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Bifurcating Spatial Patterns Arising from Travelling Waves in a Tissue Interaction Model

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Abstract—We analyse a tissue interaction model recently proposed to account for pattern formation in the morphogenesis of skin organ primordia. We show that the model can exhibit travelling wave solutions which leave in their wake a spatially nonuniform, steady state solution.

Keywords—Travelling waves, Tissue interaction, Steady state patterns.

1. INTRODUCTION

The formation of spatial pattern is one of the central processes in developmental biology. Many models have been proposed for this process and they are capable of producing a wide range of spatial patterns. In most of these, spatial pattern formation occurs simultaneously on the whole domain (see [1] for review). However, for several development processes, pattern is actually formed sequentially and travelling wavefronts are the precursors to patterning. This is the case in, for example, the pattern formation of feather germ primordia on the chick back: the pattern propogates outwards from the dorsal midline. Recently, a mechanochemical tissue interaction model has been proposed to account for this phenomenon [2]. It has been shown that this model can give rise to travelling waves in one dimension [3] and to propagating patterns in one and two dimensions [4]. In this paper, we show that this tissue-interaction model, in one dimension, can give rise to a different type of patterning, in which a propagating travelling front leaves a spatial pattern behind.

The model is based on the interaction of the epidermis and the dermis within the chick skin. The epidermis is considered to be a viscoelastic material at low Reynolds number which under-

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goes contraction due to cell traction [1,5,6]. Epidermal cells move by convection, whereas dermal cells move randomly and are also chemotatic. The tissue interaction due to the production of two signal chemicals, one in each tissue, which diffuse into the adjacent tissue. The dermally produced chemical stimulates epidermal cell traction, whereas the epidermally produced chemical stimulates dermal cell chomotaxis. The equations for the chemical are standard are not reproduced here [2]. With the above assumptions, we can write down a viscoelastic force balance equation for the epidermal layer, and conservation equations for the epidermal and dermal cell sensitivities, respectively. The force balance equation is

$$\nabla \cdot \left\{ \frac{E}{1+v}\left[\varepsilon - \beta_1 \nabla^2 \varepsilon + \frac{v}{1-2v}(\theta - \beta_2 \nabla^2 \theta)\boldsymbol{I}\right] + \mu_1 \frac{\partial \varepsilon}{\partial t} + \mu_2 \frac{\partial \theta}{\partial t}\boldsymbol{I} + \frac{\tau s^2}{1+cs^2}\boldsymbol{I} \right\} = \rho \boldsymbol{u}, \quad (1)$$

where $\boldsymbol{u}(\mathbf{x}, t)$ is the displacement at time t of a material point in the epithelium which was initially at $\mathbf{x}$, $\theta = \nabla \cdot \boldsymbol{u}$ is the dilation, $\varepsilon = \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top}\right)$ is the strain tensor, where $\top$ denotes the transpose, and s is the concentration of the signal chemical secreted in the dermis. The parameters E and v are the Young's modulus and Poisson's ratio respectively, $\boldsymbol{I}$ is the unit tensor, and β_1 and β_2 reflect long-range elastic stresses [1].

The epidermal cell density $N(\mathbf{x}, t)$ satisfies

$$\frac{\partial N}{\partial t} = -\nabla \cdot N \frac{\partial \boldsymbol{u}}{\partial t}, \quad (2)$$

while the conservation equations for dermal cells is

$$\frac{\partial n}{\partial t} = D\nabla^2 n - \alpha \nabla \cdot n \nabla e + rn(\bar{n} - n), \quad (3)$$

where D is the coefficient of diffusion, α the chemotaxis coefficient, e the concentration of the signal chemical produced in the epithelium, and r and $\bar{n}$ are positive constants.

Assuming that the cell kinematics occur on a fast timescale and integrating the linearised epithelial cell conservation equation the model can be reduced to two equation [4] which, when nondimensionalized, take the form, in one dimension,

$$\mu \frac{\partial^3 \theta}{\partial t \partial x^2} + \frac{\partial^2 \theta}{\partial x^2} - \beta \frac{\partial^4 \theta}{\partial x^4} + \frac{\partial^2}{\partial x^2}\left\{ \frac{\tau n^2}{[1+v(1-\theta)]^2} \right\} = \rho\theta, \quad (4a)$$

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial x^2} - \alpha \frac{\partial}{\partial x}\left\{ n \frac{\partial}{\partial x}\left(\frac{1-\theta}{1+\gamma n} \right) \right\} + n(1-n), \quad (4b)$$

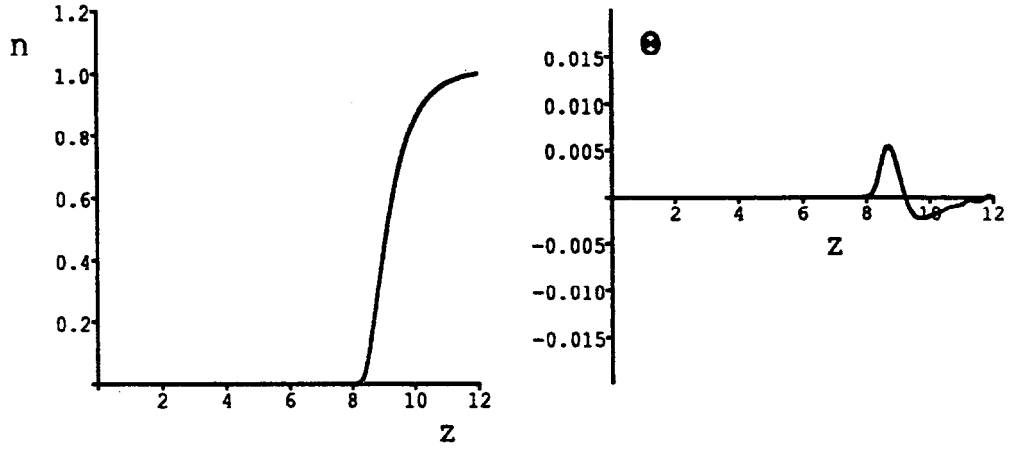
where $\theta(x, t)$ and $n(x, t)$ are the dilation in the epithelial layer and the dermal cell density, respectively, at position x and time t; $\mu, \beta, \tau, \nu, \rho, \alpha$ and γ are positive parameters.

The tissue interaction in these caricature equations is represented in (4a) by the fourth term on the left-hand side, in which cell traction in the epidermis is a function of dermal cell density, and in (4b) by the second term on the right-hand side in which dermal cell chemotaxis is a function of the dilation in the epidermis.

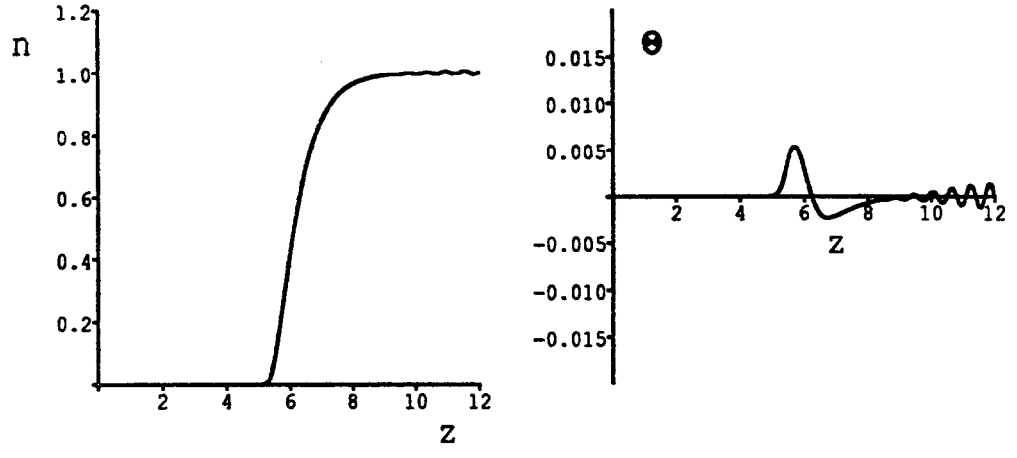
2. PROPAGATING PATTERNS

This system admits two steady states, $n = \theta = 0$, $n = 1$, $\theta = 0$. A standard linear analysis about the uniform steady state state $n = 1$, $\theta = 0$ shows that this state can be linearly unstable for certain parameter values [2]. At this steady state, the dispersion relation $\lambda(k^2)$, which is the temporal growth rate of disturbances with wave number k, satisfies

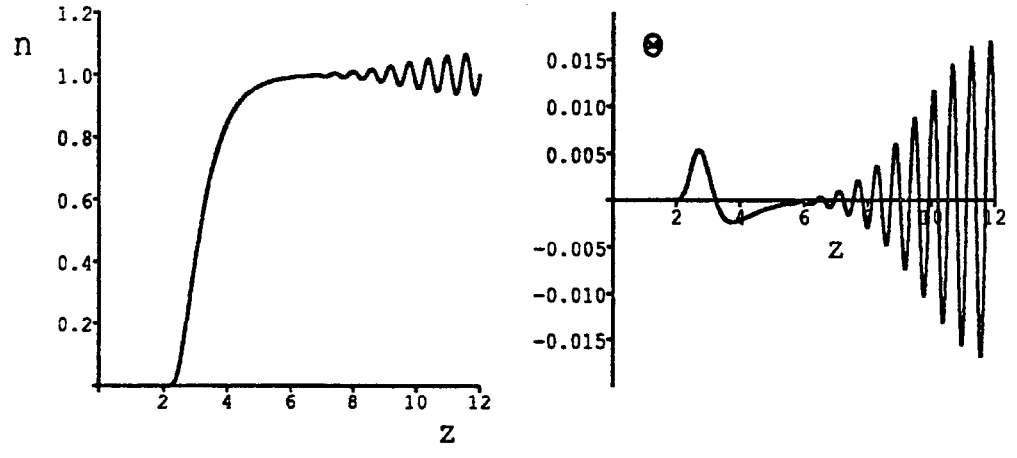
$$\mu k^2 \lambda^2 + [(\mu D + \mu Q_1 + \beta)k^4 + (P_1 + 1 + \mu)k^2 + \rho]\lambda + c(k^2) = 0, \quad (5)$$



(a)



(b)



(c)

Figure 1. The time evolution of travelling wavefronts acting as precursors to pattern formation at times $t = 1.5$, 3.0 and 4.5 for (a), (b) and (c), respectively. The parameter values are $\mu = 0.001$, $D = 0.1$, $\beta = 0.01$, $\tau = 9.73$, $\nu = 3.17$, $\rho = 100.2$, $\alpha = 5.0$, $\gamma = 0.285$, $c = 1.0$, $r = 5$.

where

$$c(k^2) = \beta(D+Q_1)k^6 - (P_2Q_2 - r\beta - DP_1 - P_1Q_1 - D - Q_1)k^4 + (r + rP_1 + \rho D + \rho Q_1)k^2 + \rho r$$

and P_1, P_2, Q_1, and Q_2 are functions of the parameters. Cruywagen and Murray [2] showed that λ can go positive for appropriate parameters values and that in this case the uniform state $(1,0)$ evolves to a spatially nonuniform steady state. Moreover, one can choose the parameters so that a particular mode is isolated and grows while all other modes decay.

Transforming the system to travelling wave coordinates, $z = x + ct$, one can show that the steady state $(0,0)$ has an unstable manifold while the steady state $(1,0)$ has a stable manifold. This suggests the possibility of travelling wave solutions connecting these two steady states. Furthermore, for biologically realistic solutions (that is, nonnegative cell densities), the wavespeed c must be greater than a minimum wavespeed, which is 2. That one can indeed obtain such travelling waves has been shown by [3]. There, we also demonstrated, that for initial conditions with compact support, the wave travels with the minimum wavespeed.

Here we show that one can, in fact, obtain both pattern formation and travelling waves concurrently since travelling waves can act as precursors to spatial pattern formation.

To illustrate this, we consider the domain $x \in [0, 12]$ and we use the logical parameter search method [7] to find values for the parameters which isolate the mode number $k = (13\pi)/2$, so that only this particular mode grows while all others decay. This parameter set (see Figure 1) also gives rise to a travelling wave solution with the minimum wavespeed, $c = 2$, for appropriate initial conditions. We choose as specimen initial conditions

$$n(x,0) = 0.1\delta(12 - x), \qquad \theta(x,0) = 0 \tag{6}$$

corresponding to an initial source of cells at the right-hand side of the domain. The results, as shown in Figure 1, confirm our prediction.

The wavenumber of the pattern agrees with the linear analysis as does the wavespeed of the travelling wave.

3. CONCLUSIONS

We have shown that the tissue interaction model (4a),(4b) can give rise to sequential pattern formation behind a travelling wavefront of cell density. The fact that cell density acts as a bifurcation variable to spatial patterning has important biological implications. Note that patterns are only possible as soon as n reaches 1. This suggests that sequential pattern formation could be set up by a travelling wave in cell density which increases cell density to appropriate bifurcation value.

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