Exceptional holonomy & gauge theory in higher dimensions

Two parts: (A) Exceptional holonomy : geometry in 6, 7, 8 dimensions

(B) Gauge theory : low & high dimensions via dimensional reduction

(A) Exceptional holonomy

6D $\quad SU(3) \quad (Z^6, \omega, \Omega)$	7D $\quad G_2 \quad (Y^7, \varphi)$	8D $\quad Spin(7) \quad (X^8, \Phi)$
$\mathbb{C}^3 \quad (z_1, z_2, z_3)$ $\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$ $\Omega_0 = dz_1 \wedge dz_2 \wedge dz_3$ Fact: $\underbrace{(\omega_0, \mathrm{Re}\,\Omega_0)}_{\mathrm{Stab} = SU(3)} \rightsquigarrow g_0, J_0, \mathrm{Im}\,\Omega_0, \mathrm{vol}_0$	$\mathbb{R}^7 = \mathbb{R}_t \times \mathbb{C}^3$ $\varphi_0 = dt \wedge \omega_0 + \mathrm{Re}\,\Omega_0 \quad$ (3-form) $*\varphi_0 = \frac{1}{2}\omega_0^2 - dt \wedge \mathrm{Im}\,\Omega_0$ Fact: $\underbrace{\varphi_0}_{\mathrm{Stab} = G_2} \rightsquigarrow g_0, \mathrm{vol}_0, *\varphi_0$	$\mathbb{R}^8 = \mathbb{R}_s \times \mathbb{R}^7 = \mathbb{C}^4$ $\Phi_0 = ds \wedge \varphi_0 + *\varphi_0 \quad$ (4-form) $= \frac{1}{2}\omega_0^2 + \mathrm{Re}\,\Omega_0$ Fact: $\underbrace{\Phi_0}_{\mathrm{Stab} = Spin(7)} \rightsquigarrow g_0, \mathrm{vol}_0, *\Phi_0 = \Phi_0$
$Z^6 \quad (\omega, \Omega) \sim (\omega_0, \Omega_0) \rightsquigarrow g, J, \mathrm{vol}$ Fact: $d\omega = 0$, $d\Omega = 0$ $\Leftrightarrow$ $\mathrm{Hol}(g) \subseteq SU(3)$ (Calabi-Yau 3-fold)	$Y^7 \quad \varphi \sim \varphi_0 \rightsquigarrow g, \mathrm{vol}, *\varphi$ Fact: $d\varphi = 0$, $d^*\varphi = 0$ $\Leftrightarrow$ $\mathrm{Hol}(g) \subseteq G_2$ (G_2-manifold)	$X^8 \quad \Phi \sim \Phi_0 \rightsquigarrow g, \mathrm{vol}, *\Phi = \Phi$ Fact: $d\Phi = 0 \Leftrightarrow \mathrm{Hol}(g) \subseteq Spin(7)$ ($Spin(7)$-manifold)
Note: $\mathbb{C}^3 = T^*\mathbb{R}^3$ L^3 oriented $\rightsquigarrow (T^*L, \omega, \Omega)$ e.g. $L = S^3 \rightsquigarrow CY3$ (Stenzel)	Note: $\mathbb{R}^7 = \Lambda^2_+ T^*\mathbb{R}^4$ (M^4, g) oriented $\rightsquigarrow (\Lambda^2_+ T^*M, \varphi)$ e.g. $M = S^4$ or $\overline{\mathbb{CP}^2} \rightsquigarrow G_2$-manifold (Bryant – Salamon)	Note: $\mathbb{R}^8 = \mathbb{S}_+(\mathbb{R}^4)$ (M^4, g) oriented, spin $\rightsquigarrow (\mathbb{S}_+(M), \Phi)$ e.g. $M = S^4 \rightsquigarrow Spin(7)$-manifold (B – S) Note: $\mathbb{R}^8 = \mathbb{R}_s \times \Lambda^2_+ T^*\mathbb{R}^4$ $\qquad\qquad \shortparallel$ $\qquad \Lambda^2_+ T^*(\mathbb{R}_s \times \mathbb{R}^4)$ (N^5, g) oriented with unit vector field $\rightsquigarrow (\Lambda^2_+ T^*N, \Phi)$

(B) Gauge theory

A connection on P, F curvature

Lower dimensions

4D (M^4, g) ASD instanton : $F = -*F$

(anti-self-dual)

M compact $\Rightarrow$ A minimises $YM = \frac{1}{2}\int_M |F|^2$

(Yang-Mills)

3D $\mathbb{R}^4 = \mathbb{R}_s \times \mathbb{R}^3 \rightsquigarrow$ connection 1-form

$\underline{A} = \Psi\, ds + A \qquad \Psi, A$ s-indep.

$$\underline{F} = d\underline{A} + \underline{A} \wedge \underline{A} = -ds \wedge d_A\Psi + F$$

$$\Rightarrow *\underline{F} = -*d_A\Psi + ds \wedge *F$$

$$\therefore \underline{F} = -*\underline{F} \Leftrightarrow F = *d_A\Psi$$

Higher dimensions

8D (X^8, Φ) $Spin(7)$-instanton : $F \wedge \Phi = -*F$

X compact $\Rightarrow$ A minimises $YM = \frac{1}{2}\int_X |F|^2$

7D $\mathbb{R}^8 = \mathbb{R}_s \times \mathbb{R}^7 \rightsquigarrow$ connection 1-form

$\underline{A} = \Psi\, ds + A \qquad \Psi, A$ s-indep.

$$\underline{F} = -ds \wedge d_A\Psi + F$$

$$\Rightarrow \underline{F} \wedge \underline{\Phi} = \underline{F} \wedge (ds \wedge \varphi_0 + *\varphi_0)$$

$$= ds \wedge (F \wedge \varphi_0 - d_A\Psi \wedge *\varphi_0) + F \wedge *\varphi_0$$

$$\therefore \underline{F} \wedge \underline{\Phi} = -*\underline{F} \Leftrightarrow \boxed{F \wedge \varphi_0 - d_A\Psi \wedge *\varphi_0 = -*F}$$

$\therefore\ \underline{F} = -*\underline{F} \iff \boxed{F = * d_A \Psi}$

(L^3, g) oriented $\rightsquigarrow (A, \Psi)$ monopole

Note: L compact $\Rightarrow d_A\Psi = 0 \Rightarrow F = 0$ $\quad A$ flat

$= ds\wedge(F\wedge\varphi_0 - d_A\Psi\wedge *\varphi_0) + F\wedge *\varphi_0$

$\therefore\ \underline{F}\wedge\underline{\Phi} = -*\underline{F} \iff \boxed{\begin{array}{l} F\wedge\varphi_0 - d_A\Psi\wedge *\varphi_0 = -*F \\ \iff F\wedge *\varphi_0 = * d_A\Psi \end{array}}$

$(Y^7, \varphi) \rightsquigarrow (A, \Psi)$ G_2-monopole

Note: Y compact $\Rightarrow d_A\Psi = 0 \Rightarrow F\wedge\varphi = -*F$

$\Updownarrow$

A G_2-instanton $\quad F\wedge *\varphi = 0$

2D

$\mathbb{R}^4 = \mathbb{R}_s\times\mathbb{R}_t\times\mathbb{R}^2 \rightsquigarrow \underline{A} = \Psi_1 ds + \Psi_2 dt + A$
s, t-indep

$\underline{F} = -ds\wedge d_A\Psi_1 - dt\wedge d_A\Psi_2 + ds\wedge dt\wedge[\Psi_1, \Psi_2] + F$

$*\underline{F} = -ds\wedge * d_A\Psi_2 + dt\wedge * d_A\Psi_1 + *[\Psi_1, \Psi_2] + ds\wedge dt\wedge *F$

$\therefore\ \underline{F} = -*\underline{F} \iff \boxed{\begin{array}{l} F = -*[\Psi_1, \Psi_2] \\ d_A\Psi_1 + * d_A\Psi_2 = 0 \end{array}}$

Two viewpoints

(a) $\mathbb{R}^2 = \mathbb{C}_z \rightsquigarrow \Psi = \frac{1}{2}(\Psi_1 + i\Psi_2)\,dz$

$\rightsquigarrow \boxed{\begin{array}{l} F = [\Psi, \Psi^*] \\ \bar\partial_A\Psi = 0 \end{array}}$

(b) $\mathbb{R}^2(x_1, x_2) \rightsquigarrow \Psi = \Psi_1 dx_1 - \Psi_2 dx_2$

$\rightsquigarrow \boxed{\begin{array}{l} F = \Psi\wedge\Psi \\ d_A\Psi = 0 \ \&\ d_A^*\Psi = 0 \end{array}}$

Σ^2 Riemann(ian) surface

$\rightsquigarrow (A, \Psi)$ Higgs bundle / Hitchin's equations

Note: $\Psi = 0 \Rightarrow F = 0$ $\quad A$ flat

6D

$\mathbb{R}^8 = \mathbb{R}_s\times\mathbb{R}_t\times\mathbb{C}^3 \rightsquigarrow \underline{A} = \Psi_1 ds + \Psi_2 dt + A$
s, t-indep.

$\underline{F} = -ds\wedge d_A\Psi_1 - dt\wedge d_A\Psi_2 + ds\wedge dt\,[\Psi_1, \Psi_2] + F$

$\underline{\Phi}_0 = ds\wedge \mathrm{Re}\,\Omega_0 - dt\wedge \mathrm{Im}\,\Omega_0 + ds\wedge dt\wedge\omega_0 + \frac{1}{2}\omega^2$

$\therefore\ \underline{F}\wedge\underline{\Phi}_0 = -*\underline{F} \iff \boxed{\begin{array}{l} F\wedge\frac{1}{2}\omega^2 = -*[\Psi_1, \Psi_2] \\ d_A\Psi_1\wedge\frac{1}{2}\omega^2 + * d_A\Psi_2 = F\wedge \mathrm{Re}\,\Omega \\ \qquad\qquad \shortparallel \\ (d_A\Psi_1 + J d_A\Psi_2)\wedge\frac{1}{2}\omega^2 \end{array}}$

(Z^6, ω, Ω) Calabi-Yau 3-fold

$\underset{\text{complex}}{\underset{\uparrow}{(A, \Psi)}}$ $\quad \boxed{\begin{array}{l} *(F\wedge\frac{1}{2}\omega^2) = i[\Psi, \Psi^*] \\ F\wedge\Omega = \partial_A\Psi\wedge\frac{1}{2}\omega^2 \end{array}}$

DT-instantons / complex / Calabi-Yau monopoles?

Note: $\Psi = 0 \Rightarrow F\wedge\frac{1}{2}\omega^2 = 0$
$\&\ F\wedge\Omega = 0$

Hermitian-Yang-Mills
(minimise YM)

1D

$\mathbb{R}^4 = \mathbb{R}_{x_1}\times\mathbb{R}_{x_2}\times\mathbb{R}_{x_3}\times\mathbb{R}_u$

$\rightsquigarrow \underline{A} = B_1 dx_1 + B_2 dx_2 + B_3 dx_3 \qquad B_j = B_j(u)$

$\underline{F} = -dx_1\wedge\dot B_1 du - dx_2\wedge\dot B_2 du - dx_3\wedge\dot B_3 du$
$\quad + [B_2, B_3]\,dx_2\wedge dx_3 + [B_3, B_1]\,dx_3\wedge dx_1 + [B_1, B_2]\,dx_1\wedge dx_2$

$\underline{F} = -*\underline{F} \iff \boxed{\begin{array}{l} \dot B_1 = [B_2, B_3] \\ \dot B_2 = [B_3, B_1] \\ \dot B_3 = [B_1, B_2] \end{array}}$

$\rightsquigarrow$ Nahm's equations

5D

$\mathbb{R}^8 = \mathbb{R}_{x_1}\times\mathbb{R}_{x_2}\times\mathbb{R}_{x_3}\times(\mathbb{R}_u\times\mathbb{R}^4)$

$\rightsquigarrow \underline{A} = B_1 dx_1 + B_2 dx_2 + B_3 dx_3 + A \qquad (x_1, x_2, x_3)$-indep.

$\rightsquigarrow$ curvature $\underline{F}$ & curvature $F = du\wedge F_0 + F_4$ of A.

$\omega_1, \omega_2, \omega_3$ standard o.n. constant self-dual 2-forms on $\mathbb{R}^4$

$\rightsquigarrow B = B_1\omega_1 + B_2\omega_2 + B_3\omega_3$

$\&\ \underline{\Phi}_0 = dx_{123}\wedge du - dx_{23}\wedge\omega_1 - dx_{31}\wedge\omega_2 - dx_{12}\wedge\omega_3$
$\quad - dx_1\wedge du\wedge\omega_1 - dx_2\wedge du\wedge\omega_2 - dx_3\wedge du\wedge\omega_3$
$\quad + \mathrm{vol}_4$

I interval $\rightsquigarrow$ Nahm's equations

$+ \mathrm{vol}_4$

$\underline{F} \wedge \underline{\Phi}_0 = - * \underline{F} \iff$

$$\boxed{\begin{aligned} &F_0 - d_A^* B = 0 \\ &F_4 + * F_4 - \nabla^A_{\partial_u} B - [B_2, B_3]\omega_1 - [B_3, B_1]\omega_2 \\ &\qquad - [B_1, B_2]\omega_3 = 0 \end{aligned}}$$

(N^5, g) oriented with unit vector field

$\rightsquigarrow$ Haydys - Witten equations