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DIFFUSION-AGGREGATION PROCESSES WITH MONO-STABLE REACTION TERMS

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Abstract. This paper analyses front propagation of the equation

$$v_\tau = \Big[D(v)v_x\Big]_x + f(v) \quad \tau \ge 0,\, x \in \mathbb{R},$$

where f is a monostable (i.e. Fisher-type) nonlinear reaction term and $D(v)$ changes its sign once, from positive to negative values, in the interval $v \in [0,1]$ where the process is studied. This model equation accounts for simultaneous diffusive and aggregative behaviors of a population dynamic depending on the population density v at time τ and position x. The existence of infinitely many traveling wave solutions is proven. These fronts are parameterized by their wave speed and monotonically connect the stationary states $v \equiv 0$ and $v \equiv 1$. In the degenerate case, i.e. when $D(0) = 0$ and/or $D(1) = 0$, sharp profiles appear, corresponding to the minimum wave speed. They also have new behaviors, in addition to those already observed in diffusive models, since they can be right compactly supported, left compactly supported, or both. The dynamics can exhibit, respectively, the phenomena of finite speed of propagation, finite speed of saturation, or both.

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1. **Introduction.** This paper investigates a special class of dynamics of the equation

$$v_\tau = \Big[D(v)v_x\Big]_x + f(v) \quad \tau \geq 0,\, x \in \mathbb{R}, \tag{1}$$

which has been proposed to model several biological phenomena (see e.g. [12]). Perhaps the most well known version is the so called mono-stable (or Fisher) model, where $f(v)$ satisfies

$$f(v) > 0 \ \text{ in } (0,1)\ , \quad f(0) = f(1) = 0. \tag{2}$$

In this case, $v \equiv 0$ and $v \equiv 1$ are solutions of (1) and it is of great interest to investigate the form of solutions $v(\tau, x)$ connecting these two stationary states. This analysis is mainly driven by searching for traveling wave solutions (t.w.s.) of (1) lying between 0 and 1. Indeed t.w.s. play a relevant role in describing the asymptotic behavior of general classes of solutions of (1) (see e.g. [7] for some recent results).

We recall that a t.w.s. is a solution $v(\tau, x)$ having a constant profile, that is such that $v(\tau, x) = u(x - c\tau) = u(t)$ for some function $u(t)$, the wave shape, and constant c, the wave speed. Notice in particular that a t.w.s. connecting 1 to 0 always satisfies the boundary value problem

$$(D(u)u')' + cu' + f(u) = 0 \tag{3}$$

$$u(-\infty) = 1, \quad u(+\infty) = 0. \tag{4}$$

The case when (1) models a reaction-diffusion process, that is when

$$D(u) > 0 \quad \text{ for all } u \in (0,1) \tag{5}$$

with possibly $D(0) = 0$ or $D(0) = D(1) = 0$, has been extensively studied. Under different regularity conditions, it has been shown that equation (1) is able to support a continuum of wave profiles connecting 0 to 1, parameterized by their speeds. A relatively full account of the results obtained and the techniques used can be found in the papers [3, 9, 14, 15] and in the monographs [4, 5].

Equation (1) can provide a model also for aggregating processes, occurring when $D(u) < 0$ in some interval in $(0,1)$. Indeed it describes the behavior of a population which reacts against the threat of extinction by clustering into groups (see [13] and references contained there). In this context the term $f(u)$ accounts for the net rate of growth.

These models originate from suitable approximations of discrete processes where attraction among individuals is combined with an underlying random walk behavior (see e.g. [18]). In the case of zero reproduction rate, the approximated dynamic precisely behaves as follows

$$v_t = -\big(vk(v)v_x\big)_x + \mu v_{xx}$$

where $\mu > 0$, $k(v)$ is a continuous function and the term μv_{xx} accounts for the underlying random walk process. Notice that, in this case, $D(v) = \mu - vk(v)$. With a low population density, the probability of conspecifics present in the vicinity can be ignored. Therefore μv_{xx} prevails and this explains why $D(u) > 0$ for u near 0.

The aim of this paper is to study the influence of aggregating phenomena. Therefore, throughout this paper we assume that

$$D(u) > 0 \ \text{ in } \ (0, \beta)\ , \quad D(u) < 0 \ \text{ in } (\beta, 1). \tag{6}$$

for some given $\beta \in (0,1)$.

We remark that the dynamics and the form of front propagation for equation (1) under condition (6), is still an open problem. We start to develop this theory in the present paper and we refer to [16] for a similar investigation in the special case when $D(u) < 0$ for all $u \in (0, 1)$.

In the interval $(\beta, 1)$ where $D(u) < 0$, the standard initial-boundary value problems associated with (1) are not well-posed (see [1] and [2]). Recall that according to Hadamard [6] a problem is said to be well-posed if a unique solution exists which depends continuously on the initial data.

The study of ill-posed problems, both with numerical (see [11]) and analytical methods, has been the object of increasing interest mainly motivated by their applications as possible models for describing a number of biological phenomena. In these frameworks, in [13] Padrón has recently investigated ill-posed initial-boundary problems associated with (1), by a Sobolev perturbation method obtaining the existence and uniqueness of a global solution. This technique, already employed for the backward heat equation, consists of introducing in (1) the additional regularizing term $(\lambda v_t - \lambda f(v))_{xx}$ with $\lambda > 0$, where $f(v)$ is of a bi-stable type, accounting for the reduction of reproductive opportunities at low population densities. Numerical simulations, carried out for small values of the parameter λ, suggest that the solutions stabilize asymptotically in time towards discontinuous steady states $v_\infty(x)$ which typically assume only the values 0 and 1.

The possible ill-posedness of our non-regularized model (1) is due to the very nature of aggregating processes. Indeed, they are characterized by movement of individuals up the population gradient, so that the gradient becomes steeper and steeper, causing the unevenness of the solution. In other words, the aggregation process can cause the appearance of clumps, with sharp borders. So, one expects that the existence of a classical solution could fail in this context. On the other hand, this simple model provides analytical insights into the aggregation processes. Moreover, notice that discrete models underlying equation (1) are well-posed and numerical computations (see [18]) reveal a good agreement between the information obtained in the discrete setting and the predictions derived from equation (1).

This paper gives a fairly complete investigation of front propagation for equation (1) under condition (6) which helps in better understanding aggregating processes. Front solutions, in fact, provide a class of regular solutions of (1) (regularity possibly failing at most at the equilibria $v = 0$ and $v = 1$). We also remark that minimum speed profiles, in particular sharp profiles of type (III) (see definition below), seem in good agreement with steady states obtained in [13].

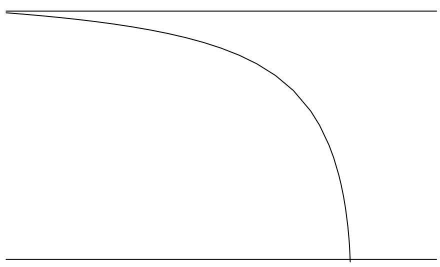
An interesting situation occurs when $D(0) = 0$. Due to the consequent lack of regularity at $u = 0$, in the diffusive case (5), the appearance of weak t.w.s. corresponding to the minimum wave speed was shown in [14, 15]. In this case, the dynamics (1) is usually said to exhibit the important phenomenon of finite speed of propagation. The investigation was then generalized in [9] to a wider class of dynamics and to a doubly-degenerate diffusivity, i.e. satisfying $D(0) = D(1) = 0$. The relevance of the latter case to the theory of biological pattern formation was discussed in [17].

The main aim of this paper is to show that, when $D(0) = D(1) = 0$, diffusion-aggregation models are able to support three different types of weak t.w.s., including the one observed in the case of diffusion. All of them appear with minimum wave speed, and can justify both right and left finite speed of propagation; further, in one case, they exhibit the compact support property. The following definition describes their behavior.

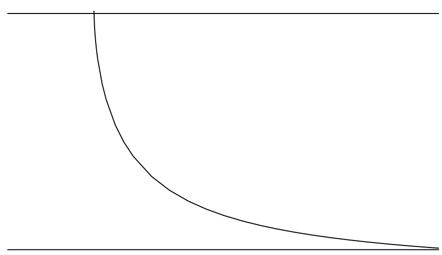
Definition 1. Consider equation (1) under conditions (2) and (6). A function $u = u(t)$ is said to be a *sharp-type* t.w.s. of (1) connecting the stationary states 0 and 1, if one of the following three cases occurs:

- $D(0) = 0$ and there exists $t_1^* \in \mathbb{R}$ such that $u(t)$ is a solution of (3) in $(-\infty, t_1^*)$
$$u(-\infty) = 1,\ u(t_1^{*-}) = 0,\ u'(t_1^{*-}) \neq 0 \text{ and } \lim_{t \to t_1^{*-}} D(u(t))u'(t) = 0. \tag{7}$$
In this case u is said to be *sharp of type (I)*.
- $D(1) = 0$ and there exists $t_0^* \in \mathbb{R}$ such that $u(t)$ is a solution of (3) in $(t_0^*, +\infty)$
$$u(t_0^{*+}) = 1,\ u'(t_0^{*+}) \neq 0,\ u(+\infty) = 0 \text{ and } \lim_{t \to t_0^{*+}} D(u(t))u'(t) = 0. \tag{8}$$
In this case u is said to be *sharp of type (II)*
- $D(0) = D(1) = 0$ and there exist $t_0^*, t_1^* \in \mathbb{R}$ such that $u(t)$ is a solution of (3) in (t_0^*, t_1^*)
$$\begin{aligned} &u(t_0^{*+}) = 1,\ u'(t_0^{*+}) \neq 0,\ u(t_1^{*-}) = 0,\ u'(t_1^{*-}) \neq 0 \text{ and} \\ &\lim_{t \to t_0^{*+}} D(u(t))u'(t) = \lim_{t \to t_1^{*-}} D(u(t))u'(t) = 0. \end{aligned} \tag{9}$$
In this case u is said to be *sharp of type (III)*

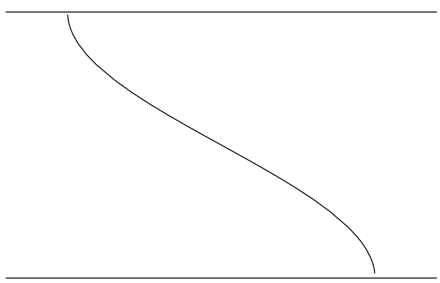
We underline that in the previous definition the values $u'(t_0^{*+})$, $u'(t_1^{*-})$ may be infinite. Classical t.w.s. will be denoted *front-type* solutions. The following figures show the various profiles of t.w.s.



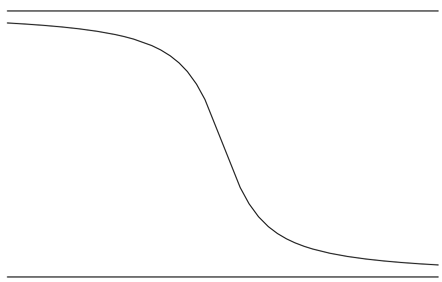
sharp of type (I) t.w.s.



sharp of type (II) t.w.s.



sharp of type (III) t.w.s.



front-type t.w.s.

In Remark 2 we will see that, whenever the t.w.s. u is sharp of type (I), with a wave speed c, and $\dot{D}(0) \neq 0$, u reaches the equilibrium 0 with the slope $-c/\dot{D}(0)$. In this case the limit involving $D(u(t))u'(t)$ is trivially satisfied hence the dynamics corresponds to the same weak behavior observed in [14] for diffusive processes. Similarly, when the t.w.s. is sharp of type (II) with wave speed c and $\dot{D}(1) \neq 0$, it leaves the equilibrium 1 with the slope $-c/\dot{D}(1)$. Instead, when $\dot{D}(0) = 0$ or $\dot{D}(1) = 0$, the sharp t.w.s. have infinite slope at the corresponding equilibrium and the validity of the limit in Definition 1 is not trivial anymore and becomes an effective requirement which characterizes sharp t.w.s.

Concerning diffusive processes, many authors pointed out (see e.g. part IV of the proof of Theorem 1) the equivalence between the existence of t.w.s. for (1) and for the constant diffusion equation

$$v_\tau = v_{xx} + D(v)f(v), \quad \tau \geq 0,\, x \in \mathbb{R}. \tag{10}$$

This explains the introduction of the following function

$$g(u) := D(u)f(u), \qquad u \in [0,1] \tag{11}$$

when treating this subject. When D satisfies conditions (6), the above equivalence implies the existence of a system of t.w.s. for equation (1). In fact, for $u \in [0,\beta]$ the function $g(u)$ represents a mono-stable reaction term, hence (10) has infinitely many t.w.s., connecting 0 to β, with the minimum wave speed $c^*_{0\beta} > 0$. On the other hand, in the interval $[\beta,1]$, equation (10) is equivalent to

$$w_\tau = w_{xx} + \tilde{D}(w)\tilde{f}(w), \quad \tau \geq 0,\, x \in \mathbb{R}$$

where $\tilde{D}(u) := -D(1-u)$ and $\tilde{f}(u) := f(1-u)$ for $u \in [\beta,1]$. Therefore, a second mono-stable reaction process is involved, in $[\beta,1]$, giving rise to other infinitely many t.w.s. from β to 1 whose wave speeds are greater than or equal to a certain threshold $c^*_{\beta 1} > 0$.

The existence and properties of front propagation between 0 and 1 in diffusion-aggregation processes highly depends on this system of waves. The following theorem, which is our main result, investigates this dependence. While sharp waves of type (I) are also encountered in diffusive processes, we remark that sharp profiles of types (II) and (III) are typical of aggregation-diffusion processes and, as far as we know, have not been previously discussed in this framework. Instead, profiles of types (II) and (III) occur in reaction-diffusion-convection processes and their appearance in that context is due to the convective effects, as shown in [10].

Before stating the main theorem, we need to recall the definition of Dini-derivatives. Given an arbitrary function $h(u)$ and a point u_0 in its domain, we recall that the upper right Dini-derivative in u_0 is given by

$$D^+h(u_0) = \limsup_{u \to u_0} \frac{h(u) - h(u_0)}{u - u_0}.$$

When replacing lim sup with lim inf one can define the lower right Dini-derivative $D^-h(u_0)$ and similar definitions hold for the upper and lower left Dini-derivatives $D_+h(u_0)$ and $D_-h(u_0)$. Of course they all reduce to $h'(u_0)$, when it exists.

Theorem 1. *Let $f \in C[0,1]$ and $D \in C^1[0,1]$ be given functions respectively satisfying* (2) *and* (6). *Assume*

$$D^+g(0) < +\infty \tag{12}$$

$$D^-g(1) < +\infty \tag{13}$$

with g defined as in (11)*. Then, there exists a value $c^* > 0$, satisfying*

$$2\sqrt{\max\{D_+g(0), D_-g(1)\}} \le c^* \le 2\sqrt{\max\Big\{\sup_{0<s\le\beta}\frac{g(s)}{s}, \sup_{\beta\le s<1}\frac{g(s)}{s-1}\Big\}} \tag{14}$$

such that equation (1) *has*
i) no t.w.s. satisfying (4) *for $c < c^*$;*
ii) a unique (up to space shifts) t.w.s. for $c = c^*$. It is:
- *sharp of type (I) if and only if $D(0) = 0$ and $c^*_{0\beta} > c^*_{\beta 1}$,*
- *sharp of type (II) if and only if $D(1) = 0$ and $c^*_{0\beta} < c^*_{\beta 1}$,*
- *sharp of type (III) if and only if $D(0) = D(1) = 0$ and $c^*_{0\beta} = c^*_{\beta 1}$,*
- *of front-type and satisfying* (4) *in the remaining cases;*

iii) a unique (up to space shifts) t.w.s. satisfying (4) *for $c > c^*$ which always has a front-type profile.*

Notice that condition (12) is satisfied both when $D(0) = 0$ and when $f'(0^+)$ exists and is finite, because $D^+g(0) = 0$ and $D^+g(0) = D(0)f'(0^+)$, respectively. Similarly, (13) is satisfied whenever $D(1) = 0$ or $f'(1^-)$ exists and is finite.

As an illustrative example, we consider the following model proposed by Turchin (see [18] and also [16]), in order to describe the aggregative movements of *Aphis varians*

$$u_t = \Big[\Big(\frac{\mu}{2} - 2k_0 u(1 - \frac{u}{w}\Big)u_x\Big]_x, \qquad t \ge 0,\ x \in \mathbb{R} \tag{15}$$

where $u = u(t,x)$ denotes the population density and μ, k_0, w are positive real constants; (15) can be considered as the first consistent model which takes into account the mutual attraction and repulsion behaviors of individuals. Indeed, when $\mu < k_0 w$ and $w\mu < 4k_0(w-1)$, the function $D(u) = \frac{\mu}{2} - 2k_0 u(1 - \frac{u}{w})$ satisfies condition (6) with $\beta = \frac{w}{2}\Big[1 - \sqrt{1 - \frac{\mu}{k_0 w}}\Big]$.

When a nonlinear rate of growth $f(u)$ with a logistic-type behavior (i.e. satisfying (2)), is included, the process belongs to the class investigated in Theorem 1. Therefore, it is able to support infinitely many t.w.s. parameterized by their wave speeds $c \ge c^* > 0$; moreover, since $D(0) = \frac{\mu}{2} > 0$, these waves are all of front-type. Finally, in the special case when $f(u) = u(1-u)$, according to (14), the minimum speed c^* satisfies $c^* \ge 2k_0(1 - \frac{1}{w}) - \frac{\mu}{2} > 0$.

The proof of Theorem 1 is developed in several parts and is contained in Section 3. Section 2 is devoted to diffusive processes, i.e. to the case when D satisfies (5). It shows the main techniques employed throughout the paper and gives some preliminary results.

For the investigation of t.w.s. in reaction-diffusion processes, a dynamical systems approach has often been successfully employed. To begin, the first order singular system

$$\begin{cases} u' = w \\ D(u)w' = -cw - \dot{D}(u)w^2 - f(u) \end{cases} \tag{16}$$

associated with (3) is written down. Notice, however, that when D satisfies (6) and $D(0) = 0$, (16) has two singularities: $u = 0$ and $u = \beta$. Therefore, the classical change of variables $\tau = \tau(t)$ defined by $\frac{d\tau}{dt} = \frac{1}{D(u(t))}$ (see e.g. [12], page 290) may not be valid on all the strip $\{(u,w) : 0 < u < 1, w \in \mathbb{R}\}$. It is mainly for this reason that it seems quite difficult to follow a similar approach, when condition (6) holds.

In this paper we reduce the existence of t.w.s. to the study of the first order singular equation which was already proposed in [3] and [4]. In particular, let $u(t)$ denote a wave profile of (1) and assume $u'(t) < 0$ for t in some interval (a, b); the latter inequality is always true, as we shall see in Proposition 1. For $u \in (u(b), u(a))$, it is possible to express $u' = u'(u)$ so we can define

$$z(u) := D(u)u'(u), \quad u(b) < u < u(a). \tag{17}$$

It is easy to see that $z(u)$ satisfies the first order equation

$$\dot{z} = -c - \frac{g(u)}{z} \tag{18}$$

and we remark that (18) is singular, when considered on all the interval $(0, 1)$. In contrast to [3, 4], here (18) will be studied with comparison-type methods.

2. **Mono-stable reaction-diffusion processes.** This section deals with the case when D is strictly positive in $(0, 1)$, including the cases $D(0) = 0$ and $D(1) = 0$. When assuming $D(0) = 0$ but $\dot{D}(0) \neq 0$, it was proved in [9] that the problem of existence of t.w.s. of (1) from 0 to 1 is equivalent to the solvability condition of the following boundary value problem

$$\begin{cases} \dot{z} = -c - \frac{g(u)}{z}, \quad u \in (0, 1) \\ z(u) < 0, \\ z(0^+) = 0, \quad z(1^-) = 0 \end{cases} \tag{19}$$

associated with (18). As it is known (see e.g. [14]), in this case the weak t.w.s. corresponding to the threshold speed is sharp of type (I). In order to simplify notation, throughout we put

$$\bar{t} := \begin{cases} +\infty & \text{if } u(t) \text{ is a front-type t.w.s.} \\ t_1^* & \text{if } u \text{ is a sharp t.w.s. of type (I)} \end{cases} \tag{20}$$

Theorem 3 (below) again investigates this equivalence, but in the more general situation $D(0) = \dot{D}(0) = 0$ and $D(1) = \dot{D}(1) = 0$.

We start with Theorem 2 concerning the solvability of problem (19). The result appeared in [8] for $D(u) > 0$ in $[0, 1]$, and then in [9] for $D(0) = 0$ but $\dot{D}(0) \neq 0$. We present it here again for the sake of completeness, as it is a preliminary result for studying mono-stable diffusion-aggregation processes.

Theorem 2. *Let $f \in C[0, 1]$, $D \in C^1[0, 1]$, respectively satisfying* (2) *and* (5) *and assume that* (12) *holds. Then there exists $c^* > 0$ satisfying*

$$2\sqrt{D_+ g(0)} \le c^* \le 2\sqrt{\sup_{s\in(0,1]} \frac{g(s)}{s}} \tag{21}$$

such that (19) *is solvable if and only if $c \ge c^*$. Moreover, for every $c \ge c^*$, the solution is unique.*

Proof. A similar result was already proved in [9, Theorem 9] when $D(0) = 0$. According to (12), when $D(0) > 0$, the value $N := \sup_{s\in(0,1]} \frac{g(s)}{s}$ defined in *step I* of the quoted result satisfies $N < +\infty$. Consequently, one can reason as in [9] and prove, in all cases, the unique solvability of (19) for every $c \ge c^*$, with $c^* > 0$ satisfying the upper bound in (21). Thus it remains to show only the lower bound of (21) when $D(0) > 0$. This follows from [8, Theorem 4.1] and the proof is complete. $\square$

The following result is needed in order to prove the equivalence stated in Theorem 3.

Proposition 1. *Let $u(t)$ be a sharp t.w.s. of type (I), or a front-type t.w.s. satisfying* (4)*, of equation* (1)*. Then $u'(t) < 0$ whenever $0 < u(t) < 1$ and the wave speed c is positive.*

Proof. Let $u(t)$ be a t.w.s. with speed c which is sharp of type (I), or of front-type satisfying (4). Integrating (3) in some interval $[t, t_1] \subset (-\infty, \bar{t})$ with $\bar{t}$ defined as in (20), one obtains

$$D(u(t_1))u'(t_1) - D(u(t))u'(t) = -c(u(t_1) - u(t)) - \int_t^{t_1} f(u(s))\, ds.$$

Since f is positive and $u(-\infty) = 1$, the limit $\lim_{t\to-\infty} D(u(t))u'(t)$ exists, hence it must be zero. When $u(t)$ is of front-type, one can similarly show that $\lim_{t\to+\infty} D(u(t))u'(t) = 0$. Consequently, from integrating equation (3) over $(-\infty, \bar{t})$ it follows that

$$-c + \int_{-\infty}^{\bar{t}} f(u(s))\, ds = 0$$

implying $c > 0$. Moreover, according to (2) and (5), from (3) it follows that if there exists $t_1 \in \mathbb{R}$ such that $0 < u(t_1) < 1$ and $u'(t_1) = 0$, then t_1 is a proper local maximum point of $u(t)$. Therefore, since $u(-\infty) = 1$, there must be another point $-\infty < t_0 < t_1$ satisfying $u(t_0) = u'(t_0) = 0$. Integrating (3) in $[t_0, \bar{t}]$, with $\bar{t}$ defined as in (20), it follows that

$$\int_{t_0}^{\bar{t}} f(u(s))\, ds = 0$$

which contradicts the positivity of f. Hence $u'(t) < 0$ whenever $0 < u(t) < 1$ and the proof is complete. $\square$

Theorem 3. *Let $f \in C[0, 1]$, $D \in C^1[0, 1]$, respectively satisfying* (2) *and* (5)*. The existence of a t.w.s. $u(t)$ of equation* (1)*, with wave speed c, satisfying* (4) *or* (7)*, is equivalent to the solvability of problem* (19)*, with the same c.*

Proof. Let $u(t)$ be a t.w.s. of (1) satisfying (4) or (7). According to Proposition 1, $z(u)$ can be defined as in (17), for $0 < u < 1$. Moreover, in the proof of Proposition 1 it is shown that $\lim_{t\to-\infty} D(u(t))u'(t) = 0$; the same is true also as $t \to +\infty$ and $u(t)$ is of front-type. Hence the necessary condition holds.

It remains to prove sufficiency. Let $z(u)$ be a solution of (19) for some $c > 0$ and denote by $u(t)$ the unique solution of

$$u'(t) = \frac{z(u)}{D(u)}, \quad u(0) = \frac{1}{2}$$

defined on its maximal existence interval (a, b). Notice that u is monotone decreasing, and $u(a^+) = 1$, $u(b^-) = 0$. Further, $u(t)$ satisfies (3) for $a < t < b$, hence it is a t.w.s. of (1). Now we show that the boundary conditions (4) or (7) are satisfied. Indeed, since

$$\lim_{t\to a^+} u'(t) = \lim_{u\to 1^-} \frac{z(u)}{D(u)} \quad \text{and} \quad \lim_{t\to b^-} u'(t) = \lim_{u\to 0^+} \frac{z(u)}{D(u)}, \tag{22}$$

if $D(0) \cdot D(1) \neq 0$, then $u(t)$ satisfies (4) also if $a > -\infty$ and/or $b < +\infty$. Consider now the case $D(1) = 0$. Reasoning as in the proof of [9, Lemma 15], it is possible to show that $\dot{z}(1)=0$. We remark that $\dot{z}(1)=0$ holds also when $\dot{D}(1)=0$. This case was not considered in the quoted paper. When $\dot{D}(1)\neq 0$, this immediately implies

$$\lim_{t\to a^+} u'(t)=\lim_{u\to 1^-}\frac{z(u)}{D(u)}=\lim_{u\to 1^-}\frac{z(u)}{u-1}\frac{u-1}{D(u)}=0.$$

Now, let $D(1)=\dot{D}(1)=0$. Since $\dot{z}(1)=0$, there must be a sequence $\{u_n\}$ such that $u_n\to 1^-$ and $\dot{z}(u_n)\to 0$ as $n\to+\infty$. According to (19), this implies

$$\frac{z(u_n)}{D(u_n)}\to 0 \quad \text{as} \quad n\to+\infty. \tag{23}$$

Given $\epsilon>0$, let $\zeta_\epsilon(u):=-\epsilon D(u)$. It follows that

$$\dot{\zeta}_\epsilon(u)\to 0, \text{ and } -c-\frac{D(u)f(u)}{\zeta_\epsilon(u)}\to -c \text{ as } u\to 1^-; \tag{24}$$

hence there exists $\delta>0$ such that

$$\dot{\zeta}_\epsilon(u)>-c-\frac{D(u)f(u)}{\zeta_\epsilon(u)}, \quad \text{for} \quad 1-\delta<u<1.$$

According to (23), it is not restrictive to assume $z(1-\delta)>\zeta_\epsilon(1-\delta)$. We claim that $z(u)>\zeta_\epsilon(u)$ for all $1-\delta<u<1$. Indeed, by contradiction, assume the existence of $1-\delta<\bar{u}<1$ such that $z(\bar{u})=\zeta_\epsilon(\bar{u})$. By (24) we have $\dot{z}(\bar{u})<\dot{\zeta}_\epsilon(\bar{u})$; moreover, $z(u)<\zeta_\epsilon(u)$ in an interval, implies $\dot{z}(u)<\dot{\zeta}_\epsilon(u)$ in the same interval again from (24). This leads to the contradictory conclusion $z(1^-)<0$. Therefore, $z(u)>\zeta_\epsilon(u)$ on all $[1-\delta,1)$, hence

$$\liminf_{t\to a^+}\ u'(t)=\liminf_{u\to 1^-}\frac{z(u)}{D(u)}\geq\lim_{u\to 1^-}\frac{\zeta_\epsilon(u)}{D(u)}=-\epsilon.$$

Since $u'(t)<0$ for every $t\in(a,b)$, the arbitrariness of ϵ implies $u'(a^+)=0$. Therefore, $u(t)$ can always be continued on all $(-\infty,b)$ and $u(-\infty)=1$.

Assume now $D(0)=0$. Reasoning as in [9, Lemma 5], it is possible to show that $\dot{z}(0)$ exists and $\dot{z}(0)=0$ or $\dot{z}(0)=-c$. Obviously $b=+\infty$ implies $u(+\infty)=0$ and $u(t)$ is a front-type t.w.s. Thus the interesting case occurs when b is finite. If $\dot{D}(0)\neq 0$ and $\dot{z}(0)=0$, then (22) yields $u'(b^-)=0$; therefore $u(t)$ can be continued on all $(-\infty,+\infty)$ and it is of front-type. Instead, when $\dot{z}(0)=-c<0$, by (22) it follows that $u'(b^-)=\dfrac{-c}{\dot{D}(0)}$. In every case $u(t)$ satisfies (7) with $t_1^*=b$, hence it is a sharp t.w.s.

Assume now $D(0)=0$ and $\dot{z}(0)=-c$. In this case we have $\lim_{t\to b^-} u'(t)=\lim_{u\to 0^+}\dfrac{z(u)}{D(u)}=-\infty$, so $b<+\infty$ and u satisfies (7), that is u is sharp of type (I).

The final case to be considered is $D(0)=\dot{D}(0)=\dot{z}(0)=0$. Since $\dot{z}(0)=0$, it is possible to find a sequence $\{u_n\}$ such that $u_n\to 0^+$ and $\dot{z}(u_n)\to 0$ as $n\to\infty$. According to (19), this implies

$$\frac{D(u_n)f(u_n)}{z(u_n)}\to -c, \quad \text{as } n\to+\infty. \tag{25}$$

Given an arbitrary $\epsilon>0$, consider again $\zeta_\epsilon(u)$ with $u\in(0,1)$. Notice that it is possible to find $\delta\in(0,1)$ satisfying

$$\dot{\zeta}_\epsilon(u)>-c-\frac{D(u)f(u)}{\zeta_\epsilon(u)} \quad \text{for } u\in(0,\delta).$$

According to (25), assuming δ sufficiently small, $z(\delta) > \zeta_\epsilon(\delta)$ holds. Consequently, it is easy to prove that $z(u) > \zeta_\epsilon(u)$ on all $(0, \delta)$, which implies

$$\liminf_{u\to 0^+} \frac{z(u)}{D(u)} \geq \lim_{u\to 0^+} \frac{\zeta_\epsilon(u)}{D(u)} = -\epsilon.$$

For the arbitrariness of $\epsilon > 0$, $u'(b) = 0$ follows and the t.w.s. is of front-type. $\square$

Remark 1. According to Theorem 2 and Proposition 1, it is possible to prove that (1) admits a unique (up to space shifts) t.w.s. of wave speed c satisfying (4) or (7) if and only if $c \geq c^*$ with c^* defined as in (21). Its profile is sharp if and only if $D(0) = 0$ and $\dot{z}(0) = -c$. Notice that $\dot{z}(0) = -c$ may occur at most when $c = c^*$, as proved in [9, Corollary 11]. Indeed $\dot{z}(0) = -c^*$, when $D(0) = 0$, as shown in [9, Theorem 2] where the proof does not depend on $\dot{D}(0)$; therefore the profile is sharp if and only if $D(0) = 0$ and $c = c^*$. Though this is an expected result, and we refer, for example, to [4] and [5, Theorem 33] for the cases when $D(u) > 0$ in $[0, 1]$, and to [9] and [15] for $D(0) = D(1) = 0$, $D(u) > 0$ in $(0, 1)$ but $\dot{D}(0)\dot{D}(1) \neq 0$, we remark that the techniques introduced in this paper are able to discuss the qualitative behavior of the minimum speed front also when $D(0) = \dot{D}(0) = 0$ and/or $D(1) = \dot{D}(1) = 0$.

3. Mono-stable diffusion-aggregation processes.

By means of the results and techniques introduced in the previous section, we now present the proof of Theorem 1 dealing with front propagation for mono-stable diffusion-aggregation dynamics.

Proof of Theorem 1. The proof contains several parts.

I - Existence of fronts for $c \geq c^*$ and estimate of c^*. Let us consider the first order equation (18) for $0 < u < \beta$. Note that $g(u)$ is a mono-stable, i.e. of Fisher-type, reaction function in the interval $[0, \beta]$. Since (12) is satisfied, Theorem 2 holds in $[0, \beta]$, and we deduce the existence of a threshold value $c_1^* > 0$, satisfying the estimate

$$2\sqrt{D_+ g(0)} \leq c_1^* \leq 2\sqrt{\sup_{s\in(0,\beta]} \frac{g(s)}{s}} \tag{26}$$

such that (18) has a unique negative solution $z(u)$ in $(0, \beta)$ with $z(0^+) = z(\beta^-) = 0$, if and only if $c \geq c_1^*$.

Given $c \geq c_1^*$, consider the Cauchy problem

$$\begin{cases} u' = \frac{z(u)}{D(u)}, & 0 < u < \beta \\ u(0) = \frac{\beta}{2}. \end{cases} \tag{27}$$

Let $u(t)$ be the unique solution of (27) defined in its maximal existence interval (t_1, t_2), with $-\infty \leq t_1 < t_2 \leq +\infty$. It is easy to see that $u(t)$ is a solution of (3) in (t_1, t_2).

Observe now that $u'(t) < 0$ for every $t \in (t_1, t_2)$, so there exists the limit $u(t_2^-) \in [0, \beta)$. Since $z(u) \neq 0$ in $(0, \beta)$, we deduce that $u(t_2^-) = 0$. Moreover, since $\dot{z}(u) = -c - \frac{g(u)}{z(u)} > -c$ for every $u \in (0, \beta)$, then $z(u) > -cu$ in $(0, \beta)$. So, denoting by $t(u)$ the inverse function of $u(t)$ in (t_1, t_2), when $D(0) \neq 0$ we have

$$t_2 - t\Big(\frac{\beta}{2}\Big) = \int_{\beta/2}^{0} \dot{t}(u)\, du = \int_0^{\beta/2} \frac{D(u)}{-z(u)}\, du \geq \int_0^{\beta/2} \frac{D(u)}{cu}\, du = +\infty, \tag{28}$$

that is, $t_2 = +\infty$.

In order to study the behavior of $u(t)$ when $t \to t_1^+$, we will use the following lemma, whose proof is left until the end of the present proof.

Lemma 1. *Under the assumptions of Theorem 1, for every $c \geq c_1^*$ there exists $\dot{z}(\beta) = \frac{1}{2}(\sqrt{c^2 - 4f(\beta)\dot{D}(\beta)} - c)$, and*

$$\lim_{u \to \beta^-} \frac{z(u)}{D(u)} = -\frac{2f(\beta)}{c + \sqrt{c^2 - 4f(\beta)\dot{D}(\beta)}}. \tag{29}$$

Taking into account that $u(t_1^+) = \beta$ and according to (29) there exists the limit

$$\lim_{t \to t_1^+} u'(t) = \lim_{u \to \beta^-} \frac{z(u)}{D(u)} \neq 0.$$

Therefore $t_1 \in \mathbb{R}$ and we obtain a solution $u(t)$ of equation (3) in (t_1, t_2) such that $t_1 \in \mathbb{R}$, $t_2 \leq +\infty$, $u(t_1) = \beta$, $u(t_2) = 0$ and $u'(t_1)$ satisfying (29). Moreover $D(0) \neq 0$ implies $t_2 = +\infty$.

Let us now consider (3) when $\beta < u < 1$. We make the following change of variable:

$$\tilde{D}(u) := -D(1-u)\ , \quad \tilde{f}(u) := f(1-u)\ , \quad \tilde{g}(u) := \tilde{D}(u)\tilde{f}(u) = -g(1-u).$$

Since $\tilde{g}(u) > 0$ for every $0 < u < 1 - \beta$, with $\tilde{g}(0) = \tilde{g}(1-\beta) = 0$ and according to (13), $D^+\tilde{g}(0) < +\infty$, we can apply Theorem 2 in $[0, 1-\beta]$ and derive the existence of a threshold value $c_2^* > 0$ satisfying

$$2\sqrt{D_+\tilde{g}(0)} \leq c_2^* \leq 2\sqrt{\sup_{s \in (0, 1-\beta]} \frac{\tilde{g}(s)}{s}},$$

that is

$$2\sqrt{D_- g(1)} \leq c_2^* \leq 2\sqrt{\sup_{s \in [\beta, 1)} \frac{g(s)}{s-1}}, \tag{30}$$

such that the equation

$$\dot{w} = -c - \frac{\tilde{g}(u)}{w}\ , \quad 0 < u < 1 - \beta, \tag{31}$$

admits negative solutions $w(u)$, satisfying $w(0^+) = w((1-\beta)^-) = 0$, if and only if $c \geq c_2^*$. Putting $z(u) := -w(1-u)$, we have

$$\dot{z}(u) = \dot{w}(1-u) = -c - \frac{\tilde{g}(1-u)}{w(1-u)} = -c - \frac{g(u)}{z(u)}\ , \quad \beta < u < 1.$$

Moreover, $z(\beta^+) = z(1^-) = 0$, and $z(u) > 0$ for every $u \in (\beta, 1)$. So, if we consider the Cauchy problem

$$\begin{cases} u' = \frac{z(u)}{D(u)}\ , & \beta < u < 1 \\ u(0) = \frac{1+\beta}{2} \end{cases} \tag{32}$$

we can repeat the same arguments developed for the case when $0 < u < \beta$, and obtain that the unique solution $u(t)$ of problem (32) is a solution of (3) in (τ_1, τ_2), with $\tau_1 \geq -\infty$, $\tau_2 \in \mathbb{R}$, $u(\tau_1^-) = 1$, $u(\tau_2^+) = \beta$ and $u'(\tau_2^+)$ satisfying (29). In fact, as in the case when $u \in (0, \beta)$, it follows that

$$u'(\tau_2^+) = -\frac{2\tilde{f}(1-\beta)}{\sqrt{c^2 - 4\dot{\tilde{g}}(1-\beta)} + c},$$

implying (29). Moreover, since $\dot{z}(u) < -c$ on $(\beta, 1)$, then $z(u) > c(1-u)$ in the same interval. Therefore, when $D(1) \neq 1$, this implies

$$t\Big(\frac{1+\beta}{2}\Big) - \tau_1 = \int_{\frac{1+\beta}{2}}^{1} \dot{t}(u)\, du = \int_{\frac{1+\beta}{2}}^{1} \frac{D(u)}{z(u)} = +\infty \tag{33}$$

that is, $\tau_1 = -\infty$. Therefore, putting $c^* := \max\{c_1^*, c_2^*\}$, by (26) and (30) we find that c^* satisfies the condition (14). Moreover, for every $c \geq c^*$ we can glue the solutions of (27) and (32) by a time-shift, obtaining a C^2 function $u(t)$ on some interval (a, b), with $-\infty \leq a < b \leq +\infty$, which is a decreasing solution of equation (3) in (a, b) and satisfies $u(a^+) = 1$ and $u(b^-) = 0$.

II) Positivity of c^*. This follows immediately from the positivity of c_1^* and c_2^*.

III) Non-existence for $c < c^*$. Let u be a t.w.s. of equation (1). Notice that $D(u) > 0$, whenever $0 < u(t) < \beta$. Therefore it is possible to reason as in the proof of Proposition 1, and show that $u'(t) < 0$ whenever $0 < u(t) < \beta$. This implies the existence of the inverse function $t(u)$ in $(0, \beta)$; defining $z(u)$ as in (17), then $z(u)$ satisfies (18) for $0 < u < \beta$, and $z(\beta^-) = D(\beta)u'(t(\beta)) = 0$. Moreover, as in the proof of Proposition 1, one can show that $\lim_{t \to \bar{t}} D(u)u'(t) = 0$, where $\bar{t}$ was defined in (20); this implies $z(0^+) = 0$.

Summarizing, if (3) has a solution $u(t)$ for some real c, satisfying any one of the conditions (4), (7), (8) and (9), then equation (18) has a negative solution $z(u)$, satisfying $z(0^+) = z(\beta^-) = 0$. Therefore, by applying Theorem 2, we deduce that $c \geq c_1^*$.

Similarly, it is possible to show that if (3) has a solution $u(t)$ for some real c, satisfying any one of the conditions (4), (7), (8) and (9), then $u'(t) < 0$ whenever $\beta < u(t) < 1$. If we define $w(u) := -D(1-u)u'(t(1-u))$ for $0 < u < 1-\beta$, it is easy to check that w is a negative solution of (31), satisfying $w(0^+) = w((1-\beta)^-) = 0$. Therefore, by applying Theorem 2, we deduce that $c \geq c_2^*$.

Summarizing, $c \geq c^*$ is a necessary condition for the existence of t.w.s. of (1) satisfying any one of the conditions (4), (7), (8) and (9).

IV) Sharp-type profile for $c = c^*$. First of all notice that the existence of t.w.s. of (10) between its stationary states 0 and β is equivalent to the solvability of (19) with 1 replaced by β. This can be easily proved following the same arguments as in the proof of Theorem 3. Therefore $c_1^* = c_{0\beta}^*$. Similarly one can show that $c_2^* = c_{\beta 1}^*$.

According to Remark 1, the solution of (27) is sharp of type (I) if and only if $D(0) = 0$ and $c = c_1^*$. Similarly, the solution of the following problem

$$\begin{cases} v' = \frac{w(v)}{-D(1-v)}, & 0 < v < 1-\beta \\ v(0) = \frac{1-\beta}{2} \end{cases} \tag{34}$$

is sharp, again of type (I), if and only if $D(1) = 0$ and $c^* = c_2^*$. Notice that the change of variable $u = 1 - v$ reduces (34) to (32); hence the sharpness phenomenon appears at the point $u = 1$. Since the t.w.s. $u^*(t)$ of (1), having wave speed c^*, is obtained by gluing the solutions of (24) and (34), after a possible time-shift, this implies the behavior of $u^*(t)$ near 0 and 1.

V - Study of the front-type profile for $c > c^*$. The argument derives from Remark 1, and it is similar to the one for $c = c^*$. The proof is then complete. $\square$

Proof of Lemma 1. For $c \geq c_1^*$, let $z(u)$ be the solution of (18) satisfying $z(0^+) = z(\beta^-) = 0$. Put $\alpha := \frac{1}{2}(\sqrt{c^2 - 4\dot{g}(\beta)} - c)$ and assume the existence of the limit

$$\lambda := \lim_{u \to \beta^-} \frac{z(u)}{u - \beta}. \tag{35}$$

Notice that $\lambda \geq 0$, since $z(u) < 0$ in $(0, \beta)$. First consider the case $\dot{D}(\beta) \neq 0$, equivalent to $\dot{g}(\beta) \neq 0$. If $\lambda = 0$, then according to (18), $\dot{z}(u) \to +\infty$ as $u \to \beta^-$, which is not possible. Consequently $\lambda \neq 0$, and since $\dot{z}(u) = -c - \dfrac{g(u)}{u-\beta}\dfrac{u-\beta}{z(u)}$, there exists also $\lim\limits_{u \to \beta^-} \dot{z}(u) = \lambda$. Then, we obtain $\lambda^2 + c\lambda + \dot{g}(\beta) = 0$, hence $\lambda = \alpha$. Assume now $\dot{g}(\beta) = 0$. By (18), we have

$$\lim_{u \to \beta^-} \frac{z(u)[\dot{z}(u) + c]}{u - \beta} = 0.$$

Notice that $\lambda \neq 0$ implies $\lambda = -c$ which is not possible since $\lambda \geq 0$; hence $\lambda = 0 = \alpha$.

We prove now that the limit in (35) exists. Indeed, assume by contradiction that

$$L := \limsup_{u \to \beta^-} \frac{z(u)}{u - \beta} > \liminf_{u \to \beta^-} \frac{z(u)}{u - \beta} := l \geq 0.$$

Let $\gamma \in (l, L)$ and let $(u_n)_n$ be an increasing sequence converging to β such that

$$\frac{z(u_n)}{u_n - \beta} = \gamma\ , \quad \text{and} \quad \frac{d}{du}\left(\frac{z(u)}{u - \beta}\right)_{|u = u_n} \leq 0.$$

Since $\dfrac{d}{du}\dfrac{z(u)}{u-\beta} = \dfrac{1}{u-\beta}\left(\dot{z}(u) - \dfrac{z(u)}{u-\beta}\right)$, we have

$$\dot{z}(u_n) = -c - \frac{g(u_n)}{\gamma\big(u_n - \beta\big)} \geq \gamma.$$

Passing to the limit as $n \to +\infty$, since $\gamma > 0$, we have $\gamma^2 + c\gamma + \dot{g}(\beta) \leq 0$, implying $\gamma \leq \alpha$. Similarly, if we choose an increasing sequence $(v_n)_n$ converging to β, such that

$$\frac{z(v_n)}{v_n - \beta} = \gamma\ , \quad \text{and} \quad \frac{d}{du}\left(\frac{z(u)}{u - \beta}\right)_{|u = v_n} \geq 0,$$

we can deduce $\gamma^2 + c\gamma + \dot{g}(\beta) \geq 0$, that is $\gamma \geq \alpha$. By the arbitrariness of $\gamma \in (l, L)$, we conclude that $l = L = \dot{z}(\beta)$.

It remains to prove (29). When $\dot{D}(\beta) \neq 0$, we obtain

$$\lim_{u \to \beta^-} \frac{z(u)}{D(u)} = \lim_{u \to \beta^-} \frac{z(u)}{u-\beta}\frac{u-\beta}{D(u)} = -\frac{2f(\beta)}{c + \sqrt{c^2 - 4\dot{g}(\beta)}} < 0,$$

so (29) is satisfied. Consider now the case when $\dot{D}(\beta) = 0$. Given $\epsilon > 0$, since $c > 0$, it is possible to find $0 < \delta_1 < \beta$ satisfying

$$-c + \frac{cf(u)}{f(\beta) + c\epsilon} + \Big(\frac{f(\beta)}{c} + \epsilon\Big)\dot{D}(u) < 0, \quad \beta - \delta_1 \leq u \leq \beta.$$

We claim that

$$z(u) > \tilde{z}(u) := -\Big(\frac{f(\beta)}{c} + \epsilon\Big)D(u), \quad \beta - \delta_1 < u < \beta. \tag{36}$$

In fact, assuming $\tilde{z}(u_0) \geq z(u_0)$ for some $u_0 \in (\beta - \delta_1, \beta)$, we obtain

$$\dot{z}(u_0) = -c - \frac{g(u_0)}{z(u_0)} \leq -c + \frac{cf(u_0)}{f(\beta) + c\epsilon} < -\Big(\frac{f(\beta)}{c} + \epsilon\Big)\dot{D}(u_0) = \dot{\tilde{z}}(u_0),$$

implying the contradictory conclusion $z(\beta^-) < \tilde{z}(\beta^-) = 0$. Hence (36) is valid. With a similar reasoning we are able to find $0 < \delta_2 < \beta$ such that

$$z(u) < -\Big(\frac{f(\beta)}{c} - \epsilon\Big)D(u), \quad \beta - \delta_2 < u < \beta, \tag{37}$$

for ϵ sufficiently small. Let $\delta := \min\{\delta_1, \delta_2\}$. According to (36) and (37),

$$-\frac{f(\beta)}{c} - \epsilon \leq \frac{z(u)}{D(u)} \leq -\frac{f(\beta)}{c} + \epsilon, \quad \beta - \delta < u < \beta.$$

Since ϵ is an arbitrary positive value

$$\lim_{u \to \beta^-} \frac{z(u)}{D(u)} = -\frac{f(\beta)}{c};$$

consequently, in this case also (29) is satisfied. $\square$

Remark 2. Assume $D(0) = 0$ and $\dot{D}(0) \neq 0$. Let $u^*(t)$ be a sharp t.w.s. of type (I). According to Theorem 1, its wave speed is $c = c^* = c_1^*$. Moreover, its corresponding $z(u)$, defined as in (17), satisfies (27) with $\dot{z}(0) = -c^*$ (see Remark 1). Hence, it follows that

$$\lim_{t \to t_1^*} {u'}^*(t) = \lim_{u \to 0^+} \frac{z(u)}{D(u)} = \lim_{u \to 0^+} \frac{z(u)}{u} \frac{u}{D(u)} = -\frac{c^*}{\dot{D}(0)}.$$

Similarly one can show that whenever $D(1) = 0$, $\dot{D}(1) \neq 0$ and $u^*(t)$ is a sharp t.w.s. of type (II), then ${u'}^*(t) = -\frac{c^*}{\dot{D}(1)}$. Instead, when $\dot{D}(0) = 0$ or $\dot{D}(1) = 0$ then the threshold t.w.s. reaches the corresponding equilibrium with infinite slope.

Remark 3. According to (28), when $D(0) > 0$, every wave profile $u(t)$ of (1) satisfies $u(t) > 0$ for all $t \in \mathbb{R}$. Similarly, when $D(1) > 0$, by (33) it follows that $u(t) < 1$ for all t.

Remark 4. Assume g differentiable in $[0, 1]$ with $g(u)$ concave in $[0, \beta]$ and convex in $[\beta, 1]$; then

$$c^* = 2\sqrt{\max\{\dot{g}(0), \dot{g}(1)\}} = 2\sqrt{\max\{D(0)\dot{f}(0), D(1)\dot{f}(1)\}}.$$

When, in particular, $\beta = \frac{1}{2}$ and g is a function symmetric with respect to $u = \frac{1}{2}$, then $c^* = c_1^* = c_2^*$. Assuming in addition $D(0) = D(1) = 0$, then sharp t.w.s. of type (III) appear corresponding to the minimum wave speed.

As an illustrative example, let $f(u) = u(1 - u)$ and $D(u) = 1 - 2u$. Then $g(u) = D(u)f(u) = 2u^3 - 3u^2 + u$ is concave in $[0, \frac{1}{2}]$, convex in $[\frac{1}{2}, 1]$; moreover, $\dot{g}(0) = \dot{g}(1) = 1$. Therefore, $c^* = 2$.

4. **Discussion.** The study of travelling wave solutions to reaction-diffusion equations has a long history and it is well known that in the case with constant positive diffusion coefficient and certain nonlinear reaction forms these equations admit smooth travelling wave constant profile solutions with constant wave speed. In this paper, we have analysed the case where the diffusion coefficient is not constant and can change sign. In some very approximate sense, this models a population which is diffusive under certain conditions, but aggregative under other conditions. We

have shown that this equation can exhibit a range of travelling wave behaviours in the degenerate case, including sharp profiles as well as what we term "finite speed of saturation" [10], that is, the dependent variable reaches its maximum value at a finite value of the (independent) spatial variable. We find conditions under which each type of solution occurs and determine the corresponding wave speed.

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