More definitions:
- Cohomological dimension:
  \( cd(G) = \sup \{n : H^n(G, M) \neq 0 \text{ for some } \mathbb{Z}G \text{ module } M \} \)
- Compactly supported cohomology:
  \( H^*_c(X, \mathbb{Z}) \) is cohomology of compactly supported cochains.

\( H^*_c(R^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{if } i \neq n \end{cases} \)

\( G \) tor. free hyperbolic, \( P_{d}(G) \) contractible, then
\( H^*(G, \mathbb{Z}G) \cong H^*_c(P_{d}(G), \mathbb{Z}) \)
\( = \lim \{ H^*_c(P_{d}(G), P_{d}(G) \setminus Y) : Y \text{ compact} \} \)
\( \cong H^*_c(\overline{\partial_d G}) \)
Theorem 9.13 (Bestvina-Mess '91)

$G$ CAT$_0$ hyp, virtually tor. free, then

$$\dim(\partial G) = \dim d(G) - 1$$

$$\rightarrow \dim(H)$$

$H \leq G$, tor. free.

Theorem 9.14 (B.-M.)

$G$ hyperbolic.

$H^i(G, \mathbb{Z}G) \simeq H^i(\partial G, \mathbb{Z}).$

Proof hints:

Key:

Theorem 9.15 (B.-M.)

$P_d(G)$ is an absolute retract, and

$\partial G \subset P_d(G)$ is a $Z$-set.

We'll skip definitions here:

ex. $S^n < B^{n+1}.$
Local connectivity of the boundary

Theorem 9.16 (Bestv.-Mass)
$\partial_G$ has no (global)
cut point $\Rightarrow$ $\partial_G$ loc. conn.
($G$ one ended)

Proof
$D \ni$ bi-infinite geodesic within $C$ of any
$xc \Gamma(G)$
2. \( \forall M \text{ large } \exists L \)

\[ S \forall \]

\[ e \to R \]

\[ \exists \text{ path of length } \leq L \]

joining such \( x, y \) without going into \( B(e, R - c) \).

Proof: If not, \( \exists x_n, y_n \)

in \( S(e, R_n) \) as above.

so that need path length \( n \)
to join outside \( B(e, R - c) \).

Shift \( x \) to base point,
take limit

horoball

\[ \text{can't be joined outside horoball!} \]

\[ y \in \text{ outside horoball} \]

\[ \text{a, b } \notin G \text{ can't be joined without going through } c, \text{ which is a cut point} \]
3. local connectivity:
   use 2 repeatedly:

Theorem 9.17 (Bowditch, Swarup)

$G$ one ended hyp.

$\Rightarrow$ no global cut points

$\Rightarrow \text{End}_G \text{ loc. conn.}$

Remark. Actually, Bowditch showed a lot more:
   local cut points ...
   (i.e. $x$ s.t. $U \ni x$, $U \text{ conn.}$, $U \setminus \{x\}$ not conn.)

local cut point $\Rightarrow$ global cut point

... come in pairs and encode the JSJ decomposition of $G$ (splittings over $\mathbb{Z}$ groups)

See also Papasoglu for $\text{fp} G$. 
10. Conformal dimension

We saw q.i. type of $G$ is $qM$ type of $D(G)$.

Def 10.1 (Variation on Pansu '89)

The (Ahlfors regular) conformal dimension of a metric space $X$ is $\text{Cdim}(X) = \inf \{ q \mid \exists \dim_q(Y) : X \approx_{q} Y, Y \text{ Ahlfors regular} \}.$

Remark - Recall $X$ is Ahlfors $Q$-regular if

$\mathcal{H}^Q(B(x, r)) \leq C Q^Q r^Q$

$\forall x \in X, \ r \leq \text{diam } X.$

$X \approx_{q} Y$ means $X, Y$ are quasi-symmetric as $qM$.

$X \approx_{q} Y \iff X \approx_{qM} Y \text{ if } X, Y \text{ bounded.}$
Why? Try to find best (nice) metric on $\mathcal{O}_G$.

For example $(X, d) \equiv (X, d^{\varepsilon})$

where $\varepsilon \in (0, 1)$,

$\dim_H(X, d^{\varepsilon}) = \frac{1}{\varepsilon} \dim_H(X, d)$

Ex. $(C_0, \mathcal{L}, d^{\log 3/\log 4})$

So can make $\dim_H$ of a visual metric on $\mathcal{O}_G$ as large as we want.

We try to make it as small as possible.

Prop 10.2 $X$ Ahlfors Q-regular.

$\dim_H (X) \leq C \dim (X) \leq Q$

$X \preceq Y \Rightarrow \dim_H (X) = \dim_H (Y)$

$C \dim (\mathcal{O}_G), G$ hyp.

is well defined.
$\mathbb{E}$

$\text{Cdim}(\mathbb{R} \times \{\infty\}) = 1$

$\$'$, usual metric

is a visual metric

$\text{Cdim}(\mathbb{R} \times (\text{closed } M^n)) = n-1$

$\text{Cdim}(\mathbb{R} \times F_2) = 0$

$\mathbb{R} F_2$ is a

Cantor set

(with positive $\text{dim}_H > 0$)

but $\mathbb{R} F_2$ has a visual

metric that is an ultrametric.

$d(x,y) \leq \max \left\{ d(x,z), \quad d(z,y) \right\}$

and $d^2$ is a metric $\forall z \in F_0$

So let $\varepsilon \to 0$

$\text{Cdim}(\mathbb{R} F_2)$

$\leq \inf_{\varepsilon} \text{dim}_H(\mathbb{R} F_2, d^2)
\leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \text{dim}_H(\mathbb{R} F_2, d)
= 0.$
In fact: \( G \) infinite.

- \( \text{Cd}(\partial_\infty G) = 0 \)
  - \( \iff \text{Cd}(\partial_\infty G) < 1 \)
  - \( \iff G \) is \( \frac{1}{v.\ Z} \) cdim attained
    - or \( v.\ is\ not\ attained \)

(Uses Cor 6.13)

- \( \text{Cd}(\partial_\infty G) = 1 \), attained
  - \( \implies G \cong H^2 \) (Cor 8.8)

\( \text{disc. cocpt., isom.} \)

Prop 10.3: \( Z \) compact.

- Ahlfors \( Q \)-regular.
  - \( E \) a family of connected sets, \( \text{diam}(E) > c > 0 \)
  - \( \forall E \in E \)

- \( \nu \) prob measure on \( E \)
  - \( \exists C, \alpha > 0 \) and
  - \( \forall B(z, r) \subset Z, r \leq \text{diam}(Z), \nu(\exists E \in E : E \cup B(z, r) \neq \emptyset) \leq C r^\alpha \)

Then: \( \text{Cd}(Z) > 1 + \frac{\alpha}{Q - \alpha} \).
Ex C = $\frac{1}{3}$ Cantor set.
\[ C_{\dim}(C \times [0,1]) = 1 + \frac{\log 2}{\log 3}. \]

\[ Q = 1 + \alpha \]

Ex \[ C_{\dim}(\frac{\partial_{\infty} CH^2}{\partial_{\infty} CH^2}) = 4. \]

CAT(1) space, curvature \(-4, -1\)
\[ \partial_{\infty} CH^2 \cong S^3, \]
but looks like \( \{(1, y_2) \} \)
with "Carnot metric" of
Hausdorff dimension 4.

Lemma gives \[ C_{\dim} > 4. \]

Open \( Q \).
e.g. $\pi_1 \left( \frac{H^4}{T} \right)$ is closed, but $\pi_1 \left( \frac{C^4}{T} \right)$ is closed (See Pansu '89 Annals.)

Proof of Prop. (modulo Lemmas) (Pansu, Bourdon, Gromov)

Suppose $f : Z' \to Z'$ is $q \leq (\text{think } qM)$, $\text{dim}_H(Z') < B = \frac{Q}{Q - \alpha}$.

For any $\varepsilon > 0$, $\exists \{B_i = B(x_i, \varepsilon_i)\}$

So that
- $\exists B_i$ covers $Z'$
- $\sum r_i B_i < \varepsilon$
- $\{B_i\}$ disjoint.

(Use definition of $\dim_H$ and 5B-lemma.)

$\exists H > 0$, balls $B_i = B(x_i, r_i)$

$s.t. B_i < f^{-1}(B'_i) < f^{-1}(B'_i) < H B_i$
Let \( X_i : \mathbb{E} \to \{0, 1\} \)
\[ X_i(E) = \begin{cases} 1 & f(E) \cap 5B_i \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \]

\[ \exists c' > 0 \forall E \in \mathbb{E} \\
0 < c' \leq \text{diam } f(E) \leq \sum_i 10c_i' \cdot X_i(E) \]

\[ \Rightarrow \frac{c'}{10} \leq \sum_i X_i(E) \text{ dvol}(E) \leq \sum_i \int_X X_i(E) \text{ dvol}(E) \leq \sum_i \int_X c_i' \cdot X_i(E) \text{ dvol}(E) \leq \sum_i \langle r_i, \beta \rangle^\beta (\sum_i 10c_i')^\alpha \]

\[ < C \cdot H^\infty \langle r_i \rangle^\alpha \]

So \( \frac{c'}{10} \leq \sum_i r_i' \cdot r_i \)

\[ \frac{1}{\beta} + \frac{\alpha}{Q} = 1 \]

\[ \leq (\sum_i r_i')^\beta (\sum_i r_i^\alpha)^\alpha \]

\[ < C \cdot H^\infty \langle r_i \rangle^\alpha \]

\[ \times \text{ for } \varepsilon \text{ small enough. } \square \]

Remark: This is a kind of "modulus" argument.
Theorem 10.4 (Bonk–Kleiner '05)

G hyperbolic, \( \partial G \cong \mathbb{S}^2 \).
\( \text{Cdim}(\partial G) \) is attained
\( \Rightarrow G \cong \mathbb{H}^3 \) disc. expo.

Remark: Cannon’s conjecture is equivalent to \( \partial_G \cong \mathbb{S}^2 \)
in then \( \text{Cdim}(\partial_G) \) is attained.

Proof ideas:

1. (Keith–Laakso '03)
\( \text{Cdim}(X) \) attained
\( \Rightarrow \) some “tangent” to \( X \)
has a curve family of positive modulus.

2. Dynamics of \( G \cap \partial G \)
\( \partial G \) is “Loewner.”
Geometry is controlled by modulus of families of curves.
3. Approximate $\partial G$ by a graph which combinatorially equivalent to a triangulation of $S^2$. On graph
"combinatorial Loewner"

4. Andrews-Koebe-Thurston: realize graph by circle packing

This gives $f_i : \partial G \to S^2$ on the vertex set, use Loewner to get
that $\lim_{i \to \infty} f_i = f : \partial G \to S^2$

Remark: $\text{dim}(\partial G)$ not always attained
(Bourdon-Pisot when $\partial G \cong \mathbb{R}^2$)

Then $L_p$ cohomology seems to be interesting. (Bourdon-Kleiner arXiv. Yesterday)