Joyce conjectures for Lagrangian mean curvature flow with circle symmetry

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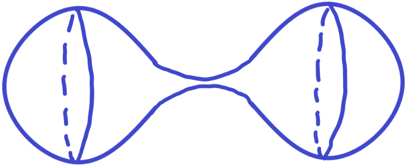
(Joint work with Goncalo Oliveira)

Lagrangian mean curvature flow (LMCF)

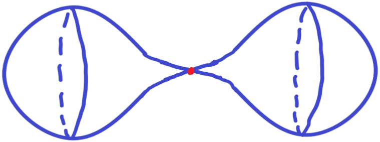
Lagrangian $L \subseteq (M, J, \omega, \Omega)$ Calabi–Yau

Examples: $M = \mathbb{C}$ (flat) $\rightsquigarrow$ L curve

$M = \mathbb{C}^2$ (flat), $T^*\mathcal{S}^2$ (Eguchi–Hanson), $K3 \rightsquigarrow L$ surface



LMCF: $\frac{\partial L_t}{\partial t} = H$



Question: How should LMCF “decompose” L?

Lagrangian angle

LMCF $L_t \subseteq (M, J, \omega, \Omega)$: $\frac{\partial L_t}{\partial t} = H = J\nabla\theta \rightsquigarrow \theta$ Lagrangian angle

- special Lagrangian (SL) $\Leftrightarrow \theta$ constant $\leftrightarrow$ critical points
  $\Rightarrow$ critical points of LMCF are minima
- graded $\Leftrightarrow$ choice of (single-valued) function θ
- almost calibrated $\Leftrightarrow \sup\theta - \inf\theta < \pi$

Thomas–Yau/Joyce conjectures: Behaviour of LMCF for almost calibrated/graded Lagrangians $\leftrightarrow$ decomposition and stability

Main result: Proof of (version of) conjectures for large class of $M^4 \rightsquigarrow \mathcal{S}^1$ symmetry and Gibbons–Hawking ansatz

Gibbons–Hawking ansatz

$(x_1, x_2, x_3) \in U \subseteq \mathbb{R}^3$, $e^{i\psi} \in \mathcal{S}^1$

- $V : U \to \mathbb{R}^+$ harmonic function, e.g.

$$U = \mathbb{R}^3 \setminus \{p_1, \ldots, p_{k+1}\}, \quad V = m + \sum_{i=1}^{k+1} \frac{1}{2|x - p_i|}$$

- ξ 1-form on U with $*\mathrm{d}\xi = \mathrm{d}V$
  $\rightsquigarrow \mathrm{d}\psi + \xi$ connection on $\mathcal{S}^1$-bundle with curvature $*\mathrm{d}V$

$\rightsquigarrow$ metric on M^4: $g = V^{-1}(\mathrm{d}\psi + \xi)^2 + V(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2)$

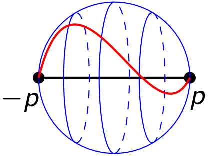
- hyperkähler $\mathrm{Hol}(g) \subseteq \mathrm{SU}(2)$ & (M^4, g) $\mathcal{S}^1$-invariant
- many ($\mathcal{S}^1$-invariant) Lagrangians $L^2 \hookrightarrow M^4$

$V = m + \sum_{i=1}^{k+1} \frac{1}{2|x-p_i|} \rightsquigarrow$ ALE ($m = 0$) and ALF ($m > 0$)

e.g. $m = 0 \rightsquigarrow$ flat $\mathbb{C}^2$ ($k = 0$), Eguchi–Hanson ($k = 1$)
& $m > 0 \rightsquigarrow$ Taub–NUT $\mathbb{C}^2$ ($k = 0$)

Circle-invariant Lagrangians

Eguchi–Hanson $T^*\mathcal{S}^2$: $V = \frac{1}{2|x-p|} + \frac{1}{2|x+p|}$



$\rightsquigarrow$ minimal Lagrangian $\mathcal{S}^2$

Gibbons–Hawking M^4: curves $\gamma \subseteq \mathbb{R}^3 \leftrightarrow \mathcal{S}^1$-invariant $L^2_\gamma \subseteq M^4$

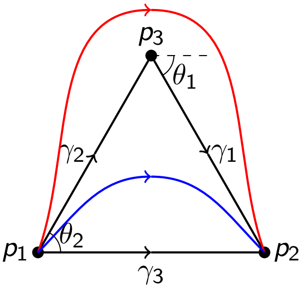
- embedded closed curve in $\mathbb{R}^3 \setminus \{p_i\} \leftrightarrow$ embedded T^2
- embedded arc endpoints $p_1, p_2 \leftrightarrow$ embedded $\mathcal{S}^2$

Lemma

- *L_γ Lagrangian $\Leftrightarrow \gamma$ planar*
- *L_γ special Lagrangian $\Leftrightarrow \gamma$ straight line*

Stability

$\gamma \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \rightsquigarrow \theta$ angle between γ and horizontal



Definition

L_γ compact Lagangian $\rightsquigarrow$ L_γ stable $\Leftrightarrow$

- *L_γ almost calibrated:* $\max\theta - \min\theta < \pi$ *and*
- *whenever* $\gamma \sim \gamma_1 \# \gamma_2$,
  $\ell(\gamma) \leq \ell(\gamma_1) + \ell(\gamma_2)$ *or* $[\theta_1, \theta_2] \not\subseteq (\min\theta, \max\theta)$

Thomas–Yau conjecture

Conjecture (Thomas–Yau)

L stable $\Rightarrow$ LMCF exists for all time and converges to SL $L_\infty \sim_{Ham} L$

Theorem (L.–Oliveira)

For S^1-invariant Lagrangian surfaces in complete Calabi–Yau from Gibbons–Hawking ansatz, Thomas–Yau conjecture is true

(Neves 2013): Any compact Lagrangian can be perturbed (and preserve graded) to non-almost calibrated L so that LMCF starting at L develops finite-time singularity (even preserving invariance)

Joyce conjectures

Conjecture (Joyce)

L graded Lagrangian in Calabi–Yau (M, J, ω, Ω) & L defines an object in (suitable derived) Fukaya category $\mathcal{F}(M)$ with HF^* unobstructed

$\Rightarrow$ LMCF through singularities and with surgeries exists for all time and converges to union of SLs $L_\infty = L_1 \cup \ldots \cup L_n \sim_{\mathcal{F}} L$ with $\theta_1 > \ldots > \theta_n$

$\Rightarrow$ Conjecture (Bridgeland, Joyce): $\exists$ Bridgeland stability condition on $\mathcal{F}(M) \rightsquigarrow$ Bridgeland stable graded Lagrangians

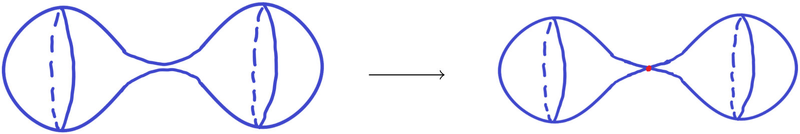
Conjecture (Joyce)

L Bridgeland stable $\Rightarrow$ LMCF through singularities and with surgeries exists for all time and converges to SL $L_\infty \sim_{\mathcal{F}} L$

Flow through singularities

$\exists$ SL cylinder $\mathbb{R} \times \mathcal{S}^{n-1} \subseteq \mathbb{C}^n$ (Lawlor neck)

(Joyce): Lawlor neck should model generic "neck pinch" singularity in LMCF

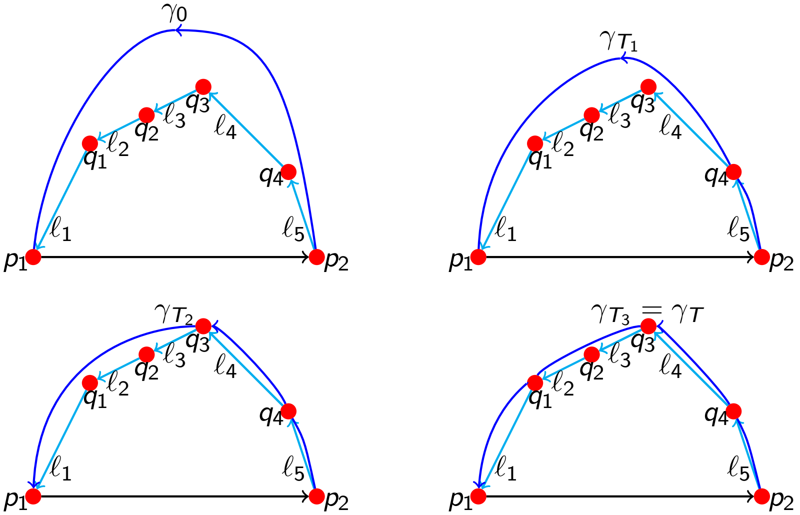


Theorem (L.–Oliveira)

L^2 compact, almost calibrated, $\mathcal{S}^1$-invariant in complete M from Gibbons–Hawking ansatz $\Rightarrow \exists$ continuous family $\{L_t\}_{t\in[0,\infty)}$ & times $0 < T_1 < \ldots < T_k = T$ such that

- *L_t satisfies LMCF for $t \notin \{T_1, \ldots, T_k\}$ & Lawlor neck pinch occurs at each T_j*
- *L_t converges to union of SL spheres $L_1 \cup \ldots \cup L_n$ as $t \to \infty$*

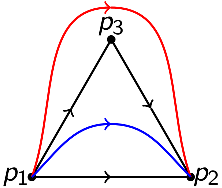
Joyce conjectures and Gibbons–Hawking ansatz



Flow of curves

$V = m + \sum_{i=1}^{k+1} \frac{1}{2|x-p_i|}$

L_γ compact almost calibrated $\Rightarrow$ γ arc in $\mathbb{R}^2$ joining p_1, p_2



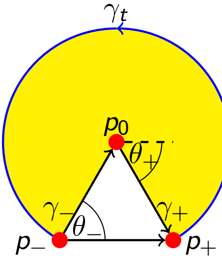
Lemma

$$LMCF\ \frac{\partial L_\gamma}{\partial t} = H \Leftrightarrow \frac{\partial \gamma}{\partial t} = V^{-1}\kappa = V^{-1}\frac{\partial^2 \gamma}{\partial s^2}$$

Key observations

- $\frac{\partial \theta}{\partial t} = \Delta\theta \Rightarrow$ variation of θ non-increasing
- almost calibrated $\Rightarrow$ singularities can only occur at p_i (blow-up analysis for curve shortening flow)

Conclusion



- $A(t)$ area of pacman disk $\rightsquigarrow$ $\dot{A} \leq \theta_+ - \theta_-$
- $\Rightarrow \gamma_t$ hits p_0 in finite time
- γ_t barrier curve $\rightsquigarrow$ finite-time singularity (p, T)

$p \in \{p_1, p_2\} \rightsquigarrow$

- singularity modelled on ancient solution $\tilde{L}_t \cong \mathbb{R}^2$ in $\mathbb{C}^2$ asymptotic to two transverse planes
- (Lambert–L.–Schulze) $\Rightarrow \tilde{L}_t =$ flat plane $\lightning$

$p = p_i$ for $i > 2 \rightsquigarrow$

- tangent flow $=$ pair of transverse planes $\rightsquigarrow$ (L.–Schulze–Székelyhidi) Lawlor neck pinch
- $\frac{\partial \theta}{\partial t} = \Delta\theta \rightsquigarrow$ finitely many singularities $+$ convergence ✓