R_N /\!\!/ G_N \quad \text{and} \quad R^s_N /\!\!/ GL(V_N) = R^s_N /\!\!/ G_N.$$

As such, the second statement follows from Proposition 2.3.6. □

4.3 Harder–Narasimhan Theory for Coherent Sheaves

4.3.1 Harder–Narasimhan Filtrations and Stratifications

The natural measure of instability for coherent sheaves is recorded via the Harder–Narasimhan filtration. This can be used to define a discrete invariant of an unstable sheaf, the Harder–Narasimhan type.

Proposition/Definition 4.3.1. *Let F be a coherent sheaf on X. Then there exists a unique filtration*

$$0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$$

of F, known as the Harder–Narasimhan filtration *of F, with the property that the subquotients $F_i := F^i/F^{i-1}$ are Gieseker semistable with strictly decreasing reduced Hilbert polynomials $p_i(t) = p(F_i, t)$:*

$$p_1 \succ p_2 \succ \cdots \succ p_\ell.$$

The Harder–Narasimhan associated graded sheaf *of F is the sheaf*

$$\mathrm{gr}^{\mathrm{HN}}(F) := \bigoplus_i F_i.$$

Proof. That the Harder–Narasimhan filtration exists and is unique is proved in [115, Corollary 28]. In the case where F is of pure dimension, a proof can also be found in [69, Section 1.3]. □

Remark. The subsheaf $F^1 \subset F$ is also known as the *maximally destabilising subsheaf* of F.

Definition 4.3.2. *Let F be a coherent sheaf on X with Harder–Narasimhan filtration*

$$0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F.$$

Write $P_i(t)$ for the Hilbert polynomial of F_i. The Harder–Narasimhan type $\tau = \tau_{\mathrm{HN}}(F)$ *of F is the tuple of Hilbert polynomials*

$$\tau = (P_1, \dots, P_\ell) \in \mathbb{Q}[t]^\ell.$$

We also say that F has Harder–Narasimhan length ℓ*.*

Remark. Observe that a sheaf F has Harder–Narasimhan length 1 if and only if F is Gieseker semistable.

In the case where X is projective over a base scheme B, the relative analogue of a Harder–Narasimhan filtration is defined as follows.

Definition 4.3.3. *Let $f : X \to B$ be a projective morphism, where B is a locally Noetherian scheme, and let $\mathcal{O}_X(1)$ be a relatively ample invertible sheaf on X. Let F be an B-flat coherent sheaf on X, such that for each geometric point $b \in B$, the sheaf F_b has Harder–Narasimhan type $\tau = (P_1, \dots, P_\ell)$.*

A relative Harder–Narasimhan filtration of F *is a filtration*

$$0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F \tag{4.3.1}$$

by coherent subsheaves of F, such that the following properties hold:

1. *For each $i = 1, \dots, \ell$, the subquotient $F_i = F^i/F^{i-1}$ is flat over B.*
2. *For each geometric point $b \in B$, the restriction of the filtration* (4.3.1) *is the Harder–Narasimhan filtration of F_b.*

Extending work of Shatz [123], there is the following result of Nitsure [104] (see also [65, Theorem 5.15]) concerning the existence of stratifications by Harder–Narasimhan type.[3]

Proposition 4.3.4 (Shatz, Nitsure). *Let $f : X \to B$ be a projective morphism, where B is a locally Noetherian scheme, and let $\mathcal{O}_X(1)$ be a relatively ample invertible sheaf on X. Let F be a B-flat coherent sheaf on X with Hilbert polynomial P. Let*

$$\mathrm{HNT}_P := \left\{ (P_1, \dots, P_\ell) : \ell \geq 1,\ \sum_{i=1}^{\ell} P_i = P,\ p_1 \succ p_2 \succ \cdots \succ p_\ell \right\} \subset \varinjlim_{\ell} \mathbb{Q}[t]^\ell$$

denote the set of Harder–Narasimhan types of coherent sheaves on X with Hilbert polynomial P (here p_i is the monic polynomial obtained by dividing P_i by its leading coefficient).

1. *There exists a partial ordering $\leq$ on HNT_P such that the map $|B| \to \mathrm{HNT}_P$ defined by $b \mapsto \tau_{\mathrm{HN}}(F_b)$ is upper-semicontinuous. In particular, each level set*

   $$B^\tau(F) = \{ b \in B : \tau_{\mathrm{HN}}(F_b) = \tau \}$$

   is a locally closed subset of the topological space $|B|$.

2. *There exists a unique scheme structure on the topological space $B^\tau(F)$, making $B^\tau(F)$ into a locally closed subscheme of B, such that $B^\tau(F)$ has the following universal property: any morphism $T \to B$ of schemes factors through $B^\tau(F)$ if and only if the pullback F_T admits a relative Harder–Narasimhan filtration of Harder–Narasimhan type τ.*

3. *A relative Harder–Narasimhan filtration of F, if one exists, is always unique.*

4. *For any morphism $T \to B$ of locally Noetherian schemes, there is a Cartesian diagram of schemes*

   $$\begin{array}{ccc} T^\tau(F_T) & \longrightarrow & B^\tau(F) \\ \big\downarrow & \lrcorner & \big\downarrow \\ T & \longrightarrow & B \end{array}$$

[3] Nitsure only considers sheaves of pure dimensions, but with minor modifications his arguments carry over to the case of considering sheaves which are not necessarily pure.

Definition 4.3.5. *The stratification*

$$B = \bigsqcup_{\tau \in \mathrm{HNT}_P} B^\tau(F)$$

of B *into the locally closed subschemes* $B^\tau(F)$ *is known as the* Harder–Narasimhan stratification *of* B *with respect to* F*.*

4.3.2 Relating HN and HKKN Stratifications

Let $f : X \to B$ be a projective morphism, where B is a finite type base scheme, and fix a choice of a relatively ample invertible sheaf $\mathcal{O}_X(1)$ on X. Fix a Hilbert polynomial P. We refer back to the notation of Subsection 4.2.2.

Associated to the universal sheaf $\mathcal{U}_N$ on X_{Q_N} is the Harder–Narasimhan stratification

$$Q_N = \bigsqcup_{\tau \in \mathrm{HNT}_P} Q_N^\tau, \quad Q_N^\tau := Q_N^\tau(\mathcal{U}_N).$$

Fix a Harder–Narasimhan type $\tau = (P_1, \ldots, P_\ell) \in \mathrm{HNT}_P$. Inside the Harder–Narasimhan stratum Q_N^τ, there is an open subscheme S_N^τ parametrising all quotients $V_N \otimes \mathcal{O}_{X_b}(-N) \to F \to 0$ for which the induced map $V_N \overset{\simeq}{\to} H^0(X_b, F(N))$ is an isomorphism.

Choose a basis for the vector space $V_N \cong \mathbb{C}^{P(N)}$. For a pair of natural numbers $(N, M) \in \mathbb{N}^2$, denote by $\beta_{N,M}(\tau)$ the (conjugacy class of) the rational 1PS of $G_N = SL(V_N) \cong SL(P(N), \mathbb{C})$ given by

$$\lambda_{\beta_{N,M}(\tau)}(t) = \mathrm{diag}(t^{\beta_1}\mathrm{id}_{P_1(N)}, \ldots, t^{\beta_\ell}\mathrm{id}_{P_\ell(N)}), \quad \beta_i := \frac{P(M)}{P(N)} - \frac{P_i(M)}{P_i(N)}.$$

Let $P_\tau = U_\tau \rtimes R_\tau \subset G_N$ be the parabolic subgroup corresponding to $\lambda_{\beta_{N,M}(\tau)}(t)$. The 1PS $\lambda_{\beta_{N,M}(\tau)}(t)$ of G_N induces a filtration

$$0 = V_N^0 \subset V_N^1 \subset \cdots \subset V_N^\ell = V_N$$

of V_N. Given an element $[V_N \otimes \mathcal{O}_{X_b}(-N) \overset{q}{\to} E \to 0]$ of Q_N, set $E^i = q(V_N^i) \subset E$ and $E_i = E^i/E^{i-1}$, We then have induced quotients

$$(V_N^i/V_N^{i-1}) \otimes \mathcal{O}_{X_b}(-N) \to E_i \to 0.$$

By [69, Lemma 4.3] the limit $\lim_{t\to 0} \lambda_{\beta_{N,M}(\tau)}(t) \cdot [V_N \otimes \mathcal{O}_{X_b}(-N) \overset{q}{\to} E \to 0]$ is given by the direct sum of quotients

$$\left[\bigoplus_{i=1}^{\ell} \left((V_N^i/V_N^{i-1}) \otimes \mathcal{O}_{X_b}(-N) \to E_i \to 0\right)\right] \in Q_N.$$

By considering the possible Hilbert polynomials of the sheaves E_i, there is an isomorphism of schemes

$$Q_N^{\lambda_{\beta_{N,M}(\tau)}(\mathbb{G}_m)} \cong \bigsqcup_{\substack{(P'_1,\dots,P'_\ell)\in\mathbb{Q}[t]^\ell \\ \sum_i P'_i=P}} \prod_{i=1}^{\ell} \mathrm{Quot}_{X/B}((V_N^i/V_N^{i-1})\otimes \mathcal{O}_X(-N), P'_i).$$

Let Z_τ^{ss} be the open subscheme of $Q_N^{\lambda_{\beta_{N,M}(\tau)}(\mathbb{G}_m)}$ corresponding under the above isomorphism to the open subscheme of $\prod_{i=1}^{\ell}\mathrm{Quot}_{X/B}((V_N^i/V_N^{i-1})\otimes\mathcal{O}_X(-N),P_i)$ where each of the coherent sheaves E_i is Gieseker semistable and where each of the induced maps $V_N^i/V_N^{i-1}\overset{\simeq}{\to} H^0(X_b,E_i(N))$ is an isomorphism. The following result of Hoskins–Kirwan [67] and Hoskins [65] relates the Harder–Narasimhan and HKKN stratifications of Q_N (see also [66, Theorem 2.8]).

Proposition 4.3.6 (Hoskins–Kirwan, Hoskins)**.** *For $M \gg N \gg 0$, the following statements hold:*

1. *For any coherent sheaf F over any geometric fibre X_b with $\tau_{\mathrm{HN}}(F)=\tau$, F is N-regular.*

2. *The G_N-orbits of closed points in S_N^τ are in bijection with isomorphism classes of coherent sheaves F over geometric fibres X_b such that F has Harder–Narasimhan type τ.*

Moreover, when $B=\mathrm{Spec}\,\mathbb{C}$ is a point, the following additional statements hold:

1. *$\beta := [\beta_{N,M}(\tau)]$ is an HKKN index for the G_N-action on $Q_N=\mathrm{Quot}_X(V_N\otimes\mathcal{O}_X(-N),P)$ with respect to $L_{N,M}$, and S_N^τ is a closed subscheme of the corresponding HKKN stratum $S_\beta\subset Q_N$.*

2. *There exists a R_τ-equivariant closed immersion $Z_\tau^{ss}\subset Z_\beta^{ss}$ such that if $Y_\tau^{ss} := p_\beta^{-1}(Z_\tau^{ss})\subset Y_\beta$ is the corresponding attractor, then*

$$S_N^\tau = G_N\cdot Y_\tau^{ss}\cong G_N\times^{P_\tau} Y_\tau^{ss}.$$

3. *The closed points of Y_τ^{ss} are given by quotients $V_N\otimes\mathcal{O}_X(-N)\to F\to 0$ such that the induced map $V_N\overset{\simeq}{\to}H^0(X,F(N))$ is an isomorphism, F has Harder–Narasimhan type τ, and the filtration*

$$0=V_N^0\subset V_N^1\subset\cdots\subset V_N^\ell=V_N$$

of $V_N = \mathbb{C}^{P(N)}$ induced by the 1PS $\lambda_{\beta_{N,M}(\tau)}$ induces the Harder–Narasimhan filtration on F. The retraction $p_\beta : Y_\tau^{ss} \to Z_\tau^{ss}$ sends a quotient $V_N \otimes \mathcal{O}_X(-N) \to F \to 0$ to the tuple whose ith entry is the quotient $(V_N^i/V_N^{i-1}) \otimes \mathcal{O}_X(-N) \to F_i \to 0$, where $F_i = F^i/F^{i-1}$ is the ith subquotient appearing in the Harder–Narasimhan filtration of F.

4. *Let $\tilde{P}_\tau$ be the parabolic subgroup of $GL(V_N)$ associated to the 1PS $\lambda_{\beta_{N,M}(\tau)}$. Then there are isomorphisms of algebraic stacks*

$$\mathcal{C}oh_X^\tau \cong [S_N^\tau/GL(V_N)] \cong [Y_\tau^{ss}/\tilde{P}_\tau],$$

where $\mathcal{C}oh_X^\tau = \mathcal{C}oh_X^\tau(\mathcal{O}_X(1), P)$ is the algebraic stack parametrising all flat, finitely presented families of coherent sheaves on X admitting a relative Harder–Narasimhan filtration of type τ (cf. [104, Theorem 8]).

4.4 Multi-Gieseker Stability

We conclude this chapter by reviewing the notion of *multi-Gieseker* stability, due to Greb–Ross–Toma [55], since we will refer to this notion in Chapter 6.

Let X be a projective scheme. Fix a *stability parameter* $\sigma = (\underline{L}, \sigma_1, \ldots, \sigma_k)$ on X; here $\underline{L}$ is a tuple of ample invertible sheaves $L_1, \ldots, L_k$, and the σ_i are non-negative rational numbers, not all zero. Given a coherent sheaf E on X of with $d = \dim \operatorname{supp} E$, the *multi-Hilbert polynomial* of E with respect to σ is the polynomial

$$P^\sigma(E,t) = \sum_{j=1}^{k} \sigma_j \chi(X, E \otimes L_j^t) = \sum_{i=0}^{d} \alpha_i^\sigma(E) \frac{t^i}{i!},$$

and the *reduced multi-Hilbert polynomial* of E is

$$p^\sigma(E,t) = \frac{P^\sigma(E,t)}{\alpha_d^\sigma(E)}.$$

Definition 4.4.1. *A coherent sheaf E on X is said to be* multi-Gieseker (semi)stable[4] *with respect to σ if for all non-zero proper subsheaves $F \subset E$,*

$$p^\sigma(F,t) \prec (\preceq)\ p^\sigma(E,t).$$

[4] Where no confusion is likely to arise, we also refer to this notion as *σ-(semi)stability.*

In the case where only one σ_i is non-zero, we recover the usual notion of Gieseker (semi)stability with respect to the ample invertible sheaf L_i. The notions of Harder–Narasimhan filtrations, Jordan–Hölder filtrations and S-equivalence carry over to the multi-Gieseker setting, and multi-Gieseker (semi)stability is an open property in flat families (cf. [55, Section 2]). If E is σ-semistable, the Jordan–Hölder associated graded sheaf is denoted $\mathrm{gr}_\sigma^{\mathrm{JH}}(E)$.

Following *loc. cit.*,[5] we make the following definitions.

Definition 4.4.2. *The coherent sheaf E is said to be* $(m, \underline{L})$-regular *if E is m-regular with respect to each L_j, that is for each $i > 0$ we have $H^i(E \otimes L_j^{m-i}) = 0$. More generally, if $\pi : X \to S$ is a projective morphism of schemes, if $\underline{\mathcal{L}} = (\mathcal{L}_1, \dots, \mathcal{L}_k)$ is a tuple of π-ample sheaves and if E is an S-flat coherent sheaf on X, we say that E is* $(m, \underline{\mathcal{L}})$-regular *if for all $j = 1, \dots, k$ and for all $i > 0$, one has $R^i\pi_*(E \otimes \mathcal{L}_j^{m-i}) = 0$.*

Definition 4.4.3. *Given a coherent sheaf E over X, the* topological type $\tau = \tau_{\mathrm{top}}(E)$ *of E (with respect to $\underline{L}$) is the tuple $(P_1(t), \dots, P_k(t))$, where $P_j(t)$ is the Hilbert polynomial $\chi(C, E \otimes L_j^t)$ of E with respect to L_j. In the case where X is projective over a base scheme B and we are given relatively ample invertible sheaves $\underline{\mathcal{L}} = (\mathcal{L}_1, \dots, \mathcal{L}_k)$ on X, we extend the notion of topological type to B-flat sheaves on X in the obvious way.*

Definition 4.4.4. *Let τ be a topological type of sheaves on X defined with respect to ample invertible sheaves $L_1, \dots, L_k$. The stability parameter $\sigma = (\underline{L}; \sigma_1, \dots, \sigma_k)$ is said to be*

1. positive *if each $\sigma_i > 0$;*
2. degenerate *if there exists some index i with $\sigma_i = 0$; and*
3. bounded (with respect to τ) *if the collection of all σ-semistable sheaves on X of topological type τ forms a bounded family.*

Remark. If E is a B-flat, $(m, \mathcal{L})$-regular sheaf over a projective B-scheme $\pi : X \to B$ and if B is locally Noetherian, then for all $m' \geq m$ and for all $j = 1, \dots, k$ the sheaves $\pi_*(E \otimes \mathcal{L}_j^{m'})$ are locally free; this follows from the cohomology and base change theorems of Grothendieck [57] as well as basic properties of Castelnuovo–Mumford regularity.

[5] The definition of topological type stated above is in fact slightly different to the notion of topological type in [55] and [56]; in these papers the topological type of a coherent sheaf is defined to be the homological Todd class $\tau_{\mathrm{Todd}}(E) \in B(X)_{\mathbb{Q}}$ of E (though see [55] Remark 1.5).

As with ordinary Gieseker stability, there exist projective moduli spaces of multi-Gieseker semistable sheaves on the projective variety X. Fix a stability parameter $\sigma = (\underline{L}; \sigma_1, \ldots, \sigma_k)$ and a topological type τ defined with respect to $\underline{L}$, and assume that σ is positive and bounded with respect to τ. Define a moduli functor

$$\mathcal{M}_{\sigma,\tau} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$$

by associating to a scheme T the set of all isomorphism classes of families of σ-semistable sheaves on X of topological type τ, parametrised by T.

Theorem 4.4.5 (Greb–Ross–Toma, [55])**.** *The moduli functor $\mathcal{M}_{\sigma,\tau}$ is universally corepresented by a projective scheme $M_{\sigma,\tau}$. Moreover, the points of $M_{\sigma,\tau}$ are in bijection with the set of S-equivalence classes of σ-semistable coherent sheaves on X of topological type τ.*

The GIT construction of the moduli space $M_{\sigma,\tau}$ is presented in Chapter 6, together with an extension of the construction to the relative setting.

Chapter 5

Compactified Jacobians of Reduced Curves

In this chapter, the final preliminary chapter of this thesis, we provide a review of the aspects of the theory of compactified Jacobians of a reduced connected projective curve C required for Chapters 6 and 8. Compactified Jacobians are moduli spaces of coherent sheaves on C, closely related to the moduli spaces considered in Chapter 4. As well as presenting existing results concerning the existence of compactified Jacobians and some of their basic properties, we also prove an Ext^2-vanishing result (Lemma 5.1.9) for torsion-free sheaves on reduced curves with locally planar singularities, which is needed for the proof of the main result of Chapter 8.

A good survey of the theory of fine compactified Jacobians of a reduced projective curve can be found in the paper [91] of Melo–Rapagnetta–Viviani.

5.1 Rank 1 Sheaves on Singular Curves

5.1.1 Jacobians of Singular Curves

Let C be a connected smooth projective curve. It is a standard result (see for instance [94]) that the set of isomorphism classes of degree 0 invertible sheaves, endowed with the tensor product, is a complete principally polarised abelian variety $\mathrm{Pic}^0(C)$, the *Picard variety* or the *Jacobian* of C. Moreover, $C \times \mathrm{Pic}^0(C)$ carries a universal invertible sheaf, also known as a *Poincaré sheaf*, making $\mathrm{Pic}^0(C)$ a fine moduli space for the moduli problem of classifying degree 0 invertible sheaves on C up to isomorphism. For each integer d, there is a complete fine moduli space $\mathrm{Pic}^d(C)$ known as the *degree d Picard* *variety* of C, which parametrises isomorphism classes of degree d invertible sheaves on C; $\mathrm{Pic}^d(C)$ is a torsor for the Jacobian $\mathrm{Pic}^0(C)$.

If C is an integral projective curve, no longer assumed to be smooth, the set $\mathrm{Pic}^d(C)$ of isomorphism classes of degree d invertible sheaves admits a natural scheme structure, however the scheme $\mathrm{Pic}^d(C)$ is no longer necessarily complete.[1] More generally, in the case where C is a reduced, connected projective curve which is reducible, the space $\mathrm{Pic}^{\underline{d}}(C)$ of isomorphism classes of invertible sheaves of multidegree $\underline{d}$ is a scheme, once again not necessarily complete. The theory of compactified Jacobians concerns the problem of compactifying $\mathrm{Pic}^{\underline{d}}(C)$ in such a way that the compactification is itself a moduli space of coherent sheaves on C.

5.1.2 Torsion-Free Sheaves on Curves

Compactified Jacobians are obtained by widening the class of sheaves being parametrised to include torsion-free sheaves of uniform rank 1.

Proposition 5.1.1. *Let F be a non-zero coherent sheaf on a connected reduced projective curve C. Then the following are equivalent:*

1. *F is a torsion-free sheaf, in the sense that for each $p \in C$, no regular element of $\mathfrak{m}_{C,p}$ is a zero divisor of F_p.*
2. *The sheaf F is* pure of dimension 1*, i.e. for all non-zero subsheaves $G \subset F$,* $\dim \operatorname{supp} G = 1$*.*
3. *F* satisfies the Serre condition S_1*; that is, for each closed point $p \in C$ we have* $\mathrm{depth}_{\mathcal{O}_{C,p}}(F_p) = 1$*.*

Though Proposition 5.1.1 is a standard result, we provide a short proof for lack of a suitable reference. This proof requires the following lemma.

Lemma/Definition 5.1.2 ([122], Septième Partie, Lemme 3)**.** *Let F be a coherent sheaf on a connected reduced projective curve C. Then there exists a unique subsheaf $T \subset F$, concentrated at a finite number of points of C, such that F/T is torsion-free. If $G \subset F$ is any subsheaf with* $\dim \operatorname{supp} G = 0$ *then $G \subset T$. T is known as the* torsion subsheaf *of F.*

---

[1] For example, if C is the nodal cubic then $\mathrm{Pic}^0(C) \cong \mathbb{G}_m$; this isomorphism arises from the choice of an isomorphism of $\mathbb{C}$-vector spaces $\mathcal{O}_{\mathbb{P}^1}|_0 \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}|_\infty$.

Proof of Proposition 5.1.1. (1) $\iff$ (2): By Lemma/Definition 5.1.2, the sheaf F is torsion-free if and only if the torsion subsheaf T of F is zero, if and only if F has no non-zero subsheaves $G \subset F$ with $\dim \operatorname{supp} G = 0$, if and only if F is pure of dimension 1.

(2) $\iff$ (3): Take a closed point $p \in C$. Since $\dim \mathcal{O}_{C,p} = 1$, the depth of F_p as an $\mathcal{O}_{C,p}$-module can either be 0 or 1. We have

$$\operatorname{depth}_{\mathcal{O}_{C,p}}(F_p) = 0 \iff \operatorname{Hom}_{\mathcal{O}_{C,p}}(\kappa(p), F_p) \neq 0 \iff p \in \operatorname{Ass}(F).$$

On the other hand, F is of pure dimension 1 if and only if the only associated points of F are of dimension 1. □

Remark. From now on, we refer to any coherent sheaf on C satisfying the equivalent statements of Proposition 5.1.1 as being *torsion-free*.

Remark. Over the smooth locus of C, a torsion-free sheaf is locally free; this follows from the fact that any finite torsion-free module over a discrete valuation ring (or more generally a PID) is free. Conversely, a locally free sheaf on C is torsion-free.

Definition 5.1.3. *Let F be a torsion-free sheaf on a connected reduced projective curve C, and let $C_1, \ldots, C_k$ denote the irreducible components of C. Let ξ_i be the generic point of C_i. The* multirank *of F is the tuple $(r_1, \ldots, r_k)$, where $r_i = r_i(F) = \operatorname{rank}_{\mathcal{O}_{C,\xi_i}} F_{\xi_i}$ is the rank of the locally free $\mathcal{O}_{C,\xi_i}$-module F_{ξ_i}. We say that F has* uniform rank r *if the multirank of F is $(r, \ldots, r)$.*

Definition 5.1.4. *Suppose $D \subset C$ is a (not necessarily connected) subcurve of C and F is a torsion-free sheaf on C. We denote by F_D the maximal torsion-free quotient of $F|_D = F \otimes_{\mathcal{O}_C} \mathcal{O}_D$.*

If F is of uniform rank 1 then the sheaf F_D may be understood as the unique quotient of F which has support D and is torsion-free along D.

Lemma 5.1.5. *Let F be a torsion-free sheaf on a connected reduced projective curve C of uniform rank 1. Then the non-zero torsion-free quotients of F are precisely the sheaves F_D, for $D \subset C$ a subcurve of C.*

Proof. Let $F \to G \to 0$ be a torsion-free quotient of F, and let D be the support of G. By restricting to D, we have a quotient $F|_D \to G \to 0$. Since G is torsion-free, this map must factor through the quotient F_D of $F|_D$, yielding a quotient $F_D \to G \to 0$. At any generic point ξ of D, the induced map $F_\xi \to G_\xi$ on stalks is an isomorphism, since a surjection of invertible sheaves is an isomorphism. It follows that the kernel K of $F_D \to G$ cannot be supported along a dimension 1 subscheme of D; since F_D is torsion-free, this forces $K = 0$, whence the map $F_D \to G$ is an isomorphism. $\square$

Definition 5.1.6. *Let F be a torsion-free coherent sheaf on C of uniform rank r.*

1. *The* degree *of F is defined by* $\deg F := \chi(F) - r\chi(\mathcal{O}_C)$.
2. *More generally, if $D \subset C$ is a subcurve, we define* $\deg_D F := \chi(F_D) - r\chi(\mathcal{O}_D)$.
3. *If $C_1, \dots, C_k$ are the irreducible components of C, the* multidegree *of F is the tuple $(d_1, \dots, d_k)$, where $d_i = d_i(F) = \deg_{C_i} F$.*

We collect some more results on torsion-free sheaves on a connected reduced projective curve C that we will need later on in this thesis.

Lemma 5.1.7 ([48], Proposition 1 and Lemma 2)**.** *Let F be a torsion-free uniform rank 1 sheaf on C.*

1. *F is simple if and only if there do not exist subcurves $D, D' \subset C$ such that D and D' share no common irreducible components, $D \cup D' = C$ and such that the canonical injection $F \to F_D \oplus F_{D'}$ is an isomorphism.*
2. *Suppose D, D' are non-empty subcurves with $D \cup D' = C$, which share at least one common irreducible component. If F_D and $F_{D'}$ are both simple, then F is simple.*
3. *Suppose F fits into an exact sequence*

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0,$$

where F' and F'' are simple torsion-free sheaves, of uniform rank 1 along their respective supports D and D' with $C = D \cup D'$. Then F is simple if and only if the sequence is not split.

Lemma 5.1.8. *Suppose D, D' are subcurves of C with D, D' sharing no common irreducible components. Let F, F' be coherent sheaves supported on D, D' respectively, with F' torsion-free. Then* $\operatorname{Hom}_C(F, F') = 0$.

Proof. It suffices to show that $\operatorname{Hom}_{\mathcal{O}_{C,p}}(F_p, F'_p) = 0$ for all $p \in D \cap D'$. Fixing p, any homomorphism $F_p \to F'_p$ of $\mathcal{O}_{C,p}$-modules must factor through $M = F_p/\mathfrak{m}_{D,p}F_p$, since $\mathfrak{m}_{D,p} \cdot F'_p = 0$. But $\mathfrak{m}_{C,p} \cdot M = 0$, so $\operatorname{supp}(M) \subset \{\mathfrak{m}_{C,p}\}$ is 0-dimensional or empty. As F'_p has depth 1 as an $\mathcal{O}_{C,p}$-module, this implies $\operatorname{Hom}_{\mathcal{O}_{C,p}}(M, F'_p) = 0$. This proves the lemma. □

Lemma 5.1.9. *Let D, D' be subcurves of C such that D and D' share no common irreducible components. Let F, F' be torsion-free sheaves supported along D, D' respectively. Assume further that for each $p \in D \cap D'$, the point $p \in C$ is a locally planar singularity. Then* $\operatorname{Ext}^2_C(F, F') = 0$.

Proof. We have $H^p(C, \mathcal{E}xt^q_C(F, F')) = 0$ for all $p \geq 2$ and $q \geq 0$ as $\dim C = 1$. We also have $H^1(C, \mathcal{E}xt^1_C(F, F')) = 0$; the latter cohomology group vanishes since $\mathcal{E}xt^1_C(F, F')$ is supported along the 0-dimensional scheme $D \cap D'$. It follows from the local-to-global Ext spectral sequence

$$H^p(C, \mathcal{E}xt^q_C(F, F')) \Rightarrow \operatorname{Ext}^{p+q}_C(F, F')$$

that $\operatorname{Ext}^2_C(F, F') = H^0(C, \mathcal{E}xt^2_C(F, F'))$. It thus suffices to show that

$$\mathcal{E}xt^2_C(F, F')_p = \operatorname{Ext}^2_{\mathcal{O}_{C,p}}(F_p, F'_p) = 0$$

for all $p \in D \cap D'$. Since this is a local problem, without loss of generality we may assume that $D \cap D'$ consists of a single singularity p. Let $R = \mathcal{O}_{C,p}$, $M = F_p$ and $N = F'_p$. We begin by working on $\operatorname{Ext}^2_R(M, N)$.

Claim 1: There exists a Grothendieck spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_R(\operatorname{Tor}^R_q(M, \operatorname{Hom}_R(N, R)), \omega) \Rightarrow \operatorname{Ext}^{p+q}_R(M, N).$$

To prove the claim, we apply [114, Theorem 10.62] with $A = M$, $B = \operatorname{Hom}_R(N, R)$ and $C = R$, noting that as R is Gorenstein and N is maximal Cohen–Macaulay (the latter is true since both the dimension and depth of N are equal to $1 = \dim R$) then there is a natural isomorphism $N \cong \operatorname{Hom}_R(B, R)$ (cf. [26, Theorem 3.3.10]). In order to apply this result, we require that for any finite projective (= free, since R is local) R-module P, $\mathrm{Ext}^i_R(B \otimes_R P, R) = 0$ for all $i > 0$. But $B \otimes_R P$ is maximal Cohen–Macaulay, so this indeed is the case by Theorem 3.3.10 of *loc. cit.* As such we may apply [114, Theorem 10.62], which proves Claim 1.

Set $N' := \mathrm{Hom}_R(N, R)$. Each $\mathrm{Tor}^R_q(M, \mathrm{Hom}_R(N, R))$ is a torsion R-module, since it is supported only at the point p, so $E_2^{0,q} = 0$ for all $q \geq 0$. We also have $E_2^{p,q} = 0$ for all $p \geq 2$, since ω has injective dimension equal to $\dim R = 1$. As such, this spectral sequence collapses, yielding an isomorphism

$$\mathrm{Ext}^2_R(M, N) \cong \mathrm{Ext}^1_R(\mathrm{Tor}^R_1(M, N'), R).$$

Claim 2: The R-module $\mathrm{Tor}^R_1(M, N')$ is trivial.

By a result of Altman–Kleiman [78], an open neighbourhood of $p \in C$ admits a closed immersion into an open neighbourhood of a smooth surface Y. In particular, there exists a finite local ring homomorphism $S := \mathcal{O}_{Y,p} \to R = \mathcal{O}_{C,p}$ from a Noetherian regular local ring S of dimension 2. By [24, Section 2, Proposition 8] the S-modules M and N' are Cohen–Macaulay. It then follows from [121, Chapter V.B, Theorem 4] that $\mathrm{Tor}^S_1(M, N') = 0$. From the change-of-rings spectral sequence

$$\mathrm{Tor}^R_p(\mathrm{Tor}^S_q(M, R), N') \Rightarrow \mathrm{Tor}^S_{p+q}(M, N')$$

(cf. [114, Theorem 10.71]), the module $\mathrm{Tor}^S_1(M, N')$ surjects onto $\mathrm{Tor}^R_1(M, N')$, whence $\mathrm{Tor}^R_1(M, N') = 0$. Consequently $\mathrm{Ext}^2_R(M, N) = \mathrm{Ext}^2_{\mathcal{O}_{C,p}}(F_p, F'_p) = 0$, which finishes the proof of the lemma. □

5.1.3 Rank 1 Torsion-Free Sheaves on Nodal Curves

The case of greatest interest in the literature is when C is a *nodal* curve and when the sheaves are torsion-free of uniform rank 1. The local structure of a torsion-free sheaf over a node admits a simple description.

Proposition 5.1.10 ([122], Huitième Partie, Propositions 2 and 3)**.** *Let F be a torsion-free sheaf over a connected nodal curve C, and let $p \in C$ be a node.*

1. *If p lies on a unique irreducible component of C, then there are unique integers a, b such that*

$$F_p \cong \mathcal{O}_{C,p}^{\oplus a} \oplus \mathfrak{m}_{C,p}^{\oplus b}.$$

The rank of F_p along this component is equal to $a + b$.

2. *If p lies on two irreducible components C_1 and C_2 of C, then there are unique integers a, b, c such that*

$$F_p \cong \mathcal{O}_{C,p}^{\oplus a} \oplus \mathcal{O}_{C_1,p}^{\oplus b} \oplus \mathcal{O}_{C_2,p}^{\oplus c}.$$

The multirank of F_p along the components C_1, C_2 is equal to $(a+b, a+c)$.

Let C be a connected nodal curve, and fix a subset S of the set of nodes of C. From the pair (C, S), we obtain the following nodal curves:

(i) $\widetilde{C}_S$ denotes the *partial normalisation* of C along S, the curve obtained by normalising only those nodes of C which belong to S.

(ii) $\widehat{C}_S$ denotes the *partial blowup*[2] of C along S; this is the curve obtained from C_S by inserting a copy E_p of $\mathbb{P}^1$ for each node $p \in S$, with E_p attached to the rest of $\widehat{C}_S$ at the two points of $\widetilde{C}_S$ lying over p.

The curves $C, \widetilde{C}_S$ and $\widehat{C}_S$ fit into the following diagram:

$$\begin{array}{ccc} \widetilde{C}_S & \xhookrightarrow{\iota_S} & \widehat{C}_S \\ & {\scriptstyle \nu_S} \searrow \quad \swarrow {\scriptstyle \pi_S} & \\ & C & \end{array} \tag{5.1.1}$$

Here ν_S is the partial normalisation map, ι_S is the evident inclusion and π_S is the extension of ν_S obtained by mapping each exceptional component E_p onto the corresponding node $p \in S$.

Using the diagram (5.1.1), it is possible to describe torsion-free sheaves of uniform rank 1 on C in terms of invertible sheaves on the partial normalisations and the partial blowups of C.

Proposition 5.1.11 (cf. [92], Proposition 1.14)**.** *Let S be a subset of the set of nodes of C. Let $\mathrm{TF}_S(C)$ be the set of all isomorphism classes of torsion-free sheaves F of uniform rank 1 on C with $\{p \in C : F_p \text{ is not free}\} = S$. Let $\mathrm{Pic}(\widehat{C}_S)'$ denote the subset of $\mathrm{Pic}(\widehat{C}_S)$ consisting of all invertible sheaves whose restriction to each exceptional component of $\widehat{C}_S$ has degree -1. The diagram* (5.1.1) *induces a commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}(\widetilde{C}_S) & \xtwoheadleftarrow{\iota_S^*} & \mathrm{Pic}(\widehat{C}_S)' \\ & {\scriptstyle (\nu_S)_*,\ \cong} \searrow \quad \swarrow\!\!\swarrow {\scriptstyle (\pi_S)_*} & \\ & \mathrm{TF}_S(C) & \end{array}$$

[2]This is not a blowup in the usual sense, however the terminology is by now commonplace in the literature.

Moreover ι_S^* and $(\pi_S)_*$ are surjective, and $(\nu_S)_*$ is a bijection; an inverse is given by sending $F \in \mathrm{TF}_S(C)$ to the quotient of $\nu_S^* F$ by its torsion subsheaf. For any subcurve $D \subset C$ and any $L \in \mathrm{Pic}(\widetilde{C}_S)$, we have

$$\deg_D(\nu_S)_* L = \deg_{\widetilde{D}_S} L + |S \cap (D \setminus D^c)|,$$

where $\widetilde{D}_S = \nu_S^{-1}(D) \subset \widetilde{C}_S$ is the partial normalisation of D along $S \cap D$ and $D^c = \overline{C \setminus D}$.

Proof. As stated, the above consists of the first and third statements of [92, Proposition 1.14]; in turn, the first statement of *loc. cit.* follows from [5, Lemmas 1.5 and 1.9]. □

5.2 Stability Conditions for Torsion-Free Uniform Rank 1 Sheaves

5.2.1 Non-Separatedness of the Moduli of Torsion-Free Sheaves

Let C be a reduced, connected, projective curve, with irreducible components $C_1, \ldots, C_k$. If the curve C is reducible, the moduli stack of all torsion-free sheaves on C of uniform rank 1 fails to be separated.

Example 5.2.1. Let C be a curve with subcurves D_1 and D_2 with $C = D_1 \cup D_2$ and with D_1, D_2 sharing no common irreducible components. Let L be an invertible sheaf on C. Let $F = L_{D_1}$ and $G = \ker(L \twoheadrightarrow F)$; then $F \oplus G$ is a torsion-free sheaf on C of uniform rank 1, not isomorphic to L.

Let $\mathcal{L} = L \otimes_{\mathbb{C}} \mathbb{C}[t]$ be the invertible sheaf on $C \times \mathbb{A}^1$ with constant fibre L, and let $\mathcal{E}$ be the coherent sheaf on $C \times \mathbb{A}^1$ given by

$$\mathcal{E} = G \oplus \bigoplus_{i \geq 1} t^i \cdot L \subset \mathcal{L}.$$

Both $\mathcal{L}$ and $\mathcal{E}$ are flat over $\mathbb{A}^1$, since each of their associated points maps to the generic point of $\mathbb{A}^1$, and both $\mathcal{L}$ and $\mathcal{E}$ restrict to $L \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ away from the origin $0 \in \mathbb{A}^1$. However, the fibre of $\mathcal{E}$ over 0 is given by

$$\mathcal{E}_0 = \frac{\mathcal{E}}{t\mathcal{E}} = \frac{G \oplus t \cdot L \oplus t^2 \cdot L \oplus \cdots}{0 \oplus t \cdot G \oplus t^2 \cdot L \oplus \cdots} \cong F \oplus G \ncong L = \mathcal{L}_0.$$

This exhibits non-separatedness of the moduli stack. ■

5.2.2 Polarisations on Reducible Curves

As with the case of forming schematic moduli spaces of coherent sheaves on projective schemes, the solution to the lack of separatedness is to choose a stability condition and restrict attention to those sheaves which satisfy the chosen stability condition.

Definition 5.2.2. *A* polarisation *on C is a pair $\nu = (\alpha, \phi)$ of ordered tuples of integers $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\phi = (\phi_1, \ldots, \phi_k)$, where each $\alpha_i > 0$.*

Given a polarisation $\nu = (\alpha, \phi)$ on C and a subcurve $D \subset C$, we denote $\alpha_D = \sum_{C_i \subset D} \alpha_i$ and $\phi_D = \sum_{C_i \subset D} \phi_i$. We also set $\nu_D = (\alpha_D, \phi_D)$.

Definition 5.2.3. *Let F be a torsion-free sheaf on C of uniform rank 1. Let $\nu = (\alpha, \phi)$ be a polarisation on C. We say that F is ν-*stable *(resp. ν-*semistable*) if for all proper non-empty subcurves $D \subset C$,*

$$\frac{\chi(F) + \phi_C}{\alpha_C} < (\textit{resp.} \leq) \; \frac{\chi(F_D) + \phi_D}{\alpha_D}. \tag{5.2.1}$$

Remark. In Definition 5.2.3, in order to check for ν-stability or ν-semistability, it suffices to restrict attention to subcurves $D \subset C$ for which D and $D^c = \overline{C \setminus D}$ are both connected.

Remark. Suppose C is Gorenstein, with dualising sheaf ω_C. By the adjunction formula for Gorenstein curves (cf. [33, Lemma 1.12]), we have

$$\deg_D \omega_C = 2g_D - 2 + k_D = -2\chi(\mathcal{O}_D) + k_D,$$

where g_D is the arithmetic genus of D and where $k_D := |D \cap D^c|$. As such, Inequality (5.2.1) may be rewritten as

$$\deg_D F > (\geq) \; \frac{\alpha_D}{\alpha_C}\left(\deg F - \frac{\deg \omega_C}{2} + \phi_C\right) + \frac{\deg_D \omega_C}{2} - \phi_D - \frac{k_D}{2}. \tag{5.2.2}$$

Given invertible sheaves L and M on C with L ample, the pair (L, M) determines a polarisation $\nu_{(L,M)}$ by setting $\deg_{C_i} L = \alpha_i$ and $\deg_{C_i} M = \phi_i$ for all $i = 1, \ldots, k$. All polarisations ν on C are of the form $\nu = \nu_{(L,M)}$ for such a pair (L, M). If $\nu = \nu_{(L,M)}$, it follows from Lemma 5.1.5 and [122, Septième Partie, Corollaire 7] that a torsion-free sheaf F on C of uniform rank 1 is $\nu = (\alpha, \phi)$-(semi)stable if and only if $F \otimes M$ is Gieseker (semi)stable with respect to the ample invertible sheaf L. In particular, if F is ν-stable then F is necessarily simple.

Remark. Our definition of a polarisation on a curve C differs to the definition commonly used in the literature on compactified Jacobians (cf. [91, Definition 2.8]). According to this latter definition, a polarisation on C is an ordered tuple $q = (q_1, \ldots, q_k)$ of rational numbers, and a torsion-free rank 1 sheaf F on C of uniform rank 1 and of Euler characteristic $\chi(F) = \sum_{i=1}^k q_i$ is said to be q-(semi)stable if $\chi(F_D) > (\geq)\ q_D$ for all non-empty proper subcurves $D \subset C$. We elect to work with polarisations as defined in Definition 5.2.3 as in Chapter 8 we study Harder–Narasimhan filtrations of unstable torsion-free rank 1 sheaves on curves; this requires working with stability conditions for sheaves on C for which the Euler characteristic is not assumed to take a fixed value.

Given a polarisation $\nu = (\alpha, \phi)$, define $q = q_\nu$ by setting

$$q_i = \left(\chi + \sum_{j=1}^{k} \phi_j\right) \frac{\alpha_i}{\sum_{j=1}^k \alpha_j} - \phi_i \tag{5.2.3}$$

for each $i = 1, \ldots, k$. It is easily seen that a torsion-free sheaf F of uniform rank 1 with $\chi(F) = \chi$ is q_ν-(semi)stable if and only if F is ν-(semi)stable. Conversely, given q with $\sum_{i=1}^k q_i = \chi$, it is possible to find a polarisation $\nu = (\alpha, \phi)$ for which (5.2.3) holds (see for instance [30, Fact 2.8]), so that $q = q_\nu$; in general the polarisation ν is not uniquely determined.

5.2.3 Multi-Gieseker Stability for Curves

As a slight digression from the theory of compactified Jacobians, we briefly discuss how the notion of multi-Gieseker stability (cf. Section 4.4) applies in the context of imposing stability conditions on torsion-free sheaves on reduced connected projective curves C, in advance of Chapter 6. Indeed, in this setting multi-Gieseker stability reduces to ordinary Gieseker stability.

Fix a stability condition $\sigma = (\underline{L}, \sigma_1, \ldots, \sigma_s)$ for coherent sheaves on C as in Section 4.4, and suppose E is a torsion-free coherent sheaf on $C = C_1 \cup \cdots \cup C_k$ of multirank $(r_1, \ldots, r_k)$. By [122, Septième Partie, Corollaire 7], we have

$$P^\sigma(E, t) := \sum_{j=1}^{s} \sigma_j \chi(X, E \otimes L_j^t) = \chi(E) \sum_{i=1}^{s} \sigma_i + t \sum_{i=1}^{s} \sigma_i \left(\sum_{j=1}^{k} r_j \deg_{C_j} L_i \right). \tag{5.2.4}$$

Consequently E is σ-(semi)stable if and only if for all non-zero proper subsheaves $F \subset E$ one has

$$\mu^\sigma(F) < (\leq)\ \mu^\sigma(E), \quad \mu^\sigma(E) := \frac{\chi(E)}{\sum_{i,j} \sigma_i r_j \deg_{C_j} L_i}. \tag{5.2.5}$$

An immediate consequence of Inequality 5.2.5 is that the stability condition is unchanged if each L_i is replaced by L_i^p, where p is a positive integer. This allows us to extend the notion of σ-stability (in particular, Gieseker stability) to the case where the L_i are ample *rational* invertible sheaves. Moreover, σ-(semi)stability coincides with Gieseker (semi)stability with respect to the $\mathbb{Q}$-ample invertible sheaf $\bigotimes_i L_i^{\sigma_i}$.

Secondly, since the collection of Gieseker stable sheaves on the curve C is bounded, the stability condition σ is always bounded.

Remark. Equation (5.2.4) implies that fixing the genus g of C and the rank r and the degree d of a uniform rank torsion-free sheaf E on C fixes the topological type $\tau_{\text{top}}(E)$ of E with respect to any finite collection $L_1, \ldots, L_s$ of ample invertible sheaves on C. Conversely, given the genus g, then the degree and rank of a torsion-free coherent sheaf E of uniform rank can be recovered from the topological type.

When C is Gorenstein and when E is of uniform rank 1, the condition of σ-stability may be rephrased in terms of the subcurves D of C, exactly as with Inequality (5.2.2); the sheaf E is σ-(semi)stable if and only if for all connected proper subcurves $D \subset C$,

$$\deg_D E > (\geq)\ \frac{\sum_i \sigma_i \deg_D L_i}{\sum_i \sigma_i \deg L_i}\left(\deg E - \frac{\deg \omega_C}{2}\right) + \frac{\deg_D \omega_C}{2} - \frac{k_D}{2}.$$

5.3 Existence and Properties of Compactified Jacobians

5.3.1 Fine Compactified Jacobians

Given a choice of polarisation q on a reduced, connected, projective curve C, we now introduce $\overline{J}_C^{\chi,ss}(\nu)$, the fine compactified Jacobian parametrising ν-semistable torsion-free sheaves on C of Euler characteristic χ. In fact we will concern ourselves with a slightly more general situation, where we have a family of curves over a base S.

Let $f : C \to S$ be a flat projective morphism over a locally Noetherian base scheme S whose geometric fibres are connected, reduced curves.

Definition 5.3.1. *We define $\overline{\mathbb{J}}_{C/S}$ to be the étale sheafification of the functor $\overline{\mathbb{J}}^*_{C/S}$: $\mathbf{Sch}_S^{\text{op}} \to \mathbf{Set}$ which associates to an S-scheme T the set of equivalence classes of T-flat,*

T-finitely presented coherent sheaves F on $C_T = C \times_S T$ whose fibres are simple torsion-free sheaves of uniform rank 1, where F, F' are declared to be equivalent if there exists an invertible sheaf M on T with $F \cong F' \otimes f_T^* M$.

Remark. As a consequence of [9, Corollary 5.3], the fibrewise simplicity of F is equivalent to the condition that for each T-scheme T', the natural morphism

$$\mathcal{O}_{T'} \to (f_{T'})_* \mathcal{H}om_{C_{T'}}(F_{T'}, F_{T'})$$

is an isomorphism.

Proposition 5.3.2 ([9], Theorem 7.4 (see also [91], Fact 2.2))**.** *$\overline{\mathbb{J}}_{C/S}$ is represented by an algebraic space locally of finite type over S, also denoted $\overline{\mathbb{J}}_{C/S}$. Moreover, there exists an open sub-algebraic space $\mathbb{J}_{C/S} \subset \overline{\mathbb{J}}_{C/S}$ parametrising those sheaves which are invertible.*

Remark. Let $\mathcal{C}oh_{C/S}^{\mathrm{simp,TFR1}}$ be the open substack of the stack of coherent sheaves $\mathcal{C}oh_{C/S}$ parametrising all flat, finitely presented families of simple torsion-free coherent sheaves on C of uniform rank 1. Then there is an isomorphism

$$\overline{\mathbb{J}}_{C/S} \cong \mathcal{C}oh_{C/S}^{\mathrm{simp,TFR1}} \mathbin{/\!\!/} \mathbb{G}_m$$

arising from the universal property of rigidification. In particular, $\overline{\mathbb{J}}_{C/S}$ is the coarse moduli space of the stack $\mathcal{C}oh_{C/S}^{\mathrm{simp,TFR1}}$.

In many cases, including when $S = \operatorname{Spec} \mathbb{C}$, the algebraic space $\overline{\mathbb{J}}_{C/S}$ is a schematic fine moduli space.

Proposition 5.3.3. *Assume there are sections $\sigma_1, \ldots, \sigma_n : S \to C$ of f, each factoring through the S-smooth locus of C, such that for each geometric point $s \in S$, each irreducible component of C_s is geometrically integral and contains $\sigma_i(s)$ for some $i = 1, \ldots, n$.*

1. *(Altman–Kleiman) The étale sheafification and the Zariski sheafification of $\overline{\mathbb{J}}^*_{C/S}$ coincide. Additionally, there exists a universal sheaf on the fibre product $C \times_S \overline{\mathbb{J}}_{C/S}$, which, up to isomorphism, is uniquely determined up to the pullback of an invertible sheaf on $\overline{\mathbb{J}}_{C/S}$.*

2. *(Esteves) $\overline{\mathbb{J}}_{C/S}$ is a scheme. In particular, if $S = \operatorname{Spec}\mathbb{C}$ then $\overline{\mathbb{J}}_C$ is always a scheme.*

   In general, there exists an étale surjection $S' \to S$ such that the base change $\overline{\mathbb{J}}_{C/S} \times_S S'$ is a scheme.

Proof. See [8, Theorem 3.4] and [48, Theorem B and Corollary 52]. □

In the case where $S = \operatorname{Spec}\mathbb{C}$, by work of Melo–Rapagnetta–Viviani [91] it is known that if the curve C has locally planar singularities then the scheme $\overline{\mathbb{J}}_C$ has many interesting geometric properties.

Proposition 5.3.4 ([91], Theorem 2.3)**.** *Suppose C is a connected reduced curve over* $\operatorname{Spec}\mathbb{C}$ *with locally planar singularities. Then:*

1. *$\overline{\mathbb{J}}_C$ is a reduced scheme, with at worst lci singularities; and*

2. *$\mathbb{J}_C$ is dense in $\overline{\mathbb{J}}_C$, and coincides with the smooth locus of $\overline{\mathbb{J}}_C$.*

We return to working with the locally Noetherian base scheme S. Since Euler characteristics are constant in flat families, there is a decomposition

$$\overline{\mathbb{J}}_{C/S} = \bigsqcup_{\chi \in \mathbb{Z}} \overline{\mathbb{J}}^{\chi}_{C/S}$$

of $\overline{\mathbb{J}}_{C/S}$ into open and closed sub-algebraic spaces, where $\overline{\mathbb{J}}^{\chi}_{C/S}$ parametrises those sheaves of Euler characteristic χ.

Definition 5.3.5. *Pick invertible sheaves L, M on C, with L relatively ample, and fix $\chi \in \mathbb{Z}$. Let $\overline{J}^{\chi,(s)s}_{C/S}(L, M)$ be the open sub-algebraic space of $\overline{\mathbb{J}}^{\chi}_{C/S}$ parametrising those sheaves F such that $F \otimes M$ is fibrewise Gieseker (semi)stable with respect to L.*

Proposition 5.3.6 ([48], Theorems A and C)**.** *The algebraic space $\overline{J}^{\chi,ss}_{C/S}(L, M)$ is of finite type over S. Moreover:*

1. *$\overline{J}^{\chi,ss}_{C/S}(L, M)$ is universally closed over S, and*

2. *$\overline{J}^{\chi,s}_{C/S}(L, M)$ is separated over S. In addition, if S is excellent (e.g. if S is locally of finite type) then $\overline{J}^{\chi,s}_{C/S}(L, M)$ is locally quasi-projective over S.*

3. In particular, if S is excellent and if $\overline{J}^{\chi,ss}_{C/S}(L,M) = \overline{J}^{\chi,s}_{C/S}(L,M)$ then $\overline{J}^{\chi,ss}_{C/S}(L,M)$ is locally projective over S.

Remark. Esteves *loc. cit.* considers stability conditions on C/S defined in terms of locally free sheaves on C; stability with respect to (L,M) can be realised as a special case of stability with respect to an Esteves polarisation. For the purposes of this thesis, it is enough to consider stability conditions arising from twisted Gieseker stability, as in Definition 5.3.5.

Suppose $S = \operatorname{Spec}\mathbb{C}$; let $\nu = \nu_{(L,M)}$ be the corresponding polarisation on C, and write $\overline{J}^{\chi,(s)s}_C(\nu) = \overline{J}^{\chi,(s)s}_{C/\operatorname{Spec}\mathbb{C}}(L,M)$. The scheme $\overline{J}^{\chi,ss}_C(\nu)$ is known as a *fine compactified Jacobian*. The open subscheme $\overline{J}^{\chi,s}_C(\nu)$ is quasi-projective; if all ν-semistable simple sheaves on C of Euler characteristic χ are ν-stable, so that $\overline{J}^{\chi,s}_C(\nu) = \overline{J}^{\chi,ss}_C(\nu)$, then $\overline{J}^{\chi,s}_C(\nu)$ is a proper quasi-projective scheme, whence a projective scheme.

5.3.2 Coarse Compactified Jacobians

Let C be a reduced connected projective curve. Given a polarisation ν on C, one has the so-called *coarse compactified Jacobian* $U^{\chi,ss}_C(\nu)$. This is a projective scheme which corepresents the moduli functor associating to each scheme T the set of isomorphism classes T-flat coherent sheaves on $C\times T$ whose fibres are ν-semistable torsion-free sheaves of uniform rank 1 with Euler characteristic χ; that $U^{\chi,ss}_C(\nu)$ corepresents this moduli functor implies the existence of a natural morphism $\rho : \overline{J}^{\chi,ss}_C(\nu) \to U^{\chi,ss}_C(\nu)$. There is an open subscheme $U^{\chi,s}_C(\nu) \subset U^{\chi,ss}_C(\nu)$ parametrising those sheaves which are ν-stable.

If $\nu = \nu_{(L,M)}$ then $U^{\chi,ss}_C(\nu)$ is isomorphic (via twisting through by M) to the good moduli space of the stack $\mathcal{C}oh^{ss}_C(L,\chi)$ of L-Gieseker semistable, uniform rank 1 coherent sheaves on C of Euler characteristic χ. In particular, $U^{\chi,ss}_C(\nu)$ admits a construction by GIT (cf. Section 4.2). A separate GIT construction of $U^{\chi,ss}_C(\nu)$ also exists, due to Seshadri (cf. [122, Septième Partie, Chapitre III]).

From the results of Seshadri *loc. cit.* and Esteves [48, Section 8], the natural morphism $\rho : \overline{J}^{\chi,ss}_C(\nu) \to U^{\chi,ss}_C(\nu)$ is surjective and is universal for morphisms from $\overline{J}^{\chi,ss}_C(\nu)$ to separated schemes; the fibres of ρ are given by S-equivalence classes of ν-semistable sheaves. The restriction of ρ to $\overline{J}^{\chi,s}_C(\nu)$ is an isomorphism $\overline{J}^{\chi,s}_C(\nu) \xrightarrow{\simeq} U^{\chi,s}_C(\nu)$; in particular, $U^{\chi,s}_C(\nu)$ is a fine moduli space of simple torsion-free uniform rank 1 coherent sheaves on C.

Chapter 6

GIT Constructions of Compactified Universal Jacobians over Stacks of Stable Maps

6.1 Introduction

This chapter studies compactified universal Jacobians over the stack of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$, along with their analogues involving semistable torsion-free sheaves of uniform rank $r > 1$, where the stability condition is defined using an ample invertible sheaf $\mathcal{L}$ on the universal curve over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The main result is that these stacks admit projective good moduli spaces which are GIT quotients, and that variation of GIT applies when the sheaf $\mathcal{L}$ is varied. Additionally we prove that, when the base is taken to be $\overline{\mathcal{M}}_{g,n}$ with $g \geq 4$, the good moduli space of a compactified universal Jacobian has canonical singularities.

Significant recent attention has been given to the study of *compactified universal Jacobians* over the stack $\overline{\mathcal{M}}_{g,n}$ of n-pointed Deligne–Mumford stable curves of genus g ([2] [4] [22] [28] [29] [30] [31] [46] [64] [71] [72] [87] [88] [89] [90] [93] [106] [107]). These are compactifications of the stack of all invertible sheaves of a given degree over smooth marked curves whose boundary objects consist of uniform rank one torsion-free sheaves over singular marked stable curves.[1] The fibres of the forgetful morphism from a compactified universal Jacobian to $\overline{\mathcal{M}}_{g,n}$ are given by moduli stacks of torsion-free sheaves on nodal curves, including the compactified Jacobians introduced in Chapter 5.

[1]Strictly speaking, the condition that the sheaves in the boundary are simple is also usually imposed, however we *do not* make this requirement in this chapter.

Compactified universal Jacobians have been successfully used to construct extensions of the universal Abel map from $\mathcal{M}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$, which in turn provides one way of defining the double ramification cycle on $\overline{\mathcal{M}}_{g,n}$ (for more details on this connection we refer the reader to [3] and the references therein).

In order to compactify the stack of all invertible sheaves (or more generally all slope semistable locally free sheaves) over marked smooth curves, in practice a stability condition is first chosen. One way of doing this is to fix a relatively ample invertible sheaf $\mathcal{L}$ on the universal curve over $\overline{\mathcal{M}}_{g,n}$ and then require that all torsion-free sheaves in the boundary are Gieseker semistable with respect to $\mathcal{L}$. Melo in [89] proved that if all $\mathcal{L}$-semistable sheaves are $\mathcal{L}$-stable then the resulting compactified universal Jacobian admits a projective coarse moduli space. Melo's argument makes use of the criterion of Kollár [79] to prove that the coarse moduli spaces are projective, and does not address the case where there are strictly $\mathcal{L}$-semistable sheaves.

If $n = 0$ and if $\mathcal{L}$ is the dualising sheaf of the universal family, the resulting compactified universal Jacobian is also known to admit a projective good moduli space (even though there are strictly semistable sheaves), known as the *universal Picard variety*; at least two prior GIT constructions of this space are known, the first due to Caporaso [27] and the second due to Pandharipande [108] (Pandharipande deals with the case where the sheaves are allowed to have any uniform rank r, not just $r = 1$). The constructions of Caporaso and Pandharipande rely on the GIT construction of Gieseker [54] of the coarse moduli space $\overline{M}_g$ of the stack $\overline{\mathcal{M}}_g$ of genus g Deligne–Mumford stable curves. In place of Gieseker's construction, we use Baldwin and Swinarski's GIT construction of the coarse moduli space $\overline{M}_{g,n}(X, \beta)$. We also use the GIT construction of Greb, Ross and Toma [55] [56] of moduli spaces of multi-Gieseker semistable sheaves on a projective scheme, which allows us to deal with multiple stability conditions at the same time. As a special case of our approach, we provide a new GIT construction of the Caporaso–Pandharipande moduli space.

As an application of our GIT construction, we use the methods of [55] and [56] to establish the existence of rational linear wall-chamber decompositions on positive-dimensional spaces of stability conditions, with the property that as the stability condition varies within the interior of the chamber, the resulting stack remains unchanged, and that crossing a wall corresponds to modifying the good moduli space by a Thaddeus flip through a moduli space corresponding to a stability condition on that wall (cf. Definition 2.5.5). Unlike the wall-chamber decomposition described by Kass and Pagani in [72] (in the case where the sheaves are all of uniform rank one and where the base stack is $\overline{\mathcal{M}}_{g,n}$), the wall-chamber decompositions described in this chapter rely on auxiliary choices of relatively ample invertible sheaves over the universal curve over the base stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$, and as such the decompositions are not intrinsic to the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

We also extend some of the results of Casalaina-Martin–Kass–Viviani [30] [31] concerning the birational geometry of the Caporaso–Pandharipande moduli space. In particular, we prove that, when working over the stack $\overline{\mathcal{M}}_{g,n}$ of stable curves, if $g \geq 4$ and if n is any non-negative integer then the good moduli space of a compactified universal Jacobian over $\overline{\mathcal{M}}_{g,n}$ has canonical singularities for any choice of stability condition, extending the result of Casalaina-Martin–Kass–Viviani that the Caporaso–Pandharipande space has canonical singularities when $g \geq 4$.

Summary of Results

Let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be the stack of genus g stable maps with n marked points to a projective variety X of class β, and let $\mathcal{M}_{g,n}(X,\beta)$ be the open substack where the source curves are non-singular. Here and throughout this chapter we assume that the discrete invariants are chosen so that the stack $\mathcal{M}_{g,n}(X,\beta)$ is non-empty.

Given a relatively ample $\mathbb{Q}$-invertible sheaf $\mathcal{L}$ on the universal curve $\mathcal{U}\overline{\mathcal{M}}_{g,n}(X,\beta)$ over the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$, define $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ to be the $\mathbb{G}_m$-rigidification of the stack parametrising flat and proper families of degree d, uniform rank r, torsion-free coherent sheaves over the underlying nodal curves C of stable maps $(C,\underline{x},f)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which are fibrewise Gieseker semistable with respect to $\mathcal{L}$ (cf. Section 6.3). Fix relatively ample $\mathbb{Q}$-invertible sheaves $\mathcal{L}_1,\ldots,\mathcal{L}_k$ on the universal curve, and set $\Sigma := (\mathbb{Q}^{\geq 0})^k \setminus \{0\}$. For each $\sigma = (\sigma_1,\ldots,\sigma_k) \in \Sigma$,[2] form the $\mathbb{Q}$-invertible sheaf $\mathcal{L}_\sigma = \bigotimes_i \mathcal{L}_i^{\sigma_i}$, and consider the resulting stacks over $\overline{\mathcal{M}}_{g,n}(X,\beta)$:

$$\overline{\mathcal{J}}(\sigma) := \overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}_\sigma).$$

We say $\sigma \in \Sigma$ is *positive* if each $\sigma_i > 0$, and *degenerate* otherwise. Next, fix a finite subset $\mathfrak{S} \subset \Sigma$. We may now state our main result, which follows from Corollary 6.5.4, Theorem 6.5.5 and Proposition 6.5.7.

[2] Since $\lambda\sigma$ defines the same stability condition as σ for each positive rational number λ, we are free to assume that $\sum \sigma_i = 1$ as necessary.

Theorem 6.1.1. *There exists a quasi-projective scheme $Z_r = Z_{r,\mathfrak{S}}$ and a reductive linear algebraic group K acting on Z_r, such that for each $\sigma \in \mathfrak{S}$, there is a K-invariant open subscheme $Z_r^{\sigma-ss} \subset Z_r$, obtained as a (relative) GIT semistable locus for an appropriate linearisation determined by σ, which admits a good quotient $Z_r^{\sigma-ss} /\!\!/ K$. Moreover:*

1. *if σ is positive, then $\overline{\mathcal{J}}(\sigma)$ is isomorphic to the quotient stack $[Z_r^{\sigma-ss}/K]$;*
2. *if σ is either positive or degenerate, the stack $\overline{\mathcal{J}}(\sigma)$ is universally closed and admits a good moduli space $\overline{J}(\sigma) = \overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}_\sigma)$, which is isomorphic to the good quotient $Z_r^{\sigma-ss} /\!\!/ K$, and admits a natural morphism to the coarse moduli space $\overline{M}_{g,n}(X,\beta)$ of $\overline{\mathcal{M}}_{g,n}(X,\beta)$; and*
3. *the fibre of $\overline{J}(\sigma)$ over $[(C,\underline{x},f)] \in M_{g,n}(X,\beta)$ is given by the geometric quotient of the moduli space of degree d, uniform rank r, slope-semistable locally free sheaves on C by the action of the group of automorphisms $\mathrm{Aut}(C,\underline{x},f)$ of the stable map $(C,\underline{x},f)$.*

Remark. In the case where σ is degenerate, it is still possible to use Theorem 6.1.1 to exhibit $\overline{\mathcal{J}}(\sigma)$ as a quotient stack of the form $[(Z'_r)^{\sigma'-ss}/K']$ for an appropriate subgroup $K' \subset K$, by first omitting the invertible sheaves $\mathcal{L}_i$ for which the corresponding entry $\sigma_i = 0$, letting σ' be the vector obtained by omitting all zero entries from σ; see the remark following Corollary 6.5.4.

As an application of the construction underlining Theorem 6.1.1 we prove, by adapting the arguments given in [56], the following result, which concerns what happens at the level of the good moduli spaces $\overline{J}(\sigma)$ as the stability condition $\sigma \in \Sigma$ is allowed to vary, after the $\mathbb{Q}$-invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_k$ have been fixed. This result follows from Proposition/Definition 6.6.4 and Theorem 6.6.5.

Theorem 6.1.2. *The set $\Sigma' = \left\{\sigma \in \Sigma : \sum_{i=1}^k \sigma_i = 1\right\}$ is cut into chambers by a finite number of rational linear walls,*[3] *such that the moduli stack $\overline{\mathcal{J}}(\sigma)$ is unchanged as σ varies in the interior of a chamber. For any $\sigma \in \Sigma'$, we have that all $\mathcal{L}_\sigma$-semistable sheaves are $\mathcal{L}_\sigma$-stable (and hence the corresponding stack $\overline{\mathcal{J}}(\sigma)$ is Deligne–Mumford) if and only if σ does not lie in any wall. Given σ_1, $\sigma_2 \in \Sigma'$, the moduli spaces $\overline{J}(\sigma_i)$*

[3]We allow for the possibility that all of Σ' is a wall; this would occur if for each of the stability conditions $\sigma \in \Sigma'$, there exists a strictly semistable sheaf.

$(i = 1, 2)$ are related by a finite number of Thaddeus flips through good moduli spaces of the form $\overline{J}(\sigma')$, $\sigma' \in \Sigma'$.

The final main result of this chapter is the following extension of [31, Theorem A] to the situation of studying the singularities of the good moduli space of a compactified universal Jacobian over $\overline{\mathcal{M}}_{g,n}$ where $n > 0$; this result is Theorem 6.7.1.

Theorem 6.1.3. *Suppose $X = \operatorname{Spec} \mathbb{C}$ and $\beta = 0$, so that we are considering compactified universal Jacobians over the stack $\overline{\mathcal{M}}_{g,n}$. If $g \geq 4$, then each of the good moduli spaces $\overline{J}(\sigma)$ has canonical singularities.*

Comparison with Other Work

In the case where $r = 1$ the stacks $\overline{\mathcal{J}}^{ss}_{g,n,d,1}(X, \beta)(\mathcal{L})$ are examples of compactified universal Jacobians. The paper [89] treats these stacks in the case where X is a point and $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is replaced with an open substack of the stack of marked reduced curves. When this substack is taken to be $\overline{\mathcal{M}}_{g,n}$, Melo considers stability conditions defined with respect to a locally free sheaf on the universal curve over $\overline{\mathcal{M}}_{g,n}$, and for explicit polarisations determines functoriality properties of the corresponding compactified universal Jacobians, such as compatibility with forgetful and clutching morphisms.

Meanwhile, Kass and Pagani [72] study stability conditions defined in terms of suitable functions ϕ defined on the space of dual graphs of stable curves in $\overline{\mathcal{M}}_{g,n}$, generalising the approach of Oda and Seshadri [105], and describe an explicit wall-chamber decomposition on the space of stability conditions, which in their notation is denoted $V^d_{g,n}$, where d is the degree of the uniform rank 1 torsion-free sheaves under consideration. We note that Kass–Pagani and Melo study the same family of stability conditions (cf. [72] Remark 4.6), and that each of the Kass–Pagani stability conditions can be realised as a universal (twisted) Gieseker stability condition with respect to a relatively ample invertible sheaf on the universal curve over $\overline{\mathcal{M}}_{g,n}$ (cf. Corollary 4.3 of *loc. cit.*), and so, by the final remark of Section 6.3.2, are covered by the results of this chapter. Kass and Pagani also successfully characterise for any $\phi \in V^d_{g,n}$ the locus of indeterminacy of the extension $\overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{J}^d_{g,n}(\phi)$ (where $\mathcal{J}^d_{g,n}(\phi)$ denotes Kass and Pagani's compactified universal Jacobian) of the Abel–Jacobi map

$$\alpha_{k,\underline{d}} : (C, \underline{x}) \mapsto \omega_C^{-k}(d_1 x_1 + \cdots + d_n x_n), \quad (C, \underline{x}) \in \mathcal{M}_{g,n},$$

and in particular describe all stability conditions ϕ for which $\alpha_{k,\underline{d}}$ extends to $\overline{\mathcal{M}}_{g,n}$.

The main difference between the decomposition of $V_{g,n}^{d}$ and the decompositions described by Theorem 6.1.2 is that the latter decompositions depend on the choice of the $\mathbb{Q}$-invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_k$, whereas the decomposition of $V_{g,n}^{d}$ only depends on the discrete invariants g, n and d. It would be interesting to determine how the wall-chamber decompositions relate to each other (given explicit $\mathbb{Q}$-invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_k$), as well as whether the decomposition of [72] can be extended to higher rank sheaves and to working over $\overline{\mathcal{M}}_{g,n}(X, \beta)$.

It is possible to construct the good moduli space of $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X, \beta)(\mathcal{L})$ via GIT in a manner which closely follows the construction of Pandharipande [108], by applying relative GIT to an appropriate Quot scheme over the parameter space considered by Baldwin–Swinarski [15] (in place of the parameter space considered by Gieseker [54]), making use of the results of Simpson [124] concerning GIT constructions of relative moduli spaces of Gieseker semistable sheaves (see Proposition 6.7.2). Indeed, it is this alternative construction that is considered when proving Theorem 6.1.3. We remark that the difficult step in Pandharipande's construction is relating GIT and moduli (semi)stability for the fibrewise action on the relative Quot scheme; once this has been established, the rest of Pandharipande's construction is essentially formal relative GIT. As observed by Pandharipande himself (cf. [108, Page 433]), the work of Simpson can instead be used to fully solve the fibrewise GIT problem.

The disadvantage of this approach, as compared with that used to prove Theorem 6.1.1, is that it is not possible to implement VGIT, since different choices of linearisations $\mathcal{L}$ necessitate working with different relative Quot schemes.

For higher ranks, there are alternative compactifications of the moduli stack of slope-semistable locally free sheaves over the moduli stack of smooth curves, which instead involve parametrising locally free sheaves on semistable curves instead of semistable torsion-free sheaves on stable curves. Examples of such compactifications have been constructed and studied by Fringuelli [50], Schmitt [117] and Teixidor i Bigas [125]. We do not consider these compactifications in this chapter.

Notation and Conventions

In addition to the conventions specified in the introduction to this thesis, in this chapter we adopt the following notation and conventions:

- Whenever we work with the stack $\overline{\mathcal{M}}_{g,n}$, we implicitly assume that $2g-2+n>0$, so that $\overline{\mathcal{M}}_{g,n}$ is non-empty and Deligne–Mumford.

- The *dual graph* of a connected nodal curve C is the graph Γ_C whose vertex set $V(\Gamma_C)$ is the set of irreducible components of C, labelled by arithmetic genera, and whose edges are given by nodes; for each node $p \in C$ incident to components C_{v_1} and C_{v_2}, there is a corresponding edge $e \in E(\Gamma_C)$ incident to v_1 and v_2 (note that we allow for the possibility that $v_1 = v_2$, which corresponds to node internal to the component C_{v_1}).

- We denote by $\mathbf{Art}_{\mathbb{C}}$ the category of Artinian local $\mathbb{C}$-algebras with residue field $\mathbb{C}$. Given $R \in \mathbf{Art}_{\mathbb{C}}$, we denote by $\operatorname{Spf} R$ the functor

$$\mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}, \quad A \mapsto \operatorname{Hom}_{\mathrm{loc}}(R, A).$$

- If $F : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ is a functor, we define the *tangent space* of F to be $TF = F(\mathbb{C}[\epsilon])$, where $\mathbb{C}[\epsilon] = \mathbb{C}[t]/t^2$ is the ring of dual numbers over $\mathbb{C}$. The vector space dual of TF is denoted $T^\vee F$. We employ standard deformation-theoretic terminology (see for instance [63] and [120]). In particular, we say that $R \in \mathbf{Art}_{\mathbb{C}}$ is a *miniversal deformation ring* for F if there exists a formally smooth natural transformation $\operatorname{Spf} R \to F$ which is an isomorphism on tangent spaces; such a ring R is unique up to (non-canonical) isomorphism.

- Let X be a normal variety. The singularities of X are said to be *rational* if for all resolutions of singularities $f : Y \to X$, $R^i f_* \mathcal{O}_Y = 0$ for all $i > 0$.

- Let X be a normal variety. Assuming that the canonical divisor K_X is $\mathbb{Q}$-Cartier, if $f : Y \to X$ is a resolution of singularities with exceptional prime divisors E_i, we may write $K_Y = f^* K_X + \sum_i a_i E_i$, where the coefficients $a_i \in \mathbb{Q}$.

  The singularities of X are said to be *canonical* (resp. *terminal*) if the canonical divisor K_X of X is $\mathbb{Q}$-Cartier and if for all resolutions of singularities $f : Y \to X$, the coefficients a_i are always non-negative (resp. positive).

Remark. Sections 6.2 to 6.6 (inclusive) of this chapter are taken from the preprint [37]. After this work was originally submitted to arχiv, Pagani and Tommasi [107] proved that, provided one first imposes the additional constraint that all semistable sheaves are stable, the only Deligne–Mumford compactified universal Jacobian over $\overline{\mathcal{M}}_g$ in degree d is the stack corresponding to the Caporaso–Pandharipande moduli space, i.e. when the dualising sheaf of the universal curve over $\overline{\mathcal{M}}_g$ is used to define the stability condition. This exists as a Deligne–Mumford stack if and only if $\gcd(2g-2, d-g+1) = 1$.

6.2 Prestable Curves and Stable Maps

We begin by recalling the definition of a prestable curve and of a stable map to a projective variety X.

6.2.1 Prestable and Stable Curves

Let g, n be non-negative integers.

Definition 6.2.1. *An n-*marked prestable curve *$(C, \underline{x}) = (C; x_1, \ldots, x_n)$ of* genus g *consists of a connected, reduced, projective nodal curve C of arithmetic genus g together with n ordered marked points $x_i \in C$, such that the markings x_i are distinct and lie in the non-singular locus of C.*

A marked prestable curve $(C, \underline{x})$ is said to be stable *if any of the following equivalent conditions hold:*

1. *$(C, \underline{x})$ has no infinitesimal automorphisms.*

2. *The invertible sheaf $\omega_C(x_1 + \cdots + x_n)$ is ample, where ω_C is the dualising sheaf of C.*

3. *Each genus 1 component of C has at least one special point, and each genus 0 component of C has at least three special points; here a* special point *is a marked point or a node.*

The notion of a family of prestable curves is given as follows.

Definition 6.2.2. *A* family *of n-marked prestable curves of genus g parametrised by a scheme S is a flat and proper morphism $\pi : C \to S$ of finite presentation, together with n sections $\sigma_1, \ldots, \sigma_n$ of π, such that for each geometric point $s \in S$, $(C_s; \sigma_1(s), \ldots, \sigma_n(s))$ is an n-marked prestable curve of genus g.*

It is well-known (see for instance [59]) that there exists an algebraic stack $\mathfrak{M}_{g,n}$, of finite presentation and with quasi-compact and separated diagonal, whose groupoid of S-points is the groupoid of all families of n-marked prestable curves of genus g parametrised by the scheme S.

6.2.2 Stable Maps

Let X be a projective variety. Following [17], a *curve class* on X is an element of the semigroup

$$H_2(X, \mathbb{Z})_+ = \{\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\, X, \mathbb{Z}) : \beta(L) \geq 0 \text{ for all ample } L\}.$$

If $f : C \to X$ is a morphism from a prestable curve C, the locally constant function $L \mapsto \deg f^*L$ on $\mathrm{Pic}\, X$ is denoted as $f_*[C] \in H_2(X, \mathbb{Z})_+$.

Let $\beta \in H_2(X, \mathbb{Z})_+$ be a fixed curve class, let g, n be non-negative integers.

Definition 6.2.3. *An n-*marked stable map *$(C, \underline{x}, f) = (C; x_1, \ldots, x_n; f : C \to X)$ of* genus g *and* curve class β *consists of a prestable curve $(C, \underline{x})$ together with a morphism of schemes $f : C \to X$ with $f_*[C] = \beta$, satisfying any of the following (equivalent) conditions:*

1. *$(C, \underline{x}, f)$ has no infinitesimal automorphisms, where an automorphism of $(C, \underline{x}, f)$ is an automorphism α of C which fixes the marked points $x_i \in C$ and satisfies $f \circ \alpha = f$.*

2. *If $X \subset \mathbb{P}^b$ is a fixed embedding of X inside a projective space, then the invertible sheaf $\omega_C(x_1 + \cdots + x_n) \otimes f^*(\mathcal{O}_{\mathbb{P}^b}(3)|_X)$ is ample.*

3. *Each genus 1 component of C mapped by f to a point has at least one special point, and each genus 0 component of C mapped by f to a point has at least three special points; here a* special point *is a marked point or a node.*

A family *of n-marked stable maps to X of genus g and curve class β parametrised by a scheme S consists of a family of n-marked genus g prestable curves $(C \to S; \sigma_1, \ldots, \sigma_n)$ and a morphism of schemes $f : C \to X$, such that for each geometric point $s \in S$, $(C_s; \sigma_1(s), \ldots, \sigma_n(s); f_s : C_s \to X)$ is a stable map of curve class β.*

It is a standard result (see for instance Theorem 3.14 of *loc. cit.*) that there exists a proper Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$, whose groupoid of S-points is the groupoid of all families of n-marked stable maps of genus g parametrised by the scheme S. There is a distinguished open substack $\mathcal{M}_{g,n}(X,\beta)$ whose objects are given by stable maps whose source curve C is non-singular. It is also well known that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ admits a *projective* coarse moduli space $\overline{M}_{g,n}(X,\beta)$. If X is a point then a stable map to X is the same as a marked stable curve, that is $\overline{\mathcal{M}}_{g,n}(\mathrm{pt},0)=\overline{\mathcal{M}}_{g,n}$. The stack $\overline{\mathcal{M}}_{g,n}$ is smooth and irreducible, and is of dimension $3g-3+n$.

As explained in [16], if X is smooth then there exists a natural perfect relative obstruction theory

$$\mathbb{E}^{\bullet}_{\overline{\mathcal{M}}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}} := R^{\bullet}(\pi_{\mathcal{U}})_*(f_{\mathcal{U}}^*T_X)\to \mathbb{L}^{\bullet}_{\overline{\mathcal{M}}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}}, \tag{6.2.1}$$

where $\mathbb{L}^{\bullet}$ denotes the relative cotangent complex. This gives rise to a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ of virtual dimension

$$\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta))=\int_{\beta}c_1(T_X)+(\dim X-3)(1-g)+n.$$

Remark. If we fix a closed embedding $i: X\hookrightarrow \mathbb{P}^b$ such that $i_*\beta=d'[\ell]$, where $[\ell]\in H_2(\mathbb{P}^b,\mathbb{Z})_+$ is the class of a line, then a necessary condition for non-singular stable maps to exist is $2g-2+n+3d'>0$.

6.3 Stacks of Semistable Sheaves over Stable Maps

In this section, we introduce the stacks $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ which are the main focus of this chapter, and explore a few of their basic properties.

6.3.1 The Stack of Torsion-Free Sheaves over $\overline{\mathcal{M}}_{g,n}(X,\beta)$

Let X be a projective variety and let $\beta\in H_2(X,\mathbb{Z})_+$ be a curve class. Fix integers g,n,d,r, with $g,n\geq 0$, $r\geq 1$ and for which the open substack $\mathcal{M}_{g,n}(X,\beta)$ is non-empty. Since there exists an (algebraic) stack of all uniform rank r, degree d, torsion-free coherent sheaves over any family of prestable curves, by [32, Corollary 13] there exists a stack $\mathfrak{M}_{g,n,d,r}$ of n-marked, genus g prestable curves with uniform rank r, degree d, torsion-free coherent sheaves.

Let[4] $\mathfrak{M}^{\mathrm{rig}}_{g,n,d,r} := \mathfrak{M}_{g,n,d,r} \mathbin{/\!\!/} \mathbb{G}_m$ denote the $\mathbb{G}_m$-rigidification with respect to the central $\mathbb{G}_m$ in the automorphism groups of the sheaves in $\mathfrak{M}_{g,n,d,r}$. Let $\mathfrak{M}^{\mathrm{simp,rig}}_{g,n,d,r}$ be the open substack of $\mathfrak{M}^{\mathrm{rig}}_{g,n,d,r}$ parametrising those objects whose underlying coherent sheaf is (fibrewise) simple. Set

$$\overline{\mathcal{J}}_{g,n,d,r}(X,\beta) := \overline{\mathcal{M}}_{g,n}(X,\beta) \times_{\mathfrak{M}_{g,n}} \mathfrak{M}^{\mathrm{rig}}_{g,n,d,r},$$

and

$$\overline{\mathcal{J}}^{\mathrm{simp}}_{g,n,d,r}(X,\beta) := \overline{\mathcal{M}}_{g,n}(X,\beta) \times_{\mathfrak{M}_{g,n}} \mathfrak{M}^{\mathrm{simp,rig}}_{g,n,d,r}.$$

The stack $\overline{\mathcal{J}}_{g,n,d,r}(X,\beta)$ is the stack associated to the category fibred in groupoids $\overline{\mathcal{J}}_{g,n,d,r}(X,\beta)^{\mathrm{pre}}$ whose fibre over a scheme S consists of all objects

$$(\pi : C \to S; \sigma_1, \dots, \sigma_n; f : C \to X; F),$$

where $(\pi : C \to S; \sigma_1, \dots, \sigma_n; f : C \to X) \in \overline{\mathcal{M}}_{g,n}(X,\beta)(S)$ is a family of stable maps parametrised by S and where F is a (S-flat, S-finitely presented) family of torsion-free coherent sheaves over $\pi : C \to S$, of relative degree d and uniform rank r; morphisms between objects $(\pi : C \to S; \sigma_1, \dots, \sigma_n; f : C \to X; F)$ and $(\pi' : C' \to S; \sigma'_1, \dots, \sigma'_n; f' : C' \to X; F')$ are given by Cartesian diagrams

$$\begin{array}{ccc} C & \xrightarrow{\overline{g}} & C' \\ \sigma_i \uparrow\downarrow \pi & \lrcorner & \pi' \downarrow\uparrow \sigma'_i \\ S & \xrightarrow{g} & S' \end{array} \tag{6.3.1}$$

which are compatible with the sections σ_i and σ'_i, together with an equivalence class of isomorphisms $F \xrightarrow{\simeq} \overline{g}^* F' \otimes \pi^* M$ (for some $M \in \operatorname{Pic} S$); isomorphisms $\psi_1 : F \xrightarrow{\simeq} \overline{g}^* F' \otimes \pi^* M_1$ and $\psi_2 : F \xrightarrow{\simeq} \overline{g}^* F' \otimes \pi^* M_2$ are equivalent if there exists an isomorphism $\eta : M_1 \xrightarrow{\simeq} M_2$ such that the diagram

$$\begin{array}{ccc} & & \overline{g}^* F' \otimes \pi^* M_1 \\ & \nearrow^{\psi_1} & \\ F & & \downarrow \mathrm{id} \otimes f^* \eta \\ & \searrow_{\psi_2} & \\ & & \overline{g}^* F' \otimes \pi^* M_2 \end{array}$$

[4] The procedure of rigidification, as described in [1], is also valid for stacks which aren't necessarily algebraic.

commutes. If

$$\overline{\mathcal{J}}ac_{g,n,d,r}(X,\beta) = \overline{\mathcal{M}}_{g,n}(X,\beta) \times_{\mathfrak{M}_{g,n}} \mathfrak{M}_{g,n,d,r}$$

(resp. $\overline{\mathcal{J}}ac^{\text{simp}}_{g,n,d,r}(X,\beta)$) is the stack whose sections coincide with those of $\overline{\mathcal{J}}_{g,n,d,r}(X,\beta)$ (resp. $\overline{\mathcal{J}}^{\text{simp}}_{g,n,d,r}(X,\beta)$) and whose morphisms are given by diagrams (6.3.1) together with isomorphisms $\mathcal{F} \xrightarrow{\sim} \overline{g}^*\mathcal{F}'$, then $\overline{\mathcal{J}}_{g,n,d,r}(X,\beta)$ is the $\mathbb{G}_m$-rigidification of $\overline{\mathcal{J}}ac_{g,n,d,r}(X,\beta)$. The stack $\overline{\mathcal{J}}_{g,n,d,r}(X,\beta)$ admits a universal curve and a universal morphism to X, obtained by pulling back the universal family over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The stack $\overline{\mathcal{J}}ac_{g,n,d,r}(X,\beta)$ admits a universal sheaf over this universal curve.

The stack $\overline{\mathcal{J}}_{g,n,d,r}(X,\beta)$ admits natural morphisms to the stacks $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and $\mathfrak{M}^{\text{rig}}_{g,n,d,r}$, obtained by forgetting respectively the sheaves and the morphisms to X.

Proposition 6.3.1. *The stack $\overline{\mathcal{J}}^{\text{simp}}_{g,n,d,r}(X,\beta)$ is Deligne–Mumford and the natural forgetful morphism to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is representable and locally of finite presentation. If X is smooth then any open Deligne–Mumford substack of $\overline{\mathcal{J}}^{\text{simp}}_{g,n,d,r}(X,\beta)$ admits a natural perfect relative obstruction theory over $\mathfrak{M}^{\text{simp,rig}}_{g,n,d,r}$.*

Proof. We first show that the stack $\mathfrak{M}^{\text{simp,rig}}_{g,n,d,r}$ is algebraic and locally of finite presentation over $\mathfrak{M}_{g,n}$. Consider a morphism $S \to \mathfrak{M}_{g,n}$, where S is a scheme, and let

$$\mathfrak{M}_S := \mathfrak{M}^{\text{simp,rig}}_{g,n,d,r} \times_{\mathfrak{M}_{g,n}} S.$$

If $f : C \to S$ is the family of prestable curves associated to $S \to \mathfrak{M}_{g,n}$ then $\mathfrak{M}_S$ is equivalent to the étale sheafification of the functor assigning to an S-scheme T the set of T-flat, T-finitely presented, simple, uniform rank r, degree d, torsion-free coherent sheaves over C_T. By [9, Theorem 7.4], $\mathfrak{M}_S$ is an algebraic space which is locally finitely presented over S, and so the natural morphism $\mathfrak{M}^{\text{simp,rig}}_{g,n,d,r} \to \mathfrak{M}_{g,n}$ is representable and locally of finite presentation. As $\mathfrak{M}_{g,n}$ is algebraic, it follows that $\mathfrak{M}^{\text{simp,rig}}_{g,n,d,r}$ is algebraic.

Base-changing to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ shows that the natural morphism $c : \overline{\mathcal{J}}^{\text{simp}}_{g,n,d,r}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ is also representable and locally of finite presentation. As $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is Deligne–Mumford, the same is therefore true for $\overline{\mathcal{J}}^{\text{simp}}_{g,n,d,r}(X,\beta)$.

In the case where X is smooth, the pullback of the perfect relative obstruction theory (6.2.1) along c defines a perfect relative obstruction theory

$$\mathbb{E}^{\bullet}_{\overline{\mathcal{J}}^{\text{simp}}_{g,n,d,r}(X,\beta)/\mathfrak{M}^{\text{simp,rig}}_{g,n,d,r}} := c^*\mathbb{E}^{\bullet}_{\overline{\mathcal{M}}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}} \to \mathbb{L}^{\bullet}_{\overline{\mathcal{J}}^{\text{simp}}_{g,n,d,r}(X,\beta)/\mathfrak{M}^{\text{simp,rig}}_{g,n,d,r}}, \tag{6.3.2}$$

which restricts to give a perfect relative obstruction theory for any open Deligne–Mumford substack of $\overline{\mathcal{J}}^{\text{simp}}_{g,n,d,r}(X,\beta)$. □

Remark. As $\overline{\mathcal{J}ac}_{g,n,d,r}(X,\beta)$ is a $\mathbb{G}_m$-gerbe over the rigidified stack $\overline{\mathcal{J}}_{g,n,d,r}(X,\beta)$, it follows from Proposition 6.3.1 that the stack $\overline{\mathcal{J}ac}^{\mathrm{simp}}_{g,n,d,r}(X,\beta)$ is also algebraic.

6.3.2 Substacks Defined by Polarisations

Let $\pi_{\mathcal{U}} : \mathcal{U}\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ be the universal family over $\overline{\mathcal{M}}_{g,n}(X,\beta)$, and fix a relatively ample $\mathbb{Q}$-invertible sheaf $\mathcal{L}$ on the universal curve $\mathcal{U}\overline{\mathcal{M}}_{g,n}(X,\beta)$. For any family of coherent sheaves F over a family of maps $(\pi : C \to S; \sigma_1, \ldots, \sigma_n; f : C \to X)$ corresponding to a morphism $S \to \overline{\mathcal{M}}_{g,n}$, by pulling back $\mathcal{L}$ we obtain a relatively ample $\mathbb{Q}$-invertible sheaf $\mathcal{L}_S$ over C.

Definition 6.3.2. *We say F is* $\mathcal{L}$-(semi)stable *if for each geometric point $s \in S$, the sheaf F_s is Gieseker (semi)stable with respect to the ample $\mathbb{Q}$-invertible sheaf $\mathcal{L}_s$ (cf. paragraph after Inequality 5.2.5).*

The conditions of being stable/semistable are both open in flat families, so there are open substacks

$$\overline{\mathcal{J}ac}^{(s)s}_{g,n,d,r}(X,\beta)(\mathcal{L}) \subset \overline{\mathcal{J}ac}_{g,n,d,r}(X,\beta) \text{ and } \overline{\mathcal{J}}^{(s)s}_{g,n,d,r}(X,\beta)(\mathcal{L}) \subset \overline{\mathcal{J}}_{g,n,d,r}(X,\beta)$$

parametrising objects whose sheaves are $\mathcal{L}$-(semi)stable. As Gieseker stable sheaves are always simple, we have open immersions

$$\overline{\mathcal{J}ac}^{s}_{g,n,d,r}(X,\beta)(\mathcal{L}) \subset \overline{\mathcal{J}ac}^{\mathrm{simp}}_{g,n,d,r}(X,\beta) \text{ and } \overline{\mathcal{J}}^{s}_{g,n,d,r}(X,\beta)(\mathcal{L}) \subset \overline{\mathcal{J}}^{\mathrm{simp}}_{g,n,d,r}(X,\beta).$$

Applying Proposition 6.3.1, the stack $\overline{\mathcal{J}ac}^{s}_{g,n,d,r}(X,\beta)(\mathcal{L})$ is algebraic, and the rigidified stack $\overline{\mathcal{J}}^{s}_{g,n,d,r}(X,\beta)(\mathcal{L})$ is Deligne–Mumford and admits a perfect relative obstruction theory over $\mathfrak{M}^{\mathrm{simp,rig}}_{g,n,d,r}$.

If $(C, \underline{x}, f)$ is a stable map in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ with C irreducible, Inequality 5.2.5 reduces to the usual inequality between the Mumford slopes $\mu(E) = \deg(E)/\mathrm{rk}(E)$. If C is additionally non-singular then all torsion-free sheaves on C are locally free. This implies that $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ contains all slope semistable locally free sheaves over stable maps whose source curve is non-singular. In particular, if $\mathcal{M}_{g,n}(X,\beta)$ is non-empty then the stack $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ is also non-empty.

Remark. It follows from Inequality 5.2.5 that replacing $\mathcal{L}$ with a positive integral power of $\mathcal{L}$ does not alter the stability condition. As such, we are free to assume as and when necessary that $\mathcal{L}$ is a genuine relatively (very) ample invertible sheaf.

Remark. Proposition 6.3.1 does not address the issue as to whether $\overline{\mathcal{J}ac}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ and $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ are algebraic in general. The algebraicity of these stacks will follow as a consequence of Theorem 6.5.5.

Remark. Given an invertible sheaf $\mathcal{N}$ on the universal curve $\mathcal{U}\overline{\mathcal{M}}_{g,n}(X,\beta)$, one may consider the stacks $\overline{\mathcal{J}ac}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L},\mathcal{N})$ and $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L},\mathcal{N})$ parametrising $(\mathcal{L},\mathcal{N})$-twisted semistable sheaves of degree d and uniform rank r over stable maps in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ (that is, in Definition 6.3.2 we impose that for each geometric point $s \in S$, $F_s \otimes \mathcal{N}_s$ is Gieseker semistable with respect to $\mathcal{L}_s$). This does not widen the class of stacks under consideration, since twisting by $\mathcal{N}$ yields isomorphisms

$$\overline{\mathcal{J}ac}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L},\mathcal{N}) \cong \overline{\mathcal{J}ac}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L},\mathcal{O})$$

and

$$\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L},\mathcal{N}) \cong \overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L},\mathcal{O}).$$

In particular, the compactified Jacobians considered by Kass–Pagani [72] and Melo [89] fit into the framework considered in this chapter, since the stability conditions considered by these authors can always be realised as twisted stability conditions.

6.3.3 Basic Properties of the Stacks

We summarise some of the basic properties of the stacks $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$.

1. The stack $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ is algebraic and of finite presentation over $\mathbb{C}$; this follows immediately from Theorem 6.1.1.

2. The stack $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ is universally closed. This will follow once we have established the projectivity of the good moduli space $\overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ of this stack, as good moduli space morphisms are universally closed.

3. If $\overline{\mathcal{J}}^{s}_{g,n,d,r}(X,\beta)(\mathcal{L}) = \overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ then the stack is Deligne–Mumford and the natural forgetful morphism to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is representable. If X is smooth then the stack admits a natural relative perfect obstruction theory over the stack $\mathfrak{M}^{\mathrm{simp,rig}}_{g,n,d,r}$; this follows from Proposition 6.3.1.

4. If $\overline{\mathcal{J}}^{s}_{g,n,d,r}(X,\beta)(\mathcal{L}) = \overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ then the stack is proper. One way to see this is to observe that every object of $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ has a finite automorphism

group scheme, or in other words the inertia stack of $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ is finite. It then follows from [35] that the coarse moduli space morphism $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}) \to \overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ is proper, so the properness of the stack $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ is a consequence of the projectivity of $\overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$.

In the case $r = 1$, we have the following additional properties:

5. If $n \geq 1$ then the universal sheaf on $\overline{\mathcal{J}ac}^{ss}_{g,n,d,1}(X,\beta)(\mathcal{L})$ descends to the rigidification $\overline{\mathcal{J}}^{ss}_{g,n,d,1}(X,\beta)(\mathcal{L})$; this follows from either [71, Lemma 3.35] or [89, Proposition 3.5].

6. If the stack of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is smooth then the stack $\overline{\mathcal{J}}^{ss}_{g,n,d,1}(X,\beta)(\mathcal{L})$ is also smooth; for instance, the argument given in the proof of [89, Proposition 3.7] carries over. If in addition the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is irreducible and if the substack $\mathcal{M}_{g,n}(X,\beta)$ is dense in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ then the stack $\overline{\mathcal{J}}^{ss}_{g,n,d,1}(X,\beta)(\mathcal{L})$ is irreducible and of dimension $\dim \overline{\mathcal{M}}_{g,n}(X,\beta) + g$, as in this case the open substack parametrising invertible sheaves over stable maps in $\mathcal{M}_{g,n}(X,\beta)$ is smooth and dense. This applies for instance when the stack of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is taken to be $\overline{\mathcal{M}}_{g,n}$ or $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^b, d')$.

6.4 GIT Constructions of Moduli Spaces of Stable Maps and of Multi-Gieseker Semistable Sheaves

The GIT construction of the moduli spaces appearing in the statement of Theorem 6.1.1 relies on two existing reductive GIT constructions, the first by Baldwin–Swinarski [15] and the second by Greb–Ross–Toma [55] [56]. In preparation for Section 6.5, where these constructions are combined to construct the moduli spaces $\overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$, we give a sketch of both of these constructions, and explain how the latter construction may be extended to the relative setting. To avoid the notation from becoming too complicated, we present the Greb–Ross–Toma construction in a general setting, largely following the notation used in [55] and [56].

6.4.1 The Construction of Baldwin and Swinarski

Let X be a projective variety, and fix the discrete invariants g, n and β. For simplicity assume there is at least one marked point (the construction for $n = 0$ marked points involves only minor changes in notation, primarily involving omitting any projective space factors corresponding to marked points). Fix a closed embedding $i : X \hookrightarrow \mathbb{P}^b$ such that $i_*\beta = d'[\ell]$, where $[\ell] \in H_2(\mathbb{P}^b, \mathbb{Z})_+$ is the class of a line. Suppose $(C, \underline{x}, f)$ is a stable map to X of class β (which we may also consider as a stable map to $\mathbb{P}^b$ of class $d'[\ell]$). For $a \in \mathbb{Z}$, set

$$e := \deg((\omega_C(x_1 + \cdots + x_n) \otimes f^*\mathcal{O}_{\mathbb{P}^b}(3))^a) = a(2g - 2 + n + 3d').$$

For any integer $a \geq 10$,[5] the invertible sheaf $(\omega_C(x_1+\cdots+x_n)\otimes f^*\mathcal{O}_{\mathbb{P}^b}(3))^a$ is non-special and very ample, and corresponds to a closed embedding of C inside $\mathbb{P}(H^0(C, (\omega_C(x_1 + \cdots + x_n) \otimes f^*\mathcal{O}_{\mathbb{P}^b}(3))^a)^\vee)$. Let

$$W := \mathbb{C}^{e-g+1}.$$

Choose an isomorphism $W \cong H^0(C, (\omega_C(x_1 + \cdots + x_n) \otimes f^*\mathcal{O}_{\mathbb{P}^b}(3))^a)^\vee$; this yields a closed embedding $C \hookrightarrow \mathbb{P}(W)$. The graph of f gives in turn a closed embedding $C \hookrightarrow \mathbb{P}(W) \times \mathbb{P}^b$. After appending the markings $(x_i, f(x_i))$, we have an associated point of the scheme

$$\mathcal{H} = \mathrm{Hilb}(\mathbb{P}(W) \times \mathbb{P}^b, P_0) \times (\mathbb{P}(W) \times \mathbb{P}^b)^{\times n},$$

where $\mathrm{Hilb}(\mathbb{P}(W) \times \mathbb{P}^b, P_0)$ is the Hilbert scheme parametrising curves $C' \subset \mathbb{P}(W) \times \mathbb{P}^b$ with Hilbert polynomial

$$P_0(m_W, m_b) = \chi(\mathcal{O}_{\mathbb{P}(W)}(m_W) \otimes \mathcal{O}_{\mathbb{P}^b}(m_b) \otimes \mathcal{O}_{C'}) = em_W + dm_b - g + 1.$$

Let $(\phi_{\mathcal{U}} : \mathcal{UH} \to \mathcal{H}; \tau_1, \ldots, \tau_n)$ denote the universal family over $\mathcal{H}$, given by taking the product of the universal family over $\mathrm{Hilb}(\mathbb{P}(W) \times \mathbb{P}^b, P_0)$ with n copies of the identity map on $\mathbb{P}(W) \times \mathbb{P}^b$. Let $\mathcal{H}' \subset \mathcal{H}$ denote the closed subscheme consisting of all points $(h; x_1, \ldots, x_n) \in \mathcal{H}$ where the marked points x_i lie on the curve $\mathcal{UH}_h \subset \mathbb{P}(W) \times \mathbb{P}^b$. As explained in [52, Sections 2.3 and 5.1], there exists a locally closed subscheme $I = I_{X,\beta} \subset \mathcal{H}'$ consisting of all points $(h; x_1, \ldots, x_n) \in \mathcal{H}'$ such that the following conditions are satisfied:[6]

(i) $(\mathcal{UH}_h; x_1, \ldots, x_n)$ is a prestable marked curve;

[5] $a = 10$ is the smallest choice of a for which the statement of [15, Theorem 5.21] holds for any choice of valid discrete invariants g, n, d'; we refer the reader to the remark following *loc. cit.*

[6] The subscheme $\mathcal{H}'$ is what Baldwin and Swinarski denote as I and the subscheme I is what Baldwin and Swinarski denote as J; we have elected to change the notation so as not to confuse notation with that for the moduli spaces of the stacks $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X, \beta)(\mathcal{L})$.

(ii) the projection $\mathcal{UH}_h \to \mathbb{P}(W)$ is a closed embedding with non-degenerate image;

(iii) the projection $p_b : \mathcal{UH}_h \to \mathbb{P}^b$ factors through the inclusion $i : X \hookrightarrow \mathbb{P}^b$;

(iv) there is an equality of homology classes $(p_b)_*[\mathcal{UH}_h] = \beta \in H_2(X, \mathbb{Z})_+$; and

(v) there exists an isomorphism of invertible sheaves on the curve $\mathcal{UH}_h$

$$\mathcal{O}_{\mathbb{P}(W)}(1) \otimes \mathcal{O}_{\mathbb{P}^b}(1) \otimes \mathcal{O}_{\mathcal{UH}_h} \cong (\omega_{\mathcal{UH}_h}(x_1 + \cdots + x_n))^a \otimes \mathcal{O}_{\mathbb{P}^b}(3a+1) \otimes \mathcal{O}_{\mathcal{UH}_h}$$

(more formally, if $S \to \mathcal{H}'$ is a morphism from a scheme, then this morphism factors through I only if the invertible sheaves $\mathcal{O}_{\mathbb{P}(W)}(1) \otimes \mathcal{O}_{\mathbb{P}^b}(1) \otimes \mathcal{O}_{\mathcal{UH}'_S}$ and $(\omega_{\mathcal{UH}'_S/S}(\tau_1(S) + \cdots + \tau_n(S)))^a \otimes \mathcal{O}_{\mathbb{P}^b}(3a+1) \otimes \mathcal{O}_{\mathcal{UH}'_S}$ differ by the pullback of an invertible sheaf on S).

Let $\overline{I}$ denote the closure of I in $\mathcal{H}'$, and consider the restriction of the universal family to I, denoted as $(\phi_{\mathcal{U}} : \mathcal{U}I \to I; \tau_1, \ldots, \tau_n)$. There is a universal morphism $\mathcal{U}I \to X$ given by the composition of the closed embedding

$$\mathcal{U}I \hookrightarrow I \times \mathbb{P}(W) \times \mathbb{P}^b \times (\mathbb{P}(W) \times \mathbb{P}^b)^{\times n}$$

with the projection onto the first $\mathbb{P}^b$ factor. The restriction of this morphism over each point of I defines a stable map to X of class β. It follows that the scheme I admits a natural forgetful morphism to $\overline{\mathcal{M}}_{g,n}(X, \beta)$, given by forgetting the embedding of the stable map inside $\mathbb{P}(W) \times \mathbb{P}^b$.

Letting $GL(W)$ act in the usual way on $\mathbb{P}(W)$ and act trivially on $\mathbb{P}^b$, the corresponding diagonal action on $\mathbb{P}(W) \times \mathbb{P}^b$ gives rise to an induced action of $GL(W)$ on $\mathcal{H}$, under which the subschemes $\mathcal{H}'$, I and $\overline{I}$ are all invariant. This action descends to an action of $PGL(W)$. The family $(\phi_{\mathcal{U}} : \mathcal{U}I \to I; \tau_1, \ldots, \tau_n)$ has the local universal property for $\overline{\mathcal{M}}_{g,n}(X, \beta)$, and two points $i_1, i_2 \in I$ correspond to isomorphic stable maps if and only if i_1 and i_2 lie in the same $GL(W)$-orbit (cf. [15, Proposition 3.4]).

For sufficiently large positive integers m_W and m_b, there is a closed immersion

$$\mathrm{Hilb}(\mathbb{P}(W) \times \mathbb{P}^b, P_0) \hookrightarrow \mathbb{P}(\Lambda^{P_0(m_W, m_b)} Z_{m_W, m_b}),$$

where

$$Z_{m_W, m_b} = H^0(\mathbb{P}(W) \times \mathbb{P}^b, \mathcal{O}_{\mathbb{P}(W)}(m_W) \otimes \mathcal{O}_{\mathbb{P}^b}(m_b)).$$

We extend this to a closed embedding

$$\mathcal{H} = \mathrm{Hilb}(\mathbb{P}(W) \times \mathbb{P}^b, P_0) \times (\mathbb{P}(W) \times \mathbb{P}^b)^{\times n} \hookrightarrow \mathbb{P}(\Lambda^{P_0(m_W, m_b)} Z_{m_W, m_b}) \times (\mathbb{P}(W) \times \mathbb{P}^b)^{\times n}$$

by taking the identity on $(\mathbb{P}(W) \times \mathbb{P}^b)^{\times n}$. After choosing a positive integer m_{pts}, we have a $GL(W)$-equivariant closed embedding of $\mathcal{H}$ inside a single projective space:

$$\theta_{m_W, m_b, m_{\mathrm{pts}}} : \mathcal{H} \hookrightarrow \mathbb{P}\left(\Lambda^{P_0(m_W, m_b)} Z_{m_W, m_b} \otimes \bigotimes_{i=1}^{n} (\mathrm{Sym}^{m_{\mathrm{pts}}} W \otimes \mathrm{Sym}^{m_{\mathrm{pts}}} \mathbb{C}^{b+1})\right). \tag{6.4.1}$$

Pulling back the twisting sheaf $\mathcal{O}(1)$ along this embedding gives a very ample linearisation $L_{m_W, m_b, m_{\mathrm{pts}}}$ for the action of $GL(W)$ on both $\mathcal{H}$ and the closed subscheme $\overline{I}$. We may now state the main result of Baldwin and Swinarski.

Theorem 6.4.1 ([15], Corollary 6.2)**.** *Suppose $(m_W, m_b, m_{\mathrm{pts}})$ is a triple of positive integers satisfying the inequalities in the statement of Theorem 6.1 of loc. cit. (such triples always exist). Then for the induced action of $SL(W)$ on $\overline{I}$ there are equalities*

$$\overline{I}^{ss}(L_{m_W, m_b, m_{\mathrm{pts}}}) = \overline{I}^{s}(L_{m_W, m_b, m_{\mathrm{pts}}}) = I.$$

Moreover, the coarse moduli space $\overline{M}_{g,n}(X, \beta)$ of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is isomorphic to the GIT quotient $\overline{I} /\!\!/_{L_{m_W, m_b, m_{\mathrm{pts}}}} SL(W)$. In particular $\overline{M}_{g,n}(X, \beta)$ is a geometric quotient of I.

6.4.2 Quiver Representation Stability

Next, we consider the construction of Greb–Ross–Toma [55] [56] of the moduli space of multi-Gieseker semistable sheaves on a projective scheme.

Given $k \in \mathbb{N}$, consider the labelled quiver

$$\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, h, t : \mathcal{Q}_1 \to \mathcal{Q}_0, H : \mathcal{Q}_1 \to \mathbf{Vect}_{\mathbb{C}}^{\mathrm{fd}})$$

with vertex set $\mathcal{Q}_0 = \{v_i, w_j : i, j = 1, \ldots, k\}$, arrow set $\mathcal{Q}_1 = \{\alpha_{ij} : i, j = 1, \ldots, k\}$ and with heads and tails given by

$$h(\alpha_{ij}) = v_i, \quad t(\alpha_{ij}) = w_j$$

(the functor $H : \mathcal{Q}_1 \to \mathbf{Vect}_{\mathbb{C}}^{\mathrm{fd}}$ will be specified momentarily). Let H_{ij} be the vector space $H(\alpha_{ij})$.

Definition 6.4.2. *A* representation *of the labelled quiver $\mathcal{Q}$ consists of the data $\psi = (\{V_i\}_{i=1}^k, \{W_j\}_{j=1}^k, \{\psi_{ij}\}_{i,j=1}^k)$, where the V_i and W_j are finite-dimensional $\mathbb{C}$-vector spaces and where each ψ_{ij} is a linear map $V_i \otimes H_{ij} \to W_j$.*

Morphisms of representations of $\mathcal{Q}$ are defined in analogy with morphisms of ordinary quiver representations (cf. [75]); in particular, a subrepresentation ψ' of ψ is specified by choices of subspaces $V_i' \subset V_i$ and $W_j' \subset W_j$, such that each ψ_{ij} sends $V_i' \otimes H_{ij}$ to W_j'.

The dimension vector *of ψ is the vector $\underline{d} = (\dim V_1, \dim W_1, \ldots, \dim V_k, \dim W_k)$.*

Write the dimension vector of $M = \bigoplus_j V_j \oplus W_j$ as $\underline{d} = (d_{11}, d_{12}, \ldots, d_{k1}, d_{k2})$. Given a tuple $\sigma = (\sigma_1, \ldots, \sigma_k)$ of non-negative rational numbers, not all of which are zero, define the vector $\theta_\sigma = (\theta_{11}, \theta_{12}, \ldots, \theta_{k1}, \theta_{k2})$ by setting

$$\theta_{j1} := \frac{\sigma_j}{\sum_i \sigma_i d_{i1}}, \quad \theta_{j2} := -\frac{\sigma_j}{\sum_i \sigma_i d_{i2}}.$$

Then, for any representation $M' = \bigoplus_j V_j' \oplus W_j'$, define

$$\theta_\sigma(M') = \sum_{j=1}^k (\theta_{j1} \dim V_j' + \theta_{j2} \dim W_j').$$

Definition 6.4.3. *Let M be a $\mathcal{Q}$-representation with dimension vector $\underline{d}$. M is said to be* (semi)stable with respect to σ *if for all proper subrepresentations $M' \subset M$, we have $\theta_\sigma(M') < (\leq)\ 0$.*

Analogously to ordinary quiver representations, every σ-semistable $\mathcal{Q}$-representation admits a Jordan–Hölder filtration, and hence a *Jordan–Hölder associated graded* object $\mathrm{gr}_\sigma^{\mathrm{JH}}(M)$. This gives a notion of *S-equivalence* for σ-semistable representations; two σ-semistable representations M and M' are S-equivalent if and only if $\mathrm{gr}_\sigma^{\mathrm{JH}}(M) \cong \mathrm{gr}_\sigma^{\mathrm{JH}}(M')$.

Fix a dimension vector $\underline{d}$. Let R be the vector space

$$R = \mathrm{Rep}(\mathcal{Q}, \underline{d}) = \bigoplus_{i,j=1}^k \mathrm{Hom}(\mathbb{C}^{d_{i1}} \otimes H_{ij}, \mathbb{C}^{d_{j2}}).$$

Each point of R corresponds to a representation of $\mathcal{Q}$ of dimension vector $\underline{d}$. There is a natural linear left action on R by

$$\widetilde{G} = \prod_{j=1}^k (GL(d_{j_1}, \mathbb{C}) \times GL(d_{j2}, \mathbb{C})),$$

where $\widetilde{G}$ acts by base change automorphisms. Let

$$\Delta = \left\{(t \cdot \mathrm{id}_{d_{11}}, \ldots, t \cdot \mathrm{id}_{d_{k2}}) \in \widetilde{G} : t \in \mathbb{G}_m\right\}$$

be the kernel of the representation $\widetilde{G} \to GL(R)$ and let $G = \widetilde{G}/\Delta$. Given an integral vector $\theta = (\theta_{j1}, \theta_{j2}, \ldots, \theta_{k1}, \theta_{k2}) \in \mathbb{Z}^{2k}$, let χ_θ be the character of $\widetilde{G}$ given by

$$\chi_\theta : g \mapsto \prod_{j=1}^{k} (\det(g_{j1})^{-\theta_{j1}} \cdot \det(g_{j2})^{-\theta_{j2}}).$$

Assume $\underline{d}$ and θ are chosen so that $\sum_j(\theta_{j1}d_{j1} + \theta_{j2}d_{j2}) = 0$, so that we may (and do) view χ_θ as a character of G. Let $R^{\theta-(s)s} \subset R$ denote the GIT (semi)stable loci for the action of G on R with the linearisation $\mathcal{O}_R(\chi_\theta)$, the trivial bundle on R twisted by the character χ_θ, and let $p : R^{\theta-ss} \to R /\!\!/_\theta G$ denote the resulting good quotient. In this setting, the results of King [75] concerning moduli of representations of a quiver translate as follows.

Proposition 6.4.4 ([55], Theorem 5.5)**.** *Suppose $\theta = c\theta_\sigma$ for some σ and some positive integer c, where c is chosen such that θ is integral.*

1. *A point of R corresponding to a $\mathcal{Q}$-representation M is GIT (semi)stable with respect to θ if and only if M is (semi)stable with respect to σ.*
2. *If $r_1, r_2 \in R^{\theta-ss}$ are points corresponding to representations M_1 and M_2 respectively, then $p(r_1) = p(r_2)$ if and only if M_1 and M_2 are S-equivalent as σ-semistable representations.*

6.4.3 The Functorial Approach to Moduli Spaces of Sheaves

Let X be a projective scheme, and fix a stability parameter $\sigma = (\underline{L}, \sigma_1, \ldots, \sigma_k)$ on X (cf. Section 4.4). Fix a topological type τ of coherent sheaves on X defined with respect to $\underline{L}$, and let $P_j(t)$ be the corresponding Hilbert polynomials. Set $H_{ij} = H^0(X, L_i^{-m_1} \otimes L_j^{m_2}) = \mathrm{Hom}_X(L_j^{-m_2}, L_i^{-m_1})$, where $m_2 > m_1 > 0$ are integers to be determined. Let

$$T = \bigoplus_{j=1}^{k} L_j^{-m_1} \oplus L_j^{-m_2},$$

and consider the algebra

$$A = L \oplus H \subset \mathrm{End}_X(T),$$

where L is generated by the projection operators onto the summands $L_i^{-m_1}$ and $L_j^{-m_2}$ of T and where $H = \bigoplus_{i,j=1}^k H_{ij}$. T is a left A-module and H is an L-bimodule. The category of representations of $\mathcal{Q}$ is equivalent to the category of finite-dimensional, right A-modules M; from now on we identify these two categories.

Given a coherent sheaf E on X, the vector space $\mathrm{Hom}_X(T, E)$ has a natural A-module structure, given by the decomposition

$$\mathrm{Hom}_X(T,E) = \bigoplus_{j=1}^{k} H^0(E \otimes L_j^{m_1}) \oplus H^0(E \otimes L_j^{m_2})$$

together with the multiplication maps $H^0(E \otimes L_i^{m_1}) \otimes H_{ij} \to H^0(E \otimes L_j^{m_2})$. This defines a functor

$$\mathrm{Hom}_X(T, -) : \mathbf{Coh}(X) \to \mathbf{mod}(A)$$

to the category of finitely generated A-modules. The functor $\mathrm{Hom}_X(T, -)$ admits a left adjoint, denoted $- \otimes_A T$. If $\mu : H \otimes_L T \to T$ is the left A-module structure map, then $M \otimes_A T$ is the cokernel of the map

$$1 \otimes \mu - \alpha \otimes 1 : M \otimes_L H \otimes_L T \to M \otimes_L T.$$

If $m_2 \gg m_1$, by [55, Theorem 3.4] the functor $\mathrm{Hom}_X(T, -)$ is fully faithful on the full subcategory of $(m_1, \underline{L})$-regular coherent sheaves on X of topological type τ.

Now consider the case where X is a scheme which is projective over a finite type scheme Y. Fix a projective morphism (of finite type) $\pi : X \to Y$,[7] and replace $\underline{L}$ with $\underline{\mathcal{L}}$, a tuple of π-very ample invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_k$. Continue to fix a topological type τ of Y-flat sheaves on X (this time defined with respect to the $\mathcal{L}_i$); let $P_1(t), \ldots, P_k(t)$ be the associated Hilbert polynomials. Replace H_{ij} with the sheaf $\pi_*(\mathcal{L}_i^{-m_1} \otimes \mathcal{L}_j^{m_2})$, so that A is now a sheaf of $\mathcal{O}_Y$-algebras, acting naturally on T.

Definition 6.4.5. *Let $g : S \to Y$ be a scheme over Y. A* flat family of A-modules over S *consists of a sheaf $\mathcal{M}$ of right modules over the sheaf of algebras $A_S = g^*A$, which is locally free of finite rank as an $\mathcal{O}_S$-module.*

If S is a Y-scheme and if E is a coherent sheaf on $X_S = X \times_Y S$, set

$$\mathcal{H}om_X(T, E) := (\pi_S)_*(\mathcal{H}om_{X_S}(T_S, E)) \in \mathbf{Coh}(S).$$

[7]In Section 6.5 this projective morphism will be taken to be the universal family $\mathcal{U}I \to I$.

This is naturally a coherent sheaf of right A_S-modules. The functor $- \otimes_A T$ extends to give a left adjoint of $\mathcal{H}om_X(T, -)$.

Remark. In Proposition 6.4.6 and in all of the statements that follow, $m_2 - m_1$ can and should be chosen large enough such that each of the sheaves $H_{ij} = \pi_*(\mathcal{L}_i^{-m_1} \otimes \mathcal{L}_j^{m_2})$ are locally free and the formation of the pushforwards all commute with base change.

Proposition 6.4.6 ([55] Propositions 5.8, 5.9, [10] Propositions 4.1, 4.2 in the relative setting)**.** *Let m_1 be a natural number. Then for $m_2 \gg m_1$, the following holds.*

1. *For any Y-scheme S, the functor $\mathcal{H}om_X(T, -)$ is a fully faithful functor from the full subcategory of S-flat, $(m_1, \underline{\mathcal{L}})$-regular coherent sheaves on X_S of topological type τ to the full subcategory of* $\mathbf{mod}(A_S)$ *consisting of flat families of A-modules over S.*

2. *If B is any Y-scheme and if $\mathcal{M}$ is any flat family of A-modules over B of dimension vector*
$$\underline{d} = (P_1(m_1), P_1(m_2), \ldots, P_k(m_1), P_k(m_2)),$$
then there exists a unique locally closed subscheme $\iota : B_\tau^{[reg]} \hookrightarrow B$ with the following properties:
   - *(i) $\iota^*\mathcal{M} \otimes_A T$ is a $B_\tau^{[reg]}$-flat, $B_\tau^{[reg]}$-finitely presented family of $(m_1, \underline{\mathcal{L}})$-regular sheaves on X of topological type τ, and the unit map $\iota^*\mathcal{M} \to \mathcal{H}om_X(T, \iota^*\mathcal{M} \otimes_A T)$ is an isomorphism.*
   - *(ii) If $g : S \to B$ is such that there exists an S-flat, S-finitely presented family E of $(m_1, \underline{\mathcal{L}})$-regular sheaves on X of topological type τ and an isomorphism $g^*\mathcal{M} \cong \mathcal{H}om_X(T, E)$, then g factors through $\iota : B_\tau^{[reg]} \hookrightarrow B$ and $E \cong g^*\mathcal{M} \otimes_A T$.*

Proof. This follows from making minor modifications to the proofs of [55, Propositions 5.8 and 5.9], these modifications being analogous to those indicated in [10, Section 6.5]. □

6.4.4 Comparison of Semistability - Positive Case

Given a coherent sheaf E of topological type τ on a projective scheme X, we have the notion of multi-Gieseker stability for the sheaf E (cf. Section 4.4) and the notion of stability for the A-module $\mathrm{Hom}_X(T, E)$, both with respect to a common stability parameter $\sigma = (\underline{L}, \sigma_1, \dots, \sigma_k)$. Provided σ is bounded, the following result relates these two notions.

Theorem 6.4.7 ([10] Section 5, [55] Theorem 8.1)**.** *Let $\pi : X \to Y$ be a projective morphism, where Y is a scheme of finite type. Let $\underline{\mathcal{L}} = (\mathcal{L}_1, \dots, \mathcal{L}_k)$ be a tuple of π-very ample invertible sheaves and let τ be a topological type defined with respect to $\underline{\mathcal{L}}$. Let $\sigma = (\underline{\mathcal{L}}, \sigma_1, \dots, \sigma_k)$ be a stability parameter which is bounded with respect to τ. Then for integers $m_2 \gg m_1 \gg m_0 \gg 0$, the following holds: let E be a Y-flat coherent sheaf on X and let $y \in Y$ be a geometric point. Then:*

1. *The sheaf E_y is multi-Gieseker (semi)stable with respect to σ if and only if E_y is pure, $(m_0, \underline{\mathcal{L}_y})$-regular and $\mathrm{Hom}_{X_y}(T_y, E_y)$ is a σ-(semi)stable A-module.*
2. *Suppose in addition that σ is positive, and suppose E_y is σ-semistable. Then*

$$\mathrm{Hom}_{X_y}(T_y, \mathrm{gr}_\sigma^{\mathrm{JH}}(E_y)) \cong \mathrm{gr}_\sigma^{\mathrm{JH}}(\mathrm{Hom}_{X_y}(T_y, E_y)),$$

where $\mathrm{gr}_\sigma^{\mathrm{JH}}$ denotes the σ-associated graded object arising from a Jordan–Hölder filtration of E_y or of $\mathrm{Hom}_{X_y}(T_y, E_y)$ respectively. In particular, $\mathrm{Hom}_{X_y}(T_y, -)$ preserves S-equivalence with respect to σ.

6.4.5 The GIT Construction - Positive Case

We continue to fix the projective morphism $\pi : X \to Y$, the π-very ample invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_k$ and the topological type τ of Y-flat coherent sheaves on X. Choose natural numbers m_0, m_1 and m_2 such that Proposition 6.4.6 and Theorem 6.4.7 apply. Set the dimension vector to be

$$\underline{d} = (P_1(m_1), P_1(m_2), \dots, P_k(m_1), P_k(m_2)).$$

If E and F are coherent sheaves on Y with F locally free, let $\underline{\mathrm{Hom}}_Y(E, F)$ denote the finite type affine Y-scheme representing the functor assigning to a Y-scheme $g : S \to Y$ the set $\mathrm{Hom}_S(g^*E, g^*F)$ (such a scheme exists and is affine over Y by, for instance, [127, Tag 08JY]). With $H_{ij} = \pi_*(\mathcal{L}_i^{-m_1} \otimes \mathcal{L}_j^{m_2})$, form the Y-scheme

$$R = \prod_{i,j=1}^{k} \underline{\mathrm{Hom}}_Y(\mathcal{O}_Y^{d_{i1}} \otimes H_{ij}, \mathcal{O}_Y^{d_{j2}}).$$

Let $p : R \to Y$ denote the structure morphism. The scheme R carries a tautological family $\mathbb{M}$ of A-modules. By Proposition 6.4.6, there exists a locally closed subscheme

$$\iota_0 : R_\tau^{[reg]} \hookrightarrow R$$

parametrising the modules which appear in the image of the $\mathcal{H}om_X(T, -)$-functor on the full subcategory of all $(m_1, \underline{\mathcal{L}})$-regular coherent sheaves of topological type τ. Then $\iota_0^*\mathbb{M} \otimes_A T$ is a flat, finitely presented family of $(m_1, \underline{\mathcal{L}})$-regular sheaves of topological type τ. Let

$$\iota : Q \hookrightarrow R_\tau^{[reg]}$$

denote the open subscheme parametrising those sheaves which are also $(m_0, \underline{\mathcal{L}})$-regular, and let $\mathbb{F} = \iota^*\mathbb{M} \otimes_A T$. Let

$$Q^{[\sigma-ss]} \subset Q$$

denote the open subscheme where the fibres of $\mathbb{F}$ are σ-semistable. Under the natural fibrewise (over Y) action of G on R, where $G = \prod_{j=1}^k (GL(d_{j_1}, \mathbb{C}) \times GL(d_{j2}, \mathbb{C}))/\Delta$, the subschemes Q and $Q^{[\sigma-ss]}$ are invariant. Endow R with the linearisation $\mathcal{O}_R(\chi_\theta)$, where $\theta = c\theta_\sigma$ for some positive integer c chosen so that θ is integral, and let

$$R^{\sigma-ss} := R^{G-ss}(\chi_\theta/Y)$$

be the corresponding relative GIT semistable locus (cf. Definition 2.4.1). Let $\overline{Q^{[\sigma-ss]}}$ be the closure of $Q^{[\sigma-ss]}$ in R, and set

$$(\overline{Q^{[\sigma-ss]}})^{\sigma-ss} := \overline{Q^{[\sigma-ss]}} \cap R^{\sigma-ss}.$$

Theorem 6.4.8 ([10] Section 6, [55] Sections 9-10)**.** *Suppose σ is a positive stability parameter which is bounded with respect to τ. Then there is an equality of schemes $(\overline{Q^{[\sigma-ss]}})^{\sigma-ss} := \overline{Q^{[\sigma-ss]}} \cap R^{\sigma-ss} = Q^{[\sigma-ss]}$, and there exists a good quotient $q : Q^{[\sigma-ss]} \to M_{\sigma,\tau}$ of Y-schemes for the action of G on $Q^{[\sigma-ss]}$, where*

$$M_{\sigma,\tau} = \overline{Q^{[\sigma-ss]}} /\!\!/_{\theta_\sigma} G = Q^{[\sigma-ss]} /\!\!/ G = \mathbf{Proj}_Y \left(\bigoplus_{m=0}^{\infty} (p_*(\mathcal{O}_R(\chi_{\theta_\sigma}^m)|_{\overline{Q^{[\sigma-ss]}}}))^G \right).$$

The scheme $M_{\sigma,\tau}$ is projective over Y. Moreover, the closed points of $M_{\sigma,\tau}$ are in 1-1 correspondence with S-equivalence classes of σ-semistable sheaves over geometric fibres of $\pi : X \to Y$.

Proof. Suppose first $Y = \operatorname{Spec}\mathbb{C}$ is a point (that is, we are working with a fixed fibre over some geometric point $y \in Y$). That there is an equality $\overline{Q^{[\sigma-ss]}} \cap R^{\sigma-ss} = Q^{[\sigma-ss]}$, and that there is a good quotient $M_{\sigma,\tau}$ of $Q^{[\sigma-ss]}$ given by

$$M_{\sigma,\tau} = \operatorname{Proj}\left(\bigoplus_{m=0}^{\infty} H^0(\overline{Q^{[\sigma-ss]}}, \mathcal{O}_R(\chi_{\theta_\sigma}^m)|_{\overline{Q^{[\sigma-ss]}}})^G\right),$$

both follow from [55, Theorem 10.1]. The projectivity (over $\operatorname{Spec}\mathbb{C}$) of $M_{\sigma,\tau}$ is Theorem 9.6 of *loc. cit.* (which in turn is proved in the same way as [10, Proposition 6.6], via establishing that the valuative criterion for properness holds), and the identification of the closed points of $M_{\sigma,\tau}$ with S-equivalence classes of semistable sheaves is [55, Theorem 9.4]. This yields the result in the fibrewise setting.

In order to pass from the fibrewise setting to working over the quasi-projective scheme Y, we apply Proposition 2.4.5. Since the points of the relative semistable locus $R^{\sigma-ss} = R^{G-ss}(\chi_\theta/Y)$ are determined by the fibrewise semistable loci, we have an equality $\overline{Q^{[\sigma-ss]}} \cap R^{\sigma-ss} = Q^{[\sigma-ss]}$. We also have the existence of a good quotient $M_{\sigma,\tau}$ of $Q^{[\sigma-ss]}$, given by the relative projective spectrum

$$Q^{[\sigma-ss]} /\!\!/ G = \mathbf{Proj}_Y\left(\bigoplus_{m=0}^{\infty} (p_*(\mathcal{O}_R(\chi_{\theta_\sigma}^m)|_{\overline{Q^{[\sigma-ss]}}}))^G\right).$$

The description of the closed points of $M_{\sigma,\tau}$ is immediate from the fibrewise case. That $M_{\sigma,\tau}$ is proper, and thus projective, over Y follows from the same valuative criterion argument used in the proof of [10, Proposition 6.6], which carries over to the relative case, exactly as in Section 6.5 of *loc. cit.*. □

6.4.6 The GIT Construction - Degenerate Case

Let $\pi : X \to Y$, $\mathcal{L}_1, \ldots, \mathcal{L}_k$ and τ be as in Section 6.4.5, and fix a bounded stability parameter $\sigma = (\underline{\mathcal{L}}, \sigma_1, \ldots, \sigma_k)$. Suppose instead that σ is degenerate, that is some $\sigma_i = 0$. In this case, the second conclusion of Theorem 6.4.7 no-longer applies; we instead proceed as follows.

For integers $m_2 \gg m_1 \gg m_0 \gg 0$, the result of Proposition 6.4.6 and the first conclusion of Theorem 6.4.7 still applies to Y-flat coherent sheaves on X of topological type τ. Fix such integers m_i. Relabelling indices if necessary, we may assume there exists a positive integer $k' < k$ with the property that $\sigma_j > 0$ for all $j \le k'$ and $\sigma_j = 0$ for all

$j > k'$. Let $\underline{\mathcal{L}}' = (\mathcal{L}_1, \ldots, \mathcal{L}_{k'})$ and $\sigma' = (\underline{\mathcal{L}}', \sigma_1, \ldots, \sigma_{k'})$. Then σ' is a positive stability parameter, though with respect to a proper subset of the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_k$.

Form a subquiver $\mathcal{Q}'$ of the original quiver $\mathcal{Q}$ by taking the full subquiver with vertices $\mathcal{Q}'_0 = \{v_i, w_j : i, j = 1, \ldots, k'\}$. Let $T', A', \underline{d}', R', G'$ and Q' be the objects associated with the quiver $\mathcal{Q}'$ (with R' and G' considered as schemes over Y), so that if E is a Y-flat, $(m_1, \underline{\mathcal{L}}')$-regular sheaf on X of topological type τ then $\mathcal{H}om_X(T', E)$ is a flat family of A'-modules of dimension vector $\underline{d}'$. After possibly increasing the m_i, we may also assume that Proposition 6.4.6 and Theorem 6.4.7 apply to the positive stability parameter σ'.

The inclusion $\mathcal{Q}' \subset \mathcal{Q}$ gives rise to a projection

$$\phi' : R \to R'.$$

Continue to set

$$\theta_{j1} = \frac{\sigma_j}{\sum_i \sigma_i d_{i1}}, \quad \theta_{j2} = -\frac{\sigma_j}{\sum_i \sigma_i d_{i2}}$$

for all $j = 1, \ldots, k$. The group G acts on R as $G' \times G''$, where G'' corresponds to the rows $j = k' + 1, \ldots, k$ of $\mathcal{Q}$. Letting G'' act on R' trivially, the projection $\phi' : R \to R'$ is G-equivariant. Consider the subschemes

$$D := \overline{Q^{[\sigma-ss]}} \subset R, \quad D^{\sigma-ss} := D \cap R^{\sigma-ss},$$

where $Q^{[\sigma-ss]} \subset R_\tau^{[reg]}$ is as defined in Section 6.4.5, as well as the subschemes

$$D' := \overline{(Q')^{[\sigma'-ss]}} \subset R', \quad (D')^{\sigma'-ss} := D' \cap (R')^{\sigma'-ss}.$$

Proposition 6.4.9 ([56] Proposition 2.3)**.** *The projection $\phi' : R \to R'$ induces a G'-equivariant, G''-invariant map $D^{\sigma-ss} \to (D')^{\sigma'-ss}$, and the induced map $\hat{\phi}' : D^{\sigma-ss} /\!\!/ G'' \to (D')^{\sigma'-ss} = (Q')^{[\sigma'-ss]}$ is a G'-equivariant isomorphism.*

Proof. In the case $Y = \operatorname{Spec} \mathbb{C}$, this is [56, Proposition 2.3]. The result in the relative setting then follows from the fibrewise setting by invoking Proposition 2.4.5, as in the proof of Theorem 6.4.8. □

Corollary 6.4.10 ([56] Corollary 2.4)**.** *Suppose σ is a degenerate bounded stability parameter. Then there exists a good quotient $M_{\sigma,\tau} := D^{\sigma-ss} /\!\!/ G \cong (D')^{\sigma'-ss} /\!\!/ G'$ of Y-schemes for the action of G on $D^{\sigma-ss}$. The scheme $M_{\sigma,\tau}$ is projective over Y. Moreover, the closed points of $M_{\sigma,\tau}$ are in 1-1 correspondence with S-equivalence classes of σ-semistable sheaves over geometric fibres of $\pi : X \to Y$.* □

6.4.7 The Master Space Construction

Instead of considering a single stability parameter, suppose we have a finite set of bounded stability parameters $\mathfrak{S}$, with each $\sigma \in \mathfrak{S}$ defined with respect to the invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_k$. We allow for some of these stability parameters to be degenerate.

Fix positive integers $m_2 \gg m_1 \gg m_0 \gg 0$ such that for each stability parameter $\sigma \in \mathfrak{S}$, as well as for any positive stability parameter σ' obtained by truncating a degenerate stability condition $\sigma \in \mathfrak{S}$, the conclusions of Proposition 6.4.6 and Theorem 6.4.7 all hold; suppose further that m_0, m_1 and m_2 are chosen so that the formation of the locally free sheaves $H_{ij} = \pi_*(\mathcal{L}_i^{-m_1} \otimes \mathcal{L}_j^{m_2})$ is always compatible with base change. Form the affine Y-scheme

$$Z = Z_{\mathfrak{S}} := \bigcup_{\sigma \in \mathfrak{S}} \overline{Q^{[\sigma-ss]}}. \tag{6.4.2}$$

For each $\sigma \in \mathfrak{S}$, let

$$Z^{\sigma-ss} := Z \cap R^{\sigma-ss}.$$

The schemes $M_{\sigma,\tau}$, for $\sigma \in \mathfrak{S}$, are all GIT quotients of the master space Z.

Theorem 6.4.11 ([55] Theorem 10.1, [56] Theorem 4.2)**.** *For each $\sigma \in \mathfrak{S}$, there is an equality $Z^{\sigma-ss} = (\overline{Q^{[\sigma-ss]}})^{\sigma-ss} = \overline{Q^{[\sigma-ss]}} \cap R^{\sigma-ss}$. In particular, $M_{\sigma,\tau}$ is equal to the good quotient of Y-schemes*

$$Z^{\sigma-ss} /\!\!/ G = \mathbf{Proj}_Y\left(\bigoplus_{m=0}^{\infty} (p_* \mathcal{O}_Z(\chi_\sigma^m))^G\right), \quad \mathcal{O}_Z(\chi_\sigma^m) := \mathcal{O}_R(\chi_\sigma^m)|_Z.$$

Proof. In the case where Y is a point, this is [55, Theorem 10.1] (when σ is positive) and [56, Theorem 4.2] (when σ is degenerate). The result in the relative setting then follows as a consequence of Proposition 2.4.5, as in the proof of Theorem 6.4.8. □

6.4.8 Existence of Further Quotients

Assume now that X and Y both admit actions of a reductive linear algebraic group H, with π an H-equivariant morphism. As in the hypotheses of Proposition 2.4.6, assume there exists an equivariant open immersion $\iota : Y \hookrightarrow \overline{Y}$ into a projective scheme $\overline{Y}$ acted on by H, with an ample H-linearisation L such that (over $\operatorname{Spec} \mathbb{C}$)

$$\overline{Y}^{ss}(L) = \overline{Y}^{s}(L) = Y.$$

Assume in addition that $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are H-linearised. Fix a topological type τ, and fix a finite set of bounded stability parameters $\mathfrak{S}$ defined with respect to the $\mathcal{L}_i$.

As in Section 6.4.5, form the Y-scheme

$$R = \prod_{i,j=1}^{k} \underline{\mathrm{Hom}}_Y(\mathcal{O}_Y^{d_{i1}} \otimes H_{ij}, \mathcal{O}_Y^{d_{j2}}), \quad H_{ij} = \pi_*(\mathcal{L}_i^{-m_1} \otimes \mathcal{L}_j^{m_2}),$$

with the natural numbers m_0, m_1 and m_2 chosen such that the formation of the locally free sheaves $H_{ij} = \pi_*(\mathcal{L}_i^{-m_1} \otimes \mathcal{L}_j^{m_2})$ is always compatible with base change and such that Theorem 6.4.11 applies for the fibrewise action of G on R, where as before $G = \prod_{j=1}^{k}(GL(d_{j_1}, \mathbb{C}) \times GL(d_{j2}, \mathbb{C}))/\Delta$. For each $\sigma \in \mathfrak{S}$, fix a positive integer $c = c_\sigma$ such that the vector $c\theta_\sigma$ is integral, and let $\chi_\sigma = \chi_{c\theta_\sigma}$ denote the corresponding character of G. The actions of H on X, Y and the $\mathcal{L}_i$ give rise to an induced action of H on the sheaves H_{ij} lifting the action of H on Y. This in turn gives rise to an H-action on R, with the natural projection $p : R \to Y$ being H-equivariant; the actions of G and H on R commute. In particular, we may consider each $\mathcal{O}_R(\chi_\sigma)$ as a $(G \times H)$-linearisation on R.

Let $Z = Z_{\mathfrak{S}}$ be as in (6.4.2), and let $p_\sigma : Z^{\sigma-ss} \to M_{\sigma,\tau}$ be the good quotient over Y given by Theorem 6.4.11; recall this is explicitly given as

$$M_{\sigma,\tau} = Z^{\sigma-ss} /\!\!/ G = \mathbf{Proj}_Y \left(\bigoplus_{m=0}^{\infty} (p_* \mathcal{O}_Z(\chi_\sigma^m))^G \right), \quad \mathcal{O}_Z(\chi_\sigma^m) := \mathcal{O}_R(\chi_\sigma^m)|_Z.$$

Since each $\mathcal{O}_Z(\chi_\sigma^m)$ is naturally H-linearised and as the G and H-actions on R commute, $M_{\sigma,\tau}$ admits a residual H-action, and the relatively ample twisting sheaf $\mathcal{O}_M(1)$ on $M_{\sigma,\tau}$ admits an induced H-linearisation. If $q_\sigma : M_{\sigma,\tau} \to Y$ denotes the H-equivariant structure morphism, Proposition 2.4.6 is then applicable to q_σ; in particular there exists a geometric quotient $M_{\sigma,\tau} /\!\!/ H$ of $M_{\sigma,\tau}$ and a projective morphism $\hat{q}_\sigma : M_{\sigma,\tau} /\!\!/ H \to Y /\!\!/ H$ induced by q_σ. By Lemma 2.1.3, the composition $Z^{\sigma-ss} \to M_{\sigma,\tau} \to M_{\sigma,\tau} /\!\!/ H$ is then a good quotient for the $(G \times H)$-action on $Z^{\sigma-ss}$. This proves the following result.

Proposition 6.4.12. *Let H be a reductive linear algebraic group. Let $\pi : X \to Y$ be an equivariant projective morphism between quasi-projective schemes acted on by H. Assume there exists an equivariant open immersion $\iota : Y \hookrightarrow \overline{Y}$ into a projective scheme $\overline{Y}$ acted on by H, with an ample linearisation L such that $\overline{Y}^{ss}(L) = \overline{Y}^{s}(L) = Y$. Fix H-linearised, π-very ample invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_k$, fix a topological type τ of flat* *sheaves on the fibres of π and fix a finite set of bounded stability parameters $\mathfrak{S}$, with both τ and $\mathfrak{S}$ defined with respect to the $\mathcal{L}_i$. Form the subscheme $Z = Z_{\mathfrak{S}}$ given by Equation 6.4.2. Finally, fix $\sigma \in \mathfrak{S}$.*

There then exists a projective good quotient $Z^{\sigma-ss} /\!/ G \times H = M_{\sigma,\tau} /\!/ H$ of $Z^{\sigma-ss}$, and there exists a natural projective morphism $\hat{q}_\sigma : Z^{\sigma-ss} /\!/ G \times H \to Y /\!/ H = \overline{Y}^s(L)/H$. If $y \in Y$ is a closed point, the fibre of $\hat{q}_\sigma$ over y is isomorphic to the geometric quotient $(M_{\sigma,\tau})_y/\mathrm{Stab}_H(y)$. □

The good quotient $Z^{\sigma-ss} /\!/ G \times H$ can be understood as the quotient of a relative GIT semistable locus for a $G \times H$-linearisation on Z.

Proposition 6.4.13. *There exist positive integers $a, N > 0$ such that for all $\sigma \in \mathfrak{S}$, there is an equality*

$$Z^{\sigma-ss} = Z^{G\times H-ss}(p^*(L^{aN}|_Y) \otimes \mathcal{O}_Z(\chi_\sigma^N)/Y)$$

of open subschemes of Z.

Proof. Fix $\sigma \in \mathfrak{S}$. From the proof of Proposition 2.4.6, for all $a \gg 0$, each point of $M_{\sigma,\tau}$ is H-stable with respect to the linearisation $q_\sigma^*(L^a|_Y) \otimes \mathcal{O}_M(1)$. By Proposition 2.4.2 some power $\mathcal{O}_M(N_\sigma)$ of $\mathcal{O}_M(1)$ pulls back along the good quotient p_σ to $\mathcal{O}_Z(\chi_\sigma^{N_\sigma})|_{Z^{\sigma-ss}}$; we may assume without loss of generality that $N_\sigma = N$ is the same for each $\sigma \in \mathfrak{S}$.

Fixing $a > 0$ sufficiently large, we have equalities

$$Z^{\sigma-ss} = Z^{G-ss}(\mathcal{O}_Z(\chi_\sigma^N)/Y) = Z^{G-ss}(p^*(L^{aN}|_Y) \otimes \mathcal{O}_Z(\chi_\sigma^N)/Y);$$

the first equality follows from $Z^{\sigma-ss} = Z \cap R^{\sigma-ss} = Z \cap R^{G-ss}(\mathcal{O}_R(\chi_\sigma^N)/Y)$ and the second equality holds since the relative G-semistable locus is determined fibrewise (cf. Proposition 2.4.5). From this and from the observation that $G \times H$-semistability implies G-semistability, it suffices to show that there is an inclusion

$$Z^{G-ss}(p^*(L^{aN}|_Y) \otimes \mathcal{O}_Z(\chi_\sigma^N)/Y) \subset Z^{G\times H-ss}(p^*(L^{aN}|_Y) \otimes \mathcal{O}_Z(\chi_\sigma^N)/Y)$$

of open subschemes of Z.

If $z \in Z^{G-ss}(p^*(L^{aN}|_Y) \otimes \mathcal{O}_Z(\chi_\sigma^N)/Y)$ lies over $\overline{z} \in M_{\sigma,\tau}$, as $\overline{z}$ is H-stable there exists $k > 0$ and a section $f \in H^0(M_{\sigma,\tau}, q_\sigma^*(L^{akN}|_Y) \otimes \mathcal{O}_M(kN))^H$ such that the non-vanishing locus $(M_{\sigma,\tau})_f$ is affine and $\overline{z} \in (M_{\sigma,\tau})_f$. If $f' := p_\sigma^*(f)$, then f' is a $G \times H$-invariant section of $p^*(L^{aNk}|_Y) \otimes \mathcal{O}_Z(\chi_\sigma^{Nk})$ with affine non-vanishing locus $p_\sigma^{-1}((M_{\sigma,\tau})_f)$ containing z. Hence $z \in Z^{G\times H-ss}(p^*(L^{aN}|_Y) \otimes \mathcal{O}_Z(\chi_\sigma^N)/Y)$. □

6.5 The GIT Construction of the Moduli Spaces

With the setup of the previous section, we are ready to begin proving Theorem 6.1.1.

6.5.1 Setup

Fix a projective variety X and consider the stack of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Let $\pi_{\mathcal{U}} : \mathcal{U}\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ denote the universal family over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Fix $\pi_{\mathcal{U}}$-ample $\mathbb{Q}$-invertible sheaves $\mathcal{L}_1,\ldots,\mathcal{L}_k$. For each $\sigma \in \Sigma := (\mathbb{Q}^{\geq 0})^k \setminus \{0\}$ let $\mathcal{L}_\sigma = \bigotimes_i \mathcal{L}_i^{\sigma_i}$, and consider the stacks

$$\overline{\mathcal{J}}ac(\sigma) := \overline{\mathcal{J}}ac^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}_\sigma) \text{ and } \overline{\mathcal{J}}(\sigma) := \overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}_\sigma).$$

Since replacing each $\mathcal{L}_i$ with $\mathcal{L}_i^m$ (where m is a positive integer) does not alter the stability condition, we assume without loss of generality that each $\mathcal{L}_i$ is a genuine invertible sheaf and is relatively very ample.

Lemma 6.5.1. *The collection of all degree d, uniform rank r, torsion-free coherent sheaves E over objects of the stack of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$, for which there exists some $\sigma \in \Sigma$ such that E is σ-semistable, forms a bounded family.*

Proof. Without loss of generality it is enough to consider stability conditions in the compact convex set $\Sigma' := \left\{\sigma \in \Sigma : \sum_{i=1}^k \sigma_i = 1\right\}$. Since the stable maps in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ vary in a bounded schematic family, there are only finitely many possible topological types of nodal curves appearing in such a stable map. As such, there exists a constant μ_0 such that if E is a degree d, uniform rank r, torsion-free coherent sheaf lying over a stable map $(C,\underline{x},f)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which is $\mathcal{L}_1$-semistable, then

$$\mu^{\mathcal{L}_i}(E) = \frac{\chi(E)}{r\sum_{j=1}^{\rho} \deg_{C_j} L_1} \leq \mu_0;$$

here L_i denotes the pullback of $\mathcal{L}_i$ to C, and $C_1,\ldots,C_\rho$ are the irreducible components of C.

Take $\sigma \in \Sigma'$, and suppose E is a degree d, uniform rank r, torsion-free coherent sheaf lying over a stable map $(C,\underline{x},f)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which is σ-semistable. If E is semistable with respect to $\mathcal{L}_1$ then $\mu(E) \leq \mu_0$. Otherwise, let $E' \subset E$ denote the maximally destabilising subsheaf of E with respect to $\mathcal{L}_1$ (as E is torsion-free, then E' is of pure dimension 1). For any $t \in [0,1]$, set

$$\sigma(t) := (1 - t + t\sigma_1, t\sigma_2, \ldots, t\sigma_k) \in \Sigma'.$$

Since $\mu^{\mathcal{L}_1}(E') > \mu^{\mathcal{L}_1}(E)$ and $\mu^{\sigma}(E') \leq \mu^{\sigma}(E)$, by continuity there must exist $t_0 \in [0,1]$ with $\mu^{\sigma(t_0)}(E') = \mu^{\sigma(t_0)}(E)$. This implies

$$\mu^{\mathcal{L}_1}(E') = \frac{\chi(E')}{\sum_j r_j(E') \deg_{C_j} L_1} = \frac{\sum_{i,j} \sigma(t_0)_i r_j(E') \deg_{C_j} L_i}{r(\sum_j r_j(E') \deg_{C_j} L_1)(\sum_{i,j} \sigma(t_0)_i \deg_{C_j} L_i)} \chi(E). \tag{6.5.1}$$

Since each $0 \leq r_j(E') \leq r$ and are not all zero, since $\chi(E)$ is a constant, since the curve C varies in a bounded family and since the set of stability conditions Σ' is compact, it follows from Equation (6.5.1) that there exists a constant $\mu_1 \geq \mu_0$, depending only on d, r, g, n, X, β and $\underline{\mathcal{L}}$, such that the following holds: for any degree d, uniform rank r, torsion-free coherent sheaf E lying over a stable map $(C, \underline{x}, f)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which is σ-semistable for some $\sigma \in \Sigma'$,

(i) either E is $\mathcal{L}_1$-semistable, and hence $\mu^{\mathcal{L}_1}(E) \leq \mu_1$; or

(ii) E is $\mathcal{L}_1$-unstable, and the $\mathcal{L}_1$-maximally destabilising subsheaf $E' \subset E$ satisfies $\mu^{\mathcal{L}_1}(E') \leq \mu_1$.

It follows by [69, Theorem 3.3.7] that the collection of all such sheaves E is bounded, which proves the result. □

As in Section 6.4.1, fix an embedding $X \subset \mathbb{P}^b$, set $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^b}(1)|_X$ and consider the universal family $\phi_{\mathcal{U}} : \mathcal{U}I \to I$ over the quasi-projective scheme I. Pulling back the sheaves $\mathcal{L}_i$ along the forgetful morphism $I \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ gives $\phi_{\mathcal{U}}$-very ample invertible sheaves $\mathcal{L}'_i$ on $\mathcal{U}I$, which are equivariant with respect to the $GL(W)$-action on I. Take τ to be the topological type of degree d, uniform rank r torsion-free sheaves over the fibres of $\phi_{\mathcal{U}}$ with respect to the $\mathcal{L}'_i$ (cf. remark in Section 5.2.3). By Theorem 6.4.1, there exists an equivariant open immersion $I \hookrightarrow \overline{I}$ and a linearisation $L = L_{m_W, m_b, m_{\mathrm{pts}}}$ on $\overline{I}$ for the induced $SL(W)$-action for which

$$\overline{I}^{ss}(L) = \overline{I}^{s}(L) = I,$$

with $\overline{M}_{g,n}(X,\beta) \cong I /\!\!/ SL(W)$ isomorphic to the resulting geometric GIT quotient. In particular, the results of Propositions 6.4.12 and 6.4.13 are applicable to the universal family $\phi_{\mathcal{U}} : \mathcal{U}I \to I$ and the linearised sheaves $\mathcal{L}'_i$.

Fix a finite subset $\mathfrak{S} \subset \Sigma$. We now restrict attention to stability parameters $\sigma \in \mathfrak{S}$. As the collection of sheaves in question is bounded (cf. Lemma 6.5.1), we may pick natural numbers m_0, m_1 and m_2 such that Theorem 6.4.11 applies for the projective morphism $\phi_{\mathcal{U}} : \mathcal{U}I \to I$ and the sheaves $\mathcal{L}'_i$; from our choice of m_i the formation of the locally free sheaves $H_{ij} = (\phi_{\mathcal{U}})_*((\mathcal{L}'_i)^{-m_1} \otimes (\mathcal{L}'_j)^{m_2})$ is always compatible with base change, and any sheaf appearing in $\overline{\mathcal{J}}_{g,n,d,r}(X,\beta)$ which is semistable with respect to any $\sigma \in \mathfrak{S}$ is $(m_0, \underline{\mathcal{L}'})$-regular.

Form the I-schemes

$$Z = Z_{\mathfrak{S}} = \bigcup_{\sigma \in \mathfrak{S}} \overline{Q^{[\sigma-ss]}} \subset R = \prod_{i,j=1}^{k} \underline{\mathrm{Hom}}_I(\mathcal{O}_I^{d_{i1}} \otimes H_{ij}, \mathcal{O}_I^{d_{j2}}),$$

with the dimension vector given by

$$\underline{d} = (d_{11}, d_{12}, \ldots, d_{k1}, d_{k2}) = (P_1(m_1), P_1(m_2), \ldots, P_k(m_1), P_k(m_2)).$$

We endow R with the usual fibrewise actions of the groups G and $\widetilde{G}$, where

$$\widetilde{G} = \prod_{j=1}^{k} (GL(d_{j_1}, \mathbb{C}) \times GL(d_{j2}, \mathbb{C})), \quad G = \widetilde{G}/\Delta.$$

As in Section 6.4.8, the $\widetilde{G}$-action on R extends to an action of $\widetilde{G} \times GL(W)$. The $\widetilde{G} \times GL(W)$-action naturally descends to an action of $G \times PGL(W)$, since the diagonal one-parameter subgroups of both $\widetilde{G}$ and $GL(W)$ act trivially.

We now restrict attention further to sheaves of uniform rank r.

Lemma 6.5.2. *Let $C \to S$ be a flat, projective family of genus g prestable curves. Let F be an S-flat torsion-free coherent sheaf on C, such that for each geometric point $s \in S$, the sheaf F_s has Hilbert polynomial P with respect to a fixed choice of an S-ample invertible on C. Then there exists an open and closed subscheme $S_r \subset S$ such that, if $g : T \to S$ is any morphism of schemes, then g factors through $S_r \subset S$ if and only if for all geometric points $t \in T$, the sheaf $F_{g(t)}$ is of uniform rank r over the curve $C_{g(t)}$.*

Proof. This is essentially the content of [108, Lemma 8.1.1]. □

Fix a stability parameter $\sigma \in \mathfrak{S}$. If σ is degenerate, let $\phi' = \phi'_\sigma : R \to R' = R'_\sigma$ be the projection map described in Section 6.4.6, and let σ' be the corresponding positive stability parameter. By Theorem 6.4.8 and Proposition 6.4.9 we have equalities

$$Z^{\sigma-ss} = \begin{cases} Q^{[\sigma-ss]} & \text{if } \sigma \text{ is positive,} \\ (\phi')^{-1}((Q')^{[\sigma'-ss]}) & \text{if } \sigma \text{ is degenerate.} \end{cases}$$

Both $Q^{[\sigma-ss]}$ and $(Q')^{[\sigma'-ss]}$ parametrise flat families of torsion-free sheaves on prestable curves, so by Lemma 6.5.2 there are open and closed subschemes, respectively denoted $Q_r^{[\sigma-ss]}$ and $(Q'_r)^{[\sigma'-ss]}$, parametrising those sheaves in the tautological family $\mathbb{F}$ which are of uniform rank r. Let

$$Z_r^{\sigma-ss} = \begin{cases} Q_r^{[\sigma-ss]} & \text{if } \sigma \text{ is positive,} \\ (\phi')^{-1}((Q'_r)^{[\sigma'-ss]}) & \text{if } \sigma \text{ is degenerate.} \end{cases}$$

In both cases, the scheme $Z_r^{\sigma-ss}$ is open and closed in $Z^{\sigma-ss}$.

6.5.2 Stacks of σ-semistable Sheaves as Quotient Stacks

In the case where σ is positive, $Z_r^{\sigma-ss} = Q_r^{[\sigma-ss]}$ gives a smooth presentation of both $\overline{\mathcal{J}}ac(\sigma)$ and $\overline{\mathcal{J}}(\sigma)$, exhibiting both of these as quotient stacks.

Proposition 6.5.3. *Suppose $\sigma \in \mathfrak{S}$ is positive. Then there is an isomorphism of stacks over $\overline{\mathcal{M}}_{g,n}(X,\beta)$*

$$\overline{\mathcal{J}}ac(\sigma) := \overline{\mathcal{J}}ac^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}_\sigma) \cong [Z_r^{\sigma-ss}/\widetilde{G} \times PGL(W)].$$

Proof. Let $[Z_r^{\sigma-ss}/\widetilde{G} \times PGL(W)]^{\text{pre}}$ denote the quotient pre-stack associated to the $\widetilde{G} \times PGL(W)$-action on $Z_r^{\sigma-ss} = Q_r^{[\sigma-ss]}$, so that $[Z_r^{\sigma-ss}/\widetilde{G} \times PGL(W)]$ is the stack associated to $[Z_r^{\sigma-ss}/\widetilde{G} \times PGL(W)]^{\text{pre}}$. Given a morphism of schemes $S \to Z_r^{\sigma-ss}$ over I, the pullback of the tautological family over $Q_r^{[\sigma-ss]}$ gives rise to a flat family of uniform rank r, degree d, σ-semistable torsion-free coherent sheaves over the family of n-pointed genus g stable maps to X embedded in $\mathbb{P}(W) \times \mathbb{P}^r$, obtained by pulling back the universal family $\phi_{\mathcal{U}} : \mathcal{U}I \to I$ to S giving an object of $\overline{\mathcal{J}}ac(\sigma)(S)$. This defines a morphism $Z_r^{\sigma-ss} \to \overline{\mathcal{J}}ac(\sigma)$ of stacks over $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which is invariant with respect to the action of $\widetilde{G} \times PGL(W)$, giving a morphism of categories fibred in groupoids

$$\Phi : [Z_r^{\sigma-ss}/\widetilde{G} \times PGL(W)]^{\text{pre}} \to \overline{\mathcal{J}}ac(\sigma).$$

By the uniqueness of stackifications, it suffices to show that the stackification $\tilde{\Phi}$ of Φ is an equivalence of categories fibred in groupoids.

$\tilde{\Phi}$ *is fully faithful:* Since stackification is fully faithful, it suffices to show that Φ is fully faithful. Suppose we are given morphisms $\gamma_i : S \to Z_r^{\sigma-ss}$ $(i = 1, 2)$ such that the resulting families of semistable sheaves over stable maps

$$(h_i : C_i \to S; \sigma_1^i, \ldots, \sigma_n^i; f_i : C_i \to X; F_i) \in \overline{\mathcal{J}}ac(\sigma)(S)$$

are isomorphic. Fix such an isomorphism; that is, fix an isomorphism of S-schemes $g : C_1 \to C_2$ which is compatible with the sections σ_j^i and the morphisms to X and fix an isomorphism of coherent sheaves $\alpha : F_1 \to g^* F_2$.

The isomorphism g induces an isomorphism of invertible sheaves $\mathcal{N}_1 \cong g^*\mathcal{N}_2$, where

$$\mathcal{N}_i = \omega_{C_i/S}(\sigma_1^i(S) + \cdots + \sigma_n^i(S)) \otimes f_i^*(\mathcal{O}_X(3)).$$

Choose $a \geq 10$ as in Section 6.4.1. As the restriction of $\mathcal{N}_i^a$ to a geometric fibre is non-special, the sheaves $(h_i)_*(\mathcal{N}_i^a)$ are locally free of rank $e - g + 1 = \dim W$. By the universal property of I there are invertible sheaves $M_i \in \operatorname{Pic} S$ and isomorphisms

$$\mathcal{O}_{\mathbb{P}(W)}(1) \otimes \mathcal{O}_{\mathbb{P}^b}(1) \otimes \mathcal{O}_{C_i} \cong (\omega_{C_i/S}(\sigma_1^i(S) + \cdots + \sigma_n^i(S)))^a \otimes f_i^*(\mathcal{O}_X(3a+1)) \otimes \mathcal{O}_{C_i} \otimes h_i^* M_i$$

of invertible sheaves on C_i. Pull back these isomorphisms along the projection $p_W : \mathbb{P}(W) \times \mathbb{P}^b \to \mathbb{P}(W)$ then push forward to S to obtain isomorphisms

$$(h_i)_*(\mathcal{O}_{\mathbb{P}(W)}(1) \otimes \mathcal{O}_{p_W(C_i)}) \cong (h_i)_*(\mathcal{N}_i^a) \otimes M_i.$$

As each point of I corresponds (after composing with the projection p_W) to a non-degenerate curve in $\mathbb{P}(W)$, the natural morphisms

$$H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)) \otimes \mathcal{O}_S \to (h_i)_*(\mathcal{O}_{\mathbb{P}(W)}(1) \otimes \mathcal{O}_{p_W(C_i)})$$

are both isomorphisms. The sequence of isomorphisms

$$\begin{aligned} H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)) \otimes \mathcal{O}_S &\cong (h_1)_*(\mathcal{O}_{\mathbb{P}(W)}(1) \otimes \mathcal{O}_{p_W(C_1)}) \\ &\cong (h_1)_*(\mathcal{N}_1^a) \otimes M_1 \\ &\cong (h_2)_*(\mathcal{N}_2^a) \otimes M_1 \\ &\cong (h_2)_*(\mathcal{O}_{\mathbb{P}(W)}(1) \otimes \mathcal{O}_{p_W(C_2)}) \otimes M_1 \otimes M_2^{-1} \\ &\cong H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1)) \otimes M_1 \otimes M_2^{-1} \end{aligned}$$

yields an automorphism of $S \times \mathbb{P}(W)$ sending the curve $p_W(C_1)$ to $p_W(C_2)$, along with a corresponding S-valued point of $PGL(W)$, depending only on g, γ_1 and γ_2.

On the other hand, the isomorphism $\alpha : F_1 \to g^*F_2$ gives rise to an isomorphism of A-modules $\mathcal{H}om_X(T, F_1) \cong \mathcal{H}om_X(T, g^*F_2)$. By Proposition 6.4.6, the A-module $\mathcal{H}om_X(T, F_1)$ is canonically isomorphic to the pullback of the tautological family $\mathbb{M}|_{Z_r^{\sigma-ss}}$ along the morphism γ_1, and similarly for the A-module $\mathcal{H}om_X(T, g^*F_2)$, so there is an induced isomorphism between the (free) A-modules $\gamma_1^*(\mathbb{M}|_{Z_r^{\sigma-ss}})$ and $\gamma_2^*(\mathbb{M}|_{Z_r^{\sigma-ss}})$. Such an isomorphism determines an S-valued point of $\widetilde{G}$, depending only on α, γ_1 and γ_2. It follows that Φ is fully faithful.

$\tilde{\Phi}$ *is essentially surjective:* Take an object

$$(h : C \to S; \sigma_1, \dots, \sigma_n; f : C \to X; F) \in \overline{\mathcal{J}}ac(\sigma)(S). \tag{6.5.2}$$

Let $\mathcal{N}$ be the invertible sheaf

$$\mathcal{N} = \omega_{C/S}(\sigma_1(S) + \cdots + \sigma_n(S)) \otimes f^*(\mathcal{O}_X(3)),$$

and consider the locally free sheaf $h_*\mathcal{N}^a$. From our choice of natural numbers m_i the sheaves $h_*(F \otimes (\mathcal{L}'_i)_S^{m_1})$, $h_*(F \otimes (\mathcal{L}'_j)_S^{m_2})$ $(i, j = 1, \dots, k)$ appearing in the A-module $\mathcal{H}om_X(T, F)$ are also all locally free $\mathcal{O}_S$-modules. Choose an open cover $\{S_i\}$ of S which simultaneously trivialises the sheaves

$$h_*\mathcal{N}^a, \quad h_*(F \otimes (\mathcal{L}'_i)_S^{m_1}), \quad i = 1, \dots, k, \quad h_*(F \otimes (\mathcal{L}'_j)_S^{m_2}), \quad j = 1, \dots, k. \tag{6.5.3}$$

As $\overline{\mathcal{J}}ac(\sigma)$ is a stack, it suffices to show that the restriction of the object (6.5.2) to each $S_i \subset S$ lies in the essential image of $Z_r^{\sigma-ss} \to \overline{\mathcal{J}}ac(\sigma)$. Without loss of generality, we may therefore assume that all of the sheaves in (6.5.3) are free $\mathcal{O}_S$-modules.

Pick an isomorphism $W^\vee \otimes \mathcal{O}_S \cong h_*\mathcal{N}^a$. Pulling back to C, we obtain a quotient

$$W^\vee \otimes \mathcal{O}_C \cong h^*h_*\mathcal{L}^a \to \mathcal{L}^a \to 0.$$

This quotient embeds C inside $S \times \mathbb{P}(W)$ as a family of non-degenerate curves; in turn C embeds inside $S \times \mathbb{P}(W) \times \mathbb{P}^b$ via the graph of $f : C \to X \subset \mathbb{P}^b$. From the universal property of I we obtain a morphism $\gamma' : S \to I$. On the other hand, by Theorem 6.4.7 the free A-module $\mathcal{H}om_X(T, F)$ is (fibrewise) σ-semistable, so from the universal property of $Z_r^{\sigma-ss} = Q_r^{[\sigma-ss]}$ we obtain a morphism $\gamma : S \to Z_r^{\sigma-ss}$ which lifts the morphism $\gamma' : S \to I$. This proves that $\tilde{\Phi}$ is essentially surjective, completing the proof of the result. $\square$

Corollary 6.5.4. *Suppose $\sigma \in \mathfrak{S}$ is positive. Then there is an isomorphism of stacks over $\overline{\mathcal{M}}_{g,n}(X,\beta)$*

$$\overline{\mathcal{J}}(\sigma) := \overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}_\sigma) \cong [Z_r^{\sigma-ss}/G \times PGL(W)].$$

In particular, each of the stacks $\overline{\mathcal{J}}(\sigma)$ is algebraic and of finite type over $\mathbb{C}$.

Proof. Under the isomorphism of Proposition 6.5.3, the diagonal one-parameter subgroup $\mathbb{G}_m \cong \Delta \subset \widetilde{G}$ corresponds to the standard central copy of $\mathbb{G}_m$ in the automorphism groups of the sheaves appearing in $\overline{\mathcal{J}}ac(\sigma)$. The desired isomorphism is then obtained by rigidifying the isomorphism of Proposition 6.5.3 with respect to these copies of $\mathbb{G}_m$. □

Remark. We only claim that $Z_r^{\sigma-ss}$ gives a smooth presentation when σ is positive. However, in the degenerate case the scheme $(Q'_r)^{[\sigma'-ss]}$ can be used to exhibit $\overline{\mathcal{J}}ac(\sigma)$ and $\overline{\mathcal{J}}(\sigma)$ as quotient stacks, after replacing $\widetilde{G}$ and G with $\widetilde{G}'$ and G' respectively.

6.5.3 The Good Moduli Spaces are GIT Quotients

With the notation as above, we now show that the stack $\overline{\mathcal{J}}(\sigma)$ admits a projective good moduli space, which is a good quotient of the parameter space $Z_r^{\sigma-ss}$.

Theorem 6.5.5. *Let $\sigma \in \mathfrak{S}$ be either positive or degenerate. The stack $\overline{\mathcal{J}}(\sigma)$ admits a projective good moduli space*

$$\overline{J}(\sigma) = \overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}_\sigma),$$

which is a tame moduli space if all σ-semistable sheaves appearing in $\overline{\mathcal{J}}(\sigma)$ are σ-stable. Moreover, $\overline{J}(\sigma)$ is a good quotient of the scheme $Z_r^{\sigma-ss}$ under the action of the group $G \times SL(W)$:

$$\overline{J}(\sigma) \cong Z_r /\!\!/_{\theta_\sigma} G \times SL(W) := Z_r^{\sigma-ss} /\!\!/ G \times SL(W).$$

This good quotient coincides with the quotient of a relative (over I) GIT semistable locus for the action of $G \times SL(W)$ on Z_r.

Proof. First note that if all σ-semistable sheaves appearing in $\overline{\mathcal{J}}(\sigma)$ are σ-stable then $\overline{\mathcal{J}}(\sigma)$ is a Deligne–Mumford stack (cf. Section 6.3.2), hence any good moduli space is automatically a coarse/tame moduli space.

Since $\overline{I}^{SL(W)-ss}(L) = \overline{I}^{SL(W-s)}(L) = I$, after first replacing Z with the open and closed subscheme Z_r, the results of Propositions 6.4.12 and 6.4.13 are applicable to the morphism $Z_r \to I$. As such, there exists a $G \times SL(W)$-linearisation $\tilde{L}_\sigma$ on Z_r, an equality $Z_r^{\sigma-ss} = Z^{G\times SL(W)-ss}(\tilde{L}_\sigma/I)$ of semistable loci, and a projective good quotient $Z_r^{\sigma-ss} /\!\!/ G \times SL(W)$ for the action of $G \times SL(W)$ on $Z_r^{\sigma-ss}$.

Since the group $GL(W)$ acts on $Z_r^{\sigma-ss}$ through the quotient $PGL(W)$, there is a canonical identification of good quotients

$$Z_r^{\sigma-ss} /\!\!/ G \times SL(W) \equiv Z_r^{\sigma-ss} /\!\!/ G \times PGL(W).$$

If σ is positive, the isomorphism $\overline{J}(\sigma) \cong Z_r /\!\!/_{\theta_\sigma} G\times SL(W)$ follows by combining Corollary 6.5.4 with Proposition 2.3.6. If instead σ is degenerate, we apply Corollary 6.5.4 with Proposition 2.3.6 to $Z_r^{\sigma'-ss}$ in place of $Z_r^{\sigma-ss}$ and G' in place of G, then apply Corollary 6.4.10 to obtain an isomorphism of good quotients

$$Z_r^{\sigma-ss} /\!\!/ G \times SL(W) \cong Z_r^{\sigma'-ss} /\!\!/ G' \times SL(W) \equiv Z_r^{\sigma'-ss} /\!\!/ G' \times PGL(W).$$

This completes the proof. □

6.5.4 Closed Points of the Moduli Spaces

Continue to fix the stability condition σ. Let $(C, \underline{x}, f)$ be a stable map in $\overline{\mathcal{M}}_{g,n}(X, \beta)$, and let $[(C, \underline{x}, f)]$ be the corresponding point in the coarse moduli space $\overline{M}_{g,n}(X, \beta)$. Denote the pullback of $\overline{\mathcal{J}}(\sigma)$ along $(C, \underline{x}, f) : \operatorname{Spec} \mathbb{C} \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ as $\overline{\mathcal{J}}_{(C,\underline{x},f)}(\sigma)$.

Fix a point $j_0 \in I$ corresponding to an embedded stable map whose underlying stable map is $(C, \underline{x}, f)$. If $(Z_r)_{j_0}^{\sigma-ss}$ denotes the pullback of the master space $Z_r^{\sigma-ss}$ along j_0, the good quotient $(Z_r)_{j_0}^{\sigma-ss} /\!\!/ G$ is isomorphic to the good moduli space $\overline{J}_{(C,\underline{x},f)}(\sigma)$ of $\overline{\mathcal{J}}_{(C,\underline{x},f)}(\sigma)$. By Theorem 6.4.8 and Corollary 6.4.10, the closed points of $\overline{J}_{(C,\underline{x},f)}(\sigma)$ are in 1-1 correspondence with S-equivalence classes of σ-semistable degree d, uniform rank r, torsion-free sheaves on the nodal curve C underlying $(C, \underline{x}, f)$. The automorphism group $\operatorname{Aut}(C, \underline{x}, f)$ of the stable map $(C, \underline{x}, f)$ acts on the moduli space $\overline{J}_{(C,\underline{x},f)}(\sigma)$, with the action corresponding to taking pullbacks of sheaves.

Proposition 6.5.6. *The fibre of the map $\overline{J}(\sigma) \to \overline{M}_{g,n}(X, \beta)$ over the closed point $[(C, \underline{x}, f)] \in \overline{M}_{g,n}(X, \beta)$ is isomorphic to the geometric quotient $\overline{J}_{(C,\underline{x},f)}(\sigma)/\operatorname{Aut}(C, \underline{x}, f)$.*

Proof. Applying the final statement of Proposition 6.4.12, it is enough to show that the image of the stabiliser subgroup $\mathrm{Stab}_{SL(W)}(j_0)$ of the point j_0 under the $SL(W)$ action on I inside $PGL(W)$ can be identified with $\mathrm{Aut}(C, \underline{x}, f)$. But this follows by the same argument used to prove that Φ is fully faithful in the proof of Proposition 6.5.3, taking the base S to be $\mathrm{Spec}\,\mathbb{C}$. □

This yields the following description of the closed points of $\overline{J}(\sigma)$, analogous to the result [108, Theorem 8.2.1].

Proposition 6.5.7. *Let $(C, \underline{x}, f)$ be a stable map in $\overline{\mathcal{M}}_{g,n}(X, \beta)$, and let $[(C, \underline{x}, f)]$ be the corresponding point in the coarse moduli space $\overline{M}_{g,n}(X, \beta)$. Then the closed points of $\overline{J}(\sigma)$ lying over the closed point $[(C, \underline{x}, f)]$ are in 1-1 correspondence with equivalence classes of σ-semistable uniform rank r, degree d, torsion-free sheaves E over C, where sheaves E and E' are equivalent if and only if the Jordan–Hölder factors of these sheaves differ by an automorphism of the stable map $(C, \underline{x}, f)$.* □

Remark. As a consequence of Proposition 6.5.7, the open substack $\overline{\mathcal{J}}^s(\sigma) \subset \overline{\mathcal{J}}(\sigma)$ parametrising σ-stable sheaves is saturated with respect to the good moduli space morphism $\pi_\sigma : \overline{\mathcal{J}}(\sigma) \to \overline{J}(\sigma)$, in the sense that

$$\overline{\mathcal{J}}^s(\sigma) = \pi_\sigma^{-1}(\pi_\sigma(\overline{\mathcal{J}}^s(\sigma))).$$

Therefore $\overline{J}^s(\sigma) := \pi_\sigma(\overline{\mathcal{J}}^s(\sigma))$ is an open subscheme of $\overline{J}(\sigma)$ which is a coarse/tame moduli space for $\overline{\mathcal{J}}^s(\sigma)$ (cf. [6, Remark 6.2]).

6.6 Wall and Chamber Decompositions

We now turn to proving Theorem 6.1.2, which concerns what happens as the parameter σ varies in the space of stability conditions.

6.6.1 Existence of Stability Decompositions

Here we closely follow [55, Section 4]. We keep the notation of Section 6.5. Fix $\pi_{\mathcal{U}}$-ample $\mathbb{Q}$-invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_k$ on the universal curve, and denote by $\Sigma' = \left\{\sigma \in (\mathbb{Q}^{\geq 0})^k \setminus \{0\} : \sum_{i=1}^k \sigma_i = 1\right\}$ the resulting space of stability conditions.

Definition 6.6.1. *A* chamber structure *on* Σ' *is given by a collection* $\{W_i : i \in I\}$ *of* walls*, where each wall is either a real hypersurface in the convex space* $\Sigma'_{\mathbb{R}} = \left\{\sigma \in (\mathbb{R}^{\geq 0})^k \setminus \{0\} : \sum_{i=1}^k \sigma_i = 1\right\}$ *(the* proper walls*), or all of* $\Sigma'_{\mathbb{R}}$*. A subset* $\mathcal{C} \subset \Sigma'_{\mathbb{R}}$ *which is maximally connected with respect to the property that for each* $i \in I$*, either* $\mathcal{C} \subset W_i$ *or* $\mathcal{C} \cap W_i = \emptyset$*, is called a* chamber*. The chamber structure is said to be* rational linear *if each proper wall* W_j *is a rational hyperplane, and* proper *if all walls are proper.*

Consider the collection of sheaves $\mathcal{S}$ consisting of all sheaves F for which there exists $\sigma \in \Sigma'$ and a σ-semistable, degree d, uniform rank r, torsion-free coherent sheaf E over some stable map $(C, \underline{x}, f)$ in $\overline{\mathcal{M}}_{g,n}(X, \beta)$, such that F is a saturated subsheaf of E and such that for some $\sigma' \in \Sigma'$ (not necessarily equal to σ), one has $\mu^{\sigma'}(F) \geq \mu^{\sigma'}(E)$.

Lemma 6.6.2 (cf. [55], Lemma 4.5)**.** *The collection* $\mathcal{S}$ *is a bounded family of sheaves.*

Proof. Suppose $E \supset F \in \mathcal{S}$ with E σ-semistable and $\mu^{\sigma'}(F) \geq \mu^{\sigma'}(E)$. By Lemma 2.11 of *loc. cit.* there exists $j \in \{1, \ldots, k\}$ with $\mu^{\mathcal{L}_j}(F) \geq \mu^{\sigma'}(E)$. Since by Lemma 6.5.1 E varies in a bounded family, and since the quotient E/F is torsion-free or zero, the boundedness of $\mathcal{S}$ follows by applying Grothendieck's lemma [69, Lemma 1.7.9], as in the proof of [55, Lemma 4.5]. □

Now suppose $E \supset F \in \mathcal{S}$ is a sheaf over some stable map $(C, \underline{x}, f)$ in $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Suppose the underlying curve C has irreducible components $C_1, \ldots, C_\rho$. Set

$$W_{F,(C,\underline{x},f)} := \left\{ \sigma \in \Sigma'_{\mathbb{R}} : \sum_{i=1}^{k} \sigma_i \left(r\chi(F) \sum_{j=1}^{\rho} \deg_{C_j} L_i - \chi(E) \sum_{j=1}^{\rho} r_j(F) \deg_{C_j} L_i \right) = 0 \right\}.$$

Each $W_{F,(C,\underline{x},f)}$ is empty, the whole of Σ' or a rational hyperplane in $\Sigma'_{\mathbb{R}}$. In the first case the wall $W_{F,(C,\underline{x},f)}$ is discarded. As the wall $W_{F,(C,\underline{x},f)}$ depends only on the Euler characteristic and the multirank of F, and as $\mathcal{S}$ is bounded, letting $F \in \mathcal{S}$ vary yields finitely many possibilities for the proper walls which arise in this way, each of which is a rational hyperplane. This gives rise to a finite rational linear chamber structure on the set Σ'.

Lemma 6.6.3. *Suppose* E *is a degree* d *torsion-free coherent sheaf of uniform rank* r *over some stable map* $(C, \underline{x}, f)$*, such that* E *is semistable with respect to some* $\sigma \in \Sigma'$*. Suppose further that* F *is a saturated subsheaf of* E *with* $F \in \mathcal{S}$*. Suppose additionally that* $\sigma', \sigma'' \in \Sigma'$ *lie in a common chamber* $\mathcal{C} \subset \Sigma'$*. Then* $\mu^{\sigma'}(F) < (\leq)\ \mu^{\sigma'}(E)$ *if and only if* $\mu^{\sigma''}(F) < (\leq)\ \mu^{\sigma''}(E)$*.*

Proof. This proof proceeds on similar lines to the proof of Lemma 4.6 of *loc. cit.* We deal with the non-strict inequality case; the strict inequality case follows by essentially the same argument. Swapping σ' and σ'' if necessary, suppose for a contradiction that

$$\mu^{\sigma'}(F) \leq \mu^{\sigma'}(E) \quad \text{and} \quad \mu^{\sigma''}(F) > \mu^{\sigma''}(E).$$

Consider the linear function $h : \Sigma'_{\mathbb{R}} \to \mathbb{R}$ given by

$$h(\sigma) = \sum_{i=1}^{k} \sigma_i \left(r\chi(F) \sum_{j=1}^{\rho} \deg_{C_j} L_i - \chi(E) \sum_{j=1}^{\rho} r_j(F) \deg_{C_j} L_i \right).$$

By definition $W_{F,(C,\underline{x},f)}$ is the zero set of h. By assumption, we have $h(\sigma') \leq 0$ and $h(\sigma'') > 0$. Since σ' and σ'' are both contained in the chamber $\mathcal{C}$, it follows that $\mathcal{C} \cap W_{F,(C,\underline{x},f)} = \emptyset$. On the other hand there exists $\sigma''' \in \Sigma'$ on the line segment joining σ' and σ'' with $h(\sigma''') = 0$. As the chamber $\mathcal{C}$ is convex, $\mathcal{C}$ must contain this line segment, which implies $\mathcal{C} \cap W_{F,(C,\underline{x},f)} \neq \emptyset$, a contradiction. □

Proposition/Definition 6.6.4. *There exists a rational linear chamber structure on Σ' cut out by finitely many walls, such that if $\sigma', \sigma'' \in \Sigma'$ belong to the same chamber, then:*

1. *if E is a degree d, uniform rank r torsion-free coherent sheaf over a stable map in $\overline{\mathcal{M}}_{g,n}(X,\beta)$, then E is σ'-semistable if and only if E is σ''-semistable; and*

2. *if E and E' are degree d, uniform rank r torsion-free coherent sheaves over the same stable map $(C,\underline{x},f)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$, which are both semistable with respect to both σ' and σ'', then E and E' are S-equivalent with respect to σ' if and only if they are S-equivalent with respect to σ''.*

Moreover, if $\sigma \in \Sigma'$ does not lie in any wall then all σ-semistable sheaves are σ-stable. We refer to this chamber structure on Σ' as the stability chamber decomposition *(with respect to $\mathcal{L}_1, \ldots, \mathcal{L}_k$), and to the corresponding chambers as* stability chambers.

Proof. The proof of the first two assertions proceeds along very similar lines to the proof of Proposition 4.2 of *loc. cit.* Suppose $\sigma', \sigma'' \in \Sigma'$ lie in the same chamber. Let E be a σ'-semistable, degree d, uniform rank r, torsion-free coherent sheaf over a stable map $(C,\underline{x},f)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$, and let $F \subset E$ be a saturated subsheaf. If $\mu^{\sigma''}(F) < \mu^{\sigma''}(E)$ then F does not destabilise E; otherwise $E \supset F \in \mathcal{S}$ and so $\mu^{\sigma''}(F) \leq \mu^{\sigma''}(E)$ by Lemma 6.6.3. As such, E is also σ''-semistable; this establishes the first assertion.

To establish the second assertion, suppose E is strictly σ'-semistable, so that there exists a saturated subsheaf $F \subset E$ with $\mu^{\sigma'}(F) = \mu^{\sigma'}(E)$. By Lemma 6.6.3, for any such subsheaf F we have $\mu^{\sigma''}(F) = \mu^{\sigma''}(E)$. In particular, a maximal length Jordan–Hölder filtration of E with respect to σ' must also be a maximal length Jordan–Hölder filtration of E with respect to σ'' and vice versa, so $\mathrm{gr}^{\mathrm{JH}}_{\sigma'}(E) \cong \mathrm{gr}^{\mathrm{JH}}_{\sigma''}(E)$; this proves the second assertion.

It remains to show that if $\sigma \in \Sigma'$ is not contained in any wall $W_{F,(C,\underline{x},f)}$ then all σ-semistable sheaves are σ-stable. However, if E is a σ-semistable degree d, uniform rank r torsion-free coherent sheaf over the stable map $(C, \underline{x}, f)$ then for all saturated subsheaves $F \subset E$ we must have $\mu^{\sigma}(F) < \mu^{\sigma}(E)$, which implies that E is σ-stable. □

6.6.2 Changing the Stability Condition and VGIT

As a consequence of the master space construction of Section 6.5 and the existence of a finite stability decomposition on the space of stability conditions Σ', the moduli spaces $\overline{J}(\sigma)$ are always related by finite sequences of Thaddeus flips (cf. Definition 2.5.5).

Theorem 6.6.5. *With the notation as in Section 6.5.1, let $\sigma_0, \sigma_1 \in \Sigma'$ be stability parameters defined with respect to the relatively ample $\mathbb{Q}$-invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_k$. Let $\sigma(t) = (1-t)\sigma_0 + t\sigma_1$ for $t \in \mathbb{Q} \cap [0,1]$. Then the good moduli spaces $\overline{J}(\sigma_0)$ and $\overline{J}(\sigma_1)$ are related by a finite number of Thaddeus flips of the form*

$$\begin{array}{ccc} \overline{J}(\sigma(t_i)) & & \overline{J}(\sigma(t_{i+1})) \\ & \searrow \quad \swarrow & \\ & \overline{J}(\sigma(t_i')) & \end{array} \tag{6.6.1}$$

for some $t_i, t_i' \in \mathbb{Q} \cap [0,1]$. In particular, $\overline{J}(\sigma_0)$ and $\overline{J}(\sigma_1)$ are related by a finite number of Thaddeus flips through moduli spaces of rank r, degree d, torsion-free coherent sheaves over stable maps in $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

Proof. The proof proceeds along similar lines to that of [56, Theorem 5.2]. If σ_0 and σ_1 define the same stability condition then there is nothing to prove. Suppose instead that σ_0 and σ_1 lie in different stability chambers. As per usual, without loss of generality we may assume that the $\mathcal{L}_i$ are genuine relatively very ample invertible sheaves. Let $\mathfrak{S} \subset \Sigma' \cap \{\sigma(t) : t \in \mathbb{Q} \cap [0,1]\}$ be a set of representative stability conditions for each of the stability chambers the line segment $\sigma(t)$ intersects, with exactly one representative in $\mathfrak{S}$ for each such chamber; for the chambers containing σ_0 and σ_1, we choose these stability conditions as representatives. As the stability decomposition is cut out by finitely many walls, the set $\mathfrak{S}$ is finite.

As in Section 6.5.1 form the master space $Z = Z_{\mathfrak{S}}$; the moduli spaces $\overline{J}(\sigma_t)$ obtained as $t \in [0,1]$ varies are all GIT quotients of this master space. Every time a wall W_i separating chambers $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ with respective representative stability conditions $\sigma(t_i)$ and $\sigma(t_{i+1})$ is crossed by $\sigma(t)$, taking limits in the inequalities (5.2.5) give inclusions

$$Z_r^{\sigma(t_i)-ss} \subset Z_r^{\sigma(t_i')-ss} \supset Z_r^{\sigma(t_{i+1})-ss}.$$

Here $\sigma(t_i')$ is the stability condition between $\sigma(t_i)$ and $\sigma(t_{i+1})$ lying on the wall W_i. The moduli spaces $\overline{J}(\sigma(t_i))$ and $\overline{J}(\sigma(t_{i+1}))$ are then related by the Thaddeus flip[8] (6.6.1) through the moduli space $\overline{J}(\sigma(t_i'))$. As there are finitely many wall crossings between σ_0 and σ_1, the good moduli spaces $\overline{J}(\sigma_0)$ and $\overline{J}(\sigma_1)$ are related by a finite number of Thaddeus flips. □

6.7 Singularities of the Moduli Spaces over $\overline{M}_{g,n}$

In this final part of Chapter 6 we prove Theorem 6.1.3, which extends [31, Theorem A] to the case where there are $n > 0$ marked points. Evidently, as far as stability conditions are concerned it is enough to work with a single relatively (very) ample invertible sheaf $\mathcal{L}$ on the universal curve over $\overline{\mathcal{M}}_{g,n}$. As such, we will prove the following theorem.

Theorem 6.7.1. *Fix a relatively very ample invertible sheaf $\mathcal{L}$ on the universal curve over $\overline{\mathcal{M}}_{g,n}$. Assume $g \geq 4$. Then the good moduli space $\overline{J}^{ss}_{g,n,d,1}(\operatorname{Spec}\mathbb{C}, 0)(\mathcal{L})$ has canonical singularities.*

This section closely follows the papers [30] and [31].

6.7.1 An Alternative GIT Construction of Compactified Universal Jacobians

The construction presented in Section 6.5 was carried out in such a way as to be able to relate changes in stability conditions to variation of GIT (cf. Theorem 6.1.2). However,

[8]This is a Thaddeus flip in the sense of Definition 2.5.5, since the above loci are relative GIT semistable loci, cf. Theorem 6.5.5.

in order to be able to prove Theorem 6.7.1, it will be helpful to instead work with an alternative GIT construction of the good moduli space $\overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$, based on Simpson's construction (cf. Section 4.2); this construction is similar to the construction given by Pandharipande in [108].

Fix a relatively very ample invertible sheaf $\mathcal{L}$ on the universal curve over $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Once again, consider the universal family $(\phi_{\mathcal{U}} : \mathcal{U}I \to I; \tau_1,\ldots,\tau_n)$ from Section 6.4.1, and let $L_{\text{curve}} = L_{m_W,m_b,m_{\text{pts}}}$, with the triple $(m_W, m_b, m_{\text{pts}})$ as in the statement of Theorem 6.4.1, so that we have $\overline{I}^{SL(W)-ss}(L_{\text{curve}}) = \overline{I}^{SL(W)-s}(L_{\text{curve}}) = I$.

This time, we apply Simpson's construction to the projective morphism $\phi_{\mathcal{U}} : \mathcal{U}I \to I$, along with the pullback $\mathcal{L}_I$ of $\mathcal{L}$ to $\mathcal{U}I$. For $M \gg N \gg 0$, this yields an open subscheme $R' = R_N$ of $Q_N = \text{Quot}_{\mathcal{U}I/I}(V_N \otimes \mathcal{L}_I^{-N}, \widetilde{P})$, where $\widetilde{P}$ is the Hilbert polynomial of degree d, uniform rank r torsion-free sheaves over the fibres of the universal curve $\phi_{\mathcal{U}}$ with respect to $\mathcal{L}_I$. We also obtain a fibrewise action of $GL(V) = GL(V_N)$ (which factors through the quotient $PGL(V_N)$) and a relatively ample (with respect to the structure morphism $\overline{R'} \to I$) linearisation $L_{\text{sheaf}} = L_{N,M}$ on $\overline{R'}$, the closure of R' in Q_N, for which $\overline{R'}^{SL(V)-ss}(L_{\text{sheaf}}) = R'$. By Lemma 6.5.2, there exists an open and closed subscheme $R'_r \subset R'$ parametrising those points of R' whose underlying coherent sheaf is of uniform rank r. The schemes R' and R'_r admit actions by the product group $GL(V) \times GL(W)$; both actions factor through the quotient $PGL(V) \times PGL(W)$.

Proposition 6.7.2. *There are isomorphisms of stacks over $\overline{\mathcal{M}}_{g,n}(X,\beta)$*

$$\overline{\mathcal{J}}ac^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}) \cong [R'_r/GL(V) \times PGL(W)]$$

and

$$\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}) \cong [R'_r/PGL(V) \times PGL(W)].$$

Moreover, there is a projective good quotient $R'_d /\!\!/ SL(V) \times SL(W) = R'_d /\!\!/ PGL(V) \times PGL(W)$, and this is isomorphic to the good moduli space $\overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ of both $\overline{\mathcal{J}}ac^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ and $\overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$.

Proof. After making the necessary modifications to take into account the use of Simpson's construction, the argument used to prove the first assertion proceeds along the same lines as the ones used to prove Proposition 6.5.3 and Corollary 6.5.4. That is, there is an equivariant morphism $R'_r \to \overline{\mathcal{J}}ac^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$, inducing a morphism of categories fibred in groupoids

$$\Phi : [R'_r/GL(V) \times PGL(W)]^{\text{pre}} \to \overline{\mathcal{J}}ac^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L}).$$

This morphism is fully faithful, hence so is its stackification $\tilde{\Phi}$. The morphism $\tilde{\Phi}$ is then an isomorphism of algebraic stacks $[R'_r/GL(V) \times PGL(W)] \xrightarrow{\simeq} \overline{\mathcal{J}}ac^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$. The $\mathbb{G}_m$-rigidification of this morphism yields an isomorphism $[R'_r/PGL(V) \times PGL(W)] \xrightarrow{\simeq} \overline{\mathcal{J}}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$.

That there exists a projective good quotient $R'_d /\!\!/ SL(V) \times SL(W) = R'_d /\!\!/ PGL(V) \times PGL(W)$ isomorphic to the good moduli space $\overline{J}^{ss}_{g,n,d,r}(X,\beta)(\mathcal{L})$ now follows from Proposition 2.3.6 (which asserts that if a quotient stack $[Y/G]$ has a schematic good moduli space M then M is a good quotient of Y by G) and the uniqueness of good moduli spaces. □

The existence of a projective good quotient $R'_d /\!\!/ SL(V) \times SL(W)$ can also be proved using relative GIT, in a way similar to Pandharipande's construction [108]. The existence of a relatively projective good quotient of R'_d over I with respect to the fibrewise $SL(V)$ action is given by Theorem 4.2.5. The formation of the quotient with respect to the residual $SL(W)$-action can then be carried out using Proposition 2.4.6. The resulting good quotient coincides with the quotient of a relative GIT semistable locus $\overline{R}'^{SL(V)\times SL(W)-ss}_r(\tilde{L}/I)$ for some ample $SL(V) \times SL(W)$-linearisation $\tilde{L}$ on the closure $\overline{R}'_r$ of R_r, analogously to the result of Proposition 6.4.13. Since each point of I is $SL(W)$-stable, each point of R'_r is (relatively) stable under the action of $SL(W)$.

Lemma 6.7.3. *Let $p \in R'_r$ be a point corresponding to a $\mathcal{L}$-semistable sheaf F over the stable map $(C, \underline{x}, f)$. Then the point p is $SL(V) \times SL(W)$-polystable if and only if F is a polystable sheaf (with respect to $\mathcal{L}$).*

Proof. The point p is polystable if and only if $\pi^{-1}(\pi(p))$ equals the $SL(V)\times SL(W)$-orbit of p in R'_r (as opposed to the orbit closure), where $\pi : R'_r \to R'_r /\!\!/ SL(V)\times SL(W)$ denotes the GIT quotient. Since the $SL(W)$-quotient $R'_r /\!\!/ SL(V) \to (R'_r /\!\!/ SL(V)) /\!\!/ SL(W) = R'_r /\!\!/ SL(V)\times SL(W)$ is a geometric quotient, and in particular an orbit space morphism, this is true if and only if $(\pi')^{-1}(\pi'(p)) = SL(V)\cdot p$ where $\pi' : R'_r \to R'_r /\!\!/ SL(V)$ is the GIT quotient with respect to $SL(V)$, i.e. p is $SL(V)$-polystable. But p is $SL(V)$-polystable if and only if F is $\mathcal{L}$-polystable, by Theorem 4.2.5. □

For the rest of this chapter, we restrict attention to the setting of considering torsion-free uniform rank $r = 1$ sheaves over marked stable curves $(C, \underline{x}) \in \overline{\mathcal{M}}_{g,n}$ which are (fibrewise) semistable with respect to the choice of relatively (very) ample invertible sheaf $\mathcal{L}$. As such, for the rest of this chapter we introduce the following shorthand notation.

Notation 6.7.4. *We denote* $\overline{\mathcal{J}}_{g,n} := \overline{\mathcal{J}}^{ss}_{g,n,d,1}(\operatorname{Spec}\mathbb{C}, 0)(\mathcal{L})$*, and denote its good moduli space as* $\overline{J}_{g,n}$*.*

6.7.2 Deformation Functors

Before we can begin proving Theorem 6.7.1, we first need to study the deformation theory of torsion-free rank 1 sheaves on marked stable curves. The content of this section closely follows [30, Section 3].

Let $(C, \underline{x})$ be a marked curve (not necessarily complete), and let F be a coherent sheaf on C. Let A be a local $\mathbb{C}$-algebra with residue field $\mathbb{C}$.

Definition 6.7.5. *A* deformation of $(C, \underline{x}, F)$ *over* A *is a quintuple* $(C_A, \underline{x}_A, F_A, i, j)$ *where:*

1. C_A *is a flat* A*-scheme, and* $\underline{x}_A$ *is a tuple of sections* $x_{k,A} : \operatorname{Spec} A \to C_A$ *(for* $k = 1, \dots, n$*);*
2. F_A *is an* A*-flat coherent sheaf on* C_A *of finite presentation;*
3. i *is an isomorphism* $C_A \otimes_A \mathbb{C} \xrightarrow{\simeq} C$*, compatible with the sections* $\underline{x}$ *and* $\underline{x}_A$*; and*
4. $j : i_*(F_A \otimes_A \mathbb{C}) \xrightarrow{\simeq} F$ *is an isomorphism of* $\mathcal{O}_C$*-modules.*

The trivial deformation of $(C, \underline{x}, F)$ *over* A *is the quintuple* $(C \otimes_{\mathbb{C}} A, \underline{x} \otimes_{\mathbb{C}} A, F \otimes_{\mathbb{C}} A, i_{\text{can}}, j_{\text{can}})$*, where* i_{can} *and* j_{can} *are the canonical maps.*

If $(C'_A, \underline{x}'_A, F'_A, i', j')$ *is a second deformation of* $(C, \underline{x}, F)$ *over* A*, then an* isomorphism of deformations from $(C_A, \underline{x}_A, F_A, i, j)$ to $(C'_A, \underline{x}'_A, F'_A, i', j')$ *is a pair* (η, λ)*, where:*

1. η *is an isomorphism of* A*-schemes* $C_A \xrightarrow{\simeq} C'_A$*, compatible with the sections* $\underline{x}_A$ *and* $\underline{x}'_A$*, such that* $i' \circ (\eta \otimes 1) = i$*; and*
2. λ *is an isomorphism of* $\mathcal{O}_{C'_A}$*-modules* $\eta_*(F_A) \xrightarrow{\simeq} F'_A$ *such that* $j' \circ i'_*(\lambda \otimes 1) = j$*.*

A deformation of $(C, \underline{x})$ over A is defined by omitting the data of F_A and j from the definition of a deformation of a triple $(C, \underline{x}, F)$; the trivial deformation *of $(C, \underline{x})$ over A and* isomorphisms of deformations *of $(C, \underline{x})$ over A are defined analogously.*

A deformation of F over A *is a pair (F_A, j) such that $(C \otimes_{\mathbb{C}} A, \underline{x} \otimes_{\mathbb{C}} A, F_A, i_{\mathrm{can}}, j)$ is a deformation of $(C, \underline{x}, F)$ over A. The* trivial deformation of F over A *is the pair $(F \otimes_{\mathbb{C}} A, j_{\mathrm{can}})$. If (F'_A, j') is a second deformation of F over A, an* isomorphism of deformations *from (F_A, j) to (F'_A, j') is an isomorphism $\lambda : F_A \xrightarrow{\cong} F'_A$ such that $j = j' \circ (\lambda \otimes 1)$.*

Definition 6.7.6. *The functor $\mathrm{Def}_{(C,\underline{x},F)} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ is defined by setting $\mathrm{Def}_{(C,\underline{x},F)}(A)$ to be the set of all isomorphism classes of deformations of $(C, \underline{x}, F)$ over A. The functors $\mathrm{Def}_{(C,\underline{x})}$ and Def_F are defined analogously.*

Recall from Section 6.3.1 that an automorphism of the triple $(C, \underline{x}, F)$ is a pair (σ, τ), where $\sigma \in \mathrm{Aut}(C, \underline{x})$ and τ is an isomorphism of $\mathcal{O}_C$-modules $\sigma_* F \xrightarrow{\cong} F$. The group $\mathrm{Aut}(C, \underline{x}, F)$ acts in a natural way on $\mathrm{Def}_{(C,\underline{x},F)}$, via

$$(\sigma, \tau) : (C_A, \underline{x}_A, F_A, i, j) \mapsto (C_A, \underline{x}_A, F_A, \sigma \circ i, \tau \circ \sigma_*(j)). \tag{6.7.1}$$

In a similar manner, there are actions of $\mathrm{Aut}(C, \underline{x})$ on $\mathrm{Def}_{(C,\underline{x})}$ and $\mathrm{Aut}(F)$ on Def_F.

Notation 6.7.7. *Given a marked prestable curve $(C, \underline{x})$, a local $\mathbb{C}$-algebra A with residue field $\mathbb{C}$ and a deformation $(C_A, \underline{x}_A)$ of $(C, \underline{x})$ over A, we denote*[9]

$$\mathcal{L}_{(C_A,\underline{x}_A)} := \omega_{C_A/A}^{10}\left(10\left(\sum_{i=1}^{n} x_{i,A}\right)\right).$$

We also denote $\mathcal{L}_{(C,\underline{x})} := \mathcal{L}_{(C_{\mathbb{C}},\underline{x}_{\mathbb{C}})} = \omega_C^{10}(10(x_1 + \cdots + x_n))$.

We now introduce the deformation functors arising from the parameter scheme

$$Q := Q_N = \mathrm{Quot}_{\mathcal{U}I/I}(V_N \otimes \mathcal{L}_I^{-N}, \widetilde{P}),$$

where $V = V_N$ is a finite dimensional $\mathbb{C}$-vector space. Suppose we are given a quotient $q : V \otimes \mathcal{L}_{(C,\underline{x})}^{-N} \to F$ and a closed embedding $p : (C, \underline{x}) \to \mathbb{P}(W) \times \mathbb{P}(W)^{\times n}$ which on C restricts to a 10-log-canonical[10] non-degenerate closed embedding $C \to \mathbb{P}(W)$.

[9] From now on we fix $a = 10$.

[10] That is, the pullback of $\mathcal{O}_{\mathbb{P}(W)}(1)$ to C is isomorphic to $\mathcal{L}_{(C,\underline{x})} = \omega_C^{10}(10(x_1 + \cdots + x_n))$.

Definition 6.7.8. *A* deformation of the pair (p, q) over *A is a septuple of the form* $(C_A, \underline{x}_A, F_A, i, j, p_A, q_A)$ *such that:*

1. $(C_A, \underline{x}_A, F_A, i, j)$ *is a deformation of* $(C, \underline{x}, F)$ *over A;*

2. $p_A : (C_A, \underline{x}_A) \to \mathbb{P}_A(W) \times \mathbb{P}_A(W)^{\times n}$ *is a closed embedding such that* $\mathcal{O}_{C_A}(1)$ *and* $\omega^{10}_{C_A/A}(10\underline{x}_A)$ *are isomorphic and such that* $p_A \otimes 1 = p \circ i$*; and*

3. $q_A : V \otimes \mathcal{L}^{-N}_{(C_A, \underline{x}_A)} \to F_A$ *is a quotient with* $q = j \circ i_*(q_A \otimes 1)$.

Given a second deformation (p', q') *over A, an* isomorphism of deformations from (p, q) to (p', q') *is given by an isomorphism* (η, λ) *of the underlying deformations of* $(C, \underline{x}, F)$ *with the additional properties that:*

1. $\lambda \circ \eta_*(q_A) = q'_A$*; and*

2. $p_A = p'_A \circ \eta$.

A deformation of the triple $(C, \underline{x}, q)$ over *A is a sextuple* $(C_A, \underline{x}_A, F_A, i, j, q_A)$*, where* $q_A : V \otimes \mathcal{L}^{-N}_{(C_A, \underline{x}_A)} \to F_A$ *is a quotient such that:*

1. $(C_A, \underline{x}_A, F_A, i, j)$ *is a deformation of* $(C, \underline{x}, F)$ *over A; and*

2. $q = j \circ i_*(q_A \otimes 1)$.

Given a second deformation $(C'_A, \underline{x}'_A, F'_A, i', j', q'_A)$ *over A, an* isomorphism of deformations from $(C_A, \underline{x}_A, F_A, i, j, q_A)$ to $(C'_A, \underline{x}'_A, F'_A, i', j', q'_A)$ *is given by an isomorphism* (η, λ) *of the underlying deformations of* $(C, \underline{x}, F)$ *with the additional property that* $\lambda \circ \eta_*(q_A) = q'_A$.

We define the functor $\mathrm{Def}_{(p,q)} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ *(resp.* $\mathrm{Def}_{(C,\underline{x},q)} : \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$*) by assigning to A the set of all isomorphism classes of deformations of* (p, q) *(resp.* $(C, \underline{x}, q)$*) over A.*

6.7.3 Miniversal Deformation Rings

We require an explicit miniversal deformation ring for the functor $\mathrm{Def}_{(C,\underline{x},F)}$. In order to obtain such a ring, we first give an explicit description for the miniversal deformation ring for the local model $(\mathcal{O}_0, F_0)$. Here we continue to follow [30, Section 3].

Definition 6.7.9. *The* standard node *is the complete local* $\mathbb{C}$*-algebra* $\mathcal{O}_0 = \mathbb{C}[[x, y]]/(x, y)$*. Denote the normalisation of* $\mathcal{O}_0$ *by* $\tilde{\mathcal{O}}_0$*; as a subring of* $\mathrm{Frac}(\mathcal{O}_0)$*, this is the ring* $\mathcal{O}_0[x/(x + y)]$*. We additionally denote by* F_0 *the ideal* $(x, y) \subset \mathcal{O}_0$*, considered as an* $\mathcal{O}_0$*-module.*

The module F_0 is the unique torsion-free uniform rank 1 $\mathcal{O}_0$-module (cf. [45, Chapter III]), and admits the following descriptions:

(i) $F_0 = \tilde{\mathcal{O}}_0$, considered as an $\mathcal{O}_0$-module; and

(ii) the $\mathcal{O}_0$-module with presentation $\langle e, f : y \cdot e = x \cdot f = 0\rangle$.

It is a standard result that a miniversal deformation space for the standard node arises from the family of curves $\operatorname{Spec} \mathbb{C}[[x, y, t]]/(xy - t) \to \operatorname{Spec} \mathbb{C}[[t]]$, realising $\mathbb{C}[[t]]$ as a miniversal deformation ring for $\mathrm{Def}_{\mathcal{O}_0}$ (see for instance [63, Example 14.0.1]). The deformation functor $\mathrm{Def}_{(\mathcal{O}_0,F_0)}$ similarly admits an explicit miniversal deformation ring.

Definition 6.7.10. *Define a deformation* $(\mathcal{O}_S, F_S, i, j)$ *of* $(\mathcal{O}_0, F_0)$ *over the ring* $S := \mathbb{C}[[u, v, t]]/(uv - t)$ *by setting:*

1. $\mathcal{O}_S = S[[x, y]]/(xy - t)$*;*

2. F_S *to be the* $\mathcal{O}_S$*-module with presentation* $\langle \tilde{e}, \tilde{f} : y \cdot \tilde{e} = -u \cdot \tilde{f}, x \cdot \tilde{f} = -v \cdot \tilde{e}\rangle$*;*

3. $i : \mathcal{O}_S \otimes_S \mathbb{C} \xrightarrow{\simeq} \mathcal{O}_0$ *to be the isomorphism which is the identity on* x *and* y*; and*

4. $j : i_*(F_S \otimes_S \mathbb{C}) \xrightarrow{\simeq} F_0$ *to be the isomorphism defined by* $\tilde{e} \otimes 1 \mapsto e$ *and* $\tilde{f} \otimes 1 \mapsto f$*.*

Lemma 6.7.11 ([30], Lemma 3.14)**.** *The deformation* $(\mathcal{O}_S, F_S, i, j)$ *defines a morphism* $\mathrm{Spf}(S) \to \mathrm{Def}_{(\mathcal{O}_0,F_0)}$ *which realises* $S = \mathbb{C}[[u, v, t]]/(uv - t)$ *as a miniversal deformation ring for* $\mathrm{Def}_{(\mathcal{O}_0,F_0)}$*.*

6.7.4 Deformations of Stable Curves

Before continuing, we collect a couple of standard results concerning the deformation theory of marked stable curves.

Proposition 6.7.12. *Let* $(C, \underline{x})$ *be an* n*-marked stable curve of genus* g*.*

1. *The tangent space to* $\mathrm{Def}_{(C,\underline{x})}$ *is isomorphic to* $\mathrm{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)$*; the dimension of this vector space is* $3g-3+n$.

2. *There exists a local universal deformation of* $(C, \underline{x})$ *over a smooth pointed scheme* (B, b_0) *whose Kodaira–Spencer map* $T_{b_0}B \to T\mathrm{Def}_{(C,\underline{x})} \cong \mathrm{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)$ *is an isomorphism.*

Proof. See for instance [12, Pages 182-186]. □

6.7.5 Global-to-Local Transformations

Suppose $(C, \underline{x}, F)$ is a triple consisting of a torsion-free uniform rank 1 coherent sheaf F over a stable curve $(C, \underline{x})$. Let $\Delta = \Delta_{(C,F)}$ denote the set of nodes of C where F fails to be locally free. For a node $e \in \Delta$, let $C_e = \operatorname{Spec} \widehat{\mathcal{O}}_{C,e}$ and let F_e denote the pullback of F to C_e. We have global-to-local transformations of deformation functors:

$$\mathrm{Def}_{(C,\underline{x},F)} \to \prod_{e\in\Delta} \mathrm{Def}_{(C_e,F_e)}, \quad \mathrm{Def}_F \to \prod_{e\in\Delta} \mathrm{Def}_{F_e}, \quad \mathrm{Def}_{(C,\underline{x})} \to \prod_{e\in\Delta} \mathrm{Def}_{C_e}.$$

Proposition 6.7.13. *All of these transformations are formally smooth. Moreover, the deformation functors* $\prod_{e\in\Delta} \mathrm{Def}_{C_e}$ *and* $\prod_{e\in\Delta} \mathrm{Def}_{(C_e,F_e)}$ *are both unobstructed.*

Proof. The formal smoothness of $\mathrm{Def}_{(C,\underline{x})} \to \prod_{e\in\Delta} \mathrm{Def}_{C_e}$ follows from the same argument given in [39, Proposition 1.5], after first replacing Ω_C with $\Omega_C(\underline{x})$ and making use of the isomorphism $T\mathrm{Def}_{(C,\underline{x})} \cong \mathrm{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)$. The formal smoothness of the middle transformation is a special case of [49, Proposition B.1]. From Proposition A.3 and Remark A.4 of *loc. cit.*, the functors $\prod_{e\in\Delta} \mathrm{Def}_{C_e}$ and $\prod_{e\in\Delta} \mathrm{Def}_{(C_e,F_e)}$ are both unobstructed.

Using the same argument used to prove Proposition A.1 and Lemma A.2 of *loc. cit.*, the canonical map

$$\mathrm{Def}_{(C,\underline{x},F)} \to \mathrm{Def}_{(C,\underline{x})} \times_{\prod_{e\in\Delta} \mathrm{Def}_{C_e}} \left(\prod_{e\in\Delta} \mathrm{Def}_{(C_e,F_e)} \right)$$

is formally smooth. As such, $\mathrm{Def}_{(C,\underline{x},F)} \to \prod_{e\in\Delta} \mathrm{Def}_{(C_e,F_e)}$ is a composition of formally smooth functors, whence formally smooth. □

Define $\mathrm{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x},F)} = \mathrm{Def}^{\Delta-\mathrm{loc}}_{(C,F)} := \prod_{e\in\Delta}\mathrm{Def}_{(C_e,F_e)}$, and define $\mathrm{Def}^{\Delta-\mathrm{loc}}_{F}$ and $\mathrm{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x})} = \mathrm{Def}^{\Delta-\mathrm{loc}}_{C}$ analogously. Set $\mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}$ to be the kernel of $\mathrm{Def}_{(C,\underline{x},F)} \to \mathrm{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x},F)}$, and define $\mathrm{Def}^{\Delta-\mathrm{lt}}_{F}$ and $\mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})}$ analogously. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)} & \longrightarrow & \mathrm{Def}_{(C,\underline{x},F)} & \longrightarrow & \mathrm{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x},F)} \\ \downarrow{\scriptstyle \Pi^{\Delta-\mathrm{lt}}} & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Pi^{\Delta-\mathrm{loc}}} \\ \mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})} & \longrightarrow & \mathrm{Def}_{(C,\underline{x})} & \longrightarrow & \mathrm{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x})} \end{array} \tag{6.7.2}$$

where the vertical maps correspond to forgetting the sheaves. At the level of tangent spaces, we have the following diagram with exact rows (the exactness of the two rows is a consequence of Proposition 6.7.13):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T\mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)} & \longrightarrow & T\mathrm{Def}_{(C,\underline{x},F)} & \longrightarrow & T\mathrm{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x},F)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T\Pi^{\Delta-\mathrm{lt}}} & & \downarrow{\scriptstyle T\Pi} & & \downarrow{\scriptstyle T\Pi^{\Delta-\mathrm{loc}}} & & \\ 0 & \longrightarrow & T\mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})} & \longrightarrow & T\mathrm{Def}_{(C,\underline{x})} & \longrightarrow & T\mathrm{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x})} & \longrightarrow & 0 \end{array} \tag{6.7.3}$$

Since F is invertible away from the nodes in Δ, the map $T\Pi^{\Delta-\mathrm{lt}}$ is surjective. By [30, Lemma 3.16], the kernel of $T\Pi^{\Delta-\mathrm{lt}}$ is isomorphic to the tangent space $T\mathrm{Def}_L = H^1(\widetilde{C}_\Delta, \mathcal{O}^*_{\widetilde{C}_\Delta})$, where L is any invertible sheaf on the partial normalisation $\widetilde{C}_\Delta$ of C with respect to Δ whose push-forward is F. In particular, the vector space $T\mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}$, and hence $T\mathrm{Def}_{(C,\underline{x},F)}$, is finite dimensional. Exactly as in [23, Proposition 2.19] (which in turn relies on the application of Schlessinger's theorem given in [120, Theorem 3.3.11]), it follows that $\mathrm{Def}_{(C,\underline{x},F)}$ admits a miniversal deformation ring.

Denote by $R_{(C,\underline{x},F)}$ the ring $\mathbb{C}[T^\vee\,\mathrm{Def}_{(C,\underline{x},F)}]$; the miniversal deformation ring of $\mathrm{Def}_{(C,\underline{x},F)}$ can be identified with the power series ring $\hat{R}_{(C,\underline{x},F)}$, obtained by completing at the maximal ideal generated by $T^\vee\,\mathrm{Def}_{(C,\underline{x},F)}$. In particular, there is a formally smooth morphism $\Phi : \mathrm{Spf}(\hat{R}_{(C,\underline{x},F)}) \to \mathrm{Def}_{(C,\underline{x},F)}$ whose derivative $T\Phi$ is an isomorphism, realising $\hat{R}_{(C,\underline{x},F)}$ as the miniversal deformation ring. After choosing identifications of each (C_e, F_e) with $(\mathcal{O}_0, F_0)$ and choosing a splitting of the first row of (6.7.3), by Lemma

6.7.11 there is an isomorphism

$$R_{(C,\underline{x},F)} \cong \mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x},F)}] \otimes_{\mathbb{C}} \mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}] = \bigotimes_{e\in\Delta} \frac{\mathbb{C}[t_e, u_e, v_e]}{(u_e v_e - t_e)} \otimes_{\mathbb{C}} \mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}]. \tag{6.7.4}$$

Here the variable t_e is the coordinate parametrising the smoothing of the node e.

We also introduce the ring $R_{(C,\underline{x})} := \mathbb{C}[T^\vee \operatorname{Def}_{(C,\underline{x})}]$, whose completion is the miniversal deformation ring for $\operatorname{Def}_{(C,\underline{x})}$; analogously to Φ, we have a formally smooth morphism $\overline{\Phi} : \operatorname{Spf}(\hat{R}_{(C,\underline{x})}) \to \operatorname{Def}_{(C,\underline{x})}$ whose derivative is an isomorphism. After choosing a splitting of the second row of (6.7.3) compatible with the splitting of the first row, we have

$$R_{(C,\underline{x})} \cong \mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{loc}}_{(C,\underline{x})}] \otimes_{\mathbb{C}} \mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})}] = \bigotimes_{e\in\Delta} \mathbb{C}[t_e] \otimes_{\mathbb{C}} \mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})}], \tag{6.7.5}$$

where as in (6.7.4), t_e is the coordinate parametrising the smoothing of the node e.

There is an evident injection of $\mathbb{C}$-algebras $R_{(C,\underline{x})} \hookrightarrow R_{(C,\underline{x},F)}$. By taking completions followed by formal spectra, we obtain a diagram

$$\begin{array}{ccc} \operatorname{Spf}(\hat{R}_{(C,\underline{x},F)}) & \xrightarrow{\Phi} & \operatorname{Def}_{(C,\underline{x},F)} \\ \downarrow & & \downarrow \Pi \\ \operatorname{Spf}(\hat{R}_{(C,\underline{x})}) & \xrightarrow[\overline{\Phi}]{} & \operatorname{Def}_{(C,F)} \end{array} \tag{6.7.6}$$

6.7.6 Actions of Automorphism Groups

Let $(C, \underline{x}, F)$ be a triple consisting of a torsion-free uniform rank 1 coherent sheaf F over a stable curve $(C, \underline{x})$. There is an exact sequence of groups

$$1 \to \operatorname{Aut}(F) \to \operatorname{Aut}(C, \underline{x}, F) \to \operatorname{Stab}_{(C,\underline{x})}(F) \to 1,$$

where $\operatorname{Stab}_{(C,\underline{x})}(F)$ is the subgroup of $\operatorname{Aut}(C, \underline{x})$ consisting of automorphisms with $\sigma^* F = F$; since $(C, \underline{x})$ is stable, this group is finite.

As given by Equation (6.7.1), $\operatorname{Aut}(C, \underline{x}, F)$ acts naturally on $\operatorname{Def}_{(C,\underline{x},F)}$, and so acts naturally on the tangent space $T\operatorname{Def}_{(C,\underline{x},F)}$, in such a way as to preserve the top row of (6.7.3). This implies that $\operatorname{Aut}(C, \underline{x}, F)$ acts in a natural way on the ring $R_{(C,\underline{x},F)}$, preserving the decomposition (6.7.4). Similarly, $\operatorname{Aut}(C, \underline{x})$ acts in a natural way on the ring $R_{(C,\underline{x})}$, preserving the decomposition (6.7.5). These actions lift to unique actions on the miniversal deformation rings $\hat{R}_{(C,\underline{x},F)}$ and $\hat{R}_{(C,\underline{x})}$.

Proposition 6.7.14. *Let $(C, \underline{x}, F)$ consist of a triple of a torsion-free uniform rank 1 sheaf F over a stable curve $(C, \underline{x})$.*

1. *There is a unique action of* $\mathrm{Aut}(F_0)$ *on the miniversal deformation ring* $S = \mathbb{C}[[u, v, t]]/(uv - t)$ *with the properties that the map* $\mathrm{Spf}(S) \to \mathrm{Def}_{(\mathcal{O}_0, F_0)}$ *is equivariant, and that the subgroup* $1 + (x, y)\mathcal{O}_0 \subset \mathrm{Aut}(F_0)$ *acts trivially on* S.

2. *There exists a unique action of* $\mathrm{Aut}(C, \underline{x}, F)$ *on the miniversal deformation ring* $\hat{R}_{(C,\underline{x},F)}$ *which makes the morphism* $\Phi : \mathrm{Spf}(\hat{R}_{(C,\underline{x},F)}) \to \mathrm{Def}_{(C,\underline{x},F)}$ *equivariant.*

3. *There exists a unique action of* $\mathrm{Aut}(C, \underline{x})$ *on the miniversal deformation ring* $\hat{R}_{(C,\underline{x})}$ *which makes the morphism* $\overline{\Phi} : \mathrm{Spf}(\hat{R}_{(C,\underline{x})}) \to \mathrm{Def}_{(C,\underline{x})}$ *equivariant.*

Moreover, the actions in (2) and (3) are such that the two vertical maps in Diagram (6.7.6) *are equivariant with respect to the homomorphism* $\mathrm{Aut}(C, \underline{x}, F) \to \mathrm{Aut}(C, \underline{x})$.

Proof. The same argument as used in the proof of [30, Fact 5.4], which is based on applying [111, Theorem on Page 225], carries over. □

Via the inclusion $\mathrm{Aut}(F) \subset \mathrm{Aut}(C, \underline{x}, F)$, the group $\mathrm{Aut}(F)$ acts on the deformation ring $R_{(C,\underline{x},F)}$. Let $\Gamma := \Gamma_C(\Delta)$ be the dual graph of the curve obtained from C by smoothing the nodes *not* in Δ. It follows from [30, Remark 5.9] that $\mathrm{Aut}(F)$ is a torus, isomorphic to $T_\Gamma := \prod_{v \in V(\Gamma)} \mathbb{G}_m$.

Lemma 6.7.15. *Let* $\mathrm{Aut}(F)$ *act on* $R_{(C,\underline{x},F)}$ *via the inclusion* $\mathrm{Aut}(F) \subset \mathrm{Aut}(C, \underline{x}, F)$. *Then, with respect to the decomposition* (6.7.4)*:*

1. $\mathrm{Aut}(F)$ *acts trivially on* $\mathbb{C}[T^\vee \mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}]$.

2. *Given an element* $\lambda = (\lambda_v)_{v \in V(\Gamma)} \in \mathrm{Aut}(F)$, *we have*

$$\lambda \cdot u_e = \lambda_{t(e)} \lambda^{-1}_{h(e)} u_e, \quad \lambda \cdot v_e = \lambda^{-1}_{t(e)} \lambda_{h(e)} v_e, \quad \lambda \cdot t_e = t_e.$$

Proof. By the same argument used to prove Lemma 5.3 of *loc. cit.*, the subgroup $\mathrm{Aut}(F)$ acts trivially on the deformation functor $\mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}$, and hence acts trivially on $\mathbb{C}[T^\vee \mathrm{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}]$. The second statement is proved in the same way as Theorem 5.10 (ii) of *loc. cit.* □

We introduce the following notation.

Definition 6.7.16. *Let Γ be the dual graph of a connected nodal curve.*

1. *The ring $B(\Gamma)$ is the $\mathbb{C}$-algebra $\bigotimes_{e\in E(\Gamma)} \frac{\mathbb{C}[t_e,u_e,v_e]}{(u_ev_e-t_e)}$.*

2. *The ring $U(\Gamma)$ is the algebra of T_Γ-invariants of $B(\Gamma)$, for the action of T_Γ defined by specifying that for $\lambda = (\lambda_v)_{v\in V(\Gamma)}$,*

$$\lambda \cdot u_e = \lambda_{t(e)}\lambda_{h(e)}^{-1}u_e, \quad \lambda \cdot v_e = \lambda_{t(e)}^{-1}\lambda_{h(e)}v_e, \quad \lambda \cdot t_e = t_e.$$

Corollary 6.7.17. *In the situation of Lemma 6.7.15, we have an isomorphism*

$$R_{(C,\underline{x},F)}^{\mathrm{Aut}(F)} \cong U(\Gamma) \otimes_{\mathbb{C}} \mathbb{C}[T^\vee \operatorname{Def}_{(C,\underline{x},F)}^{\Delta-\mathrm{lt}}].$$

We have the following important result of Casalaina-Martin–Kass–Viviani concerning the singularities of the affine scheme $\operatorname{Spec} U(\Gamma)$.

Proposition 6.7.18 (Casalaina-Martin–Kass–Viviani)**.** *The ring $U(\Gamma)$ is a finitely generated integrally closed $\mathbb{C}$-algebra, and the singularities of* $\operatorname{Spec} U(\Gamma)$ *are Gorenstein, rational and terminal.*

Proof. In light of [31, Theorem 6.2], this is Theorem B of *loc. cit.* □

6.7.7 Completed Local Rings of the Moduli Spaces

We now relate the complete local rings of the good moduli space $\overline{J}_{g,n}$ to completed rings of invariants of the form $\hat{R}_{(C,\underline{x},F)}^{\mathrm{Aut}(C,\underline{x},F)}$. Here we closely follow [30, Section 6].

Lemma 6.7.19 (cf. [30], Lemma 6.3)**.** *Suppose $(C,\underline{x},F)$ is a triple consisting of a torsion-free uniform rank 1 coherent sheaf F over a stable curve $(C,\underline{x})$. Suppose in addition that we are given a quotient $q : V \otimes \mathcal{L}_{(C,\underline{x})}^{-N} \to F$ and a closed embedding $p : (C,\underline{x}) \to \mathbb{P}(W)\times\mathbb{P}(W)^{\times n}$ which on C is a 10-log-canonical non-degenerate closed embedding. If $H^1(C, F\otimes\mathcal{L}_{(C,\underline{x})}^N) = 0$ then the forgetful transformation* $\operatorname{Def}_{(p,q)} \to \operatorname{Def}_{(C,\underline{x},F)}$ *is formally smooth.*

Proof. Suppose we are given a surjection $B \to A$ of Artinian local $\mathbb{C}$-algebras, a deformation $(C_B, \underline{x}_B, F_B, i_B, j_B)$ over B and a deformation $(C_A, \underline{x}_A, F_A, i_A, j_A, p_A, q_A)$ of (p,q) over A such that the associated deformation of $(C,\underline{x},F)$ is isomorphic to $(C_B \otimes_B A, \underline{x}_B \otimes_B A, F_B \otimes_B A, i_B \otimes 1, j_B \otimes 1)$. We must show there exists a deformation of the form $(C_B, \underline{x}_B, F_B, i_B, j_B, p_B, q_B)$ of (p,q) over B extending the given deformation

$(C_A, \underline{x}_A, F_A, i_A, j_A, p_A, q_A)$ and inducing the deformation $(C_B \otimes_B A, \underline{x}_B \otimes_B A, F_B \otimes_B A, i_B \otimes 1, j_B \otimes 1)$.

Assume first $B \to A$ is a small extension with kernel $\kappa = \epsilon B$, i.e. $\epsilon \in \mathfrak{m}_B$ is annihilated by the maximal ideal $\mathfrak{m}_B$ of B. $C_A \to C_B$ is then a closed embedding defined by the square-zero ideal sheaf $\pi^*\kappa$, where $\pi : C_B \to \operatorname{Spec} B$ is the structure morphism; the flatness of F_B over B implies that there is a short exact sequence of $\mathcal{O}_{C_B}$-modules

$$0 \to \pi^*\kappa \otimes F_B \otimes \mathcal{L}^N_{(C_B,\underline{x}_B)} \to F_B \otimes \mathcal{L}^N_{(C_B,\underline{x}_B)} \to F_A \otimes \mathcal{L}^N_{(C_A,\underline{x}_A)} \to 0,$$

obtained by twisting the ideal sequence of $C_A \subset C_B$. The vanishing of the cohomology group $H^1(C, F \otimes \mathcal{L}^N_{(C,\underline{x})})$ implies that $H^1(C_B, \pi^*\kappa \otimes F_B \otimes \mathcal{L}^N_{(C_B,\underline{x}_B)}) = 0$, so the restriction morphism $H^0(C_B, F_B \otimes \mathcal{L}^N_{(C_B,\underline{x}_B)}) \to H^0(C_A, F_A \otimes \mathcal{L}^N_{(C_A,\underline{x}_A)})$ is surjective. Similarly, $H^0(C_B, \omega^{10}_{C_B/B}(10\underline{x}_B)) \to H^0(C_A, \omega^{10}_{C_A/A}(10\underline{x}_A))$ is surjective.

In the case where $B \to A$ is not a small extension, by factoring the morphism $B \to A$ as a sequence of small extensions we again obtain that the maps $H^0(C_B, F_B \otimes \mathcal{L}^N_{(C_B,\underline{x}_B)}) \to H^0(C_A, F_A \otimes \mathcal{L}^N_{(C_A,\underline{x}_A)})$ and $H^0(C_B, \omega^{10}_{C_B/B}(10\underline{x}_B)) \to H^0(C_A, \omega^{10}_{C_A/A}(10\underline{x}_A))$ are both surjective.

We first attempt to lift the quotient q_A. Fix a basis $e_1, \dots, e_M$ for V, and let $s_1, \dots, s_M \in H^0(C_A, F_A \otimes \mathcal{L}^N_{(C_A,\underline{x}_A)})$ be the images of the e_i under the quotient $V \otimes \mathcal{O}_{C_A} \to F_A \otimes \mathcal{L}^N_{(C_A,\underline{x}_A)}$. Pick lifts $s'_i \in H^0(C_B, F_B \otimes \mathcal{L}^N_{(C_B,\underline{x}_B)})$ of the s_i. This defines a morphism of $\mathcal{O}_{C_B}$-modules $V \otimes \mathcal{O}_{C_B} \to F_B \otimes \mathcal{L}^N_{(C_B,\underline{x}_B)}$. Let q_B be the induced map $V \otimes \mathcal{L}^{-N}_{(C_B,\underline{x}_B)} \to F_B$; by construction the morphism q_B is surjective.

We next attempt to lift the embedding $p_A : (C_A, \underline{x}_A) \to \mathbb{P}_A(W) \times \mathbb{P}_A(W)^n$. Fix a basis $e_1, \dots, e_K$ for $W^\vee$. The given embedding $C_A \to \mathbb{P}_A(W)$ corresponds to a homomorphism $W^\vee \otimes_{\mathbb{C}} A \to H^0(C_A, \omega^{10}_{C_A/A}(10\underline{x}_A))$ of A-modules such that, if t_i is the image of $e_i \otimes 1$, then the t_i generate $\omega^{10}_{C_A/A}(10\underline{x}_A)$, each open set $(C_A)_{t_i}$ is affine, and for each i the morphism $A[y_1, \dots, \hat{y}_i, \dots, y_K] \to H^0((C_A)_{t_i}, \mathcal{O}_{(C_A)_{t_i}})$ sending $y_j \mapsto t_j/t_i$ is surjective. Pick lifts $t'_i \in H^0(C_B, \omega^{10}_{C_B/B}(10\underline{x}_B))$, and define a homomorphism $W^\vee \otimes_{\mathbb{C}} B \to H^0(C_B, \omega^{10}_{C_B/B}(10\underline{x}_B))$ by sending $e_i \otimes 1 \mapsto t'_i$; this defines a morphism $C_B \to \mathbb{P}_B(W)$ which is a 10-log-canonical closed embedding. Letting $p_B : (C_B, \underline{x}_B) \to \mathbb{P}_B(W) \times \mathbb{P}_B(W)^{\times n}$ be the induced morphism, the deformation $(C_B, \underline{x}_B, F_B, i_B, j_B, p_B, q_B)$ of (p, q) over B has the desired properties. This completes the proof of the lemma. □

We now return to working with the parameter spaces introduced in Section 6.7.1. Recall that we denote $V = V_N$ and $Q = Q_N = \mathrm{Quot}_{\mathcal{U}I/I}(V_N \otimes \mathcal{L}_I^{-N}, \widetilde{P})$.

Lemma 6.7.20 (cf. [30], Lemma 6.4)**.** *Suppose $(C, \underline{x}, F)$ is a triple consisting of a torsion-free uniform rank 1 coherent sheaf F over a stable curve $(C, \underline{x})$. Suppose in addition that we are given a point $\tilde{y} \in Q$, corresponding to a quotient $q : V \otimes \mathcal{L}_{(C,\underline{x})}^{-N} \to F$ and a closed embedding $p : (C, \underline{x}) \to \mathbb{P}(W) \times \mathbb{P}(W)^{\times n}$ which on C is a 10-log-canonical non-degenerate closed embedding. Assume further that $H^1(C, F \otimes \mathcal{L}_{(C,\underline{x})}^N) = 0$, and that the induced maps $V \to H^0(C, F \otimes \mathcal{L}_{(C,\underline{x})}^N)$ and $W^\vee \to H^0(C, \omega_C^{10}(10\underline{x}))$ are both isomorphisms.*

If Z is a slice through $\tilde{y}$ for the action of $G = SL(V) \times PGL(W)$ in some invariant affine open neighbourhood $\tilde{y} \in U \subset Q$, then the completed local ring $\widehat{\mathcal{O}}_{Z,\tilde{y}}$ is a miniversal deformation ring for $\mathrm{Def}_{(C,\underline{x},F)}$.

Proof. From the universal property of Q, the complete local ring $\widehat{\mathcal{O}}_{U,\tilde{y}}$ pro-represents $\mathrm{Def}_{(p,q)}$. Let D be the deformation functor pro-represented by $\widehat{\mathcal{O}}_{Z,\tilde{y}}$. We will show that the restriction $\mathrm{Def}_{(p,q)} \to \mathrm{Def}_{(C,\underline{x},F)}$ to D is formally smooth and an isomorphism on the level of tangent spaces, which will establish the lemma. Since $\mathrm{Def}_{(p,q)} \to \mathrm{Def}_{(C,\underline{x},F)}$ is formally smooth, it suffices to show the following:

(i) $D(\mathbb{C}[\epsilon]) \to \mathrm{Def}_{(C,\underline{x},F)}(\mathbb{C}[\epsilon])$ is injective, where $\mathbb{C}[\epsilon]$ is the ring of dual numbers; and

(ii) for any $A \in \mathbf{Art}_\mathbb{C}$, the map $D(A) \to \mathrm{Def}_{(C,\underline{x},F)}(A)$ has the same image as $\mathrm{Def}_{(p,q)}(A) \to \mathrm{Def}_{(C,\underline{x},F)}(A)$.

Let $H = \mathrm{Stab}_{SL(V) \times PGL(W)}(\tilde{y})$. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the deformation functor pro-represented by the completion of the local ring of G (resp. H) at the identity. The infinitesimal action of $\mathfrak{g}$ on the trivial deformation of (p, q) defines a natural transformation $\mathfrak{g}/\mathfrak{h} \to \mathrm{Def}_{(p,q)}$. By choosing a local inverse of $Z \times^H (SL(V) \times PGL(W)) \to U$, the existence of the slice Z implies the existence of a morphism $\mathrm{Def}_{(p,q)} \to \mathfrak{g}/\mathfrak{h}$, admitting $\mathfrak{g}/\mathfrak{h} \to \mathrm{Def}_{(p,q)}$ as a section, with the property that for any $A \in \mathbf{Art}_\mathbb{C}$, the preimage of $0 \in (\mathfrak{g}/\mathfrak{h})(A)$ is $D(A) \subset \mathrm{Def}_{(p,q)}(A)$. Via utilising the morphism $\mathrm{Def}_{(p,q)} \to \mathfrak{g}/\mathfrak{h}$, the fact that the map $D(A) \to \mathrm{Def}_{(C,\underline{x},F)}(A)$ has the same image as $\mathrm{Def}_{(p,q)}(A) \to \mathrm{Def}_{(C,\underline{x},F)}(A)$ is deduced in exactly the same way as in the proof of [30, Lemma 6.4].

The image of $(\mathfrak{g}/\mathfrak{h})(\mathbb{C}[\epsilon]) \to \mathrm{Def}_{(p,q)}(\mathbb{C}[\epsilon])$ is easily seen to be contained in the kernel of $\mathrm{Def}_{(p,q)}(\mathbb{C}[\epsilon]) \to \mathrm{Def}_{(C,\underline{x},F)}(\mathbb{C}[\epsilon])$; we claim this inclusion is in fact an equality. Taking $A = \mathbb{C}[\epsilon]$, let $(C_A, \underline{x}_A, F_A, i, j, p_A, q_A)$ be a first-order deformation of (p, q), where $(C_A, \underline{x}_A, F_A, i, j)$ is the trivial first-order deformation. p_A and q_A induce morphisms of A-modules

$$V \otimes_{\mathbb{C}} A \xrightarrow{p'_A} H^0(C_A, F_A \otimes \mathcal{L}^N_{(C_A,\underline{x}_A)}), \quad W^\vee \otimes_{\mathbb{C}} A \xrightarrow{q'_A} H^0(C_A, \omega^{10}_{C_A/A}(10\underline{x}_A))$$

which restrict to isomorphisms over $\mathbb{C}$. Pick bases for the vector spaces $H^0(C, F \otimes \mathcal{L}^N_{(C,\underline{x})})$ and $H^0(C, \omega^{10}_C(10\underline{x}))$; via the identifications i and j, we obtain induced bases for $H^0(C_A, F_A \otimes \mathcal{L}^N_{(C_A,\underline{x}_A)})$ and $H^0(C_A, \omega^{10}_{C_A/A}(10\underline{x}_A))$ over A. After picking compatible bases of V and $W^\vee$, we may represent p'_A (resp. q'_A) by a matrix $[p'_A]$ (resp. $[q'_A]$) which reduces to the identity modulo ϵ. By arguing as in the penultimate paragraph of the proof of Lemma 6.4 of *loc. cit.*, we obtain that the kernel of $\mathrm{Def}_{(p,q)}(\mathbb{C}[\epsilon]) \to \mathrm{Def}_{(C,\underline{x},F)}(\mathbb{C}[\epsilon])$ is contained in the image of the orbit map. This proves the claim.

By exactly the same argument as in the final paragraph of the proof of Lemma 6.4 of *loc. cit.*, the image of the orbit map has trivial intersection with $D(\mathbb{C}[\epsilon])$. This implies that $D(\mathbb{C}[\epsilon]) \to \mathrm{Def}_{(C,\underline{x},F)}(\mathbb{C}[\epsilon])$ is injective, which finishes the proof of the lemma. □

Let $\tilde{y} \in Q$ be as in the statement of Lemma 6.7.20, and assume in addition that F is semistable with respect to the chosen (universal) stability condition $\mathcal{L}$. Let $H := \mathrm{Stab}_{SL(V)\times PGL(W)}(\tilde{y})$. There is a natural homomorphism

$$\alpha : H \to \mathrm{Aut}(C, \underline{x}, F), \quad h = (h_V, h_W) \mapsto \alpha(h) = (\alpha_1(h), \alpha_2(h)).$$

Here $\alpha(h)$ is the unique element of $\mathrm{Aut}(C, \underline{x}, F)$ such that $p \circ \alpha_1(h) = h_W \circ p$ and $\alpha_2(h) \circ \alpha_1(h)_*(q) = q \circ h_V^{-1}$.

Lemma 6.7.21 (cf. [30], Lemma 6.6)**.** *The natural homomorphism α is injective, and has image equal to the subgroup $\mathrm{Aut}_1(C, \underline{x}, F)$ of automorphisms $(\sigma, \tau) \in \mathrm{Aut}(C, \underline{x}, F)$ such that the composition*

$$H^0(C, F \otimes \mathcal{L}^N_{(C,\underline{x})}) \xrightarrow{\quad\sigma\quad} H^0(C, \sigma_* F \otimes \mathcal{L}^N_{(C,\underline{x})}) \xrightarrow{\quad\tau\quad} H^0(C, F \otimes \mathcal{L}^N_{(C,\underline{x})})$$

has determinant 1.

Proof. Taking the target X to be a point and the curve class β to be zero, recall from Proposition 6.7.2 that we have an isomorphism

$$\overline{\mathcal{J}}ac^{ss}_{g,n,d,r}(\mathcal{L}) \cong [R_1'/GL(V) \times PGL(W)],$$

where R_1' is an open subscheme of $Q = \mathrm{Quot}_{\mathcal{U}I/I}(V_N \otimes \mathcal{L}_I^{-N}, \widetilde{P})$. As F is assumed to be semistable then $\tilde{y} \in R_1'$. The restriction of the induced isomorphism of groups $\alpha' : \mathrm{Stab}_{GL(V)\times PGL(W)}(\tilde{y}) \xrightarrow{\cong} \mathrm{Aut}(C, \underline{x}, F)$ to H coincides with α, and has image equal to $\mathrm{Aut}_1(C, \underline{x}, F)$. □

We are now in a position to be able to relate the completed local rings of $\overline{J}_{g,n}$ with the miniversal deformation rings $\hat{R}_{(C,\underline{x},F)}$.

Proposition 6.7.22 (cf. [30], Theorem 6.1 and [31], Theorem 8.1)**.** *Suppose* $(C, \underline{x}, F)$ *is a triple consisting of an* $\mathcal{L}$*-semistable torsion-free uniform rank 1 coherent sheaf* F *over a stable curve* $(C, \underline{x})$*, corresponding to a closed point of the stack* $\overline{\mathcal{J}}_{g,n}$*. Suppose in addition that* F *is polystable (with respect to* $\mathcal{L}$*). Then there is an isomorphism*

$$\widehat{\mathcal{O}}_{\overline{J}_{g,n},[(C,\underline{x},F)]} \cong \hat{R}^{\mathrm{Aut}(C,\underline{x},F)}_{(C,\underline{x},F)},$$

where the action of $\mathrm{Aut}(C, \underline{x}, F)$ *on* $\hat{R}_{(C,\underline{x},F)}$ *is the one arising from Proposition 6.7.14, namely the unique action on* $\hat{R}_{(C,\underline{x},F)}$ *making* $\Phi : \mathrm{Spf}(\hat{R}_{(C,\underline{x},F)}) \to \mathrm{Def}_{(C,\underline{x},F)}$ *equivariant.*

Remark. The isomorphism $\widehat{\mathcal{O}}_{\overline{J}_{g,n},[(C,\underline{x},F)]} \cong \hat{R}^{\mathrm{Aut}(C,\underline{x},F)}_{(C,\underline{x},F)}$ is non-canonical, but this is necessarily so, as $\hat{R}_{(C,\underline{x},F)}$ itself is only defined up to a non-canonical isomorphism.

Proof. Pick a lift $\tilde{y} \in R_1' \subset Q$ of $(C, \underline{x}, F)$. Since F is polystable, by Lemma 6.7.3 the point $\tilde{y}$ is GIT polystable. In particular, we can find an invariant open affine subscheme U of R_1' containing $\tilde{y}$, for which the orbit of $\tilde{y}$ in U is closed. As such, we may apply the Luna Slice Theorem (Theorem 2.5.3) to prove the existence of a slice Z through $\tilde{y}$ in $U \subset R_1'$. In particular, there is an isomorphism of complete local rings $\widehat{\mathcal{O}}_{\overline{J}_{g,n},[(C,\underline{x},F)]} \cong \widehat{\mathcal{O}}_{Z/\!\!/H,[\tilde{y}]}$.

By Lemma 6.7.20, $\widehat{\mathcal{O}}_{Z,\tilde{y}}$ is a miniversal deformation ring for $\mathrm{Def}_{(C,\underline{x},F)}$. Moreover, the natural transformation $\mathrm{Spf}(\widehat{\mathcal{O}}_{Z,\tilde{y}}) \to \mathrm{Def}_{(C,\underline{x},F)}$ is equivariant with respect to the natural homomorphism $H = \mathrm{Stab}_{SL(V)\times PGL(W)}(\tilde{y}) \to \mathrm{Aut}(C, \underline{x}, F)$. Consequently, by Proposition 6.7.14 the identification $\widehat{\mathcal{O}}_{Z,\tilde{y}} \cong \hat{R}_{(C,\underline{x},F)}$ is $H \cong \mathrm{Aut}_1(C, \underline{x}, F)$-equivariant, whence

$$(\widehat{\mathcal{O}}_{Z,\tilde{y}})^H \cong (\hat{R}_{(C,\underline{x},F)})^{\mathrm{Aut}_1(C,\underline{x},F)}.$$

By Matsushima's criterion (Proposition 2.5.1), the group $\mathrm{Aut}_1(C, \underline{x}, F)$ is reductive. Similarly, the group $\mathrm{Aut}(C, \underline{x}, F)$ is also reductive. Invoking [30, Lemma 6.7], which asserts that taking the invariants of a (linearly) reductive group action on a local ring commutes with taking completions, we have an isomorphism $(\hat{R}_{(C,\underline{x},F)})^{\mathrm{Aut}_1(C,\underline{x},F)} \cong (R_{(C,\underline{x},F)}^{\mathrm{Aut}_1(C,\underline{x},F)})^{\wedge}$.

From the explicit description of the action on $R_{(C,\underline{x},F)}$ (cf. Lemma 6.7.15), the natural action of $\mathrm{Aut}(C, \underline{x}, F)$ factors through $\mathrm{Aut}(C, \underline{x}, F)/\mathbb{G}_m$, which is the surjective image of $\mathrm{Aut}_1(C, \underline{x}, F)$. This implies that $R_{(C,\underline{x},F)}^{\mathrm{Aut}_1(C,\underline{x},F)} = R_{(C,\underline{x},F)}^{\mathrm{Aut}(C,\underline{x},F)}$, so $\hat{R}_{(C,\underline{x},F)}^{\mathrm{Aut}(C,\underline{x},F)} \cong \hat{R}_{(C,\underline{x},F)}^{\mathrm{Aut}_1(C,\underline{x},F)}$. On the other hand, we have isomorphisms

$$\widehat{\mathcal{O}}_{\overline{J}_{g,n},[(C,\underline{x},F)]} \cong \widehat{\mathcal{O}}_{Z/\!\!/H,[\tilde{y}]} \cong (\widehat{\mathcal{O}}_{Z,\tilde{y}})^H,$$

with the second isomorphism arising from taking invariants of the reductive group H. This proves the result. □

6.7.8 Singularities of $\overline{M}_{g,n}$

Next, we require some results concerning marked stable curves $(C, \underline{x})$ for which there exists $\sigma \in \mathrm{Aut}(C, \underline{x})$ which acts on the local deformation ring as a quasi-reflection,[11] or for which the quotient of the local deformation ring by $\langle\sigma\rangle$ fails to have canonical singularities.

Let $(C, \underline{x})$ be an n-marked stable curve of genus g; recall that there is an isomorphism $T\mathrm{Def}_{(C,\underline{x})} \cong \mathrm{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C) \cong \mathbb{C}^{3g-3+n}$, and that there exists a local universal deformation of $(C, \underline{x})$ over a smooth pointed scheme. Let $\hat{R}_{(C,\underline{x})} = \mathbb{C}[[t_1, \ldots, t_{3g-3+n}]]$ be the miniversal deformation ring for $(C, \underline{x})$.

Proposition 6.7.23. *Suppose $g \geq 4$, and suppose $\sigma \in \mathrm{Aut}(C, \underline{x})$ is such that σ acts as a quasi-reflection on $\mathrm{Spec}\,\hat{R}_{(C,\underline{x})}$ or such that $\mathrm{Spec}\,\hat{R}_{(C,\underline{x})}/\langle\sigma\rangle$ fails to have canonical singularities. Then:*

1. *C has an elliptic tail E (i.e. a component of arithmetic genus 1 which meets the rest of C at a single node p), and E contains none of the markings of C.*

2. *σ is an elliptic tail automorphism: $\sigma|_{\overline{C\setminus E}} = \mathrm{id}$. The restriction of σ to E is an automorphism of order $o \in \{2, 3, 4, 6\}$ which fixes p.*

[11] An invertible $n \times n$ matrix g is said to be a *quasi-reflection* if 1 is an eigenvalue of g of geometric multiplicity $n - 1$.

3. *If $o = 4$ then E is smooth with j-invariant 1728, and if $o = 3$ or 6 then E is smooth with j-invariant 0.*

4. *If E is a singular curve with internal node q, then $o = 2$, and the action on E is given by $z \mapsto -z$ on $\widetilde{E} \cong \mathbb{P}^1$, where the fixed point ∞ lies above p and ± 1 lie above q.*

Proof. In the unpointed case, this is [61, Theorem 2]. That the result in the unpointed case implies the result holds when $n > 0$ follows from the proof of [83, Theorem 2.5]; the deformation space $H^0(C, \Omega_C \otimes \omega_C)$ of the unpointed curve C embeds into the deformation space $H^0(C, \Omega_C \otimes \omega_C(\underline{x}))$ of the pointed curve, and contracting any $\mathbb{P}^1$-components of C (when performing stable reduction after omitting the markings) does not alter the action of σ on $H^0(C, \Omega_C \otimes \omega_C)$. Hence the Reid–Tai–Shepherd-Barron age of σ viewed as an automorphism of $(C, \underline{x})$ is as least as large as the age of σ, viewed as an automorphism of C. As such, by the Reid–Tai–Shepherd-Barron criterion (cf. [97, Theorem 2.3]) and the result in the unpointed case, if σ acts as a quasi-reflection on $\operatorname{Spec} \hat{R}_{(C,\underline{x})}$ or such that $\operatorname{Spec} \hat{R}_{(C,\underline{x})}/\langle \sigma \rangle$ fails to have canonical singularities then σ is an elliptic tail automorphism of C with respect to some elliptic tail E. E cannot contain any marking x_i, since the restriction of $\Omega_C \otimes \omega_C(\underline{x})$ to E is the same as it would be if another component was attached at x_i, contradicting the result in the unpointed case. As such, the remaining statements follow from the unpointed case. □

Suppose $g \geq 4$, and suppose $\sigma \in \operatorname{Aut}(C, \underline{x})$ is such that σ acts as a quasi-reflection on $\operatorname{Spec} \hat{R}_{(C,\underline{x})}$ or such that $\operatorname{Spec} \hat{R}_{(C,\underline{x})}/\langle \sigma \rangle$ fails to have canonical singularities. We wish to understand the action of σ on $\hat{R}_{(C,\underline{x})}$ in suitable coordinates.

Lemma 6.7.24 (cf. [84], Proposition 3.2.9)**.** *For each irreducible component $C_i \subset C$, let $\nu_i : \widetilde{C}_i \to C$ be its normalisation, let $D_i \subset \widetilde{C}_i$ be the divisor corresponding to the preimages of the nodes of C lying on C_i, and let $P_i \subset \widetilde{C}_i$ be the divisor corresponding to the preimages of the marked points of C lying on C_i. Let $\Theta_{(C,\underline{x})} = \mathcal{H}om_C(\Omega_C(\underline{x}), \mathcal{O}_C)$.*

Suppose in addition that C has δ-many nodes. Then it is possible to choose coordinates $t'_1, \ldots, t'_{3g-3+n}$ of $T\mathrm{Def}_{(C,\underline{x})} \cong \operatorname{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)$ as follows:

1. *For each $i = 1, \ldots, \delta$, $t'_i = 0$ corresponds to the locus where the ith node is preserved, and the one-dimensional subspace $\{t'_j = 0 : j \neq i\}$ corresponds to the smoothing of the ith node.*

2. *For each irreducible component* C_i *of* C*, there is a subset* $I = I_{C_i} \subset \{\delta+1, \dots, 3g-3+n\}$ *such that* $\{t'_j = 0 : j \notin I\}$ *corresponds to the image of* $H^0(\widetilde{C}_i, \omega^2_{\widetilde{C}_i}(D_i+P_i))^\vee$ *in* $H^1(C, \Theta_{(C,\underline{x})}) \subset \mathrm{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)$ *under the isomorphism* $H^1(C, \Theta_{(C,\underline{x})}) \cong \bigoplus_j H^0(\widetilde{C}_j, \omega^2_{\widetilde{C}_j}(D_j + P_j))^\vee$*.*

Proof. From the local-to-global Ext spectral sequence

$$E_2^{p,q} = H^p(C, \mathcal{E}xt^q_C(\Omega_C(\underline{x}), \mathcal{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}_C(\Omega_C(\underline{x}), \mathcal{O}_C),$$

there is a short exact sequence

$$0 \to H^1(C, \Theta_{(C,\underline{x})}) \to \mathrm{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C) \to H^0(C, \mathcal{E}xt^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)) \to 0.$$

The sheaf $\mathcal{E}xt^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)$ is supported at the nodes of C; if $s \in C$ is a node, then

$$\mathcal{E}xt^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)_s = \mathrm{Ext}^1_{\mathcal{O}_{C,s}}(\Omega_{C,s}, \mathcal{O}_{C,s}) \cong \mathbb{C}_s$$

parametrises the smoothing of the node s (cf. [12, Page 181]). On the other hand, if $\nu : \widetilde{C} \to C$ is the normalisation of C and if $D \subset \widetilde{C}$ (resp. $P \subset \widetilde{C}$) is the divisor of the preimages of the nodes (resp. markings) of C, then $\Theta_{(C,\underline{x})}$ can be identified with the pushforward of $T_{\widetilde{C}}(-D-P)$ (cf. Page 186 of *loc. cit.*), whence

$$H^1(C, \Theta_{(C,\underline{x})}) \cong H^1(\widetilde{C}, T_{\widetilde{C}}(-D-P)) \cong \bigoplus_j H^0(\widetilde{C}_j, \omega^2_{\widetilde{C}_j}(D_j + P_j))^\vee,$$

where the last isomorphism is given by Serre duality. □

Corollary 6.7.25. *With* $(C, \underline{x})$ *as above, it is possible to choose coordinates* $t_1, \dots, t_{3g-3+n}$ *of* $T\mathrm{Def}_{(C,\underline{x})} \cong \mathrm{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)$ *such that:*

1. t_1 *corresponds to the smoothing of the node* p*.*

2. *If* E *is singular, then* t_2 *corresponds to the smoothing of the node* q*; otherwise,* t_2 *is a coordinate for* $T\mathrm{Def}_{(E,p)} \cong \mathbb{C}$*.*

3. *The remaining coordinates* $t_3, \dots, t_{3g-3+n}$ *correspond to nodes and components where* σ *is the identity.*

Proof. If E is smooth, we choose t_2 such that $\{t_i = 0 : i \neq 2\}$ corresponds to the image of the one-dimensional subspace $H^1(E, T_E(-p)) = H^0(E, \omega_E^2(p))^\vee = H^0(E, \mathcal{O}_E(p))^\vee$ of $H^1(C, \Theta_{(C,\underline{x})})$ in $\mathrm{Ext}^1_C(\Omega_C(\underline{x}), \mathcal{O}_C)$. Since we have $H^1(E, T_E(-p)) = \mathrm{Ext}^1_E(\Omega_E(p), \mathcal{O}_E)$, this subspace is the tangent space $T\mathrm{Def}_{(E,p)}$. Relabelling the remaining t_i'-coordinates as $t_3, \dots, t_{3g-3+n}$, by Lemma 6.7.24 these all correspond to nodes and components where σ is the identity.

If instead E is singular, the component E does not contribute to $H^1(C, \Theta_{(C,\underline{x})})$, since $H^0(\widetilde{E}, \omega^2(p + \nu^{-1}(q))) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. As such, after choosing t_1 and t_2, the remaining coordinates once again correspond to nodes and components where σ is the identity. □

We are now ready to state the extension of [84, Proposition 4.2.5] we require.

Proposition 6.7.26. *Let $\rho = e^{\pi i/3}$. With the coordinates $t_1, \dots, t_{3g-3+n}$ of $T\mathrm{Def}_{(C,\underline{x})}$ chosen as in Corollary 6.7.25, the action of σ on $T\mathrm{Def}_{(C,\underline{x})}$ is given by the matrix $M(\sigma)$, where*

$$M(\sigma) = \begin{cases} \begin{pmatrix} -1 & & \\ & 1 & \\ & & \mathrm{id} \end{pmatrix} & o = 2, \\ \begin{pmatrix} \pm i & & \\ & -1 & \\ & & \mathrm{id} \end{pmatrix} & o = 4, \\ \begin{pmatrix} \rho^2 & & \\ & \rho^4 & \\ & & \mathrm{id} \end{pmatrix} \text{ or } \begin{pmatrix} \rho^4 & & \\ & \rho^2 & \\ & & \mathrm{id} \end{pmatrix} & o = 3, \\ \begin{pmatrix} \rho^5 & & \\ & \rho^4 & \\ & & \mathrm{id} \end{pmatrix} \text{ or } \begin{pmatrix} \rho & & \\ & \rho^2 & \\ & & \mathrm{id} \end{pmatrix} & o = 6. \end{cases}$$

In particular, σ is a quasi-reflection if and only if $o = 2$.

Proof. The component E contains none of the markings of C, and the additional coordinates $t_3, \dots, t_{3g-3+n}$ correspond to nodes and components where σ is the identity, so that $\sigma : t_i \mapsto t_i$ on $T\mathrm{Def}_{(C,\underline{x})}$. As such, the necessary analysis reduces to the unmarked case, which is [84, Proposition 4.2.5]. □

6.7.9 The Moduli Space has Canonical Singularities

We are now ready to prove the generalisation of [31, Theorem A] to the case when there are $n > 0$ marked points.

Proof of Theorem 6.7.1. The proof closely follows that of Theorem 8.4 of *loc. cit.* Since the property of having canonical singularities can be checked by passing to an analytic neighbourhood, by Proposition 6.7.22 it is enough to show that for any triple $(C, \underline{x}, F)$ consisting of a polystable torsion-free uniform rank 1 coherent sheaf F over a stable curve $(C, \underline{x})$, the affine scheme $\operatorname{Spec} R_{(C,\underline{x},F)}^{\operatorname{Aut}(C,\underline{x},F)}$ has canonical singularities. From the exact sequence

$$1 \to \operatorname{Aut}(F) \to \operatorname{Aut}(C, \underline{x}, F) \to \operatorname{Stab}_{(C,\underline{x})}(F) \to 1$$

and the fact that $\operatorname{Stab}_{(C,\underline{x})}(F)$ is a finite group, we have

$$\operatorname{Spec} R_{(C,\underline{x},F)}^{\operatorname{Aut}(C,\underline{x},F)} = \operatorname{Spec} \left(R_{(C,\underline{x},F)}^{\operatorname{Aut} F} \right)^{\operatorname{Stab}_{(C,\underline{x})}(F)} = \operatorname{Spec} R_{(C,\underline{x},F)}^{\operatorname{Aut} F} / \operatorname{Stab}_{(C,\underline{x})}(F).$$

By Proposition 6.7.18, the scheme $\operatorname{Spec} R_{(C,\underline{x},F)}^{\operatorname{Aut} F}$ is a normal scheme of finite type over $\mathbb{C}$. As such, by applying Theorem A.11 of *loc. cit.*, it suffices to show that for every $\sigma \in \operatorname{Stab}_{(C,\underline{x})}(F)$, the scheme $\operatorname{Spec} R_{(C,\underline{x},F)}^{\operatorname{Aut} F} / \langle \sigma \rangle$ has canonical singularities.

Pick an invertible sheaf L on the partial normalisation $\widetilde{C}_\Delta$ whose push-forward is F, so that the kernel of $T\Pi^{\Delta-\mathrm{lt}} : T \operatorname{Def}_{(C,\underline{x},F)}^{\Delta-\mathrm{lt}} \to T \operatorname{Def}_{(C,\underline{x})}^{\Delta-\mathrm{lt}}$ is isomorphic to $T \operatorname{Def}_L$, whence

$$\mathbb{C}[T^\vee \operatorname{Def}_{(C,\underline{x},F)}^{\Delta-\mathrm{lt}}] \cong \mathbb{C}[T^\vee \operatorname{Def}_{(C,\underline{x})}^{\Delta-\mathrm{lt}}] \otimes_{\mathbb{C}} \mathbb{C}[T^\vee \operatorname{Def}_L].$$

We now split into cases depending on the automorphism σ.

Case 1: The automorphism σ does not act as a quasi-reflection on $\operatorname{Spec} R_{(C,\underline{x})}$ and $\operatorname{Spec} R_{(C,\underline{x})}/\langle \sigma \rangle$ has canonical singularities.

The inclusion $R_{(C,\underline{x})} \hookrightarrow R_{(C,\underline{x},F)}^{\operatorname{Aut}(F)}$ gives rise to the following diagram:

$$\begin{array}{ccc}
\operatorname{Spec} R_{(C,\underline{x},F)}^{\operatorname{Aut}(F)} & \xrightarrow{\quad\Psi\quad} & \operatorname{Spec} R_{(C,\underline{x})} \times \operatorname{Spec} \mathbb{C}[T^\vee \operatorname{Def}_L] \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
\operatorname{Spec} U(\Gamma) \otimes \mathbb{C}[T^\vee \operatorname{Def}_{(C,\underline{x})}^{\Delta-\mathrm{lt}}] \otimes \mathbb{C}[T^\vee \operatorname{Def}_L] & \to & \operatorname{Spec} \bigotimes_{e\in\Delta} \mathbb{C}[t_e] \otimes \mathbb{C}[T^\vee \operatorname{Def}_{(C,\underline{x})}^{\Delta-\mathrm{lt}}] \otimes \mathbb{C}[T^\vee \operatorname{Def}_L]
\end{array}$$

Here the bottom horizontal map is the map induced by the map $\bigotimes_{e\in\Delta} \mathbb{C}[t_e] \to U(\Gamma) = B(\Gamma)^{T_\Gamma}$ together with the identity on the other two factors. The vertical isomorphisms can be chosen in a way which is compatible with the action of $\langle\sigma\rangle$ on the domain and codomain of the $\langle\sigma\rangle$-equivariant morphism Ψ. As such, the argument used in Case 1 of Theorem 8.4 of *loc. cit.* carries over, yielding that $\operatorname{Spec} R^{\operatorname{Aut} F}_{(C,\underline{x},F)}/\langle\sigma\rangle$ has canonical singularities.

Case 2: σ acts as a quasi-reflection on $\operatorname{Spec} R_{(C,\underline{x})}$, or $\operatorname{Spec} R_{(C,\underline{x})}/\langle\sigma\rangle$ does not have canonical singularities.

In this case, by Proposition 6.7.23 C must have an elliptic tail (E,p), E contains none of the markings of C, and σ must be an elliptic tail automorphism.

As in Case 2 of the proof of Theorem 8.4 of *loc. cit.*, we split into two subcases:

(i) *Case 2-I:* F is not locally free at p.

(ii) *Case 2-II:* F is locally free at p.

Letting Γ_E (resp. Γ_{E^c}) denote the dual graph of E (resp. E^c), we have as in Case 2 of Theorem 8.4 of *loc. cit.* an identification

$$\operatorname{Spec} R^{\operatorname{Aut}(F)}_{(C,\underline{x},F)}/\langle\sigma\rangle = \begin{cases} \operatorname{Spec} U(\Gamma_{E^c}) \times \operatorname{Spec}\left(U(\Gamma_E)\otimes\mathbb{C}[t_p]\otimes\mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}]\right)/\langle\sigma\rangle & \text{2-I,}\\ \operatorname{Spec} U(\Gamma_{E^c}) \times \operatorname{Spec}\left(U(\Gamma_E)\otimes\mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}]\right)/\langle\sigma\rangle & \text{2-II.}\end{cases}$$

As $\operatorname{Spec} U(\Gamma_{E^c})$ has terminal singularities by Proposition 6.7.18, it suffices to show that the remaining factor, namely $\operatorname{Spec}\left(U(\Gamma_E)\otimes\mathbb{C}[t_p]\otimes\mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}]\right)/\langle\sigma\rangle$ and $\operatorname{Spec}\left(U(\Gamma_E)\otimes\mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}]\right)/\langle\sigma\rangle$ as appropriate, has canonical singularities.

We make use of the decomposition $\mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x},F)}] \cong \mathbb{C}[T^\vee \operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})}] \otimes_{\mathbb{C}} \mathbb{C}[T^\vee \operatorname{Def}_L]$. Let V_E denote the vector space

$$V_E := \begin{cases} TU(\Gamma_E)\oplus T\mathbb{C}[t_p]\oplus T\operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})}\oplus T\operatorname{Def}_L & \text{2-I,}\\ TU(\Gamma_E)\oplus T\operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})}\oplus T\operatorname{Def}_L & \text{2-II.}\end{cases}$$

We compute the matrix $R(\sigma)$ for the action of σ on V_E, with respect to a suitable basis. Comparing with the situation of Proposition 6.7.26, in Case 2-I we may take the coordinate t_1, corresponding to the smoothing of the node p, as a coordinate of $T\mathbb{C}[t_p]$, and in Case 2-II we may take t_1 as one of the coordinates of $T\operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})}$. If $o = \operatorname{ord}(\sigma) > 2$ (in which case E is smooth), we may take the coordinate t_2 as one of the coordinates of $T\operatorname{Def}^{\Delta-\mathrm{lt}}_{(C,\underline{x})}$. It follows that in both 2-I and 2-II, the upper left 2×2-block of the matrix $M(\sigma)$ of Proposition 6.7.26 will appear as a block factor of $R(\sigma)$.

Denoting by $\widetilde{E}_\Delta$ (resp. $\widetilde{E}^c_\Delta$) the partial normalisation of E (resp. E^c) at the nodes of E (resp. E^c) belonging to Δ, as in the proof of Theorem 8.4 of *loc. cit.* we have a decomposition

$$T\operatorname{Def}_L \cong T\operatorname{Def}_{L|_{\widetilde{E}_\Delta}} \oplus T\operatorname{Def}_{L|_{\widetilde{E}^c_\Delta}}.$$

Since σ is an elliptic tail automorphism, the action of σ on $T\operatorname{Def}_{L|_{\widetilde{E}^c_\Delta}}$ is trivial. Therefore, as in the unmarked case, we obtain that in Case 2, the upper left 2×2-block of the matrix $M(\sigma)$ of Proposition 6.7.26 appears as a block factor of $R(\sigma)$, and the action on $T\operatorname{Def}_L$ is determined by the action on $T\operatorname{Def}_{L|_{\widetilde{E}_\Delta}} \cong T_{L|_{\widetilde{E}_\Delta}}\operatorname{Pic}(\widetilde{E}_\Delta)$.

At this point, the rest of the argument for Case 2 involves analysing the action of σ on $T\operatorname{Def}_{L|_{\widetilde{E}_\Delta}}$. But E does not contain any markings of C; as such, the argument given in the remainder of Case 2 of Theorem 8.4 of *loc. cit.* carries over word-for-word to the marked setting, making use of the Reid–Shepherd-Barron–Tai criterion for quotient singularities. This yields that $\operatorname{Spec} R^{\operatorname{Aut} F}_{(C,\underline{x},F)}/\langle\sigma\rangle$ has canonical singularities, completing the proof of the theorem. $\square$

Chapter 7

Moduli Spaces of Hyperplanar Admissible Flags in Projective Space

7.1 Introduction

In this chapter we construct using non-reductive GIT quasi-projective coarse moduli spaces which parametrise certain classes of varying flags of subschemes $X^0 \subset X^1 \subset \cdots \subset X^n$ of a fixed projective space $\mathbb{P}(V)$ up to the action of the group $PGL(V)$ of projective automorphisms.

One of the key motivations behind Mumford's development of GIT was the construction of quasi-projective moduli spaces of projective schemes. Classical examples of such GIT moduli spaces include non-singular hypersurfaces in projective space [101], stable curves [54] [100], canonically polarised surfaces of general type [53] and canonically polarised varieties with canonical singularities of any dimension [128]. The first three of these constructions involve using the Hilbert–Mumford criterion to establish GIT stability. However, when working with higher dimensional schemes other approaches to constructing moduli spaces, involving GIT or otherwise, are often necessary; amongst other issues, it is not known how to directly implement the Hilbert–Mumford criterion to test for the GIT stability of subschemes X of projective space with $\dim X \geq 3$, even when X is smooth (Viehweg's construction relies on establishing deep positivity results to directly produce invariant sections of the linearisations on the Hilbert scheme he considers).

Related to the problem of moduli of schemes is the moduli of *flags* of schemes, such as pairs of schemes with a divisor. The main result of the chapter is that *non-reductive* GIT yields constructions of examples of non-empty quasi-projective coarse moduli spaces of flags of schemes, namely when the schemes are subschemes of a projective space $\mathbb{P}(V)$ and the flag $\underline{X} : X^0 \subset X^1 \subset \cdots \subset X^n$ is obtained by intersecting an n-dimensional variety $X^n \subset \mathbb{P}(V)$ with a flag of linear subspaces $\underline{Z} : Z^0 \subset \cdots \subset Z^n = \mathbb{P}(V)$ with $Z^i \subset \mathbb{P}(V)$, of codimension $n - i$; here the flags $\underline{X}$ and $\underline{Z}$ are allowed to vary in $\mathbb{P}(V)$ and V respectively.

Summary of Results

The objects of interest to us are known as *hyperplanar admissible flags of subschemes of* $\mathbb{P}(V)$,[1] which are flags $X^0 \subset X^1 \subset \cdots \subset X^n$ where X^n is an n-dimensional subscheme of $\mathbb{P}(V)$, and where each X^i is an i-dimensional subscheme obtained by intersecting X^n with a linear subspace $\mathbb{P}(Z^i) \subset \mathbb{P}(V)$ of codimension i, with $Z^0 \subset Z^1 \subset \cdots \subset Z^n = V$. The *degree* of this flag is given by the common degree of the subschemes $X^i \subset \mathbb{P}(V)$, and the *Hilbert type* is given by the Hilbert polynomials of the X^i. Such a flag is said to be *non-degenerate* if each $X^i \subset \mathbb{P}(Z^i)$ is non-degenerate (i.e. not contained in a hyperplane in $\mathbb{P}(Z^i)$), non-singular if X^0 is a disjoint union of reduced points and if the remaining X^i are smooth connected varieties, and *stable* if X^0 is a Chow stable length 0 subscheme of $\mathbb{P}(Z^0)$ (for more details, see Section 7.3.1). Given any non-singular non-degenerate subvariety $X^n \subset \mathbb{P}(V)$ then, provided the degree of X^n is sufficiently large, the intersection of X^n with a generic choice of a flag of linear subspaces $\mathbb{P}(Z^0) \subset \cdots \subset \mathbb{P}(Z^n) = \mathbb{P}(V)$ of the appropriate dimensions is a non-degenerate, non-singular and stable hyperplanar admissible flag (see the discussion following Definition 7.3.1).

The main result of this chapter is that there are quasi-projective coarse moduli spaces parametrising non-degenerate, non-singular and stable hyperplanar admissible flags of subschemes of $\mathbb{P}(V)$:

Theorem 7.1.1. *Let n, d be positive integers, and let V be a finite dimensional complex vector space. Assume $n + 1 < \dim V$ and $d > \dim V - n$, and in addition assume $d \notin \left\{ \frac{\dim V - n - 1 + i}{n+1-i} : i = 1, \ldots, n \right\}$. Let $\underline{\Phi} = (\Phi_0, \ldots, \Phi_n)$ be a tuple of Hilbert polynomials*

[1] The terminology here is inspired by that of [82]. The hyperplanar admissible flags of interest to us are admissible flags in the sense of Lazarsfeld–Mustaţă, with the exception that we allow multiple points in dimension 0, not just one.

of subschemes of $\mathbb{P}(V)$*, where* $\Phi_i(t) = \frac{d}{i!}t^i + O(t^{i-1})$ *for each* i*. Let* $\mathcal{F}^{\mathbb{P}(V)}_{n,d,\underline{\Phi}}$ *be the moduli functor parametrising (projective) equivalence classes of families of non-degenerate, non-singular and stable (complete) hyperplanar admissible flags of subschemes of* $\mathbb{P}(V)$ *of length* n*, degree* d *and Hilbert type* $\underline{\Phi}$ *(cf. Definition 7.3.4).*

Then there exists a quasi-projective coarse moduli space $M^{\mathbb{P}(V)}_{n,d,\underline{\Phi}}$ *for the moduli functor* $\mathcal{F}^{\mathbb{P}(V)}_{n,d,\underline{\Phi}}$*, which admits a non-reductive GIT construction.*

In Section 7.5, we explain how Theorem 7.1.1 can be extended in various ways, such as incorporating weightings to the points of X^0, or by instead considering non-degenerate and non-singular hyperplanar admissible flags of the form $X^1 \subset \cdots \subset X^n$, where in addition $X^1 \subset \mathbb{P}(Z^1)$ is required to be GIT stable.

Summary of the Proof of the Main Result

The proof of Theorem 7.1.1 proceeds as follows. In order to have a notion of a family of hyperplanar admissible flags, as part of the objects of the moduli problem we record the flag of linear subspaces $\mathbb{P}(Z^0) \subset \cdots \subset \mathbb{P}(Z^n)$, before imposing the equivalence relation of two flags being projectively equivalent. As such, we consider the diagonal action of $SL(V)$ on the product

$$\mathcal{S} := \prod_{i=0}^{n} \mathrm{Hilb}(\mathbb{P}(V), \Phi_i) \times \prod_{j=0}^{n} \mathrm{Gr}_j(V),$$

where $\mathrm{Gr}_j(V)$ is the Grassmannian of codimension $n-j$ linear subspaces of V (see Section 7.3.2). There exists an $SL(V)$-invariant locally closed subscheme $\mathcal{S}' \subset \mathcal{S}$ whose closed points correspond to non-degenerate, non-singular and stable hyperplanar admissible flags of subschemes of $\mathbb{P}(V)$ of length n, together with the data of the closed embeddings $X^i \subset \mathbb{P}(V)$ and the subspaces $Z^i \subset V$. The moduli space $M^{\mathbb{P}(V)}_{n,d,\underline{\Phi}}$ is given by the geometric quotient of $\mathcal{S}'$ by $SL(V)$.

A priori this is a problem in reductive GIT. However, since each of the subvarieties $X^i \subset \mathbb{P}(V)$ for $i < n$ are degenerate, the Hilbert points $[X^i \subset \mathbb{P}(V)] \in \mathrm{Hilb}(\mathbb{P}(V), \Phi_i)$ are unstable. The subvariety $X^n \subset \mathbb{P}(V)$ is non-degenerate; however, beyond the case of curves of degree d and genus g in $\mathbb{P}^{d-g}$ [22], in higher dimensions there is no complete algebraic classification of when $X^n \subset \mathbb{P}(V)$ has a stable or unstable Hilbert point with respect to either the Hilbert or Chow linearisations on $\mathrm{Hilb}(\mathbb{P}(V), \Phi_n)$. Thirdly, the flag of subspaces $Z^0 \subset \cdots \subset Z^{n-1} \subset Z^n = V$ defines an $SL(V)$-unstable point of

$\prod_{j=0}^{n} \mathrm{Gr}_j(V)$; if L_{Gr_i} is the ample generator of $\mathrm{Pic}(\mathrm{Gr}_i(V)) \cong \mathbb{Z}$ (for $i < n$), then for any choice of weights $w_0, \ldots, w_{n-1} \in \mathbb{Z}^{>0}$ the flag of subspaces is GIT unstable for the diagonal $SL(V)$-action on $\prod_{j=0}^{n-1} \mathrm{Gr}_j(V)$ with respect to the ample linearisation $\boxtimes_{j=0}^{n-1} L_{\mathrm{Gr}_j}^{w_j}$ (as seen by applying [40, Theorem 11.1] with $W = \mathbb{P}(Z^{n-1})$).

To rectify these issues, we instead pass to the case of looking at the action of the parabolic subgroup $P \subset SL(V)$ preserving a *fixed* flag of subspaces $Z^0 \subset Z^1 \subset \cdots \subset Z^n = V$ (where $\mathrm{codim}\, Z^i = n - i$), yielding a locally closed subscheme $\mathcal{S}'_0 \subset \mathcal{S}'$ with $\mathcal{S}' \cong SL(V) \times^P \mathcal{S}'_0$, and use NRGIT to form a geometric quotient of $\mathcal{S}'_0$ by P; this quotient by P coincides with the geometric quotient of $\mathcal{S}'$ by $SL(V)$. Rather than applying the $\widehat{U}$*-Theorem* to form this quotient in one go, we instead adopt the quotienting-in-stages procedure described in Section 3.3. We apply quotienting-in-stages directly, by verifying directly that at each stage the $\widehat{U}$-Theorem can be applied, since this is more straightforward than determining whether the quotienting-in-stages assumptions of [66] hold.

There are two key features of our quotienting-in-stages approach. The first is that, since all grading 1PS involved have only two distinct weights, it is relatively straightforward to check that the necessary weight and stabiliser conditions needed to apply the $\widehat{U}$-Theorem at each stage hold, at least for the objects of interest. The key observation is that the flat limit of $X^k \subset \mathbb{P}(Z^k)$ under any of these grading 1PS is given by X^k itself, or by the join J of X^j with a complementary linear subspace to $\mathbb{P}(Z^j) \subset \mathbb{P}(Z^k)$, where $j < k$ (cf. Lemmas 7.2.5 and 7.4.2). The geometry of these joins is what allows us to verify that the desired conditions on unipotent stabilisers hold. Since we elect to work with the Chow linearisations on the Hilbert schemes $\mathrm{Hilb}(\mathbb{P}(V), \Phi_i)$, the Hilbert–Mumford weight of J with respect to any of the grading 1PS can be explicitly computed (cf. Lemma 7.2.7); this allows us to verify the necessary minimality conditions for the weights of the grading 1PS (cf. Lemma 7.4.4).

Secondly, since we are working with complete flags $X^0 \subset \cdots \subset X^n$, we only ever have to consider the (reductive) GIT stability of subschemes of $\mathbb{P}(V)$ of dimension 0, which is classically understood.

Remark. If one imposes the additional constraint that the top-dimensional subvariety $X^n \subset \mathbb{P}(V)$ is GIT stable (for either the Chow or Hilbert linearisations), the construction of the coarse moduli space of hyperplanar flags $X^0 \subset \cdots \subset X^n \subset \mathbb{P}(V)$ can be carried out using reductive GIT, by making use of 7.4.1 to ensure that a point of $\mathcal{S}$

is $SL(V)$-stable if $[X^n \subset \mathbb{P}(V)] \in \mathrm{Hilb}(\mathbb{P}(V), \Phi_n)$ is GIT stable. If $X^n \subset \mathbb{P}(V)$ is a smooth hypersurface of degree $d > 2$, it is classically known that the corresponding point of $\mathbb{P}(H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)))$ is GIT stable. Another plentiful source of GIT stable subvarieties arises from a result of Donaldson [42], which states that if (X, L) is a smooth polarised manifold with discrete automorphism group for which there exist constant scalar curvature metrics in $c_1(L)$, then (X, L) is asymptotically Chow stable (i.e. $X \subset \mathbb{P}(H^0(X, L^k)^\vee)$ is Chow stable for all $k \gg 0$). By work of Yau [129] (and independently in the canonically polarised case by Aubin [14]), this includes canonically polarised smooth varieties and arbitrarily-polarised smooth Calabi–Yau varieties. More generally, Viehweg [128] proves that any canonically polarised variety with canonical singularities is asymptotically stable (with respect to linearisations arising as variants of the Hodge bundle), extending prior results for curves [101] [100] and surfaces [53]. Other examples of Chow stable subvarieties of projective space arise from work by Morrison [98] and Miranda [96].

Relation to Previous Work

As mentioned above, the results of this chapter rely heavily on Hoskins–Jackson's non-reductive quotienting-in-stages construction [66]. This work was also motivated in part by work of Ross [112] and Ross–Thomas [113] on the study of the Hilbert–Mumford criterion for polarised projective varieties. 1PS with two distinct weights feature prominently in these latter pieces of work, and are used to define notions of slope stability for polarised varieties, in analogy with Gieseker stability for coherent sheaves on a projective scheme (in turn, Gieseker stability is governed by 1PS with two distinct weights, as is explained in [112, Section 3.2]). Ross and Ross–Thomas prove that K-stable polarised varieties are slope stable and that asymptotically Chow stable polarised varieties are Chow-slope stable, and use slope stability to give a geometric proof of the asymptotic Chow/K-stability of smooth curves.

The relationship between [112] [113] and this chapter is as follows. If $X \subset \mathbb{P}(V)$ is a non-degenerate subvariety and λ a 1PS of $GL(V)$ with two distinct weights (for instance, the lift $\lambda^{[i]}$ of the grading 1PS λ_i considered in non-reductive quotienting-in-stages), the action of λ on $X \subset \mathbb{P}(V)$ defines a test configuration $(\mathcal{X}, \mathcal{L} = \mathcal{O}_{\mathbb{P}(V)\times\mathbb{A}^1}(1)|_{\mathcal{X}})$ of $(X, \mathcal{O}_{\mathbb{P}(V)}(1)|_X)$ whose central fibre is given by the flat limit $X_0 \subset \mathbb{P}(V)$ of X under λ. As explained in [113, Page 8] as well as in [100, Section 3 of the proof of Theorem 2.9], up to rescaling the $\lambda(\mathbb{G}_m)$-action the test configuration $(\mathcal{X}, \mathcal{L})$ is equivariantly dominated by a test configuration with total space $\mathrm{Bl}_{\mathcal{I}}(X \times \mathbb{A}^1)$, where $\mathcal{I} \subset \mathcal{O}_X[t]$ is an ideal of the form $\mathcal{I} = \mathcal{I}_Z + (t)$, with $Z \subset X$ given by the repulsive fixed-point locus for λ. The blowup $\mathrm{Bl}_{\mathcal{I}}(X \times \mathbb{A}^1)$ coincides with the total space for the deformation to the normal cone of $Z \subset X$. The notion of slope stability (resp. Chow-slope stability) of a polarised variety (X', L') arises from K-stability (resp. Chow stability) restricted to test configurations arising from deformations to normal cones of subschemes $Z' \subset X'$; when L' is very ample and has no higher cohomology, so that $X' \subset \mathbb{P}(H^0(X', L')^\vee)$, by [113, Proposition 3.7] the test configuration of (X', L') given by the deformation to the normal cone of $Z' \subset X'$ determines a 1PS of $GL(H^0(X', L'))$ with two distinct weights.

In favourable circumstances (cf. proof of Lemma 7.2.5), the geometry of the deformation to the normal cone can be used to fully determine the flat limit X_0, in terms of the join of the repulsive locus Z with a complementary linear subspace. In turn, the simple form taken by the homogeneous ideal of such a join allows for the $\lambda(\mathbb{G}_m)$-weight of $H^0(X_0, \mathcal{O}_{X_0}(k))$ (with $k \gg 0$) to be easily computed in terms of the degree and dimension of Z (cf. proof of Lemma 7.2.7), with the computations being more straightforward than those of Section 4 of *loc. cit.* (which apply for the deformation to the normal cone in general).

Motivation

This work was motivated by an attempt to construct, using GIT, a non-empty coarse moduli space of stable maps to a varying target subvariety X, with X allowed to vary inside a fixed projective space $\mathbb{P}(V)$, without any further conditions on the invertible sheaf $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}(V)}(1)|_X$ beyond its amplitude, and with the image curve in X corresponding to the intersection of X with a linear subspace L of $\mathbb{P}(V)$ of the appropriate codimension (chosen so that, at least if X is smooth and non-degenerate, the intersection of X with a generic such L is a smooth curve). This goal in turn arose as part of the more general aim of constructing moduli spaces of varieties together with morphisms from curves, as described in Chapter 1.

The attempted construction involved appending to the Baldwin–Swinarski construction of the moduli space of stable maps $\overline{M}_{g,n}(\mathbb{P}(V), \beta)$ (cf. Section 6.4.1) an additional Hilbert scheme factor, parametrising the subscheme $X \subset \mathbb{P}(V)$, followed by attempting to form a geometric GIT quotient for the diagonal action of $SL(V)$ on the enlarged parameter space. The hope was that the Baldwin–Swinarski construction could be leveraged to show that the desired moduli space existed, without having to resort to implementing (in full) the Hilbert–Mumford criterion for the $SL(V)$-action on $\mathrm{Hilb}(\mathbb{P}(V))$. The existence of this desired moduli space is closely related to the existence of a coarse moduli space parametrising partial flags $X^1 \subset X^n$ of $\mathbb{P}(V)$. In the case $n = 2$ a coarse moduli space can be constructed provided $X^1 \subset \mathbb{P}(Z^1)$ is assumed to be a non-degenerate curve of genus g and degree $d > 2(2g-2)$ (see Section 7.5.2), however the analogous construction for $n \geq 3$ has the problem that in the resulting NRGIT setup, the condition ($[\overline{R}]_0$) no-longer holds.

Future Directions

One natural follow-up question concerns the existence of natural modular compactifications of the coarse moduli spaces $M^{\mathbb{P}(V)}_{n,d,\underline{\Phi}}$. At least if one wished to utilise non-reductive GIT to construct such a compactification, this does not appear to be a feasible problem; one key feature of the setup behind Theorem 7.1.1 is that all of the flat limits appearing in the quotienting-in-stages construction can be described in terms of joins. It is not clear whether this result extends to all points in the closure of the locus $\mathcal{S}'$ in $\prod_{i=0}^{n} \mathrm{Hilb}(\mathbb{P}(V), \Phi_i) \times \prod_{j=0}^{n} \mathrm{Gr}_j(V)$ (in particular, with points in the closure corresponding to flags where not all of the schemes are reduced). As such, it is not clear whether in the closure of $\mathcal{S}'$ all points satisfy the necessary weight minimality conditions for the various grading 1PS involved, nor whether the appropriate conditions on unipotent stabilisers carry over to points in the closure. Without these conditions, the quotienting-in-stages procedure used to prove Theorem 7.1.1 breaks down.

Another, arguably more pertinent, follow-up question is whether a version of Theorem 7.1.1 can be established for *partial* flags of subschemes of $\mathbb{P}(V)$, i.e. the existence of a quasi-projective coarse moduli space parametrising flags of subschemes of $\mathbb{P}(V)$ obtained by intersecting a varying non-degenerate smooth subvariety $X^n \subset \mathbb{P}(V)$ by a partial flag of projective subspaces of $\mathbb{P}(V)$. The analogous NRGIT approach requires being able to implement the Hilbert–Mumford criterion to determine the GIT stability of smooth non-degenerate subvarieties $Y \subset \mathbb{P}(Z)$ under the action of subgroups $SL(Z') \subset SL(Z)$ with $\dim Z' \geq 2$. Even if this is understood, another issue is that limits of points under the grading 1PS can have positive-dimensional stabilisers for the action of the semisimple part of the Levi subgroup, so that the NRGIT condition ($[\overline{R}]_0$) fails to hold; this prevents us from being able to ascertain the reductive GIT stability of the images of points under taking quotients by unipotent radicals. This latter issue arises for partial flags of the form $X^1 \subset X^n$ with $n \geq 3$.

Notation and Conventions

In addition to the conventions specified in the introduction to this thesis, in this chapter we adopt the following notation and conventions:

- Given a set of points S of a projective space $\mathbb{P}(V)$, their linear span in $\mathbb{P}(V)$ is denoted $\langle S \rangle$. If $X \subset \mathbb{P}(V)$ is a closed subscheme of $\mathbb{P}(V)$ with ideal sheaf $\mathcal{I}_X$, we define $\mathbb{I}(X) := \bigoplus_{d \geq 0} H^0(\mathbb{P}(V), \mathcal{I}_X(d))$ (we also use the notation $\mathbb{I}_{\mathbb{P}(V)}(X)$ if we wish to emphasise the projective space $\mathbb{P}(V)$ containing X). $\mathbb{T}_p X$ denotes the projective tangent space to $X \subset \mathbb{P}(V)$ at a point $p \in X$.

- A closed subscheme X of $\mathbb{P}(V)$ is said to be *non-degenerate* if X is not contained in any hyperplane in $\mathbb{P}(V)$, in other words if $H^0(\mathbb{P}(V), \mathcal{I}_X(1)) = 0$, and is said to be *degenerate* otherwise.

Remark. Sections 7.2 to 7.4 (inclusive) of this chapter have been adapted from the contents of the preprint [38].

7.2 Preliminaries

7.2.1 The Chow Linearisation on Hilbert Schemes

Let $\mathrm{Hilb}(\mathbb{P}(V), \Phi)$ be the Hilbert scheme parametrising subschemes of $\mathbb{P}(V)$ with Hilbert polynomial Φ, where

$$\Phi(t) = \frac{d}{n!} t^n + O(t^{n-1}).$$

Let $\mathrm{Chow}_{n,d}(\mathbb{P}(V))$ be the Chow scheme of Angéniol [11] parametrising families of cycles in $\mathbb{P}(V)$ of dimension n and degree d. The Chow scheme has the following properties (cf. [11, Sections 6-7] and [116, Paper IV, Sections 9, 16 and 17]):

(i) There is a natural proper $GL(V)$-equivariant morphism $\Psi_{\mathrm{HC}} : \mathrm{Hilb}(\mathbb{P}(V), \Phi) \to \mathrm{Chow}_{n,d}(\mathbb{P}(V))$, known as the *Hilbert–Chow morphism.* The restriction of Ψ_{HC} to the open subscheme $\mathrm{Hilb}^{\mathrm{smth}}(\mathbb{P}(V), \Phi)$ parametrising non-singular closed subschemes of $\mathbb{P}(V)$ is an open immersion.

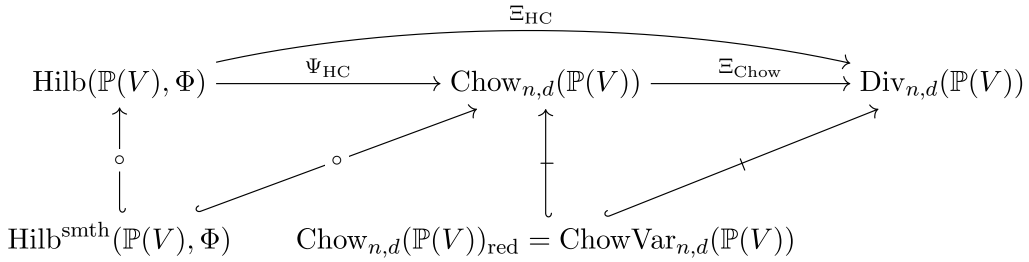
(ii) Ψ_{HC} is a local immersion over the locus parametrising equidimensional and reduced subschemes.

(iii) The underlying reduced scheme $\mathrm{Chow}_{n,d}(\mathbb{P}(V))_{\mathrm{red}}$ is the Chow variety, as introduced by Chow and van der Waerden in [34].

(iv) Let $\mathrm{Div}_{n,d}(\mathbb{P}(V))$ be the projective space of multidegree $(d,\dots,d)$ divisors in $\mathbb{P}(V^\vee)^{\times(n+1)}$. Then there is a $GL(V)$-equivariant map $\Xi_{\mathrm{Chow}} : \mathrm{Chow}_{n,d}(\mathbb{P}(V)) \to \mathrm{Div}_{n,d}(\mathbb{P}(V))$ whose restriction to $\mathrm{Chow}_{n,d}(\mathbb{P}(V))_{\mathrm{red}}$ is a closed immersion; the composite $\Xi_{\mathrm{HC}} = \Xi_{\mathrm{Chow}} \circ \Psi_{\mathrm{HC}}$ assigns to a subvariety $X \subset \mathbb{P}(V)$ its Chow form Ξ_X. The morphism Ξ_{Chow} is finite, and so $\mathcal{L}_{\mathrm{Chow}_{n,d}(\mathbb{P}(V))} := \Xi^*_{\mathrm{Chow}}\mathcal{O}_{\mathrm{Div}_{n,d}(\mathbb{P}(V))}(1)$ is an ample linearisation for the $GL(V)$-action on the projective scheme $\mathrm{Chow}_{n,d}(\mathbb{P}(V))$, known as the *Chow linearisation*.

(v) As the Chow form Ξ_X of a subvariety $X \subset \mathbb{P}(V)$ uniquely determines X, all points of the fibres $\Xi^{-1}_{\mathrm{HC}}(\Xi_X)$ and $\Xi^{-1}_{\mathrm{Chow}}(\Xi_X)$ have underlying reduced scheme X.

The following diagram summarises the relationship between the Hilbert and Chow schemes:



Suppose $\lambda : \mathbb{G}_m \to SL(V)$ is a 1PS, and $X \subset \mathbb{P}(V)$ is a subvariety fixed by the action of λ. Let $R(X) = \mathrm{Sym}^\bullet(V^\vee)/\mathbb{I}(X) = \bigoplus_m R(X)_m$ be the homogeneous coordinate ring of X. Let $w([X]_m, \lambda)$ be the λ-weight of the vector space $R(X)_m$.[2]

Proposition 7.2.1 ([100], Proposition 2.11)**.** *For large m, $w([X]_m, \lambda)$ is is given by a polynomial of the form $\frac{-a_X}{(n+1)!}m^{n+1} + O(m^n)$. The Hilbert–Mumford weight of $\Psi_{\mathrm{HC}}(X)$ is given by*

$$\mu^{\mathcal{L}_{\mathrm{Chow}_{n,d}(\mathbb{P}(V))}}(\Psi_{\mathrm{HC}}(X), \lambda) = \mu^{\mathcal{O}_{\mathrm{Div}_{n,d}(\mathbb{P}(V))}(1)}(\Xi_{\mathrm{HC}}(X), \lambda) = a_X.$$

[2] If $W = \bigoplus_{i\in\mathbb{Z}} W_i$ is the weight space decomposition of a finite dimensional $\mathbb{G}_m$-representation W, the weight of W is given by $\sum_i i \dim W_i$.

In the case where $\mathrm{Hilb}(\mathbb{P}(V),d)$ is the Hilbert scheme of 0-dimensional length d subschemes of $\mathbb{P}(V)$, Ξ_{HC} factors through the natural morphism

$$\Xi_{\mathrm{HC}} : \mathrm{Hilb}(\mathbb{P}(V),d) \to \mathrm{Sym}^d(\mathbb{P}(V)) = \prod_d \mathbb{P}(V)/\mathfrak{S}_d,$$

and this restricts to an open immersion on the open subscheme parametrising d disjoint unordered reduced points in $\mathbb{P}(V)$.

Proposition 7.2.2. *Fix a splitting $V = W \oplus W'$, so that we may regard $SL(W)$ as a subgroup of $SL(V)$ by declaring that $SL(W)$ acts trivially on vectors in W'. Suppose $Y \in \mathrm{Hilb}(\mathbb{P}(V),d)$ corresponds to a set of distinct unordered points $\Xi_{\mathrm{HC}}(Y) = \{p_1,\ldots,p_d\}$, with each $p_i \in \mathbb{P}(W)$. Then $\Psi_{\mathrm{HC}}(Y) \in \mathrm{Chow}_{0,d}(\mathbb{P}(V))$ is $SL(W)$-(semi)stable with respect to the Chow linearisation if and only if for all proper linear subspaces $Z \subset \mathbb{P}(W)$,*

$$\frac{\#(Y\cap Z)}{d} < (\leq)\ \frac{\dim Z+1}{\dim W}. \tag{7.2.1}$$

Proof. There is an $SL(W)$-equivariant closed immersion of Chow schemes

$$\mathrm{Chow}_{0,d}(\mathbb{P}(W)) \hookrightarrow \mathrm{Chow}_{0,d}(\mathbb{P}(V)),$$

under which the Chow linearisation on $\mathrm{Chow}_{0,d}(\mathbb{P}(V))$ pulls back to the Chow linearisation on $\mathrm{Chow}_{0,d}(\mathbb{P}(W))$. As such, we may work instead with the $SL(W)$-action on $\mathrm{Chow}_{0,d}(\mathbb{P}(W))$. The result then follows from [99, Proposition 7.27]. □

In the case where λ is a 1PS of $SL(V)$ with two distinct weights and where $Y \subset \mathbb{P}(V)$ is a closed subvariety contained entirely within a single weight space, the Chow weight of Y can be computed in a very straightforward manner.

Lemma 7.2.3. *Suppose we are given a decomposition $V = U \oplus W$, and let $Y \subset \mathbb{P}(W)$ be a closed subvariety of dimension r and degree d. Let $\lambda : \mathbb{G}_m \to SL(V)$ be the 1PS defined by declaring that each $u \in U$ has weight a and each $w \in W$ has weight b, where $a \dim U + b \dim W = 0$ and where neither a nor b are zero. Then the Hilbert–Mumford weight of the point $\Psi_{\mathrm{HC}}(Y)$ (where Y is considered as a closed subvariety of $\mathbb{P}(V)$) is equal to $bd(r+1)$.*

Proof. Pick bases $x_1,\ldots,x_n$ and $y_1,\ldots,y_m$ for $U^\vee$ and $W^\vee$ respectively, and let $I_Y = \mathbb{I}_{\mathbb{P}(W)}(Y) \subset \mathrm{Sym}^\bullet W^\vee = \mathbb{C}[y_1,\ldots,y_m]$. The homogeneous ideal of $Y \subset \mathbb{P}(V)$ is given by

$$\mathbb{I}_{\mathbb{P}(V)}(Y) = (I_Y, x_1,\ldots,x_n)\cdot R \subset R := \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_m],$$

so for all k sufficiently large we have

$$H^0(Y, \mathcal{O}_{\mathbb{P}(V)}(k)|_Y) = \frac{R_k}{\mathbb{I}_{\mathbb{P}(V)}(Y)_k} = \frac{\mathbb{C}[y_1, \ldots, y_m]_k}{(I_Y)_k}, \quad h^0(Y, \mathcal{O}_{\mathbb{P}(V)}(k)|_Y) = \frac{d}{r!}k^r + O(k^{r-1}).$$

All elements of $\mathbb{C}[y_1, \ldots, y_m]_k/(I_Y)_k$ have weight $-kb$, so for all k sufficiently large

$$w([Y]_k, \lambda) = -kbh^0(Y, \mathcal{O}_{\mathbb{P}(V)}(k)|_Y) = \frac{-bd(r+1)}{(r+1)!}k^{r+1} + O(k^r).$$

Applying Proposition 7.2.1 completes the proof of the lemma. □

7.2.2 Joins of Varieties with Linear Subspaces

We recall the definition of the join of a subvariety $Y \subset \mathbb{P}(V)$ with a complementary linear subspace $L \subset \mathbb{P}(V)$.

Definition 7.2.4. *Let $Y \subset \mathbb{P}(V)$ be a subvariety and let $L \subset \mathbb{P}(V)$ be a linear subspace with the property that $Y \cap L = \emptyset$. The* join *of Y and L is the subvariety of $\mathbb{P}(V)$ obtained as the union of all lines joining points of Y with points of L:*

$$J(Y, L) = \bigcup_{\substack{y \in Y \\ q \in L}} \langle y, q \rangle.$$

Suppose Y is contained in a linear subspace $L' = \mathbb{P}(W) \subset \mathbb{P}(V)$ disjoint from L with the property that $\mathbb{P}(V) = \langle L, L' \rangle$. The following facts are all standard (see for instance [60, Examples 3.1 and 6.18, Proposition 11.36 and Exercise 16.14]) and/or are straightforward to check.

(i) There is an equality $\mathbb{I}_{\mathbb{P}(V)}(J(Y, L)) = \mathbb{I}_{\mathbb{P}(W)}(Y) \cdot R$ of ideals of $R = \mathrm{Sym}^\bullet(V^\vee)$.

(ii) For each $q \in L$, $\mathbb{T}_q J(Y, L) = \mathbb{P}(V)$.

(iii) Suppose $p \in J(Y, L) \setminus L$ lies on the line joining $y \in Y$ to $q \in L$. Then $\mathbb{T}_p J(Y, L) = \langle L, \mathbb{T}_y Y \rangle$. If $y \in Y$ is a non-singular point then $\dim \mathbb{T}_p J(Y, L) = \dim_y Y + \dim L + 1$.

Let $X \subset \mathbb{P}(V)$ be a closed subscheme, and suppose $V = U \oplus W$ for non-zero subspaces $U, W \subset V$. Assume that $X \not\subset \mathbb{P}(W)$. Define a 1PS λ of $GL(V)$ by declaring that all vectors $u \in U$ have weight a and all vectors $w \in W$ have weight b, where $a < b$. Let $Y = X \cap \mathbb{P}(W)$, and let $X_0 = \lim_{t \to 0} \lambda(t) \cdot X \subset \mathbb{P}(V)$ be the flat limit of X under λ.

Lemma 7.2.5. *In the above situation, assume in addition that the rational map* $\mathrm{pr}_{\mathbb{P}(U)} : X \dashrightarrow \mathbb{P}(U)$ *is dominant, that* Y *is reduced and that the closed immersion* $Y \hookrightarrow X$ *is regular. Then the flat limit* X_0 *is given by the join* $J(Y, \mathbb{P}(U))$.

Before we give the proof of Lemma 7.2.5, we require the following lemma.

Lemma 7.2.6. *Let* $f : Y \to Z$ *be a morphism between schemes which are flat over* $\mathbb{A}^1$. *Let* $f_0 : Y_0 \to Z_0$ *be the base change of* f *along* $\operatorname{Spec} \mathbb{C} \xrightarrow{0} \mathbb{A}^1$. *Then there is an equality of schemes* $\mathrm{im}(f_0) = \mathrm{im}(f)_0$, *where* im *denotes the scheme-theoretic image.*

Proof. We have an immersion of schemes $\mathrm{im}(f_0) \subset \mathrm{im}(f)_0$, which we claim is an isomorphism. Without loss of generality we may assume $Y = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B$ are affine, with f corresponding to a homomorphism of $\mathbb{C}[t]$-algebras $\psi : B \to A$. Since $\mathrm{im}(\psi) \subset A$ contains no t-torsion, we have $\mathrm{Tor}_1^{\mathbb{C}[t]}(\mathrm{im}(\psi), \mathbb{C}[t]/t) = 0$. As such, applying $- \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/t$ to the short exact sequence $0 \to \ker \psi \to B \to \mathrm{im}(\psi) \to 0$ gives the short exact sequence

$$0 \to \frac{\ker \psi}{t \ker \psi} \to \frac{B}{tB} \to \frac{\mathrm{im}(\psi)}{t\ \mathrm{im}(\psi)} \to 0.$$

In turn, there is a monomorphism $\ker \psi / t \ker \psi \hookrightarrow \ker \overline{\psi} \subset B/tB$. This induces a surjection

$$\frac{B/tB}{\ker \overline{\psi}} \twoheadrightarrow \frac{B/tB}{\ker \psi / t \ker \psi} = \frac{\mathrm{im}(\psi)}{t\ \mathrm{im}(\psi)} = \frac{B/\ker \psi}{t(B/\ker \psi)}.$$

It follows that there is a closed immersion $\mathrm{im}(f)_0 \hookrightarrow \mathrm{im}(f_0)$, which by construction is inverse to $\mathrm{im}(f_0) \subset \mathrm{im}(f)_0$. $\square$

Proof of Lemma 7.2.5. Consider the rational map

$$\Lambda' : X \times \mathbb{A}^1 \dashrightarrow \mathbb{P}(V) \times \mathbb{A}^1, \quad ([u + w], t) \mapsto ([u + t^{b-a} w], t),$$

and let $\mathcal{I} \subset \mathcal{O}_{X \times \mathbb{A}^1} = \mathcal{O}_X[t]$ denote the ideal sheaf of the base locus of Λ'. Blowing up $\mathcal{I}$, we have a diagram

$$\begin{array}{ccc} B = \mathrm{Bl}_{\mathcal{I}}(X \times \mathbb{A}^1) & & \\ \downarrow & \searrow^{\Lambda} & \\ X \times \mathbb{A}^1 & \overset{\Lambda'}{\dashrightarrow} & \mathbb{P}(V) \times \mathbb{A}^1 \end{array}$$

Since $X \times \mathbb{A}^1$ is flat over $\mathbb{A}^1$, the same is true of B (cf. [51, Appendix B6.7]). By [100, Lemma 2.13], the scheme-theoretic image $\mathrm{im}(\Lambda) \subset \mathbb{P}(V) \times \mathbb{A}^1$ is flat over $\mathbb{A}^1$, and the flat limit X_0 is given by the fibre of $\mathrm{im}(\Lambda)$ over $0 \in \mathbb{A}^1$. By Lemma 7.2.6, this coincides with the scheme-theoretic image of B_0.

Without loss of generality, we may assume that $b - a = 1$ (rescaling the weights of λ will not change the flat limit as a subscheme of $\mathbb{P}(V)$, only the induced $\lambda(\mathbb{G}_m)$-action on it). We have an equality

$$\mathcal{I} = \mathcal{I}_Y + (t)$$

of ideals of $\mathcal{O}_X[t]$, where $\mathcal{I}_Y$ is the ideal sheaf of $Y \subset X$. As such, $B = \mathrm{Bl}_{\mathcal{I}}(X \times \mathbb{A}^1) = \mathrm{Bl}_{Y \times 0}(X \times \mathbb{A}^1)$ is the total space for the deformation to the normal cone of $Y \subset X$ (cf. [51, Chapter 5]). The fibre of B over $0 \in \mathbb{A}^1$ is given by

$$B_0 = \mathrm{Bl}_Y X \cup_{E_Y X} \mathbb{P}_Y(N_{Y \subset X} \oplus \mathcal{O}_Y),$$

noting that since $Y \subset X$ is a regular immersion, the normal cone $C_{Y \subset X} = N_{Y \subset X}$ coincides with the normal bundle, and is a (geometric) vector bundle. Here $\mathrm{Bl}_Y X$ and $\mathbb{P}_Y(N_{Y \subset X} \oplus \mathcal{O}_Y)$ intersect along the exceptional divisor $E_Y X$ and the section at infinity respectively. Since X and Y are reduced, the scheme B_0 is also reduced, and so X_0 must also be reduced (as scheme-theoretic images of reduced schemes are reduced). In other words, the scheme structure on X_0 coincides with the reduced induced scheme structure on the underlying space $|X_0|$.

On the other hand, since Λ_0 is the identity on $Y \subset B_0$, since Λ_0 is equivariant with respect to the $\mathbb{G}_m$-actions on X_0 and $\mathbb{P}_Y(N_{Y \subset X} \oplus \mathcal{O}_Y)$, and since Λ_0 coincides with the dominant morphism $\mathrm{pr}_{\mathbb{P}(U)}$ on $\mathrm{Bl}_Y X \setminus E_Y X = X \setminus Y$, we must have $|X_0| = |J(Y, \mathbb{P}(U))|$. It follows that $X_0 = J(Y, \mathbb{P}(U))$ as subschemes of $\mathbb{P}(V)$, completing the proof of the lemma. $\square$

We now calculate the Hilbert–Mumford weight of the Chow point of a join $J(Y, \mathbb{P}(U))$, using Proposition 7.2.1.

Lemma 7.2.7. *Suppose $V = U \oplus W$, and let $Y \subset \mathbb{P}(W)$ be a closed subvariety of dimension r and degree d. Let $J = J(Y, \mathbb{P}(U))$ be the join of Y and $\mathbb{P}(U)$ in $\mathbb{P}(V)$.*

Let $\lambda : \mathbb{G}_m \to SL(V)$ be the 1PS defined by declaring that each $u \in U$ has weight a and each $w \in W$ has weight b, where $a \dim U + b \dim W = 0$ and where neither a nor b are *zero. If* $\dim Y = \dim \mathbb{P}(U)$*, assume further that* $a + bd \neq 0$*. Then the Hilbert–Mumford weight of the point* $\Psi_{\mathrm{HC}}(J)$ *with respect to* λ *is equal to*

$$\begin{cases} a(\dim \mathbb{P}(U)+1) & \text{if } \dim Y < \dim \mathbb{P}(U), \\ bd(\dim Y + 1) & \text{if } \dim Y > \dim \mathbb{P}(U), \\ (a+bd)(\dim Y + 1) & \text{if } \dim Y = \dim \mathbb{P}(U). \end{cases}$$

Proof. It is possible to use [113, Theorem 4.8] to prove this result, however we elect give a more direct, albeit similar calculation of the Hilbert–Mumford weight. As in the proof of Lemma 7.2.3, pick bases $x_1, \dots, x_n$ and $y_1, \dots, y_m$ for $U^\vee$ and $W^\vee$ respectively. Let $I_Y = \mathbb{I}_{\mathbb{P}(W)}(Y) \subset \mathrm{Sym}^\bullet W^\vee = \mathbb{C}[y_1, \dots, y_m]$, and set

$$I_J := \mathbb{I}_{\mathbb{P}(V)}(J) = I_Y \cdot R \subset R := \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m].$$

For all k sufficiently large, we have

$$H^0(J, \mathcal{O}_{\mathbb{P}(V)}(k)|_J) = \frac{R_k}{(I_J)_k} \cong \bigoplus_{i+j=k} \mathbb{C}[x_1, \dots, x_n]_i \otimes_{\mathbb{C}} \frac{\mathbb{C}[y_1, \dots, y_m]_j}{(I_Y)_j} \tag{7.2.2}$$

Elements of $\mathbb{C}[x_1, \dots, x_n]_i$ all have weight $-ia$ and elements of $\mathbb{C}[y_1, \dots, y_m]_j/(I_Y)_j$ all have weight $-jb$. As such, by Equation (7.2.2) we have

$$\begin{aligned} w([J]_k, \lambda) &= -\sum_{i=0}^{k} \left(ia\binom{n+i-1}{i} + (k-i)b\psi_Y(k-i) \right) \\ &= -\sum_{i=1}^{k} i \left(a\binom{n+i-1}{i} + b\psi_Y(i) \right), \end{aligned}$$

where $\psi_Y(i) := \dim_{\mathbb{C}}(\mathbb{C}[y_1, \dots, y_m]_i/(I_Y)_i)$. For all sufficiently large k, we have

$$\psi_Y(k) = h^0(Y, \mathcal{O}_{\mathbb{P}(W)}(k)|_Y) = \frac{d}{r!}k^r + O(k^{r-1}).$$

Therefore, for $k \gg 0$,

$$\begin{aligned} w([J]_k, \lambda) &= -\sum_{i=1}^{k} i \left(a\binom{n+i-1}{i} + b\left(\frac{d}{r!}i^r + O(i^{r-1}) \right) \right) + O(1) \\ &= -\sum_{i=1}^{k} ia\binom{n+i-1}{i} - b\frac{d}{r!(r+2)}k^{r+2} + O(k^{r+1}) \\ &= \frac{-bd(r+1)}{(r+2)!}k^{r+2} - \sum_{i=1}^{k} ia\binom{n+i-1}{i} + O(k^{r+1}), \end{aligned}$$

where in the second line we used $\sum_{i=1}^{k} i^p = \frac{k^{p+1}}{p+1} + O(k^p)$. On the other hand we have $\sum_{i=1}^{k} ia\binom{n+i-1}{i} = \frac{an}{(n+1)!}k^{n+1} + O(k^n)$. The lemma then follows from Proposition 7.2.1. □

7.3 Moduli of Hyperplanar Admissible Flags

7.3.1 The Moduli Functor

Fix a finite dimensional vector space V and positive integers $d, n > 0$.

Definition 7.3.1. *A* (complete) hyperplanar admissible flag $(\underline{X}, \underline{Z})$ *of subschemes of* $\mathbb{P}(V)$ *(of* length n *and* degree d*) is a tuple* $\underline{X} = (X^0, X^1, \ldots, X^n)$ *of closed subschemes* $X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n$ *of* $\mathbb{P}(V)$*, together with a tuple* $\underline{Z} = (Z^0, Z^1, \ldots, Z^n)$ *of linear subspaces* $Z^0 \subset Z^1 \subset \cdots \subset Z^{n-1} \subset Z^n = V$ *such that:*

(i) $\dim X^i = i$ *for all* $i = 0, 1, \ldots, n$*;*

(ii) $\operatorname{codim}_V Z^i = n - i$ *for each* $i = 0, 1, \ldots, n$*;*

(iii) $\deg X^n = d$*; and*

(iv) $X^i = X^n \cap \mathbb{P}(Z^i)$ *for each* $i = 0, 1, \ldots, n$*.*[3]

The flag $(\underline{X}, \underline{Z})$ *is* non-degenerate *if each* X^i *is not contained in a proper linear subspace of* Z^i*. The flag* $(\underline{X}, \underline{Z})$ *is* non-singular *if each of* $X^1, \ldots, X^n$ *is a non-singular connected projective variety, and if* X^0 *is the disjoint union of* d *reduced points. The flag* $(\underline{X}, \underline{Z})$ *is* stable *if* $X^0 \subset \mathbb{P}(Z^0)$ *is Chow stable, in the sense that for each proper linear subspace* $Z \subset \mathbb{P}(Z^0)$*, Inequality* (7.2.1) *holds strictly, with* W *taken to be* Z^0 *and* Y *to be* X^0*.*

Remark. If $(\underline{X}, \underline{Z})$ is non-degenerate, the linear subspace Z^i can be recovered from $\underline{X}$ by taking the linear span of X^i in $\mathbb{P}(V)$. In addition, each inclusion $X^i \subset X^{i+1}$ is a regular embedding.

Of course, length n hyperplanar admissible flags of subschemes of $\mathbb{P}(V)$ make sense only when $\dim \mathbb{P}(V) \geq n$, and if $n = \dim \mathbb{P}(V)$ then the only possibilities are complete flags of linear subspaces, which are all projectively equivalent. As such, whenever we discuss length n hyperplanar admissible flags of subschemes of $\mathbb{P}(V)$, we assume that $n + 1 < \dim V$. A necessary condition for there to exist non-degenerate, non-singular and stable hyperplanar admissible flags of degree d is that $d > \dim Z^0 = \dim V - n$. Indeed, if k of the points of a Chow stable configuration $X^0 \subset \mathbb{P}(Z^0)$ have a projective linear span of dimension $\ell \leq k - 1$, Inequality 7.2.1 implies that $d > \frac{k}{\ell+1} \dim Z^0$.

[3] In particular, $\deg X^i = d$ for all $i = 0, \ldots, n$.

On the other hand, if $d > \dim Z^0$ then a generic configuration of points $X^0 \subset \mathbb{P}(Z^0)$ obtained by intersecting a non-degenerate curve $X^1 \subset \mathbb{P}(Z^1)$ of degree d with a generic hyperplane $\mathbb{P}(Z^0) \subset \mathbb{P}(Z^1)$ will be Chow stable, as such a generic intersection will satisfy the property that any subset of X^0 of size $\dim \mathbb{P}(Z^0)$ spans a hyperplane in $\mathbb{P}(Z^0)$ by the so-called *general position theorem* of [13, Chapter III, §1]. The non-triviality of the moduli of non-degenerate, non-singular and stable hyperplane admissible flags for such degrees is then implied by Bertini's theorem (cf. [62, Theorem II.8.18]), together with the non-degenericity of generic hyperplane sections of irreducible non-degenerate projective subvarieties of dimension at least 2 (cf. [60, Proposition 18.10]).

Definition 7.3.2. *The* Hilbert type $\underline{\Phi} = (\Phi_0, \ldots, \Phi_n)$ *of the flag* $(\underline{X}, \underline{Z})$ *is the tuple whose entries are the Hilbert polynomials of the* X^i*:*

$$\Phi_i(t) = \chi(X^i, \mathcal{O}_{\mathbb{P}(V)}(t)|_{X^i}).$$

In order to have a moduli functor, we need to specify what is meant by a *family* of hyperplanar admissible flags, and when two families are considered to be equivalent.

Definition 7.3.3. *Let* S *be a scheme. A* family of hyperplanar admissible flags $(\underline{\mathcal{X}}, \underline{\mathcal{Z}}, L)$ *of subschemes of* $\mathbb{P}(V)$ *(of length* n*, degree* d *and Hilbert type* $\underline{\Phi}$*)* parametrised by S *is given by the following data:*

1. *an invertible sheaf* L *on* S*;*
2. *a tuple* $\underline{\mathcal{X}} = (\mathcal{X}^0, \ldots, \mathcal{X}^n)$ *of closed subschemes* $\mathcal{X}^0 \subset \cdots \subset \mathcal{X}^n \subset \mathbb{P}_S(V \otimes L)$ *which are flat and finitely presented over* S*; and*
3. *a tuple* $\underline{\mathcal{Z}} = (\mathcal{Z}^0, \ldots, \mathcal{Z}^n)$ *of locally free subsheaves* $\mathcal{Z}^0 \subset \cdots \subset \mathcal{Z}^n = V \otimes L$*,*

such that the following properties hold:

(i) *each* $\mathcal{Z}^i$ *is of corank* $n - i$*; and*

(ii) *for each geometric point* $s \in S$*, the fibre* $(\underline{\mathcal{X}}, \underline{\mathcal{Z}})_s = (\underline{\mathcal{X}}_s, \underline{\mathcal{Z}}_s)$ *is a hyperplanar admissible flag of subschemes of* $\mathbb{P}(V \otimes L_s) \cong \mathbb{P}(V)$ *of length* n*, degree* d *and Hilbert type* $\underline{\Phi}$*, as in Definition 7.3.1.*

The family $(\underline{\mathcal{X}}, \underline{\mathcal{Z}}, L)$ *is said to be* non-degenerate *(resp.* non-singular*,* stable*) if for each geometric point* $s \in S$*, the flag* $(\underline{\mathcal{X}}, \underline{\mathcal{Z}})_s$ *is non-degenerate (resp. non-singular, stable).*

Definition 7.3.4. *Let $(\underline{\mathcal{X}}, \underline{\mathcal{Z}}, L)$ and $(\underline{\mathcal{X}}', \underline{\mathcal{Z}}', L')$ be families of hyperplanar admissible flags of subschemes of* $\mathbb{P}(V)$ *over a common base scheme S. These families are said to be* (projectively) equivalent *if there exists an isomorphism of invertible sheaves $\phi : L \to L'$ and an element $g \in GL(V)$ such that:*

(i) $g \otimes \phi : V \otimes L \xrightarrow{\sim} V \otimes L'$ *sends* $\mathcal{Z}^i$ *to* $(\mathcal{Z}^i)'$ *for all* $i = 0, \dots, n$, *and*

(ii) $\mathbb{P}_S(g \otimes \phi) : \mathbb{P}_S(V \otimes L) \xrightarrow{\sim} \mathbb{P}_S(V \otimes L')$ *sends* $\mathcal{X}^i$ *to* $(\mathcal{X}^i)'$ *for all* $i = 0, \dots, n$.

Given a morphism of schemes $T \to S$, the pullback of a family of hyperplanar admissible flags $(\underline{\mathcal{X}}, \underline{\mathcal{Z}}, L)$ parametrised by S is a family of hyperplanar admissible flags parametrised by T, and projectively equivalent families over S pull back to projectively equivalent families over T. As such, there is a well-defined moduli functor

$$\mathcal{F}^{\mathbb{P}(V)}_{n,d,\underline{\Phi}} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$$

which associates to a scheme S the set of all equivalence classes of families of non-degenerate, non-singular and stable hyperplanar admissible flags of subschemes of $\mathbb{P}(V)$ of length n, degree d and Hilbert type $\underline{\Phi}$ parametrised by S.

Remark. Suppose $(\underline{X}, \underline{Z})$, $(\underline{X}', \underline{Z}')$ are non-degenerate hyperplanar admissible flags of subschemes of $\mathbb{P}(V)$ with the closed embeddings $X^n, (X')^n \subset \mathbb{P}(V)$ being linearly normal, i.e. $H^1(\mathbb{P}(V), \mathcal{I}_{X^n}(1)) = H^1(\mathbb{P}(V), \mathcal{I}_{(X')^n}(1)) = 0$, so that the restriction maps $V^\vee \to H^0(X^n, \mathcal{O}_{X^n}(1))$ and $V^\vee \to H^0((X')^n, \mathcal{O}_{X'^n}(1))$ are both isomorphisms. Then there exists an isomorphism of polarised varieties $(X^n, \mathcal{O}_{\mathbb{P}(V)}(1)|_{X^n}) \cong ((X')^n, \mathcal{O}_{\mathbb{P}(V)}(1)|_{(X')^n})$ taking each X^i to $(X')^i$ if and only if $(\underline{X}, \underline{Z})$ and $(\underline{X}', \underline{Z}')$ are projectively equivalent. Indeed, any such isomorphism $(X^n, \mathcal{O}_{\mathbb{P}(V)}(1)|_{X^n}) \cong ((X')^n, \mathcal{O}_{\mathbb{P}(V)}(1)|_{(X')^n})$ gives rise to an automorphism of the overlying projective space $\mathbb{P}(V)$, by considering the induced isomorphism on global sections.

7.3.2 The Local Universal Property

Let n, d be positive integers and let V be a complex vector space of finite dimension; assume $\dim V > n + 1$ and $d > \dim V - n$. We now exhibit a family which has the local universal property for the moduli functor $\mathcal{F} = \mathcal{F}^{\mathbb{P}(V)}_{n,d,\underline{\Phi}}$.

Let $\mathcal{H}_i = \mathrm{Hilb}(\mathbb{P}(V), \Phi_i)$ be the Hilbert scheme parametrising closed subschemes of $\mathbb{P}(V)$ with Hilbert polynomial Φ_i, and let $\mathrm{Gr}_i = \mathrm{Gr}_i(V)$ be the Grassmannian parametrising subspaces of V of *codimension* $n - i$. Let

$$\mathcal{S} := \prod_{i=0}^{n} \mathcal{H}_i \times \prod_{j=0}^{n} \mathrm{Gr}_j.$$

We endow $\mathcal{S}$ with the diagonal action of $GL(V)$ induced by the natural $GL(V)$-actions on the $\mathcal{H}_i$ and Gr_j.[4] Over $\mathcal{H}$ are the following universal objects:

(i) $\mathcal{S}$-flat, $\mathcal{S}$-finitely presented closed subschemes $\mathcal{Y}^i \subset \mathbb{P}(V) \times \mathcal{S}$ for each $i = 0, \dots, n$, with $\mathcal{Y}^i$ given by the pullback of the universal family over $\mathcal{H}_i$.

(ii) Corank $n - j$ locally free subsheaves $\mathcal{W}^j \subset V \otimes \mathcal{O}_{\mathcal{S}}$ for each $j = 0, \dots, n$, with $\mathcal{W}^j$ given by the pullback of the universal subbundle over Gr_j.

Lemma 7.3.5. *There exists a locally closed, $GL(V)$-invariant subscheme $\mathcal{S}' \subset \mathcal{S}$ with the following universal property: suppose S is a scheme and $S \to \mathcal{S}$ is a morphism, corresponding to the closed subschemes $\mathcal{Y}^i_S \subset \mathbb{P}(V) \times S$ and the locally free subsheaves $\mathcal{W}^j_S \subset V \otimes \mathcal{O}_S$. Then $S \to \mathcal{S}$ factors through $\mathcal{S}'$ if and only if for each geometric point $s \in S$:*

1. *$\mathcal{Y}^0_s \subset \cdots \subset \mathcal{Y}^n_s$ as subschemes of $\mathbb{P}(V)$, and $\mathcal{W}^0_s \subset \cdots \subset \mathcal{W}^n_s = V$ as subspaces of V;*
2. *$\mathcal{Y}^i_s = \mathcal{Y}^n_s \cap \mathbb{P}(\mathcal{W}^i_s)$ for all $i = 0, \dots, n$;*
3. *$\dim \mathcal{Y}^i_s = i$ for all $i = 0, \dots, n$;*
4. *$\deg \mathcal{Y}^i_s = d$ for all $i = 0, \dots, n$;*
5. *$\mathcal{Y}^i_s \subset \mathbb{P}(\mathcal{W}^i_s)$ is a non-degenerate, non-singular, connected projective subvariety for each $i = 1, \dots, n$; and*
6. *$\mathcal{Y}^0_s \subset \mathbb{P}(\mathcal{W}^0_s)$ is non-degenerate, $\mathcal{Y}^0_s$ is the disjoint union of d reduced points and $\mathcal{Y}^0_s$ is Chow stable.*

[4] Note that the action of $GL(V)$ factors through the quotient $PGL(V)$.

Proof. The first two conditions are incidence correspondences; these conditions cut out a closed subscheme of $\mathcal{S}$. After first imposing the first two conditions, the remaining conditions are all open in flat families, so define an open subscheme $\mathcal{S}'$ of this closed subscheme. □

Let $(\underline{\mathcal{Y}}', \underline{\mathcal{W}}')$ be the restriction of $(\underline{\mathcal{Y}}, \underline{\mathcal{W}})$ to $\mathcal{S}' \subset \mathcal{S}$. Note that $(\underline{\mathcal{Y}}', \underline{\mathcal{W}}', \mathcal{O}_{\mathcal{S}'})$ is a family of non-degenerate, non-singular and stable hyperplanar admissible flags of subschemes of $\mathbb{P}(V)$ of length n, degree d and Hilbert type $\underline{\Phi}$ over $\mathcal{S}'$.

Lemma 7.3.6. *The family $(\underline{\mathcal{Y}}', \underline{\mathcal{W}}', \mathcal{O}_{\mathcal{H}'})$ has the local universal property for the moduli functor $\mathcal{F} = \mathcal{F}^{\mathbb{P}(V)}_{n,d,\underline{\Phi}}$.*

Proof. Given a family $(\underline{\mathcal{X}}, \underline{\mathcal{Z}}, L)$ over a scheme S and a point $s \in S$, by passing to an open neighbourhood $U \subset S$ of s where L is trivial we may assume without loss of generality that $L = \mathcal{O}_S$. The result then follows from Lemma 7.3.5 together with the universal properties of the schemes $\mathcal{H}_i$ and Gr_j. □

Corollary 7.3.7. *The moduli functor $\mathcal{F}$ is corepresentable if and only if there exists a categorical quotient $q : \mathcal{S}' \to \mathcal{S}' /\!\!/ SL(V)$ of $\mathcal{S}'$ by $SL(V)$, and that a coarse moduli space of $\mathcal{F}$ exists if and only if q is an orbit space morphism.*

Proof. If s_0, s_1 are points of $\mathcal{S}'$ and if $(\underline{\mathcal{Y}}_0, \underline{\mathcal{W}}_0)$, $(\underline{\mathcal{Y}}_1, \underline{\mathcal{W}}_1)$ are the restrictions of the universal family $(\underline{\mathcal{Y}}, \underline{\mathcal{W}})$ to s_0 and s_1 respectively, the flags $(\underline{\mathcal{Y}}_0, \underline{\mathcal{W}}_0)$ and $(\underline{\mathcal{Y}}_1, \underline{\mathcal{W}}_1)$ are equivalent if and only if s_0 and s_1 lie in the same $SL(V)$-orbit in $\mathcal{S}'$. The result then follows by Proposition 2.3.3. □

7.4 Proof of the Main Result

Let n, d be positive integers and let V be a complex vector space of dimension $n + 1 < \dim V$. Assume in addition that $d > \dim V - n$ and that $d \notin \left\{ \frac{\dim V - n - 1 + i}{n + 1 - i} : i = 1, \ldots, n \right\}$. Let $\underline{\Phi} = (\Phi_0, \ldots, \Phi_n)$ be a tuple of Hilbert polynomials of subschemes of $\mathbb{P}(V)$, where $\Phi_i(t) = \frac{d}{i!} t^i + O(t^{i-1})$ for each $i = 0, 1, \ldots, n$. We will construct a categorical quotient for the action of the group $G = SL(V)$ on the subscheme $\mathcal{S}' \subset \prod_{i=0}^n \mathcal{H}_i \times \prod_{j=0}^n \mathrm{Gr}_j$.

7.4.1 Reduction to a Parabolic Action

Fix a decomposition

$$V = W \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_n$$

where $W \subset V$ is a subspace of codimension n. For each $i = 0, 1, \ldots, n$ set $Z^i = W \oplus \bigoplus_{j \leq i} \mathbb{C}v_j$. Let p_0 be the point of $\prod_{i=0}^n \mathrm{Gr}_i$ corresponding to $(Z^0, \ldots, Z^n)$, and let $P = \mathrm{Stab}_G(p_0)$ be the parabolic subgroup of G preserving the flag $Z^0 \subset \cdots \subset Z^n$. With respect to a basis of V compatible with the flag $Z^0 \subset \cdots \subset Z^n$ we have

$$P = \left\{ \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} & A_{1,n+1} \\ 0 & a_{2,2} & \cdots & a_{2,n} & a_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n,n} & a_{n,n+1} \\ 0 & 0 & \cdots & 0 & a_{n+1,n+1} \end{pmatrix} \in SL(V) : \begin{array}{c} A_{11,} \in \mathbb{C}^{\dim W \times \dim W}, \\ A_{1,i} \in \mathbb{C}^{\dim W} \text{ for } i = 2, \ldots, n+1, \\ a_{i,j} \in \mathbb{C} \text{ for } i, j > 1 \end{array} \right\}.$$

In particular, in the notation of Section 3.3 we have $P = P(\tilde{\lambda})$, where $\tilde{\lambda}$ is any 1PS of G with decreasing weights $\beta_1 > \beta_2 > \cdots > \beta_{n+1}$ whose multiplicities are $(\dim W, 1, \ldots, 1)$. Let $\mathcal{S}'_0 \subset \mathcal{S}'$ be the preimage of p_0. Then $\mathcal{S}' = G \cdot \mathcal{S}'_0$, so by Lemma 2.1.6 we have

$$\mathcal{S}' \cong G \times^P \mathcal{S}'_0.$$

It follows that a categorical quotient of $\mathcal{S}'$ by G exists (and is an orbit space morphism) if and only if a categorical quotient of $\mathcal{S}'_0$ by P exists (and is an orbit space morphism). Indeed, if a categorical P-quotient $\mathcal{S}'_0 /\!\!/ P$ exists, then the action morphism $G \times \mathcal{S}'_0 \to \mathcal{S}'$ induces an isomorphism of categorical quotients $\mathcal{S}' /\!\!/ G \cong \mathcal{S}'_0 /\!\!/ P$.

7.4.2 Linearising the Parabolic Action

From now on, we regard $\mathcal{S}'_0$ as a locally closed P-invariant subscheme of $\mathcal{H} := \prod_{i=0}^n \mathcal{H}_i$ in the obvious way. For each $i = 0, \ldots, n$, let

$$\Psi_i : \mathrm{Hilb}(\mathbb{P}(V), \Phi_i) \to \mathrm{Ch}_i := \mathrm{Chow}_{i,d}(\mathbb{P}(V))$$

be the corresponding Hilbert–Chow morphism. Let

$$\Psi = \prod_{i=0}^n \Psi_i : \mathcal{H} \to \mathrm{Ch} := \prod_{i=0}^n \mathrm{Ch}_i$$

be the product of the Ψ_i. Given a tuple of positive integers $\underline{a} = (a_0, \ldots, a_n)$, let $\mathcal{L}_{\mathrm{Ch}}^{\underline{a}} := \boxtimes_{i=0}^n \mathcal{L}_{\mathrm{Ch}_i}^{a_i}$, where $\mathcal{L}_{\mathrm{Ch}_i}$ is the Chow linearisation on the Chow scheme Ch_i.

In the notation of Section 3.3 we have $R'_1 = R'^{(1)} = SL(W)$ and $R'^{(i)} = \{e\}$ for all $i > 1$. We choose $\underline{a}$ to be of the form $\underline{a} = (a'_0 m, m, \ldots, m)$ for some sufficiently large integers $m > 0$ and $a'_0 > 0$, where m is chosen such that each $\mathcal{L}^m_{\mathrm{Ch}_i}$ is very ample, and a'_0 is chosen according to the following result (where the reductive group K is taken to be $R'^{(1)} = SL(W)$):

Lemma 7.4.1. *Let K be a reductive linear algebraic group acting on a product of projective schemes $\prod_{i=0}^n Y_i$, where each Y_i is endowed with a very ample K-linearisation L_i. Then there exists a positive integer $a' > 0$ such that for all $a'_0 > a'$, for every point $p = (p_0, \ldots, p_n) \in \prod_{i=0}^n Y_i$, if p_0 is K-stable with respect to the linearisation L_0 then p is K-stable with respect to the linearisation $L_0^{a'_0} \boxtimes L_1 \boxtimes \cdots \boxtimes L_n$.*

Proof. By using each L_i to embed Y_i inside a projective space, we may assume each Y_i is a projective space $\mathbb{P}(W_i)$. The lemma now follows from [118, Proposition 1.7.3.1]. □

7.4.3 Upstairs Stabilisers and Weights

Fix integers $\beta_1 > 0 > \beta_2 > \cdots > \beta_{n+1}$ with $\beta_1 \dim W + \sum_{i=2}^{n+1} \beta_i = 0$.[5] Define a 1PS $\lambda : \mathbb{G}_m \to SL(V)$ by setting

$$\lambda(t) \cdot v = \begin{cases} t^{\beta_1} v & \text{if } v \in W, \\ t^{\beta_{i+1}} v & \text{if } v = v_i. \end{cases}$$

P is the parabolic subgroup of $SL(V)$ associated to λ, and λ grades the unipotent radical of the parabolic P. We will freely make use of the notation of Section 3.3 as it applies to the parabolic group P. In particular, by averaging the weights of λ we obtain length two 1PS $\lambda^{[1]}, \ldots, \lambda^{[n]}$. We also obtain unipotent groups $U^{[1]}, \ldots, U^{[n]}$ by considering the row filtration of the unipotent radical of P, with $U^{[i]}$ graded by $\lambda^{[i]}$. The 1PS $\lambda^{[i]}$ has weight space decomposition $V = Z^{i-1} \oplus V^{i-1}$ where $V^j = \bigoplus_{k>j} \mathbb{C} v_k$; vectors in Z^{i-1} have $\lambda^{[i]}$-weight $\beta_{\leq i}$ and vectors in V^{i-1} have $\lambda^{[i]}$-weight $\beta_{>i}$.

Fix a point p of $\mathcal{S}'_0$ corresponding to a hyperplanar admissible flag $(X^0, X^1, \ldots, X^n)$ with $X^i = X^n \cap \mathbb{P}(Z^i)$ and $\langle X^i \rangle = \mathbb{P}(Z^i) = \mathbb{P}\left(W \oplus \bigoplus_{j \leq i} \mathbb{C} v_j\right)$ for all i. For all $1 \leq i \leq j \leq n$, let $V^{i,j} = \bigoplus_{k=i}^{j} \mathbb{C} v_k$; we also set $p^{[i]}(x) := \lim_{t \to 0} \lambda^{[i]}(t) \cdot x$ for a point $x \in \mathcal{H}$.

[5] For the purposes of this chapter, the specific choices of the β_i's doesn't matter; what matters is that they satisfy the given constraints.

Lemma 7.4.2. *The following statements hold:*

1. *For each $i = 1, \dots, n$, the point $p^{[i]}(p) \in \mathcal{H}$ is given by the flag*

$$p^{[i]}(p) = (X^0, \dots, X^{i-1}, J(X^{i-1}, \mathbb{P}(V^{i,i})), J(X^{i-1}, \mathbb{P}(V^{i,i+1})), \dots, J(X^{i-1}, \mathbb{P}(V^{i,n}))).$$

2. *For all $j \le i$ we have $p^{[j]}(p^{[i]}(p)) = p^{[j]}(p)$.*

Proof. $\lambda^{[i]}$ fixes pointwise the subschemes $X^0, \dots, X^{i-1}$. On the other hand, for each $k \ge i$ the projection $X^k \dashrightarrow \mathbb{P}(V^{i,k})$ is dominant, since $X^k \subset \mathbb{P}(Z^k)$ is non-degenerate. The first assertion now follows from Lemma 7.2.5. For the second assertion, note that $Z^{j-1} \cap V^{i,k} = 0$ for all $k \ge i$, so

$$J(X^{i-1}, \mathbb{P}(V^{i,k})) \cap \mathbb{P}(Z^{j-1}) = X^{i-1} \cap \mathbb{P}(Z^{j-1}) = X^{j-1}.$$

The second assertion then follows from a second application of Lemma 7.2.5. □

Lemma 7.4.3. *For all $i = 1, \dots, n$, we have* $\mathrm{Stab}_{U^{[i]}}(p^{[i]}(p)) = \{e\}$.

Proof. If $k \ge i$, then any projective automorphism of $\mathbb{P}(V)$ arising from the unipotent group $U^{[i]}$ which preserves $J(X^{i-1}, \mathbb{P}(V^{i,k}))$ must preserve $\mathbb{P}(V^{i,k})$, as seen by considering dimensions of projective tangent spaces; points of $\mathbb{P}(V^{i,k})$ have projective tangent space equal to all of $\mathbb{P}(V)$, whereas points of $J(X^{i-1}, \mathbb{P}(V^{i,k}))$ off $\mathbb{P}(V^{i,k})$ have projective tangent spaces of dimension $(i-1) + (k-i) + 1 = k < \dim \mathbb{P}(V)$.

It follows that any element $u = (A_{pq})_{p,q=1}^{n+1} \in \mathrm{Stab}_{U^{[i]}}(p^{[i]}(p))$ must preserve each of $\mathbb{P}(V^{i,i}), \dots, \mathbb{P}(V^{i,n})$, which implies that if $p \ne q$ then $A_{pq} = 0$ whenever $p \le i$ and $q > i$. On the other hand $u \in U^{[i]}$, so for $p \ne q$ we have $A_{pq} = 0$ if $q \le i$ or if $p > i$. Consequently $u = e$ must be the identity matrix. □

Lemma 7.4.4. *For each $i = 1, \dots, n$, the Hilbert–Mumford weight $\mu^{\mathcal{L}^a_{\mathrm{Ch}}}(\Psi(p), \lambda^{[i]})$ is the same for all points $p \in \mathcal{S}'_0$.*

Proof. First of all, since $d > \dim V - n = \dim W$, since $\beta_2, \dots, \beta_{n+1} < 0$ and since $d \notin \left\{ \frac{\dim W + i - 1}{n+1-i} : i = 1, \dots, n \right\}$ then for each $i = 1, \dots, n$ we have

$$\beta_{>i} + d\beta_{\le i} = \left(\frac{1}{n+1-i} - \frac{d}{\dim W + i - 1} \right) (\beta_{i+1} + \dots + \beta_{n+1}) \ne 0.$$

Therefore Lemma 7.2.7 is applicable.

Choose a basis $W^\vee$ and let $x_i \in V^\vee$ be dual to v_i. For a point $p \in \mathcal{S}_0'$, up to a positive scalar multiple the Hilbert–Mumford weight of $\Psi(p)$ is given by

$$\begin{aligned}\mu^{\mathcal{L}_{\mathrm{Ch}}^{\underline{a}}}(\Psi(p), \lambda^{[i]}) &= a_0' \mu^{\mathcal{L}_{\mathrm{Ch}_1}}(\Psi_0(X^0), \lambda^{[i]}) \\ &\quad + \sum_{j=1}^{i-1} \mu^{\mathcal{L}_{\mathrm{Ch}_j}}(\Psi_j(X^j), \lambda^{[i]}) + \sum_{j=i}^{n} \mu^{\mathcal{L}_{\mathrm{Ch}_j}}(\Psi_j(J(X^{i-1}, \mathbb{P}(V^{i,j}))), \lambda^{[i]}).\end{aligned}$$

The contribution $\mu^{\mathcal{L}_{\mathrm{Ch}_j}}(\Psi_j(X^j), \lambda^{[i]})$ coming from X^j (where $j < i$) is equal to $\beta_{\leq i} d(j+1)$ by Lemma 7.2.3. For the contribution of the join $J_j := J(X^{i-1}, \mathbb{P}(V^{i,j}))$ (where $j \geq i$), we have that the homogeneous ideal of $J_j \subset \mathbb{P}(V)$ is given by

$$\mathbb{I}_{\mathbb{P}(V)}(J_j) = \langle \mathbb{I}_{\mathbb{P}(Z_j)}(J_j), x_{j+1}, \ldots, x_n \rangle,$$

so the homogeneous coordinate ring of $J_j \subset \mathbb{P}(V)$ is the same as that for $J_j \subset \mathbb{P}(Z^j)$. Applying Lemma 7.2.7 then gives

$$\mu^{\mathcal{L}_{\mathrm{Ch}_j}}(\Psi_j(J_j), \lambda^{[i]}) = \begin{cases} \beta_{>i}(j-i+1) & \text{if } j > 2i-1, \\ \beta_{\leq i} di & \text{if } j < 2i-1, \\ i(\beta_{>i} + d\beta_{\leq i}) & \text{if } j = 2i-1. \end{cases}$$

It is now clear that the resulting expression for $\mu^{\mathcal{L}_{\mathrm{Ch}}^{\underline{a}}}(\Psi(p), \lambda^{[i]})$ is the same for all points $p \in \mathcal{S}_0'$. □

7.4.4 The Quotienting-in-Stages Procedure

We now construct a quotient of $\mathcal{S}_0'$ in stages, by implementing Construction 3.3.1. After n stages, we will obtain a quotient for the action of P, the parabolic group in Section 7.4.1.

For the base step of the quotienting-in-stages procedure, as Ψ restricts to an isomorphism $\mathcal{S}_0' \xrightarrow{\cong} \Psi(\mathcal{S}_0')$, we can regard $\mathcal{S}_0' \subset \mathrm{Ch}$. Let $\mathcal{Q}_1$ be the projective scheme obtained by taking the closure of $\mathcal{S}_0'$ in Ch. For some $s_0 > 0$, the $GL(V)$-linearisation $\mathcal{L}_{\mathrm{Ch}}^{s_0\underline{a}}$ is very ample; let L_1 be the restriction of this linearisation to $\mathcal{Q}_1$, twisted by a suitable character $\epsilon_1\chi_1$ (to ensure that the corresponding $\hat{H}_1$-linearisation is adapted). As per Construction 3.3.1, we need to consider the action $\widehat{H}_1 = U_1 \rtimes (SL(W) \times \lambda_1(\mathbb{G}_m)) \circlearrowleft \mathcal{Q}_1$.

Fix a point p of $\mathcal{S}_0'$ corresponding to a hyperplanar admissible flag $(X^0, X^1, \ldots, X^n)$ with $X^i = X^n \cap \mathbb{P}(Z^i)$ and $\langle X^i \rangle = \mathbb{P}(Z^i) = \mathbb{P}\left(W \oplus \bigoplus_{j \leq i} \mathbb{C}v_j\right)$ for all i. Let $x = \Psi(p) \in \mathcal{Q}_1$. We introduce the following notation:

1. Set $\mathcal{Q}_{(1)} := (\mathcal{Q}_1)_{\min}^{\widehat{H}_1-vs}$.
2. For all $i = 1, \ldots, n$, set $\mathcal{Q}_1^{[i]} := \mathcal{Q}_1(\lambda^{[i]})_{\min}$ and set $Z_1^{[i]} := Z(\mathcal{Q}_1, \lambda^{[i]})_{\min}$.
3. Denote $p^{[i]}$ the $\lambda^{[i]}$-retraction $p^{[i]} : \mathcal{Q}_1^{[i]} \to Z_1^{[i]}$.

Lemma 7.4.5. *The following statements hold:*

1. *the point x lies in $\mathcal{Q}_{(1)}$, and for all $i \geq 1$ we have $x \in \mathcal{Q}_1^{[i]}$;*
2. *for all $i > 1$, we have $p^{[i]}(x) \in \mathcal{Q}_{(1)}$; and*
3. *for all $i \geq 1$, we have $p^{[i]}(x) \in Z_1^{[i]}$.*

In particular, for the action $\widehat{H}_1 \circlearrowright \mathcal{Q}_1$ linearised by L_1, Theorem 3.2.9 is applicable.

Proof. We begin by considering the third statement. From Lemma 7.4.4, the Hilbert–Mumford weight of $p^{[i]}(x)$ with respect to the linearisation L_1 and the 1PS $\lambda^{[i]}$ is constant across the dense open subscheme $\mathcal{S}_0' \equiv \Psi(\mathcal{S}_0') \subset \mathcal{Q}_1$. This common weight must be the minimal weight, and so $p^{[i]}(x) \in Z_1^{[i]}$ as claimed.

Let us now show that the points $x, p^{[2]}(x), \ldots, p^{[n]}(x)$ are in $\mathcal{Q}_{(1)} = (\mathcal{Q}_1)_{\min}^{\widehat{H}_1-vs}$; this involves establishing the following for all elements $y \in \{x, p^{[2]}(x), \ldots, p^{[n]}(x)\}$:

(i) $y \in \mathcal{Q}_1^{[1]} = \mathcal{Q}_1(\lambda^{[1]})_{\min}$, that is $p^{[1]}(y)$ has minimal $\lambda^{[1]}$-weight.

(ii) $p^{[1]}(y)$ has trivial $U^{[1]}$-stabiliser and is stable under the action of $R_1' = R'^{(1)} = SL(W)$ on $Z_1^{[1]}$.

(iii) y is not contained in the $U^{[1]}$-sweep of $Z_1^{[1]}$.

However, by Lemma 7.4.2 we have for all such points y that

$$p^{[1]}(y) = p^{[1]}(x) = \Psi(X^0, J(X^0, \mathbb{P}(V^{1,1})), \ldots, J(X^0, \mathbb{P}(V^{1,n}))).$$

We have already shown that this point has minimal $\lambda^{[1]}$-weight, so each such point y is in $\mathcal{Q}_1^{[1]}$. By Lemma 7.4.1, the Chow stability of $X^0 \subset \mathbb{P}(Z^0)$ together with Proposition 7.2.2 implies that the point $p^{[1]}(x)$ is $SL(W)$-stable with respect to L_1. Lemma 7.4.3 implies that $\mathrm{Stab}_{U^{[1]}}(p_{[1]}(x))$ is trivial. Finally, points in $U^{[1]}Z_1^{[1]}$ are fixed by the retraction $p^{[1]}$, whereas no point $y \in \{x, p^{[2]}(x), \ldots, p^{[n]}(x)\}$ is fixed by $p^{[1]}$, as the dimension 1 component of a flag corresponding to such a point y (namely, X^1) does not coincide with that of $p^{[1]}(y)$. Consequently each such point y cannot lie in $U^{[1]}Z_1^{[1]}$. This completes the proof of the Lemma. □

We now consider the induction step of Construction 3.3.1. Suppose $1 \leq i < \ell - 1 = n$, and suppose we have constructed successive quotients q_j of the form

$$\begin{array}{ccc}
(\mathcal{Q}_j)^{\widehat{H}_j - vs}_{\min} & \xrightarrow{\quad q_j \quad} & (\mathcal{Q}_j)^{\widehat{H}_j - vs}_{\min} / \widehat{H}_j \\
\downarrow & & \downarrow \\
\mathcal{Q}_j & \overset{q_j}{\dashrightarrow} & \mathcal{Q}_{j+1} \\
\downarrow & & \downarrow \\
\mathbb{P}(H^0(\mathcal{Q}_j, L_j^{s_j})^\vee) & \underset{H^0(\mathcal{Q}_j, L_j^{s_j})^{\widehat{H}_j} \subset H^0(\mathcal{Q}_j, L_j^{s_j})}{\dashrightarrow} & \mathbb{P}((H^0(\mathcal{Q}_j, L_j^{s_j})^{\widehat{H}_j})^\vee)
\end{array}$$

for each $j = 1, \ldots, i$, where the top horizontal arrow is a geometric $\widehat{H}_j$-quotient, where $\mathcal{Q}_{j+1}$ is the closure of the locally closed subscheme

$$(\mathcal{Q}_j)^{\widehat{H}_j - vs}_{\min} / \widehat{H}_j \subset \mathbb{P}((H^0(\mathcal{Q}_j, L_j^{s_j})^{\widehat{H}_j})^\vee),$$

and where each $\mathcal{Q}_j$ is endowed with the very ample linearisation L_j obtained by restricting the $\mathcal{O}(1)$ of $\mathbb{P}((H^0(\mathcal{Q}_{j-1}, L_{j-1}^{s_{j-1}})^{\widehat{H}_{j-1}})^\vee)$ to $\mathcal{Q}_j$ and then twisting by a suitable character $\epsilon_j \chi_j$ of $\widehat{H}_j$ to ensure adaptedness.

The very ample invertible sheaf L_{i+1} on the projective scheme $\mathcal{Q}_{i+1}$ carries a linearisation for a residual action of $P/\widehat{H}^{(i)}$, and in particular for $\widehat{H}_{i+1}$. By twisting L_{i+1} by a character of $\widehat{H}_{i+1}$ of the form $\epsilon_{i+1}\chi_{i+1}$ if necessary, we may assume that this $\widehat{H}_{i+1}$-linearisation is adapted.

For the induction step, we need to show that Theorem 3.2.9 can be applied to this linearised action of $\widehat{H}_{i+1}$. In order to establish this, we introduce the following notation:

1. For $j = 1, \ldots, i$, set

$$q_{(j)} := q_j \circ q_{j-1} \circ \cdots \circ q_1 : \mathcal{Q}_1 \dashrightarrow \mathcal{Q}_{j+1}.$$

2. We inductively define open subschemes $\mathcal{Q}_{(j)} \subset \mathcal{Q}_1$ for $j = 1, \ldots, i+1$ by setting $\mathcal{Q}_{(1)} := (\mathcal{Q}_1)^{\widehat{H}_1 - vs}_{\min}$ (as before) and setting for $j > 1$

$$\mathcal{Q}_{(j)} := \mathcal{Q}_{(j-1)} \cap q_{(j-1)}^{-1}((\mathcal{Q}_j)^{\widehat{H}_j - vs}_{\min}),$$

so that for each j, the morphism $q_{(j)} : \mathcal{Q}_{(j)} \to q_{(j)}(\mathcal{Q}_{(j)}) = \mathcal{Q}_{(j)}/\widehat{H}^{(j)} \subset \mathcal{Q}_{j+1}$ is a well-defined geometric $\widehat{H}^{(j)}$-quotient.

3. For all $k = 1, \ldots, i+1$ and for all $k \leq j < \ell = n+1$, let $\mathcal{Q}_k^{[j]} = \mathcal{Q}_k(\lambda_k^{[j]})_{\min}$, let $Z_k^{[j]} = Z(\mathcal{Q}_k, \lambda_k^{[j]})_{\min}$, and let $p_k^{[j]} : \mathcal{Q}_k^{[j]} \to Z_k^{[j]}$ be the retraction under $\lambda_k^{[j]}$. As a special case of the above, we continue to denote $p^{[j]} = p_1^{[j]}$ for the $\lambda_1^{[j]} = \lambda^{[j]}$ retraction on $\mathcal{Q}_1$.

4. Set $\mathcal{Q}_{(0)} := \mathcal{Q}_1$ and $q_{(0)} := \mathrm{id}_{\mathcal{Q}_1}$.

Lemma 7.4.6. *For each $j = 0, 1, \ldots, i$, the following statements hold:*

1. *$x \in \mathcal{Q}_{(j+1)}$, and for all $k \geq j+1$ we have $q_{(j)}(x) \in \mathcal{Q}_{j+1}^{[k]}$;*

2. *for all $j+1 < k \leq n$, we have $p^{[k]}(x) = p_1^{[k]}(x) \in \mathcal{Q}_{(j+1)}$; and*

3. *for all $j+1 \leq k \leq n$, $q_{(j)}(p^{[k]}(x)) \in Z_{j+1}^{[k]}$.*

Proof. We argue by induction on j, noting that the base case is Lemma 7.4.5. Assume that the assertions of Lemma 7.4.6 holds for each $j = 0, \ldots, r-1$, where $r \leq i$. We first show that for each $k \geq r+1$, the point $q_{(r)}(p^{[k]}(x)) = p_{r+1}^{[k]}(q_{(r)}(x))$ is contained in $Z_{r+1}^{[k]}$. For each $k \geq r+1$, the induction hypothesis implies that $q_{(r-1)}(p^{[k]}(x)) = p_r^{[k]}(q_{(r-1)}(x)) \in Z_r^{[k]} \cap (\mathcal{Q}_r)_{\min}^{\widehat{H}_r - vs}$. In particular, we have $Z_r^{[k]} \cap (\mathcal{Q}_r)_{\min}^{\widehat{H}_r - vs} \neq \emptyset$. From Construction 3.3.1 the map $q_r : \mathcal{Q}_r \dashrightarrow \mathcal{Q}_{r+1}$ is given by the projection corresponding to the inclusion $B := H^0(\mathcal{Q}_r, L_r^{s_r})^{\widehat{H}_r} \subset A := H^0(\mathcal{Q}_r, L_r^{s_r})$, and the twist of L_{r+1} by the character $-\epsilon_{r+1}\chi_{r+1}$ pulls back under q_r to give $L_r^{s_r}$. As such, the arguments given in the proof of [66, Lemma 5.4] carry over to the map $q_r : \mathcal{Q}_r \dashrightarrow \mathcal{Q}_{r+1}$. Namely, that $Z_r^{[k]} \cap (\mathcal{Q}_r)_{\min}^{\widehat{H}_r - vs} \neq \emptyset$ implies that the maximal weights for $\lambda_r^{[k]}(\mathbb{G}_m)$ acting on both A and B coincide, so we may choose a basis $\mathcal{B}_A$ for A of the form

$$\mathcal{B}_A = \{\alpha_\iota, \alpha'_\kappa, \beta_\mu, \beta'_\nu\},$$

where the $\alpha_\iota, \alpha'_\kappa$ are a basis for B and the α_ι, β_μ are a basis for the maximal weight space for $\lambda_r^{[k]}(\mathbb{G}_m)$ in A. With respect to this basis, the morphism q_r corresponds to projecting onto the $\alpha_\iota, \alpha'_\kappa$-coordinates, and the retraction $p_r^{[k]}$ on $\mathcal{Q}_r$ corresponds to projecting onto the α_ι, β_μ-coordinates. Setting $z := q_{(r-1)}(p^{[k]}(x))$, since $z \in Z_r^{[k]}$ then all $\alpha'_\kappa, \beta'_\nu$-coordinates of z vanish, and since $z \in (\mathcal{Q}_r)_{\min}^{\widehat{H}_r - vs}$ then at least one α_ι-coordinate of z does not vanish. Applying q_r to z (that is, projecting onto the α_ι-coordinates), we obtain that the point $q_r(z) = q_{(r)}(p^{[k]}(x)) \in \mathcal{Q}_{r+1}$ is fixed by $\lambda_{r+1}^{[k]}$ and has all coordinates in the maximal weight space, whence $z \in Z_{r+1}^{[k]}$. Consequently, we find that for all $k \geq r+1$,

$$p_{r+1}^{[k]}(q_{(r)}(x)) = q_{(r)}(p^{[k]}(x)) = q_r(q_{(r-1)}(p^{[k]}(x))) \in Z_{r+1}^{[k]}.$$

In turn this implies that $q_{(r)}(x) \in \mathcal{Q}_{r+1}^{[k]}$ for all $k \geq r+1$.

Let us now show that the points $x, p^{[r+2]}(x), \ldots, p^{[n]}(x)$ lie in $\mathcal{Q}_{(r+1)}$, that is their images under $q_{(r)}$ lie in $(\mathcal{Q}_{r+1})_{\min}^{\widehat{H}_{r+1}-vs}$. Note that the semisimple part R'_{r+1} of the Levi subgroup R_{r+1} of $\widehat{H}_{r+1}$ is trivial, so R'_{r+1}-stability is vacuous. As such, it suffices to establish the following for any point $y \in \{q_{(r)}(x), q_{(r)}(p^{[r+2]}(x)), \ldots, q_{(r)}(p^{[n]}(x))\}$:

(i) $p_{r+1}^{[r+1]}(y) \in Z_{r+1}^{[r+1]}$.

(ii) $p_{r+1}^{[r+1]}(y)$ has trivial U_{r+1}-stabiliser.

(iii) y is not contained in the U_{r+1}-sweep of $Z_{r+1}^{[r+1]}$.

By Lemma 7.4.2 we have for all such points y

$$p_{r+1}^{[r+1]}(y) = p_{r+1}^{[r+1]}(q_{(r)}(x)) = q_{(r)}(p^{[r+1]}(x)),$$

which is represented by the $\widehat{H}^{(r)}$-orbit of the configuration

$$p^{[r+1]}(p) = (X^0, \ldots, X^r, J(X^r, \mathbb{P}(V^{r+1,r+1})), \ldots, J(X^r, \mathbb{P}(V^{r+1,n}))).$$

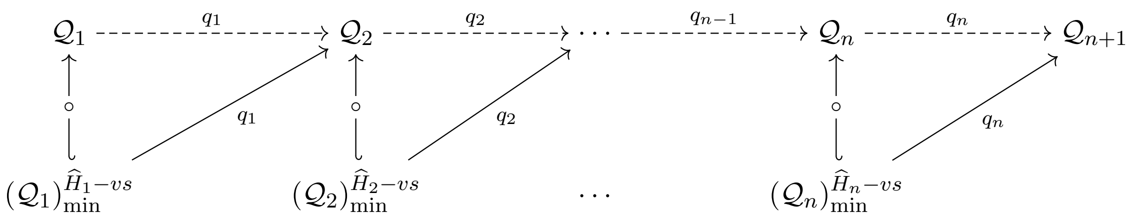
of subschemes of $\mathbb{P}(V)$. We have already established that $q_{(r)}(p^{[r+1]}(x))$ is in $Z_{r+1}^{[r+1]}$. Combining Lemma 7.4.3 with [66, Lemma 5.12], the triviality of $\mathrm{Stab}_{U^{[r+1]}}(p^{[r+1]}(p))$ implies that $q_{(r)}(p^{[r+1]}(x))$ has trivial U_{r+1}-stabiliser. Any point in $q_{(r)}(\mathcal{Q}_{(r)}) \cap U_{r+1} Z_{r+1}^{[r+1]}$ is fixed under the retraction $p_{r+1}^{[r+1]}$. However none of the points y are fixed by $p_{r+1}^{[r+1]}$, as they are all represented by configurations whose dimension $(r+1)$ component is the non-singular variety X^{r+1}, whereas $p_{r+1}^{[r+1]}(y)$ is represented in dimension $(r+1)$ by the singular variety $J(X^r, \mathbb{P}(V^{r+1,r+1}))$.

Consequently none of these points y can lie in $U_{r+1} Z_{r+1}^{[r+1]}$. It follows that each $y \in \{q_{(r)}(x), q_{(r)}(p^{[r+2]}(x)), \ldots, q_{(r)}(p^{[n]}(x))\}$ lies in $(\mathcal{Q}_{r+1})_{\min}^{\widehat{H}_{r+1}-vs}$. This concludes the induction step and in turn concludes the proof of the lemma. □

Corollary 7.4.7. *For the action $\widehat{H}_{i+1} \circlearrowright \mathcal{Q}_{i+1}$ linearised by L_{i+1}, Theorem 3.2.9 is applicable.*

Proof. This immediately follows from Lemma 7.4.6. □

Arguing by induction on i, it follows that Construction 3.3.1 can be carried out in full, yielding a diagram of the form



where for each i the restriction $q_i : (\mathcal{Q}_i)^{\widehat{H}_i-vs}_{\min} \to q_i((\mathcal{Q}_i)^{\widehat{H}_i-vs}_{\min}) = (\mathcal{Q}_i)^{\widehat{H}_i-vs}_{\min}/\widehat{H}_i$ is a geometric $\widehat{H}_i$-quotient. With

$$q_{(n)} := q_n \circ q_{n-1} \circ \cdots \circ q_1 : \mathcal{Q}_1 \dashrightarrow \mathcal{Q}_{n+1}$$

and

$$\mathcal{Q}_{(n)} := \mathcal{Q}_{(n-1)} \cap q^{-1}_{(n-1)}((\mathcal{Q}_n)^{\widehat{H}_n-vs}_{\min}),$$

the morphism $q_{(n)} : \mathcal{Q}_{(n)} \to q_{(n)}(\mathcal{Q}_{(n)}) \subset \mathcal{Q}_{n+1}$ is a well-defined geometric $\widehat{H}^{(n)}$-quotient of $\mathcal{Q}_{(n)}$. If x is a point of $\mathcal{S}'_0 \equiv \Psi(\mathcal{S}'_0)$, by iterating Lemma 7.4.6 one has $x \in \mathcal{Q}_{(n)}$.

7.4.5 Completing the Proof

By tying everything together, we complete the proof of Theorem 7.1.1.

Proof of Theorem 7.1.1. By Corollary 7.3.7, the moduli functor $\mathcal{F}^{\mathbb{P}(V)}_{n,d,\underline{\Phi}}$ admits a coarse moduli space if and only if the scheme $\mathcal{S}'$ admits a categorical quotient for the action of $SL(V)$ which is an orbit space morphism. From Section 7.4.1, this is equivalent to $\mathcal{S}'_0 \equiv \Psi(\mathcal{S}'_0)$ admitting a categorical quotient for the action of P which is an orbit space morphism.

From Section 7.4.4, the scheme $\mathcal{Q}_{(n)}$ admits a quasi-projective geometric $\widehat{H}^{(n)}$-quotient. Invoking Proposition 3.3.2, it follows that $\mathcal{Q}_{(n)}$ admits a quasi-projective geometric P-quotient. There is an inclusion $\Psi(\mathcal{S}'_0) \subset \mathcal{Q}_{(n)}$ of open subschemes of $\mathcal{Q}_{(1)}$. A geometric P-quotient of $\mathcal{S}'_0$ can then be obtained by restricting the geometric P-quotient $q_{(n)} : \mathcal{Q}_{(n)} \to q_{(n)}(\mathcal{Q}_{(n)}) \subset \mathcal{Q}_{n+1}$, by applying Lemma 2.1.5 to the U_i-quotients (which are all Zariski-locally trivial) and Lemma 2.1.4 to the quotients by the $\lambda_i(\mathbb{G}_m)$ and by $R'_1 = SL(W)$ (which all have the property of being geometric quotients). In particular,

$\mathcal{S}_0' \equiv \Psi(\mathcal{S}_0')$ admits a categorical quotient for the action of P which is an orbit space morphism. This completes the proof of Theorem 7.1.1. $\square$

7.5 Extending the Main Result

To conclude this chapter, we consider a couple of ways in which Theorem 7.1.1 can be extended.

7.5.1 Weighting the Points

The construction presented above treats X^0 as an unordered collection of points in the linear subspace $Z^0 \subset \mathbb{P}(V)$. It is possible to consider a modified setup, where we instead consider X^0 to be an *ordered* collection of d points in Z^0, by replacing the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}(V), d)$ of 0-dimensional length d subschemes of $\mathbb{P}(V)$ with the d-fold product $\mathbb{P}(V)^{\times d}$. In place of the Chow linearisation $\mathcal{L}_{\mathrm{Ch}_0}$, one may take the linearisation

$$\mathcal{O}_{\mathbb{P}(V)}(\underline{w}) := \boxtimes_{i=1}^{d} \mathcal{O}_{\mathbb{P}(V)}(w_i), \quad \underline{w} = (w_1, \ldots, w_d) \in (\mathbb{Z}^{>0})^d.$$

With respect to the induced action of $SL(W)$ arising from a given decomposition $V = W \oplus W'$, it follows from [40, Theorem 11.1] that if $p = (p_1, \ldots, p_d) \in \mathbb{P}(W)^{\times n} \subset \mathbb{P}(V)^{\times n}$ then p is $SL(W)$-(semi)stable with respect to the linearisation $\mathcal{O}_{\mathbb{P}(V)}(\underline{w})$ if and only if for all proper linear subspaces $Z \subset \mathbb{P}(W)$,

$$\frac{\sum_{p_i \in Z} w_i}{\sum_{i=1}^{d} w_i} < (\leq) \; \frac{\dim Z + 1}{\dim W} \tag{7.5.1}$$

Applying the same quotienting-in-stages construction used to prove Theorem 7.1.1 with this modified setup, one obtains (for each $\underline{w} \in (\mathbb{Z}^{\geq 0})^d$) a coarse moduli space parametrising all non-degenerate, non-singular and stable hyperplanar admissible flags $(\underline{X}, \underline{Z})$ together with a labelling of the points of X^0, where the appropriate notion of stability is given by requiring that the labelled points in $X^0 \subset \mathbb{P}(Z^0)$ satisfy Inequality 7.5.1 strictly.

7.5.2 Omitting the Points

Another variant of Theorem 7.1.1 can be obtained in which the flags being parametrised are of the form

$$X^1 \subset \cdots \subset X^{n-1} \subset X^n, \quad Z^1 \subset \cdots \subset Z^{n-1} \subset Z^n = V \tag{7.5.2}$$

where $X^1 \subset \mathbb{P}(Z^1)$ is a smooth, non-degenerate connected projective curve which is now required to be GIT stable with respect to the Chow linearisation on $\mathrm{Hilb}(\mathbb{P}(Z^1), \Phi_1)$ (and all other subvarieties $X^i \subset \mathbb{P}(Z^i)$ are non-degenerate, smooth and connected). As above, the construction proceeds along very similar lines to the quotienting-in-stages construction used to prove Theorem 7.1.1; the locus $\mathcal{S}'$ being quotiented is taken to be the analogously-defined locally closed subscheme of $\prod_{i=1}^n \mathcal{H}_i \times \prod_{j=1}^n \mathrm{Gr}_j$, and the group P is taken to be the parabolic subgroup of $SL(V)$ preserving the flag $Z^1 \subset \cdots \subset Z^n = V$. In the base step of the quotienting-in-stages procedure (cf. Lemma 7.4.5), the GIT stability of $X^1 \subset \mathbb{P}(Z^1)$ is used to argue that the points $p^{[1]}(y)$, $y \in \{x, p^{[2]}(x), \ldots, p^{[n-1]}(x)\}$ are all stable with respect to the reductive linear algebraic group R_1'; otherwise the quotienting-in-stages construction proceeds as in Section 7.4.4.

From the results of [22], it is known that if $g \geq 2$ and if $d > 2(2g-2)$ then smooth non-degenerate connected projective curves in $\mathbb{P}^{d-g}$ of degree d and genus g are Chow stable; in particular, the Chow stable locus of the corresponding Hilbert scheme of curves is non-empty. Imposing the requirement that the degree and genus of X^1 satisfy these constraints imposes constraints on the higher-dimensional X^i fitting into a flag of the form (7.5.2). Indeed, fixing $k \in \{2, \ldots, n\}$, it follows from the Hirzebruch–Riemann–Roch theorem (as applied in [51, Example 18.3.5]) that if $H \subset X^k$ is the hyperplane class then

$$\begin{aligned} 1 - g &= \chi(X^1, \mathcal{O}_{X^1}) \\ &= \int_{X^k} \left(\prod_{i=1}^{k-1} (1 - \exp(-H)) \cap \mathrm{Td}(X^k) \right) \\ &= \int_{X^k} \left(H^{k-1} \cdot \left(-\frac{K_{X^k}}{2} \right) - \frac{k-1}{2} H^k \right) \\ &= -\frac{k-1}{2} \cdot d - \frac{K_{X^k} \cdot X^1}{2}. \end{aligned}$$

Here $\mathrm{Td}(X^k) = 1 - \frac{1}{2} K_{X^k} + \cdots$ is the Todd class of X^k. The inequality $d > 2(2g-2)$ implies that if $g \geq 2$ then

$$K_{X^k} \cdot X^1 < -(2k-3)(2g-2) \leq 0.$$

In particular, each canonical divisor K_{X^k} cannot be nef or numerically trivial.

For similar reasons to when $k = 1$, it is also possible to obtain a variant of Theorem 7.1.1 concerning flags of the form

$$X^k \subset \cdots \subset X^{n-1} \subset X^n, \quad Z^k \subset \cdots \subset Z^{n-1} \subset Z^n = V$$

where $X^k \subset \mathbb{P}(Z^k)$ is a smooth, non-degenerate, *Chow stable* subvariety of $\mathbb{P}(Z^k)$.

Chapter 8

Moduli Spaces of Unstable Rank 1 Torsion-Free Sheaves on Reducible Curves

8.1 Introduction

In this chapter, we study the moduli of Gieseker unstable rank 1 torsion-free sheaves on reducible curves. The main result that we prove is the existence of *quasi-projective* fine moduli spaces of simple unstable torsion-free uniform rank 1 sheaves on certain reduced, connected, projective, reducible curves C; in many cases these fine moduli spaces are projective. In particular, this gives the first examples of *projective* moduli spaces of unstable sheaves on projective schemes which admit fully modular descriptions, without any restriction on the Harder–Narasimhan length.

As explained in Chapter 4, GIT gives a method to construct projective moduli spaces of semistable sheaves on a projective scheme X. Conversely, no such general approach is known to exist for the analogous problem of constructing moduli spaces of unstable sheaves on X of a given Harder–Narasimhan type. Even though this is a problem in non-reductive GIT (cf. Section 4.3.2), the $\widehat{U}$-Theorem of Bérczi–Doran–Hawes–Kirwan can only be applied in certain cases. The most well-behaved case concerns the moduli of unstable sheaves F of Harder–Narasimhan length 2 on a projective scheme X whose Harder–Narasimhan filtration is non-split, whose Harder–Narasimhan subquotients are Gieseker stable and with fixed $\dim \operatorname{End}_X(F)$; the $\widehat{U}$-Theorem can be used to construct a quasi-projective coarse moduli space parametrising all such sheaves, as shown by Jackson [70].

For Harder–Narasimhan lengths $\ell > 2$, there are two current NRGIT approaches to constructing moduli spaces of unstable sheaves F of Harder–Narasimhan length ℓ, whose Harder–Narasimhan filtration is non-split and whose Harder–Narasimhan subquotients are stable. The first approach is the non-reductive quotienting-in-stages procedure of Hoskins and Jackson [66] (cf. Section 3.3); this involves fixing, in addition to fixing the Harder–Narasimhan type, the quantities

$$\dim \mathrm{Hom}_X(F/F^i, F^i), \quad \dim \mathrm{Hom}_X(F/F^i, F^{i-1}),$$

for all i, where $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ is the Harder–Narasimhan filtration of F. The second approach of Qiao [110] involves choosing a sequence $s_0, \ldots, s_N$ of non-increasing functions $\{1, \ldots, \ell\} \to \{0, \ldots, \ell - 1\}$, and fixing the dimensions of the spaces

$$\mathrm{End}_X(F)_{s_k} := \left\{ \phi \in \mathrm{End}_X(F) : \phi(F^i) \subset F^{s_k(i)} \text{ for all } i = 1, \ldots, \ell \right\}, \quad k = 1, \ldots, N.$$

With either approach, the resulting moduli spaces parametrise sheaves of a fixed *refined* Harder–Narasimhan type (though the respective notions of refinements are different); for either approach to work, there must exist an unstable sheaf F whose Harder–Narasimhan associated graded has the same refined Harder–Narasimhan type as F. Hoskins–Jackson can only provide examples of such sheaves whose Harder–Narasimhan length is 2 (and this case is covered by Jackson's Harder–Narasimhan length 2 result), whereas Qiao is able to use his approach to construct quasi-projective coarse moduli spaces of unstable locally free sheaves on a smooth projective curve of genus g which, in addition to having fixed $\dim \mathrm{End}_X(F)_{s_k}$ for all $k = 0, \ldots, N$ (for a chosen sequence $s_0, \ldots, s_N$), are required to satisfy $\mu_i - \mu_{i+2} > 2g - 2$ for all i, where $\mu_1 > \cdots > \mu_\ell$ are the slopes of the Harder–Narasimhan subquotients.

A separate non-GIT construction exists of the moduli space of unstable vector bundles F on X of fixed Harder–Narasimhan type τ of length 2 whose Harder–Narasimhan filtration is non-split, whose Harder–Narasimhan subquotients are stable and with fixed $\dim \mathrm{End}_X(F)$; this construction is due to Brambila-Paz and Rios Sierra [25]. If $\tau = ((r_1, d_1), (r_2, d_2))$ is the Harder–Narasimhan type, the moduli space is constructed from the fine moduli spaces $M_i = M_X^s(r_i, d_i)$ $(i = 1, 2)$ of stable locally free sheaves on X of ranks r_i and degrees d_i by passing to an appropriate union of strata of a flattening stratification of $M_1 \times M_2$ for a relative Ext^1-sheaf defined using the universal bundles on the M_i, and then taking the projective bundle associated to this sheaf over the union of the strata.

As is shown in this chapter, an adaptation of the construction of Brambila-Paz–Rios Sierra can be used to prove the existence of *quasi-projective* schematic fine moduli spaces of simple unstable torsion-free coherent sheaves of uniform rank 1 on certain reducible curves C, building on from the existence of fine moduli spaces of simple semistable rank 1 torsion-free sheaves supported along given subcurves $D \subset C$, i.e. the fine compactified Jacobians of the subcurves D. This is a setting where *all* unstable sheaves F have the property that both F and the Harder–Narasimhan associated graded of F have the same Hoskins–Jackson refined Harder–Narasimhan type (cf. remark after Definition 8.2.2), and so an alternative proof of the quasi-projectivity of the open subscheme where the Harder–Narasimhan subquotients are stable can be given using non-reductive GIT. However, non-reductive GIT reveals much less about the geometry of the fine moduli spaces of unstable sheaves on C compared to the approach taken in this chapter.

Summary of Results

Let C be a reduced connected projective curve. Fix a polarisation $\nu = (\alpha, \phi)$ on C. Let $\tau = (\theta_i, \upsilon_i)_{i=1}^{\ell}$ be a Harder–Narasimhan type for simple torsion-free sheaves on C of Euler characteristic χ and uniform rank 1, defined with respect to ν (cf. Definition 8.2.1). Let $\overline{J}_C^{\tau} \subset \overline{\mathbb{J}}_C^{\chi}$ be the fine schematic moduli space parametrising those sheaves F on C of Harder–Narasimhan type τ; the existence of such a subscheme is a consequence of Proposition 4.3.4. The space $\overline{J}_C^{\tau}$ decomposes as a disjoint union of open and closed subschemes of the form $\overline{J}_C^{\tau_0}$, where $\tau_0 = (\theta_i, \upsilon_i, D_i)_{i=1}^{\ell}$ is a *(support-) refined* Harder–Narasimhan type refining τ; $\overline{J}_C^{\tau_0}$ parametrises those sheaves $F \in \overline{J}_C^{\tau}$ such that, if $0 = F^0 \subset F^1 \subset \cdots \subset F^{\ell} = F$ is the ν-Harder–Narasimhan filtration of F, then $\mathrm{supp}(F_i)$ is equal to the subcurve $D_i \subset C$. In turn, there is an open subscheme $\overline{J}_C^{\tau_0 - sp}$ of $\overline{J}_C^{\tau_0}$ which parametrises those sheaves $F \in \overline{J}_C^{\tau_0}$ which are τ_0-*simple*, meaning that:

(i) each sheaf F_i is simple (viewed as a sheaf on D_i); and

(ii) each inclusion $F^i \subset F^{i+1}$ is non-split.

In order to ensure that $\overline{J}_C^{\tau_0 - sp}$ is non-empty, we assume that the subcurves D_i are all connected and that $D_i \cap D_{i+1} \neq \emptyset$ for all $i = 1, \ldots, \ell - 1$. Let $\overline{J}_i = \overline{J}_{D_i}^{\chi_i, ss}(\nu_{D_i})$ be the fine compactified Jacobian parametrising simple ν_{D_i}-semistable sheaves on D_i of Euler characteristic $\chi_i = \theta_i - \phi_{D_i}$. There is a map

$$\mathrm{gr} : \overline{J}_C^{\tau_0} \dashrightarrow \prod_{i=1}^{\ell} \overline{J}_i$$

which associates to an unstable sheaf F its Harder–Narasimhan subquotients $F_i = F^i/F^{i-1}$. This restricts to a well-defined morphism $\mathrm{gr} : \overline{J}_C^{\tau_0-sp} \to \prod_{i=1}^{\ell} \overline{J}_i$. We may now state the main result of this chapter.

Theorem 8.1.1. *Assume that for all $i \neq j$, any point $p \in D_i \cap D_j$ is a locally planar singularity of C. Then there exists a diagram*

$$\begin{array}{ccc}
\overline{J}_C^{\tau_0-sp} & \xrightarrow{\simeq} & P^{\ell} \\
 & & \downarrow{\scriptstyle \pi_\ell} \\
 & & P^{\ell-1} \\
 & & \downarrow \\
 & & \vdots \\
 & \searrow^{\mathrm{gr}} & \downarrow \\
 & & P^2 \\
 & & \downarrow{\scriptstyle \pi_2} \\
 & & P^1 = \prod_{i=1}^{\ell} \overline{J}_i
\end{array}$$

where each $\pi_i : P^i \to P^{i-1}$ is a Zariski-locally trivial projective bundle.

The projective bundles $\pi_i : P^i \to P^{i-1}$ are constructed from universal spaces of extensions, as described in the following subsection.

By combining Theorem 8.1.1 with the properties of fine compactified Jacobians given in Section 5.3, we have the following immediate corollary.

Corollary 8.1.2. *The morphism $\mathrm{gr} : \overline{J}_C^{\tau_0-sp} \to \prod_{i=1}^{\ell} \overline{J}_i$ is projective (in particular proper), surjective and smooth. In particular, if C has locally planar singularities then the scheme $\overline{J}_C^{\tau_0-sp}$ is reduced, has lci singularities, and the smooth locus coincides with the locus in $\overline{J}_C^{\tau_0-sp}$ where all subquotients F_i are invertible.*

Moreover, the open subscheme $\overline{J}_C^{\tau_0-s} \subset \overline{J}_C^{\tau_0-sp}$ parametrising those sheaves $F \in \overline{J}_C^{\tau_0-sp}$ whose Harder–Narasimhan subquotients $F_i = F^i/F^{i-1}$ are all stable is a quasi-projective scheme. In particular, if $\overline{J}_{D_i}^{\chi_i,ss}(\nu_{D_i}) = \overline{J}_{D_i}^{\chi_i,s}(\nu_{D_i})$ for all $i = 1, \ldots, \ell$, then the fine moduli space $\overline{J}_C^{\tau_0-sp}$ is a projective scheme. □

In addition to presenting the proof of Theorem 8.1.1, we also compute in this chapter the relative dimension of the associated graded morphism $\mathrm{gr} : \overline{J}_C^{\tau_0-sp} \to \prod_{i=1}^{\ell} \overline{J}_i$ in terms of the Harder–Narasimhan length ℓ and the subcurves D_i of C in the case where each point $p \in D_i \cap D_j$ is a node of C (cf. Corollary 8.4.3). We conclude by indicating how Theorem 8.1.1 can be generalised to the relative setting.

Summary of the Proof of the Main Result

The proof proceeds along similar lines to the construction of Brambila-Paz–Rios Sierra [25]. Suppose $F \in \overline{J}_C^{\tau_0-sp}$ is an unstable sheaf of refined Harder–Narasimhan type τ_0, with Harder–Narasimhan filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ and subquotients $F_i = F^i/F^{i-1} \in \overline{J}_i$. We may reconstruct F from the subquotients F_i by taking a successive sequence of non-split extensions

$$0 \longrightarrow F^{i-1} \longrightarrow F^i \longrightarrow F_i \longrightarrow 0.$$

The space of such extensions, up to equivalence of extensions and up to scalar multiplication, is given by the projective space $\mathbb{P}(\mathrm{Ext}^1_C(F_i, F^{i-1}))$. Recalling that we have universal sheaves $\mathcal{U}_i$ on $D_i \times \overline{J}_i \subset C \times \overline{J}_i$, this suggests that the moduli space $\overline{J}_C^{\tau_0-sp}$ can be constructed from the fine moduli spaces $\overline{J}_i$ by taking iterated (relative) moduli spaces of nowhere-split extensions of the $\mathcal{U}_i$. This approach is the one used to prove Theorem 8.1.1.

The space P^2 is constructed from P^1 by taking a projective bundle Q^2 over $\overline{J}_1 \times \overline{J}_2$ and setting $P^2 = Q^2 \times \prod_{i=3}^{\ell} \overline{J}_i$ (with the morphism $P^2 \to P^1$ being given by the identity on the factor $\prod_{i=3}^{\ell} \overline{J}_i$). The fibre of Q^2 over the point $(F_1, F_2) \in \overline{J}_1 \times \overline{J}_2$ is given by $\mathbb{P}(\mathrm{Ext}^1_C(F_2, F_1))$. The projective bundle Q^2 is a fine moduli space for nowhere-split (twisted) extensions of $\mathcal{U}_2$ by $\mathcal{U}_1$, up to a suitable notion of equivalence. In particular, Q^2 carries a universal sheaf $\mathcal{U}^2$, the middle term of the universal short exact sequence parametrised by Q^2.

In order to show that such a fine moduli space exists and is a projective bundle over $\overline{J}_1 \times \overline{J}_2$, we make use of results of Lange [80] concerning universal spaces of extensions of coherent sheaves. Letting p_{12}, p_{13} and p_{23} be the projections from $C \times \overline{J}_1 \times \overline{J}_2$ onto $C \times \overline{J}_1$, $C \times \overline{J}_2$ and $\overline{J}_1 \times \overline{J}_2$ respectively, we set

$$Q^2 = \mathbb{P}_{\overline{J}_1 \times \overline{J}_2} \left(\mathcal{E}xt^1_{p_{23}}(p_{13}^* \mathcal{U}_2, p_{12}^* \mathcal{U}_1) \right).$$

The relative Ext sheaf $\mathcal{E}xt^1_{p_{23}}(p_{13}^*\mathcal{U}_2, p_{12}^*\mathcal{U}_1)$ is locally trivial; the proof of this uses the base change results presented in *loc. cit.* together with the fibrewise vanishing of $\mathrm{Ext}^i(p_{13}^*\mathcal{U}_2, p_{12}^*\mathcal{U}_1)$ for $i = 0, 2$. The local triviality of the relative Ext sheaf is also used to show that Q^2 has the desired universal properties.

The space P^3 is constructed from P^2 in a similar manner, by taking the projective bundle

$$Q^3 = \mathbb{P}_{Q^2 \times \overline{J}_3}\left(\mathcal{E}xt^1_{p_{23}}(p_{13}^*\mathcal{U}_3, p_{12}^*\mathcal{U}^2)\right)$$

over $Q^2 \times \overline{J}_3$ and setting $P^3 = Q^3 \times \prod_{i=4}^{\ell} \overline{J}_i$. As with Q^2, the space Q^3 carries a universal short exact sequence with middle term $\mathcal{U}^3$, which is used to construct $P^4 = Q^4 \times \prod_{i=5}^{\ell} \overline{J}_i$ from Q^3 and $\overline{J}_4$.

The construction continues inductively, terminating after $\ell - 1$ stages with the construction of the projective bundle

$$P^\ell = Q^\ell = \mathbb{P}_{Q^{\ell-1} \times \overline{J}_\ell}\left(\mathcal{E}xt^1_{p_{23}}(p_{13}^*\mathcal{U}_\ell, p_{12}^*\mathcal{U}^{\ell-1})\right)$$

over $P^{\ell-1} = Q^{\ell-1} \times \overline{J}_\ell$; the space P^ℓ is a fine moduli space for nowhere-split (twisted) extensions of $\mathcal{U}_\ell$ by $\mathcal{U}^{\ell-1}$, up to equivalence. The universal properties of $\overline{J}_C^{\tau_0-sp}$ and P^ℓ are then used to define explicit inverse morphisms $\overline{J}_C^{\tau_0-sp} \to P^\ell$ and $P^\ell \to \overline{J}_C^{\tau_0-sp}$.

A priori, each scheme Q^i parametrises nowhere-split *extensions* of coherent sheaves, up to a suitable notion of equivalence. However, the notion of τ_0-simplicity has the important consequence that any τ_0-simple sheaf F with Harder–Narasimhan filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ has the property that each sheaf F^i is simple. This is used to show that each scheme Q^i parametrises (isomorphism classes of) coherent sheaves on the nose; in effect, each extension equivalence class in each Q^i is uniquely determined by the isomorphism class of the middle term (see Corollary 8.3.5). This is crucial to proving the existence of an isomorphism $P^\ell \cong \overline{J}_C^{\tau_0-sp}$, since $\overline{J}_C^{\tau_0-sp}$ is a moduli space of coherent sheaves (as opposed to a moduli space of sequences of extensions of coherent sheaves).

Future Directions

There are two main ways in which one could potentially generalise Theorem 8.1.1 (in the non-relative setting). The first is to allow for some of the inclusions $F^i \subset F^{i+1}$ in the Harder–Narasimhan filtration to be split, as opposed to asking that all such inclusions are non-split. The second is to possibly allow for unstable sheaves with multiranks other than $(1, \dots, 1)$; for instance, if each subcurve D_i in a given refined Harder–Narasimhan type is non-singular, one can make use of the existence of universal sheaves on the moduli spaces of stable locally free sheaves on the curves D_i of ranks $r_i > 0$, in place of the universal sheaves on the fine compactified Jacobians $\overline{J}_i$.

The main subtlety in either case which would need addressing (and which does not arise in the situation of Theorem 8.1.1) is that, concerning the extension space $\mathbb{P}_S(\mathcal{E}xt^1_{X/S}(F_2, F_1))$, if either of F_1 or F_2 are not fibrewise simple, then Corollary 8.3.5 no-longer applies, meaning that it is no-longer clear that $\mathbb{P}_S(\mathcal{E}xt^1_{X/S}(F_2, F_1))$ is a moduli space of coherent sheaves. In higher ranks, the middle terms of non-split extensions of simple sheaves on curves need not be simple.[1] As such, it is not clear in either case that taking the appropriate iterated universal extension spaces results in a moduli space of (isomorphism classes of) coherent sheaves, especially when working with sheaves of Harder–Narasimhan length $\ell > 2$.

8.2 Harder–Narasimhan Filtrations for Rank 1 Sheaves on Curves

Let $f : C \to S$ be a flat projective morphism over a locally Noetherian base scheme S whose geometric fibres are connected, reduced curves. Pick invertible sheaves L, M on C, with L relatively ample. As a special case of the existence of Harder–Narasimhan stratifications for families of coherent sheaves (cf. Section 4.3.1), one has the Harder–Narasimhan stratification

$$\overline{\mathbb{J}}^{\chi}_{C/S} = \bigsqcup_{\tau \in \mathrm{HNT}(L,M,\chi)} \overline{J}^{\tau}_{C/S}$$

of $\overline{\mathbb{J}}^{\chi}_{C/S}$ into locally closed sub-algebraic spaces $\overline{J}^{\tau}_{C/S} = \overline{J}^{\tau}_{C/S}(L, M)$, with $\overline{J}^{\tau}_{C/S}$ parametrising all sheaves $F \in \overline{\mathbb{J}}^{\chi}_{C/S}$ with the property that $F \otimes M$ admits a relative Harder–Narasimhan filtration, necessarily of the form

$$0 = F^0 \otimes M \subset F^1 \otimes M \subset \cdots \subset F^{\ell} \otimes M = F \otimes M$$

[1] For example, let C be a smooth curve of genus $g \geq 1$, and let $0 \to \mathcal{O}_C \to F \to \mathcal{O}_C \to 0$ correspond to a non-trivial element of $\mathrm{Ext}^1_C(\mathcal{O}_C, \mathcal{O}_C) = H^1(C, \mathcal{O}_C)$ with F locally free. Then $h^0(F), h^0(F^\vee) > 0$; by tensoring together non-trivial sections of F and $F^\vee$, one can produce an element of $H^0(C, F \otimes F^\vee) = \mathrm{End}_C(F)$ corresponding to a non-scalar endomorphism of F.

for a unique filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ of F with flat subquotients $F_i = F^i/F^{i-1}$, of Harder–Narasimhan type τ. In particular, each sheaf $F_i \otimes M$ is Gieseker semistable with respect to L.

If $S = \operatorname{Spec} \mathbb{C}$ is a point and if F_i has support D_i, there are inequalities

$$\frac{\chi(F_1) + \deg_{D_1} M}{\deg_{D_1} L} > \cdots > \frac{\chi(F_\ell) + \deg_{D_\ell} M}{\deg_{D_\ell} L}$$

arising from the inequalities of reduced Hilbert polynomials $p(F_1 \otimes M, t) \succ \cdots \succ p(F_\ell \otimes M, t)$ defined with respect to the ample invertible sheaf L. Moreover, the filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ depends only on the polarisation $\nu_{(L,M)}$ defined by (L, M). This leads to the following definition.

Definition 8.2.1. *Suppose $S = \operatorname{Spec} \mathbb{C}$ is a point, and let F be a torsion-free sheaf on C of uniform rank 1. Let $\nu = (\alpha, \phi)$ be a polarisation on C. The* ν-Harder–Narasimhan type *of F is given by the ordered tuple of pairs*

$$\tau_\nu(F) = (\chi(F_i) + \phi_{D_i}, \alpha_{D_i})_{i=1}^{\ell},$$

where, if L and M are any invertible sheaves on C with L ample and $\nu = \nu_{(L,M)}$, the sheaves F^i are those appearing in the Harder–Narasimhan filtration $0 = F^0 \otimes M \subset F^1 \otimes M \subset \cdots \subset F^\ell \otimes M = F \otimes M$ of $F \otimes M$ with respect to L, and the subcurve D_i is the support of F_i. We refer to the filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ as the ν-Harder–Narasimhan filtration *of F. The notion of a* relative ν-Harder–Narasimhan filtration *for families of sheaves on C is defined analogously.*

From now on, we take $S = \operatorname{Spec} \mathbb{C}$. Suppose $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ is the ν-Harder–Narasimhan filtration of a torsion-free uniform rank 1 sheaf F. Since Gieseker semistable sheaves are of pure dimension, each F_i is a non-zero torsion-free quotient of the subsheaf $F^{i-1} \subset F^i$, hence is of the form $F_i = (F^i)_{D_i}$ for some non-empty proper subcurve $D_i \subset \operatorname{supp}(F^i)$. Each quotient F/F^i is torsion-free, as seen by considering the short exact sequences

$$0 \longrightarrow F^j/F^i \longrightarrow F^{j+1}/F^i \longrightarrow F_j \longrightarrow 0$$

for $j = \ell - 1, \ldots, i+1, i$, and observing that the torsion subsheaf of F^{j+1}/F^i must be contained in the image of the inclusion $F^j/F^i \subset F^{j+1}/F^i$. It follows that $F^i = F^{\operatorname{supp}(F^i)^c} = \ker(F \twoheadrightarrow F_{\operatorname{supp}(F^i)^c})$, where $\operatorname{supp}(F^i)^c = \overline{C \setminus \operatorname{supp}(F^i)} = \operatorname{supp}(F/F^i)$.

We introduce a refinement of the notion of ν-Harder–Narasimhan types.

Definition 8.2.2. *Fix subcurves $D_1, \dots, D_\ell \subset C$ with the property that $C = \bigcup_{i=1}^{\ell} D_i$, and that if $i \neq j$ then D_i, D_j share no common irreducible components. Let $\tau = (\theta_i, \upsilon_i)_{i=1}^{\ell}$ be a Harder–Narasimhan type for torsion-free uniform rank* 1 *sheaves on C with respect to ν, and let τ_0 be the ordered tuple*

$$\tau_0 = (\theta_i, \upsilon_i, D_i)_{i=1}^{\ell}.$$

If F is a torsion-free sheaf on C of uniform rank 1*, we say that F has* (support-) refined ν-Harder–Narasimhan type τ_0 *if $\tau_\nu(F) = \tau$ and if* $\mathrm{supp}(F_i) = D_i$ *for all $i = 1, \dots, \ell$.*

Remark. The refinement of ν-Harder–Narasimhan types introduced in Definition 8.2.2 is different to the refinement of ν-Harder–Narasimhan types introduced by Hoskins–Jackson (cf. [66, Definition 6.1]). After first fixing (L, M) with $\nu = \nu_{(L,M)}$, this latter refinement involves fixing the quantities

$$\dim \mathrm{Hom}_C(F/F^i \otimes M, F^i \otimes M), \quad \dim \mathrm{Hom}_C(F/F^i \otimes M, F^{i-1} \otimes M)$$

for all i. However, all of these quantities are zero, a consequence of Lemma 5.1.8, so the Hoskins–Jackson refinement trivially refines an ν-Harder–Narasimhan type of unstable sheaves on the curve C. For the rest of this chapter, the notion of a *refined Harder–Narasimhan type* refers exclusively to Definition 8.2.2.

As a consequence of the following Lemma 8.2.3, there exists a decomposition

$$\overline{J}_C^{\tau} = \bigsqcup_{\tau_0'} \overline{J}_C^{\tau_0'}$$

of $\overline{J}_C^{\tau}$ into open and closed subschemes $\overline{J}_C^{\tau_0'}$, where the disjoint union is taken over all refined Harder–Narasimhan types τ_0' which refine τ.

Lemma 8.2.3. *Let B be a scheme, and let F be a B-flat family of torsion-free coherent sheaves on $C \times B$, where C is a connected, reduced, projective curve. Suppose C has irreducible components $C_1, \dots, C_k$. Let $r = (r_1, \dots, r_k) \in (\mathbb{Z}^{\geq 0})^k$ be a tuple of non-negative integers. Then there exists an open and closed subscheme $B_r \subset B$ such that a morphism $S \to B$ factors through B_r if and only if the multirank of F_S is equal to r over each point $s \in S$.*

Proof. Fix an ample invertible sheaf $\mathcal{O}_C(1)$ on C. Suppose $b_0 \in B$ is such that F_{b_0} has multirank r. Let T be an irreducible curve, and consider a morphism $(T, t_0) \to (B, b_0)$. For each $t \in T$, let $r_j(t)$ be the rank of F_t along C_j. Then $r_j(t_0) = r_j$, and since the F_t have the same Hilbert polynomial with respect to $\mathcal{O}_C(1)$, it follows from the Hilbert polynomial formula in [122, Septième Partie, Corollaire 7] that

$$t \mapsto \sum_{j=1}^{k} r_j(t) \deg_{C_j} \mathcal{O}_C(1)$$

is constant along T. But each map $t \mapsto r_j(t)$ is upper-semicontinuous and each quantity $\deg_{C_j} \mathcal{O}_C(1) > 0$, so $t \mapsto \sum_{j=1}^{k} r_j(t) \deg_{C_j} \mathcal{O}_C(1)$ must be locally constant on T. This proves the lemma. □

Fix a refined Harder–Narasimhan type $\tau_0 = (\theta_i, \upsilon_i, D_i)_{i=1}^{\ell}$ for torsion-free coherent sheaves of uniform rank 1 on C, defined with respect to the polarisation ν. The T-valued points of $\overline{J}_C^{\tau_0}$ are given by equivalence classes of T-flat, T-finitely presented coherent sheaves F on $C_T = C \times T$ whose fibres are simple torsion-free sheaves of uniform rank 1, for which F admits a relative ν-Harder–Narasimhan filtration

$$0 = F^0 \subset F^1 \subset \cdots \subset F^{\ell} = F$$

of relative Harder–Narasimhan type τ, with the additional property that for each geometric point $t \in T$, the sheaf $(F_i)_T = (F^i/F^{i-1})_T$ is of uniform rank 1 along the subcurve $D_i \subset C$; sheaves F and F' are declared equivalent if there exists an invertible sheaf N on T with $F' \cong F \otimes \mathrm{pr}_T^* N$. We say that the filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^{\ell} = F$ is the *relative refined ν-Harder–Narasimhan filtration* of the sheaf F.

Over $C \times \overline{J}_C^{\tau_0}$, there is a universal relative Harder–Narasimhan filtration

$$0 = \mathcal{V}^0 \subset \mathcal{V}^1 \subset \cdots \subset \mathcal{V}^{\ell}.$$

This filtration is unique up to simultaneously twisting each $\mathcal{V}^i$ by the pullback of an invertible sheaf $N \in \mathrm{Pic}(\overline{J}_C^{\tau_0 - sp})$.

For each i, let $\chi_i = \theta_i - \phi_{D_i}$, and let $\overline{J}_i = \overline{J}_{D_i}^{\chi_i, ss}(\nu_{D_i})$. There is a map

$$\mathrm{gr} : \overline{J}_C^{\tau_0} \dashrightarrow \prod_{i=1}^{\ell} \overline{J}_i,$$

which sends a sheaf F with ν-Harder–Narasimhan filtration $0 = F^0 \subset F^1 \cdots \subset \cdots \subset F^\ell = F$ to the tuple $(F_1, \dots, F_\ell)$ given by the Harder–Narasimhan subquotients; the domain of definition of gr is given by the locus where each sheaf F_i is (fibrewise) simple.

Motivated by [66, Definition 4.18], we define the following open subschemes of $\overline{J}_C^{\tau_0}$.

Definition 8.2.4. *A sheaf $F \in \overline{J}_C^{\tau_0}$ with Harder–Narasimhan filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ is said to be* τ_0-simple *if:*

1. *each sheaf F_i is simple (viewed as a sheaf on D_i); and*

2. *each inclusion $F^i \subset F^{i+1}$ is non-split.*

If in addition each $F_i \in \overline{J}_{D_i}^{\chi_i,s}(\nu_{D_i})$ is stable with respect to ν_{D_i}, we say that F is τ_0-stable. *We denote the open subscheme of $\overline{J}_C^{\tau_0}$ parametrising all τ_0-simple (resp. τ_0-stable) sheaves by $\overline{J}_C^{\tau_0-sp}$ (resp. $\overline{J}_C^{\tau_0-s}$).*

Remark. Suppose $F_i \in \overline{\mathbb{J}}_{D_i}$ $(i = 1, \dots, \ell)$ are simple torsion-free sheaves, and suppose we have for each $i = 2, \dots, \ell$ non-split short exact sequences

$$0 \longrightarrow F^{i-1} \longrightarrow F^i \longrightarrow F_i \longrightarrow 0$$

of coherent sheaves on C, where $F^1 = F_1$. By inductively applying Lemma 5.1.7, each of the sheaves F^i are simple. In particular, $F = F^\ell$ is a *simple* torsion-free sheaf on C of uniform rank 1.

8.3 Proof of the Main Result

We now turn to proving Theorem 8.1.1. Let $\tau_0 = (\theta_i, \upsilon_i, D_i)_{i=1}^\ell$ be a ν-refined Harder–Narasimhan type for unstable simple torsion-free coherent sheaves of uniform rank 1 on the reduced, connected, projective curve C. Assume that the subcurves D_i are all connected and that $D_i \cap D_{i+1} \neq \emptyset$ for all $i = 1, \dots, \ell - 1$ (otherwise the scheme $\overline{J}_C^{\tau_0-sp}$ is empty). Assume further that for all $i \neq j$, each point $p \in D_i \cap D_j$ is a locally planar singularity of C.

8.3.1 Universal Spaces of Extensions

We recall some results of Lange [80] that we require.

Let $f : X \to S$ be a flat projective morphism of Noetherian schemes, and let F_1, F_2 be coherent sheaves on X, flat over S.[2]

Definition 8.3.1. *The ith* relative Ext-sheaf *is the coherent sheaf on S defined by*

$$\mathcal{E}xt^i_f(F_2, F_1) := R^i(f_*\mathcal{H}om_X(F_2, -))(F_1).$$

By Lemma 4.1 of *loc. cit.*, if E is a locally free $\mathcal{O}_S$-module of finite rank, there are canonical isomorphisms

$$\mathcal{E}xt^i_f(F_2, F_1) \otimes E \xrightarrow{\simeq} \mathcal{E}xt^i_f(F_2, F_1 \otimes f^*E) \cong \mathcal{E}xt^i_f(F_2 \otimes f^*E^\vee, F_1).$$

Proposition 8.3.2 ([80], Section 4). *Assume $\mathcal{E}xt^0_f(F_2, F_1) = 0$ and $\mathcal{E}xt^1_f(F_2, F_1)$ commutes with base change (in particular, $\mathcal{E}xt^1_f(F_2, F_1)$ is locally free of finite rank). Let $\mathbb{P} := \mathbb{P}_S(\mathcal{E}xt^1_f(F_2, F_1))$, with projection $\pi : \mathbb{P} \to S$. Then:*

1. *Given a morphism $g : T \to S$, the set of T-valued points of $\mathbb{P}$ are given by equivalence classes of pairs $(L, 0 \to (F_1)_T \otimes f_T^*L \to E \to (F_2)_T \to 0)$, where L is an invertible sheaf on T and where $0 \to (F_1)_T \otimes f_T^*L \to E \to (F_2)_T \to 0$ is a nowhere-split short exact sequence of sheaves on $X_T = X \times_S T$, with $(L, 0 \to (F_1)_T \otimes f_T^*L \to E \to (F_2)_T \to 0)$ and $(L, 0 \to (F_1)_T \otimes f_T^*L' \to E' \to (F_2)_T \to 0)$ being declared equivalent if there are isomorphisms $\phi : L \xrightarrow{\simeq} L'$ and $\psi : E \xrightarrow{\simeq} E'$ which fit into the following diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (F_1)_T \otimes f_T^*L & \longrightarrow & E & \longrightarrow & (F_2)_T & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_{(F_1)_T} \otimes f_T^*\phi} & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \mathrm{id}_{(F_2)_T}} & & \\ 0 & \longrightarrow & (F_1)_T \otimes f_T^*L' & \longrightarrow & E' & \longrightarrow & (F_2)_T & \longrightarrow & 0 \end{array}$$

2. *There exists a universal extension*

$$0 \longrightarrow (F_1)_{\mathbb{P}} \otimes f_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(1) \longrightarrow \mathcal{U} \longrightarrow (F_2)_{\mathbb{P}} \longrightarrow 0$$

on $X_{\mathbb{P}}$, which is nowhere-split.

[2]By invoking the base change results of [9, Section 1] in place of the base change results of [80], the results of this subsection remain true in the case where f is a proper flat finitely presented morphism and the sheaves F_1, F_2 are S-flat and locally finitely presented.

The universal extension in Proposition 8.3.2 corresponds to the tautological quotient $\pi^*\mathcal{E}xt^1_f(F_2,F_1)^\vee \to \mathcal{O}_{\mathbb{P}}(1) \to 0$ over $\mathbb{P}$ under the identifications

$$\begin{aligned}\mathrm{Hom}_{\mathbb{P}}(\pi^*\mathcal{E}xt^1_f(F_2,F_1)^\vee,\mathcal{O}_{\mathbb{P}}(1)) &= H^0(\mathbb{P},\pi^*\mathcal{E}xt^1_f(F_2,F_1)\otimes\mathcal{O}_{\mathbb{P}}(1))\\ &= H^0(\mathbb{P},\mathcal{E}xt^1_{f_{\mathbb{P}}}((F_2)_{\mathbb{P}},(F_1)_{\mathbb{P}})\otimes\mathcal{O}_{\mathbb{P}}(1))\\ &= \mathrm{Ext}^1_{X_{\mathbb{P}}}((F_2)_{\mathbb{P}},(F_1)_{\mathbb{P}}\otimes f_{\mathbb{P}}^*\mathcal{O}_{\mathbb{P}}(1)),\end{aligned}$$

where the final equality follows from the paragraph succeeding Corollary 4.4 in *loc. cit.*

The compatibility of $\mathcal{E}xt^1_f(F_2,F_1)$ with base change is guaranteed by the fibrewise vanishing of Hom and Ext^2.

Lemma 8.3.3. *Let F,G be S-flat coherent sheaves on X, and suppose that for each geometric point $s\in S$, $\mathrm{Hom}_{X_s}(F_s,G_s)=\mathrm{Ext}^2_{X_s}(F_s,G_s)=0$. Then $\mathcal{E}xt^0_f(F,G)=\mathcal{E}xt^2_f(F,G)=0$ and $\mathcal{E}xt^1_f(F,G)$ is locally free and commutes with base change.*

Proof. This follows from a straightforward application of Theorem 1.4 of *loc. cit.* For each point $s\in S$ and for $i=0,2$, the base change morphism $\tau^i(s):\mathcal{E}xt^i_f(F,G)\otimes_S\kappa(s)\to\mathrm{Ext}^i_{X_s}(F_s,G_s)=0$ is trivially surjective, so $\mathcal{E}xt^i_f(F,G)\otimes_S\kappa(s)=0$ for all points $s\in S$, which implies by Nakayama's lemma that $\mathcal{E}xt^i_f(F,G)$ vanishes. In particular $\mathcal{E}xt^2_f(F,G)$ is locally free, so we may apply Theorem 1.4 of *loc. cit.* again to show that $\tau^1(s)$ is surjective for each point $s\in S$, and in turn to obtain that $\mathcal{E}xt^1_f(F,G)$ is locally free. □

Suppose that $\mathcal{E}xt^0_f(F_2,F_1)=0$ and $\mathcal{E}xt^1_f(F_2,F_1)$ is compatible with base change, so that we are in the setting of Proposition 8.3.2. Suppose as well that we are given a morphism $g:T\to S$ and nowhere-split short exact sequences

$$0\longrightarrow (F_1)_T\otimes f_T^*L\xrightarrow{\ \gamma\ } E\xrightarrow{\ \delta\ }(F_2)_T\longrightarrow 0,$$

$$0\longrightarrow (F_1)_T\otimes f_T^*L'\xrightarrow{\ \gamma'\ } E'\xrightarrow{\ \delta'\ }(F_2)_T\longrightarrow 0$$

of coherent sheaves on X_T.

Lemma 8.3.4. *In the situation above, assume in addition that the sheaves F_1 and F_2 are fibrewise simple*[3] *and that for each geometric point $s\in S$,*

$$\mathrm{Hom}_{X_s}((F_1)_s,(F_2)_s)=0.$$

[3]This is equivalent, by [9, Corollary 5.3], to the condition that for each $i=1,2$ the natural morphism $\mathcal{O}_S\to\mathcal{E}xt^0_f(F_i,F_i)=f_*\mathcal{H}om_X(F_i,F_i)$ is an isomorphism and is an isomorphism after any base change $T'\to S$.

Then the sheaves E and E' are isomorphic if and only if the pairs $(L, 0 \to (F_1)_T \otimes f_T^* L \to E \to (F_2)_T \to 0)$ *and* $(L', 0 \to (F_1)_T \otimes f_T^* L' \to E' \to (F_2)_T \to 0)$ *are equivalent (and hence correspond to the same T-valued point of* $\mathbb{P}_S(\mathcal{E}xt_f^1(F_2, F_1))$*).*

Proof. For any invertible sheaf $M \in \operatorname{Pic}(T)$ we have $\operatorname{Hom}_{X_T}((F_1)_T \otimes f_T^* M, (F_2)_T) = 0$, since this is the space of global sections of the sheaf $\mathcal{E}xt^0_{f_T}((F_1)_T, (F_2)_T) \otimes M^\vee$, and $\mathcal{E}xt^0_{f_T}((F_1)_T, (F_2)_T) = 0$ by Theorem 1.4 of *loc. cit.* together with the fibrewise vanishing $\operatorname{Hom}_{X_s}((F_1)_s, (F_2)_s) = 0$.

In particular, given an isomorphism $\psi : E \to E'$ of coherent sheaves, the compositions $\delta' \circ \psi \circ \gamma$ and $\delta \circ \psi^{-1} \circ \gamma'$ are both zero; it follows from this and the universal properties of (co)kernels that there are isomorphisms $\zeta : (F_1)_T \otimes f_T^* L \to (F_1)_T \otimes f_T^* L'$ and $\eta : (F_2)_T \to (F_2)_T$ which fit into the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (F_1)_T \otimes f_T^* L & \xrightarrow{\gamma} & E & \xrightarrow{\delta} & (F_2)_T & \longrightarrow & 0 \\ & & \zeta \downarrow & & \downarrow \psi & & \downarrow \eta & & \\ 0 & \longrightarrow & (F_1)_T \otimes f_T^* L' & \xrightarrow{\gamma'} & E' & \xrightarrow{\delta'} & (F_2)_T & \longrightarrow & 0. \end{array}$$

As F_2 is fibrewise simple, the isomorphism η is given by multiplication by a unit $\eta \in H^0(T, \mathcal{O}_T^\times)$; without loss of generality, we may assume $\eta = 1$. On the other hand, the fibrewise simplicity of F_1 together with Theorem 1.4 of *loc. cit.* implies that

$$\mathcal{E}xt^0_{f_T}((F_1)_T, (F_1)_T) = g^* \mathcal{E}xt^0_f(F_1, F_1) = \mathcal{O}_T.$$

As such, we have equalities

$$\operatorname{Hom}_{X_T}((F_1)_T \otimes f_T^* L, (F_1)_T \otimes f_T^* L') = H^0(T, L' \otimes L^\vee) = \operatorname{Hom}_T(L, L').$$

In particular, the invertible sheaves L and L' are isomorphic, and we may take ζ to be of the form $\operatorname{id}_{(F_1)_T} \otimes f_T^* \zeta'$, where $\zeta' : L \xrightarrow{\simeq} L'$ is an isomorphism. This shows that the pairs $(L, 0 \to (F_1)_T \otimes f_T^* L \to E \to (F_2)_T \to 0)$ and $(L', 0 \to (F_1)_T \otimes f_T^* L' \to E' \to (F_2)_T \to 0)$ are equivalent. □

Corollary 8.3.5. *In the situation of Lemma 8.3.4, the set of T-valued points of the projective bundle* $\mathbb{P}_S(\mathcal{E}xt_f^1(F_2, F_1))$ *is given by the set of isomorphism classes of coherent sheaves* $E \in \mathbf{Coh}(X_T)$ *for which there exists an invertible sheaf* $L \in \operatorname{Pic}(T)$ *and a nowhere-split short exact sequence*

$$0 \longrightarrow (F_1)_T \otimes f_T^* L \longrightarrow E \longrightarrow (F_2)_T \longrightarrow 0.$$

Proof. This immediately follows from Proposition 8.3.2 and Lemma 8.3.4. □

8.3.2 Constructing the Tower of Projective Bundles

We now construct the tower of projective bundles $P^\ell \to P^{\ell-1} \to \cdots \to P^2 \to P^1$ appearing in the statement of Theorem 8.1.1. For each $i = 1, \ldots, \ell$ let $\overline{J}_i = \overline{J}^{\chi_i, ss}_{D_i}(\nu_{D_i})$ be the fine compactified Jacobian parametrising ν_{D_i}-semistable coherent sheaves on D_i of Euler characteristic $\chi_i = \theta_i - \phi_{D_i}$. Fix a choice of universal sheaf $\mathcal{U}_i$ on $D_i \times \overline{J}_i$. Via the inclusion $D_i \subset C$, we may regard $\mathcal{U}_i$ as a flat family of simple torsion-free sheaves on C, of uniform rank 1 along the (fibrewise) supports D_i.

The projective bundles P^i are constructed inductively. For each $i = 1, \ldots, \ell$, P^i will be a product of the form $P^i = Q^i \times \prod_{j=i+1}^{\ell} \overline{J}_i$, with $Q^1 := \overline{J}_1$, and for each $i > 1$, Q^i will be a Zariski-locally trivial projective bundle over $Q^{i-1} \times \overline{J}_i$; the morphism $\pi_i : P^i \to P^{i-1}$ will be given as the product of the projective bundle $Q^i \to Q^{i-1} \times \overline{J}_i$ with the identity on the remaining factors. The schemes Q^i are constructed as follows.

Construction 8.3.6. *Base step:* Set $Q^1 := \overline{J}_1$, and set $\mathcal{U}^1 := \mathcal{U}_1$, the chosen universal sheaf on $C \times Q^1 = C \times \overline{J}_1$.

Induction step: Suppose we have constructed for each $i = 1, \ldots, k$ (where $1 \leq k < \ell$) the Zariski-locally trivial bundle $Q^i \to Q^{i-1} \times \overline{J}_i$ (where $Q^0 := \operatorname{Spec} \mathbb{C}$), together with a Q^i-flat coherent sheaf $\mathcal{U}^i$ on $C \times Q^i$, which along each geometric fibre is simple, torsion-free and of uniform rank 1 along the supporting subcurve $D^i := \bigcup_{j \leq i} D_j$. Consider the diagram

$$\begin{array}{ccccc} & & C \times Q^i \times \overline{J}_{i+1} & \xrightarrow{\quad p_{23} \quad} & Q^i \times \overline{J}_{i+1} \\ & \swarrow^{p_{12}} & & \searrow^{p_{13}} & \\ C \times Q^i & & & & C \times \overline{J}_{i+1} \end{array}$$

Since D^i and D_{i+1} share no common irreducible components and since each $p \in D^i \cap D_{i+1}$ is a locally planar singularity of C, for each geometric point $s \in Q^i \times \overline{J}_{i+1}$ we have

$$\operatorname{Hom}_C((p_{13}^*\mathcal{U}_{i+1})_s, (p_{12}^*\mathcal{U}^i)_s) = 0 = \operatorname{Ext}^2_C((p_{13}^*\mathcal{U}_{i+1})_s, (p_{12}^*\mathcal{U}^i)_s)$$

by Lemmas 5.1.8 and 5.1.9. As such, we may apply Lemma 8.3.3 to obtain that the sheaf $\mathcal{E}xt^0_{p_{23}}(p_{13}^*\mathcal{U}_{i+1}, p_{12}^*\mathcal{U}^i) = 0$ and that the sheaf $\mathcal{E}xt^1_{p_{23}}(p_{13}^*\mathcal{U}_{i+1}, p_{12}^*\mathcal{U}^i)$ is locally free and commutes with base change. Set

$$Q^{i+1} := \mathbb{P}_{Q^i \times \overline{J}_{i+1}}(\mathcal{E}xt^1_{p_{23}}(p_{13}^*\mathcal{U}_{i+1}, p_{12}^*\mathcal{U}^i)).$$

Consider the following Cartesian diagram:

$$\begin{array}{ccc} C \times Q^{i+1} & \xrightarrow{\quad p'_{23} \quad} & Q^{i+1} \\ {\scriptstyle \rho'_{i+1}} \downarrow & \lrcorner & \downarrow {\scriptstyle \rho_{i+1}} \\ C \times Q^i \times \overline{J}_{i+1} & \xrightarrow[\quad p_{23} \quad]{} & Q^i \times \overline{J}_{i+1} \end{array}$$

By Proposition 8.3.2 there is a nowhere-split universal extension

$$0 \longrightarrow (\rho'_{i+1})^* p_{12}^* \mathcal{U}^i \otimes (p'_{23})^* \mathcal{O}_{Q^{i+1}}(1) \longrightarrow \mathcal{U}^{i+1} \longrightarrow (\rho'_{i+1})^* p_{13}^* \mathcal{U}_{i+1} \longrightarrow 0$$

of coherent sheaves on $C \times Q^{i+1}$. Take $\mathcal{U}^{i+1}$ to be the middle term of this extension. The sheaf $\mathcal{U}^{i+1}$ is flat over Q^{i+1}, and along each geometric fibre is torsion-free and of uniform rank 1 along the supporting subcurve D^{i+1}. That $\mathcal{U}^{i+1}$ is fibrewise simple follows from Lemma 5.1.7. This completes the induction step of Construction 8.3.6. ■

It also follows from Lemma 5.1.8 that for all i, we have $\mathrm{Hom}_C((p_{12}^* \mathcal{U}^i)_s, (p_{13}^* \mathcal{U}_{i+1})_s) = 0$ for each geometric point $s \in Q^i \times \overline{J}_{i+1}$, and so Corollary 8.3.5 is applicable to the scheme Q^{i+1}. In particular, the set of T-valued points of Q^{i+1} is given by the set of all isomorphism classes of T-flat coherent sheaves $F \in \mathbf{Coh}(C_T)$ such that, if $\mathcal{U}_T^i$ (resp. $(\mathcal{U}_{i+1})_T$) denotes the pullback of the sheaf $\mathcal{U}_i$ (resp. $\mathcal{U}_{i+1}$) to C_T, then *there exists* a nowhere-split short exact sequence

$$0 \longrightarrow \mathcal{U}_T^i \otimes \mathrm{pr}_T^* M \longrightarrow F \longrightarrow (\mathcal{U}_{i+1})_T \longrightarrow 0$$

for some invertible sheaf $M \in \mathrm{Pic}(T)$. Each such sheaf F is a family of fibrewise simple, torsion-free sheaves of uniform rank 1 supported along the subcurve $D^{i+1} \subset C$.

8.3.3 The Isomorphism $\overline{J}_C^{\tau_0 - sp} \cong P^\ell$

We are now in a position to complete the proof of the main result of this chapter.

Proof of Theorem 8.1.1. We first exhibit a well-defined morphism $\Phi : \overline{J}_C^{\tau_0 - sp} \to P^\ell$ of schemes over $P^1 = \prod_{i=1}^{\ell} \overline{J}_i$.

Let T be a scheme, and consider a T-valued point of $\overline{J}_C^{\tau_0 - sp}$ represented by a coherent sheaf F with relative refined ν-Harder–Narasimhan filtration

$$0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F.$$

From the universal properties of the fine compactified Jacobians $\overline{J}_i = \overline{J}_{D_i}^{\chi_i,ss}(\nu_{D_i})$, there are unique morphisms $T \to \overline{J}_i$ such that, if $(\mathcal{U}_i)_T$ denotes the pullback of $\mathcal{U}_i$ along this morphism, viewed as a coherent sheaf on C_T with fibrewise support $D_i \subset C$, then there are invertible sheaves $M_1, \ldots, M_\ell \in \mathrm{Pic}(T)$ and isomorphisms

$$F_i \cong (\mathcal{U}_i)_T \otimes \mathrm{pr}_T^* M_i$$

of coherent sheaves on C_T; in this case, the morphism $T \to \overline{J}_i$ coincides with the composition $T \to \overline{J}_C^{\tau_0 - sp} \to \prod_{j=1}^{\ell} \overline{J}_j \to \overline{J}_i$. Since the F_i are all (fibrewise) simple, the isomorphism class of the invertible sheaf M_i is uniquely determined by the isomorphism class of F_i (cf. proof of Lemma 8.3.4). However, any other representative of the given T-point is isomorphic to $F \otimes \mathrm{pr}_T^* N$ for some $N \in \mathrm{Pic}(T)$, and so the $M_i \in \mathrm{Pic}(T)$ themselves are only uniquely determined up to simultaneous translation by N. In order to have a distinguished representative of the given T-valued point of $\overline{J}_C^{\tau_0 - sp}$, we impose the condition that $M_\ell \cong \mathcal{O}_T$ is trivial, so that the final subquotient F_ℓ is isomorphic to the pullback along T of the universal sheaf $\mathcal{U}_\ell$.

From the inclusion $F^1 \subset F^2$ we have a nowhere-split short exact sequence

$$0 \longrightarrow (\mathcal{U}_1)_T \otimes \mathrm{pr}_T^*(M_1 \otimes M_2^\vee) \longrightarrow F^2 \otimes \mathrm{pr}_T^* M_2^\vee \longrightarrow (\mathcal{U}_2)_T \longrightarrow 0,$$

from which we obtain a morphism $T \to Q^2$ over $T \to \overline{J}_1 \times \overline{J}_2$, with the pullback $\mathcal{U}_T^2$ of $\mathcal{U}^2$ along $T \to Q^2$ coinciding with $F^2 \otimes \mathrm{pr}_T^* M_2^\vee$ (up to isomorphism), and the pullback of $\mathcal{O}_{Q^2}(1)$ coinciding with $M_1 \otimes M_2^\vee$.

For each $i \geq 2$, we have a nowhere-split short exact sequence

$$0 \longrightarrow F^i \otimes \mathrm{pr}_T^* M_{i+1}^\vee \longrightarrow F^{i+1} \otimes \mathrm{pr}_T^* M_{i+1}^\vee \longrightarrow (\mathcal{U}_{i+1})_T \longrightarrow 0.$$

By induction, there is a morphism $T \to Q^i$ for which $F^i \cong \mathcal{U}_T^i \otimes \mathrm{pr}_T^* M_i$. As such, the above sequence yields a morphism $T \to Q^{i+1}$, compatible with the morphism $T \to Q^i \times \overline{J}_{i+1}$, for which the pullback $\mathcal{U}_T^{i+1}$ of $\mathcal{U}^{i+1}$ along $T \to Q^{i+1}$ is isomorphic to $F^{i+1} \otimes \mathrm{pr}_T^* M_{i+1}^\vee$, and the pullback of $\mathcal{O}_{Q^{i+1}}(1)$ along $T \to Q^{i+1}$ is isomorphic to $M_i \otimes M_{i+1}^\vee$.

After $\ell - 1$ stages, we obtain a well-defined morphism $T \to Q^\ell = P^\ell$ for which the composition $T \to P^\ell \to P^1 = \prod_{i=1}^{\ell} \overline{J}_i$ coincides with the composition $T \to \overline{J}_C^{\tau_0 - sp} \to \prod_{i=1}^{\ell} \overline{J}_i$. In particular, we may take $T = \overline{J}_C^{\tau_0 - sp}$; the identity morphism $T \to \overline{J}_C^{\tau_0 - sp}$ corresponds to any choice of universal relative Harder–Narasimhan filtration

$$0 = \mathcal{V}^0 \subset \mathcal{V}^1 \subset \cdots \subset \mathcal{V}^\ell$$

over $C \times \overline{J}_C^{\tau_0-sp}$. Without loss of generality, we may assume that the universal filtration has been chosen such that the sheaf $\mathcal{V}_\ell = \mathcal{V}^\ell/\mathcal{V}^{\ell-1}$ is isomorphic to the pullback $(\mathcal{U}_\ell)_{\overline{J}_C^{\tau_0-sp}}$ of the universal sheaf $\mathcal{U}_\ell$. From this filtration we obtain our desired morphism of P^1-schemes $\Phi : \overline{J}_C^{\tau_0-sp} \to P^\ell$.

Next, we exhibit a well-defined morphism $\Psi : P^\ell \to \overline{J}_C^{\tau_0-sp}$ of schemes over P^1. By an abuse of notation, we identify the universal sheaves $\mathcal{U}^i$ and $\mathcal{U}_i$ with their pullbacks to $C \times P^\ell$. Writing $\mathcal{U}^1 = \mathcal{U}_1$, and denoting by L_i the pullback of $\mathcal{O}_{Q^{i+1}}(1)$ to P^ℓ, we have over $C \times P^\ell$ nowhere-split short exact sequences

$$0 \longrightarrow \mathcal{U}^i \otimes \mathrm{pr}_{P^\ell}^* L_i \longrightarrow \mathcal{U}^{i+1} \longrightarrow \mathcal{U}_{i+1} \longrightarrow 0,$$

and hence a filtration

$$0 = \mathcal{W}^0 \subset \mathcal{W}^1 \subset \cdots \subset \mathcal{W}^\ell, \quad \mathcal{W}^i := \mathcal{U}^i \otimes \mathrm{pr}_{P^\ell}^* \left(\bigotimes_{j=i}^{\ell-1} L_j \right),$$

whose inclusions are nowhere-split, with the subquotients given by $\mathcal{W}_i = \mathcal{W}^i/\mathcal{W}^{i-1} = \mathcal{U}_i \otimes \mathrm{pr}_{P^\ell}^* \left(\bigotimes_{j=i}^{\ell-1} L_j \right)$. The sheaf $\mathcal{W}_i$ corresponds to a flat, finitely presented family of simple ν-semistable torsion-free coherent sheaves on C, of uniform rank 1 along the supporting subcurve $D_i \subset C$; the filtration $0 \subset \mathcal{W}^1 \subset \cdots \subset \mathcal{W}^\ell$ is a relative refined Harder–Narasimhan filtration of refined Harder–Narasimhan type τ_0. As such, this filtration gives rise to a morphism $\Psi : P^\ell \to \overline{J}_C^{\tau_0-sp}$ of schemes over P^1.

We claim that the morphisms $\Phi : \overline{J}_C^{\tau_0-sp} \to P^\ell$ and $\Psi : P^\ell \to \overline{J}_C^{\tau_0-sp}$ are inverse. We have uniquely determined classes $M_i \in \mathrm{Pic}(\overline{J}_C^{\tau_0-sp})$ (with $M_\ell = \mathcal{O}_{\overline{J}_C^{\tau_0-sp}}$) for which $\mathcal{V}_i = \mathcal{V}^i/\mathcal{V}^{i-1}$ is isomorphic to the pullback of $\mathcal{U}_i$ twisted by $\mathrm{pr}^*_{\overline{J}_C^{\tau_0-sp}}(M_i)$, for each $i = 1, \dots, \ell$. From the construction of Φ and Ψ, there are isomorphisms

$$\mathcal{V}^i \cong (\mathrm{id}_C \times \Phi)^* \mathcal{W}^i, \quad \mathcal{V}_i \cong (\mathrm{id}_C \times \Phi)^* \mathcal{W}_i, \quad M_i \cong \Phi^* \left(\bigotimes_{j=i}^{\ell-1} L_j \right) \quad \text{if } i \leq \ell - 1,$$

and

$$\mathcal{W}^i \cong (\mathrm{id}_C \times \Psi)^* \mathcal{V}^i, \quad \mathcal{W}_i \cong (\mathrm{id}_C \times \Psi)^* \mathcal{V}_i, \quad L_i \cong \begin{cases} \Psi^*(M_i \otimes M_{i+1}^\vee) & \text{if } i < \ell - 1, \\ \Psi^* M_{\ell-1} & \text{if } i = \ell - 1, \end{cases}$$

In particular, both compositions $\Phi \circ \Psi$ and $\Psi \circ \Phi$ preserve the respective universal objects parametrised by P^ℓ and $\overline{J}_C^{\tau_0-sp}$, whence both compositions must be the identity. This completes the proof. □

Remark. It is possible to give another proof that the scheme $\overline{J}_C^{\tau_0-s}(\nu)$ is quasi-projective, by making use of non-reductive quotienting in stages. Fix (L,M) with $\nu = \nu_{(L,M)}$. Without loss of generality we may assume $M = \mathcal{O}_C$, since twisting by M yields an isomorphism $\overline{J}_C^{\tau_0-s}(L,M) \cong \overline{J}_C^{\tau_0-s}(L,\mathcal{O}_C)$. Recall that if $\mathcal{C}oh_C^{\mathrm{simp,TFR1}}$ is the open substack of the stack of coherent sheaves $\mathcal{C}oh_C$ parametrising all flat families of simple torsion-free coherent sheaves on C of uniform rank 1, then there is an isomorphism

$$\overline{\mathbb{J}}_C \cong \mathcal{C}oh_C^{\mathrm{simp,TFR1}} \mathbin{/\!\!/} \mathbb{G}_m.$$

In particular, for sufficiently large integers $M' \gg N \gg 0$ as in the statement of Proposition 4.3.6 applied to the polarised scheme (C,L), there is an isomorphism of algebraic stacks

$$\overline{J}_C^{\tau_0-s}(L,\mathcal{O}_C) \cong [S_N^{\tau_0-s}/GL(V_N)] \mathbin{/\!\!/} \mathbb{G}_m = [S_N^{\tau_0-s}/PGL(V_N)],$$

where $S_N^{\tau_0-s}$ is the open subscheme of the Harder–Narasimhan stratum S_N^{τ} parametrising those quotients whose underlying sheaf is τ_0-stable. With the groups $P_\tau \subset SL(V_N)$ and $\tilde{P}_\tau \subset GL(V_N)$ as in the statement of Proposition 4.3.6, there is also an equivariant isomorphism $S_N^{\tau_0-s} \cong SL(V_N) \times^{P_\tau} Y^{\tau_0-s}$, where $Y^{\tau_0-s} \subset Y_\tau^{ss}$ consists of those quotients whose underlying sheaf is τ_0-stable, and an isomorphism of quotient stacks $[S_N^{\tau_0-s}/GL(V_N)] \cong [Y^{\tau_0-s}/\tilde{P}_\tau]$. Consequently, there is an isomorphism of stacks

$$\overline{J}_C^{\tau_0-s} \cong [Y^{\tau_0-s}/\tilde{P}_\tau] \mathbin{/\!\!/} \mathbb{G}_m = [Y^{\tau_0-s}/\overline{P}_\tau],$$

where $\overline{P}_\tau$ is the common image of P_τ and $\tilde{P}_\tau$ in $PGL(V_N)$. In particular, the natural morphism $Y^{\tau_0-s} \to \overline{J}_C^{\tau_0-s}$ is a geometric P_τ-quotient.

The quasi-projectivity of $\overline{J}_C^{\tau_0-s}$ can be shown by applying NRGIT to the P_τ-action on the closure of Y^{τ_0-s}, by utilising non-reductive quotienting-in-stages, making use of the results of [66, Section 4.2.4];[4] the key observation is that taking the Harder–Narasimhan associated graded preserves the Hoskins–Jackson refined Harder–Narasimhan type (cf. remark after Definition 8.2.2). Unlike with Theorem 8.1.1, NRGIT says little about the geometry of $\overline{J}_C^{\tau_0-s}$ aside from its quasi-projectivity. As such, we elect to not carry out the NRGIT construction here.

[4] NRGIT quotienting-in-stages also yields a proof, independent of the results of Section 5.3, that $\overline{J}_C^{\tau_0-s}$ is a scheme.

8.4 Nodal Refined Harder–Narasimhan Types

In the case where any two of the subcurves D_i, D_j meet only at nodes of C, the relative dimension of $\overline{J}_C^{\tau_0-sp} \to \prod_{i=1}^{\ell} \overline{J}_i$ can be expressed solely in terms of the Harder–Narasimhan length ℓ and the point counts $|D^i \cap D^{i-1}|$.

Definition 8.4.1. *Let C be a reduced connected curve, and let $\tau_0 = (\theta_i, \upsilon_i, D_i)_{i=1}^{\ell}$ be a refined Harder–Narasimhan type for simple torsion-free sheaves on C of Euler characteristic χ and uniform rank 1, with the property that each subcurve D_i is connected and that the intersections $D_i \cap D_{i+1} \neq \emptyset$ for all $i = 1, \ldots, \ell - 1$. We say that τ_0 is a* nodal refined Harder–Narasimhan type *if in addition that for all $i \neq j$, any point $p \in D_i \cap D_j$ is a nodal singularity of C.*

Fix a nodal refined Harder–Narasimhan type τ_0. Let $P^{\ell} \to P^{\ell-1} \to \cdots \to P^2 \to P^1 = \prod_{i=1}^{\ell} \overline{J}_i$ be the tower of projective bundles obtained by applying Construction 8.3.6 with the refined Harder–Narasimhan type τ_0.

Lemma 8.4.2. *Let $p \in C$ be a nodal singularity, with branches D and D'. Then*

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{O}_{C,p}}(\mathcal{O}_{D,p}, \mathcal{O}_{D',p}) = 1.$$

Remark. It follows from Lemma 8.4.2 that if

$$0 \longrightarrow \mathcal{O}_{D',p} \longrightarrow M \longrightarrow \mathcal{O}_{D,p} \longrightarrow 0$$

is any short exact sequence of $\mathcal{O}_{C,p}$-modules, then $M \cong \mathcal{O}_{D,p} \oplus \mathcal{O}_{D',p}$, corresponding to $0 \in \operatorname{Ext}^1_{\mathcal{O}_{C,p}}(\mathcal{O}_{D,p}, \mathcal{O}_{D',p})$, or $M \cong \mathcal{O}_{C,p}$, corresponding to the span of $1 \in \operatorname{Ext}^1_{\mathcal{O}_{C,p}}(\mathcal{O}_{D,p}, \mathcal{O}_{D',p}) \cong \mathbb{C}$.

Proof of Lemma 8.4.2. Since Ext^1 is finite dimensional over $\mathbb{C}$, by passing to the completion $\widehat{\mathcal{O}}_{C,p}$ of $\mathcal{O}_{C,p}$ with respect to the maximal ideal there is an equality

$$\operatorname{Ext}^1_{\mathcal{O}_{C,p}}(\mathcal{O}_{D,p}, \mathcal{O}_{D',p}) = \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C,p}}(\widehat{\mathcal{O}}_{D,p}, \widehat{\mathcal{O}}_{D',p}).$$

As $p \in C$ is a nodal singularity, there is an isomorphism $\widehat{\mathcal{O}}_{C,p} \cong \mathbb{C}[[x,y]]/(xy)$; with respect to this identification, there are isomorphisms $\widehat{\mathcal{O}}_{D,p} \cong \mathbb{C}[[x]]$ and $\widehat{\mathcal{O}}_{D',p} \cong \mathbb{C}[[y]]$ of $\mathbb{C}[[x,y]]/(xy)$-modules. A projective resolution of $\mathbb{C}[[x]]$ as an $A := \mathbb{C}[[x,y]]/(xy)$-module is given by

$$\cdots \longrightarrow A \xrightarrow{x} A \xrightarrow{y} A \xrightarrow{x} A \xrightarrow{y} A \longrightarrow \mathbb{C}[[x]] \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_A(-, \mathbb{C}[[y]])$ and taking cohomology yields

$$\mathrm{Ext}^1_A(\mathbb{C}[[x]], \mathbb{C}[[y]]) = \mathbb{C}[[y]]/y\mathbb{C}[[y]] = \mathbb{C}.$$

This proves the lemma. □

Corollary 8.4.3. *Let D, D' be subcurves of C with $D \cap D' \neq \emptyset$, which do not share in common any irreducible component of C. Suppose in addition that each point $p \in D \cap D'$ is a node of C. Let F, F' be torsion-free rank 1 sheaves supported on D, D' respectively. Then*

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_C(F, F') = |D \cap D'|.$$

Proof. As in the proof of Lemma 5.1.9, consider the local-to-global Ext spectral sequence

$$E_2^{p,q} = H^p(C, \mathcal{E}xt^q_C(F, F')) \Rightarrow \mathrm{Ext}^{p+q}_C(F, F').$$

We have $E_2^{1,0} = E_2^{2,0} = 0$, so $\mathrm{Ext}^1_C(F, F') = H^0(C, \mathcal{E}xt^1_C(F, F'))$. As the sheaf $\mathcal{E}xt^1_C(F, F')$ is supported at the nodes $p \in D \cap D'$, there is an equality

$$H^0(C, \mathcal{E}xt^1_C(F, F')) = \bigoplus_{p \in D \cap D'} \mathrm{Ext}^1_{\mathcal{O}_{C,p}}(F_p, F'_p).$$

At the node p, since D and D' are both non-singular at p there are isomorphisms $F_p \cong \mathcal{O}_{D,p}$ and $F'_p \cong \mathcal{O}_{D',p}$ (cf. Proposition 5.1.10). The corollary now follows from Lemma 8.4.2. □

Corollary 8.4.4. *Let $\tau_0 = (\theta_i, \upsilon_i, D_i)_{i=1}^{\ell}$ be a nodal refined Harder–Narasimhan type of C. The fibre dimension of the locally trivial projective bundle $\pi_k : P^k \to P^{k-1}$ is equal to $|D_k \cap D^{k-1}| - 1$, where $D^{k-1} = \bigcup_{j \leq k-1} D_j$. In particular, the relative dimension of the morphism $\mathrm{gr} : \overline{J}_C^{\tau_0 - sp} \to \prod_{i=1}^{\ell} \overline{J}_i$ is equal to $\sum_{i=2}^{\ell} |D^i \cap D^{i-1}| - (\ell - 1)$.*

Proof. From Construction 8.3.6, the relative dimension of π_k is equal to the fibre dimension of the locally trivial projective bundle

$$Q^k = \mathbb{P}_{Q^{k-1} \times \overline{J}_k}(\mathcal{E}xt^1_{p_{23}}(p_{13}^* \mathcal{U}_k, p_{12}^* \mathcal{U}^{k-1})) \to Q^{k-1} \times \overline{J}_k.$$

Fix a geometric point $s \in Q^{k-1} \times \overline{J}_k$, and set $F = (p_{13}^* \mathcal{U}_k)_s$, $F' = (p_{12}^* \mathcal{U}^{k-1})_s$; the fibre of Q^k over s is given by the projective space $Q^k_s = \mathbb{P}(\mathrm{Ext}^1_C(F, F'))$. Applying Corollary 8.4.3, we have $\dim_{\mathbb{C}} \mathrm{Ext}^1_C(F, F') = |D_k \cap D^{k-1}|$. The result follows. □

8.5 Generalising to Families of Curves

Instead of working with a fixed curve C, consider instead a flat projective morphism $f : C \to S$ of finite presentation over a locally Noetherian base scheme S, whose geometric fibres are connected, reduced curves. Fix invertible sheaves L and M on C, with L relatively ample, fix an Euler characteristic χ, and fix a Harder–Narasimhan stratum $\overline{J}^{\tau}_{C/S} = \overline{J}^{\tau}_{C/S}(L, M) \subset \overline{\mathbb{J}}^{\chi}_{C/S}$, where $\tau = (\theta_i, \upsilon_i)_{i=1}^{\ell}$ is a Harder–Narasimhan type of length ℓ. Assume there are closed subschemes $D_1, \ldots, D_\ell \subset C$, each flat over S, such that for each geometric point $s \in S$:

(i) $(D_i)_s$ is a connected subcurve of C_s;

(ii) for all $i \neq j$, $(D_i)_s$ and $(D_j)_s$ contain no common irreducible component of C_s;

(iii) for all $i \neq j$, any point $p \in (D_i)_s \cap (D_j)_s$ is a locally planar singularity of C_s; and

(iv) for all $i = 1, \ldots, \ell - 1$, $(D_i)_s \cap (D_{i+1})_s \neq \emptyset$.

Fix a choice of a sequence $D_1, \ldots, D_\ell$ of such subschemes of C.

Lemma 8.5.1. *There exists an open and closed sub-algebraic space $\overline{J}^{\tau_0}_{C/S} \subset \overline{J}^{\tau}_{C/S}$ parametrising all sheaves $F \in \overline{J}^{\tau}_{C/S}$ such that, if $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ is the relative Harder–Narasimhan filtration of F, then for each geometric point $s \in S$, the support of $(F_i)_s$ is equal to $(D_i)_s$, and $(F_i)_s$ is of uniform rank 1 along $(D_i)_s$.*

Proof. The proof proceeds along very similar lines to the proof of Lemma 8.2.3, using the fact that the rank of a flat torsion-free sheaf along any of the subcurves D_i is upper-semicontinuous. □

Let $\overline{J}^{\tau_0-sp}_{C/S} \subset \overline{J}^{\tau_0}_{C/S}$ be the open sub-algebraic space parametrising those sheaves $F \in \overline{J}^{\tau_0}_{C/S}$ such that, if $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ is the relative Harder–Narasimhan filtration of F, then

(i) each sheaf F_i is (fibrewise) simple; and

(ii) each inclusion $F^i \subset F^{i+1}$ is nowhere-split.

There is also an open sub-algebraic space $\overline{J}^{\tau_0-s}_{C/S} \subset \overline{J}^{\tau_0-sp}_{C/S}$ obtained by imposing the additional constraint that each F_i is fibrewise stable with respect to (L, M), viewed as a family of torsion-free uniform rank 1 sheaves on D_i/S. The fibres of $\overline{J}^{\tau_0-sp}_{C/S}$ over closed points of S are given by moduli spaces of simple, torsion-free, uniform rank 1 coherent sheaves of refined Harder–Narasimhan type τ_0 to which Theorem 8.1.1 is applicable. Setting $\chi_i = \theta_i - \deg_{D_i} M$, we have a well-defined morphism

$$\mathrm{gr} : \overline{J}^{\tau_0-sp}_{C/S} \to \prod_{i=1}^{\ell} \overline{J}^{\chi_i,ss}_{D_i/S}(L, M)$$

which associates to a sheaf F with relative Harder–Narasimhan filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^\ell = F$ the tuple $(F_1, \ldots, F_\ell)$ consisting of the Harder–Narasimhan subquotients (here the product should be understood as a fibre product of schemes over S).

Assume further that there are sections $\sigma_1, \ldots, \sigma_n : S \to C$ of f, each factoring through the S-smooth locus of C, such that for each geometric point $s \in S$, each irreducible component of C_s is geometrically integral and contains $\sigma_i(s)$ for some $i = 1, \ldots, n$, so that by Lemma 5.3.3 the étale and Zariski sheafifications of $\overline{\mathbb{J}}^{*}_{D_i/S}$ coincide for all $i = 1, \ldots, \ell$, and all of the fine compactified Jacobians $\overline{\mathbb{J}}_{D_i/S}$ admit universal sheaves. Construction 8.3.6 then carries over to the relative setting after making the necessary notational changes, yielding a tower of Zariski-locally trivial projective bundles $P^\ell \to P^{\ell-1} \to \cdots \to P^2 \to P^1$. There are relative analogues of the morphisms Φ and Ψ which appear in the proof of Theorem 8.1.1. The same argument used to show that these morphisms are inverse to each other carries over to the relative setting. As such, we obtain the following result, which generalises Theorem 8.1.1 to cover relative moduli spaces of unstable simple torsion-free uniform rank 1 sheaves over the fibres of C/S, of refined Harder–Narasimhan type τ_0 with respect to (L, M):

Theorem 8.5.2. *In the situation described above, there exists a diagram*

$$\begin{array}{ccc}
\overline{J}^{\tau_0-sp}_{C/S} & \xrightarrow{\ \simeq\ } & P^\ell \\
 & & \downarrow{\scriptstyle \pi_\ell} \\
 & & P^{\ell-1} \\
 & & \downarrow \\
 & & \vdots \\
 & {\scriptstyle \mathrm{gr}} \searrow & \downarrow \\
 & & P^2 \\
 & & \downarrow{\scriptstyle \pi_2} \\
 & & P^1 = \prod_{i=1}^{\ell} \overline{J}^{\chi_i,ss}_{D_i/S}(L, M)
\end{array}$$

where each $\pi_i : P^i \to P^{i-1}$ *is a Zariski-locally trivial projective bundle.* □

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