Proof of Theorem 8.5

- Let's assume $G$ countable.
- Let $E$ be round conformal structure on $S^2$.
- For $z \in S^2$, let
  $$ E(z) = \{ (D_{g_z})^* E_{g_z} : g \in G \} $$
  defined a.e.
  bounded in space of conf. structures, as each $g$ is $K$-q.
- Let $E_z = P(E(z))$ cirmun-
  (Prop. 8.2)

To see $E$ is measurable,
let $G = \{ g_0, g_1, \ldots \}$

let $E^{(m)}_z = P(\{ (D_{g_i})^* E_{g_i} : i = 0, \ldots, \})$

defined for a.e. $z \in S^2$,
measurable as $D_{g_i}$ meas.
P continuous, so $E = \lim_{n \to \infty} E^{(m)}_z$
is measurable.

Apply Thm 8.4 to find
$$ f: S^2 \to S^2 \text{ q.e. st.} $$
$$ E = (Df)^* E \text{ a.e.} $$
\[ \forall y \in G, \ a.e. \ z \in S^2 \]
\[ (Dg)_z^* E(gz) \]
\[ = \{ (Dg)_z^* (Dh)_z^* \} \}
\[ = \{ (D(hg)_z^* \} \}
\[ = E(z) \]
So \[ (Dg)_z^* \exists g_z = \exists z \ a.e. \]

\[ \Rightarrow (Dg)_z^* (Df_g)_z^* \exists f_g = (Df_g)_z^* \exists f_g \ a.e. \]

So \( f_g f^{-1} \) is \( 1 \)-qc \( \Rightarrow \) M{	ext{"}ob. \)

(True without assuming 
\( G \) countable. take a countable 
dense subgroup; limit...)
Outline of proof of theorem 8.6:

1. Define $E(z)$ as above.
2. Let $\bar{E}_z = \mathcal{P}(E(z))$.
3. As before, it's a $C^1$-b.v.
4. measurable conformal structure.
5. $\bar{E}$ is approx. cts a.e.
6. Let $p$ be such a point.
7. WLOG $p = 0$, $\bar{E}_0 = \bar{E}$.

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Pick a triple $z$.

Choose $g_i$ s.t. projection

$\pi(g_i z) \to p$ along line

as in pic.

Choose $\lambda_i \to \infty$ s.t.

$\#_{n+1}(e_{n+1}, \lambda; \pi(g_i z))$

uniformly bounded.

Let $f_i : \mathbb{R}^n \to \mathbb{R}^n$ be

$f_i(u) = \lambda_i g_i(u)$.

This keeps triple spread out

so by Prop 5.7, after
taking a subseq, 
\[ f_i \to f \text{ ac}, \text{ uniformly on } S^1 = \mathbb{R}/\mathbb{Z} \]

Show \( f \circ g \circ f^{-1} \) is \( 1 \text{-ac} \).

Basically, \( \forall \varepsilon > 0 \)

\[ f : gf_i^{-1} \text{ is } (2+\varepsilon) \text{-ac} \]

off set of measure \( m_i \) in \( S^1 \),

and \( m_i \to 0 \) as \( \varepsilon \to 0 \).

Tukia: in limit, \( f \circ g \circ f^{-1} \) is \( 1 \text{-ac} \).

Or: \( Df \times \varepsilon = \lim_{i \to \infty} (Df_i) \times \varepsilon \)

\[ = \lim_{i \to \infty} \left( \lambda_i Df_i \right) \times \varepsilon \to 0. \]

What about \( S^2 \)?

Analogue of Theorem 8.5
for unif qm groups
(Thynke toren '85). In fact:

**Theorem 8.7** (Tukia, Gabai, Casson-Jungreis '89)

If \( G \cong S^2 \), then

\( G \) is a convergence group

\( \iff \) action is topologically conjugate to Fuchsian action.

**Cor 8.8** (Tukia, Frenkel) \( G \) hyp.

\[ 0 \to G \cong S^1 \iff G \cong H^2 \text{ disc. count}. \]
Remark: Solved Seifert fibred space conjecture. \( M \) compact orientable irreducible 3-manifold, \( \pi_1 M = \pi_1 \). Then \( M \) is Seifert fibred space \( \Leftrightarrow \exists [\mathbb{B}] \cong H^1 < \pi_1 M \).

Connection: \( G \cong \text{Triples}(S') = S' \times \mathbb{R} \)

Case \( S^2 \):
Cannon's conjecture: \( G \) hyp.
\( S^4 \cong S^2 \Leftrightarrow G \cong H^3 \) geometric.

False for \( S^2, n \neq 3 \).

E.g. Mostow-Sin
Gromov-Thurston: \( \exists M^4 \) cpt. that admits Riem. metrics of curvature \( K < [-1-\epsilon, -1] \), but no hyperbolic metric.

\( G = CH^{1/2} \)
compact lattice \( \cong S^3 \)
then \( \partial G \cong S^3 \), but not \( S^3 \).
See conformal dimension later.
9. Cohomology & topological dimension

Def 9.1 $\mathbb{X}$ metric space, $R > 0$. The Rips complex $P_{R}(\mathbb{X})$ is the simplicial complex with vertex set $\mathbb{X}$, and an $n$-simplex $\{x_0, \ldots, x_n\}$ if $\text{diam} \{x_0, \ldots, x_n\} \leq R$.

- If $G$ f.g. group, $P_{R}(G) = P_{R}(G, d_{s})$.

Prop 9.2 $C_{R}$ hyp $\Rightarrow P_{R}(G)$ contractible for $R$ large enough.

Proof
- By Whitehead's theorem, $\pi_{n}(P_{R}(G)) = 0$ for all $n > 1$.
- Sufficient every finite subcomplex $\mathbb{L} \subset P_{R}(G)$ is homotopic to a point.
- Case 1 $\mathbb{L} \subset B(e, R)$.
  All $\mathbb{L}$ lies in some simplex.
Case 2: \( \exists x \in L^{(0)} \)

st. \( d(x, z) > \frac{R}{2} \) and maximal.

Then

Define \( f : L^{(0)} \rightarrow P_R(G) \)

\[
f(z) = \begin{cases} 
  w & \text{if } z = x \\ 
  z & \text{otherwise}
\end{cases}
\]

Want to extend \( f \) to \( L \).

If \( y \notin L^{(0)} \), \( d(y, x) \leq R \)

then \( d(y, w) \leq \frac{R}{2} + C < R \)

for \( R > 0 \).

So \( L = f(L) \) in \( P_R(G) \).

\( L \) finite, so continue until in case 1.

Cor 9.3: \( G \) hyp \( \iff \) fin. pres.

Proof: \( G \) f.g., then

\( G \) is fin. pres

\( \iff P_R(G) \) is simply conn.

for \( R \) large enough. (Ex.)
Cor 9.4: \( G \) hyp. then

3. Simplicial complex \( P \),
action \( G \) on \( P \) that is
(a) Simplicial, cocompact,
finite stabilisers
(b) \( P \) is fin. dim, contractible.
locally finite.
(c) \( G \) in \( \mathcal{D}_0 \), free and
transitive.

In particular,

1. If \( G \) torsion free then
\( G \backslash P \) is a \( K(\pi,1) \) so
\( G \) finite cohomological dim,
type \( F_\infty \) for all \( n \).

(2) \( G \) virtually tors. free
\( \Rightarrow \) finite virtual
cohom. dim.

(3) \( H^*(\mathcal{G}, \mathbb{Q}) \) is finite
dim.

(4) Type \( FP_\omega \) (Brown)

Remark: Open Q:
Is every Gr. hyp group
\( V \), torsion free?
\( \Leftrightarrow \) Is every Gr. hyp. group
residually finite?
(Kapovich - Wise '00).
**Def 9.5**  
Volume entropy of $G$ (with respect to fin.gen.set $S$) is \[ h(G, S) = \limsup_{r \to \infty} \frac{1}{r} \log |B_r(e, r)| \leq \log (2|S|-1) \]

**Def 9.6**  
In a metric space $X$, $\alpha > 0$, then the Hausdorff $\alpha$-measure of $A$ is

\[ H^\alpha(A) = \lim_{\delta \to 0} \inf \left\{ \sum \text{diam}(E_i)\alpha : A \subset \bigcup_i E_i, \text{diam}(E_i) < \delta \right\} \]

The Hausdorff dimension of $A$ is \[ \dim_H A = \sup \{ \alpha : H^\alpha(A) = 0 \} = \inf \{ \alpha : H^\alpha(A) = \infty \} \]

Graph of $H^\alpha(A)$:
Example: \( X = \mathbb{R}^n \) then
\[ H^v(\text{open set}) \left\{ \begin{array}{l}
\alpha > n \\
\alpha = n \\
\alpha < n
\end{array} \right. \]
(proportional to Lebesgue meas.)

(More in §10)

Prop 9.7 Let hyp. \( \varepsilon > 0 \).
\( d_\varepsilon \) is a visual metric on \( \partial \mathcal{G} \) with param. \( \varepsilon \).
then \( \dim_H (\partial \mathcal{G}, d_\varepsilon) \leq \frac{1}{\varepsilon^2} h(G, S) \).

**Proof**

If \( \alpha > \alpha' > \frac{1}{3} h(G, S) \)

\[ \exists C \text{ st. } |B_C(e, r)| \leq Ce^{\alpha r} \]

Let \( \mathcal{U}_r \) be cover
\[ \{ U_x : x \in \mathcal{S}(e, r) \} \]

\[ \text{diam}(U_x) \leq Ce^{-\varepsilon r} \]

Then
\[ \sum (\text{diam } U_x) \]

\[ \mathcal{U}_r \leq (Ce^{-\varepsilon r})^{\alpha r} e^{\varepsilon r} \rightarrow e^{-\varepsilon(\alpha r)} \rightarrow 0 \text{ as } r \rightarrow \infty. \]

I.e. \( \dim_H \leq \alpha \).
In fact
Theorem 9.8 (Coornaert '93)
There exists a Q-regular measure \( \mu \) on \( (\partial_0 G, d_\delta) \)
with \( Q = \frac{1}{2} \text{h}(G, S) \).
i.e. \( \text{dim}_H(\partial_0 G, d_\delta) = Q \).

Remark Ahlfors Q-regular \( \Rightarrow \forall r < \text{diam}(\partial_0 G, d_\delta) \)
\( \mu(B(x, r)) \leq r^Q \).
(Fact: then \( H^Q = \mu \).

Cor 9.9 \( \text{dim}\partial_0 G < \infty \)
Proof \( \text{dim}\partial_0 G < \text{dim}_H(\partial_0 G, d_\delta) \).

Prop 9.10 Can define compactification (G hyp.)
\( \overline{G} = G \cup \partial_0 G \)
(or \( \overline{G} = \overline{G} \cup \partial_0 \overline{G} \))
and \( \overline{G} \) is a compact, metrizable, fin. dim space.

Proof \( \overline{G} \leftrightarrow \{ \sigma: [0, \infty) \rightarrow \overline{G} : \}
\sigma(0) = e, \forall \text{ in st.} \}
\sigma|_{[0, \infty]} \text{ is geodesic,}
\sigma|_{[r, \infty]} \equiv \sigma(h). \)
Then $G = \bigcup_{\alpha < \Gamma} G_{\alpha}$ is compact. Has uniform structure so metrizable.

$$\dim G \leq \dim(\Pi G) + \dim(\partial G) + 1.$$ 

\[ \square \]

Recall:

**Def 9.11** $X$ top. space.

$\forall U, V$ open covers.

Say $V$ refines $U$ is

$\forall V \in U \implies V \subseteq U$.

\[ Nerve \text{ of } U \text{ is the simp. complex } N(U) \text{ with vertex set } U \text{ and an } n \text{-simplex } \{U_0, \ldots, U_n\} \iff U_0 \cap \cdots \cap U_n = \emptyset.

**Def 9.12** $X$ has $\dim(X) \leq n$ if $\forall$ every cover $U$ of $X$

has a refinement $V$ s.t.

$\dim N(V) \leq n$.

**Ex R:** 

\[ \begin{array}{c}
\end{array} \]
If $V$ refines $U$, obvious map $N(V) \to N(U)$ gets $H^*(N(U)) \to H^*(N(V))$.
Čech cohomology is $H^*(X) = \lim H^*(N(U))$.

This is nice(r): $\mathbb{S}^2$.

Then $H^3(X) \neq 0$ but $H^3(X) = 0$
$H^2(X) = \mathbb{Z}$.