Supplementary Material for "Travelling-wave analysis of a model of tumour invasion with degenerate, cross-dependent diffusion"

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This document contains supplementary material to the main paper. In Section S1, we detail calculations that motivated Conjecture 1. In Section S2, we discuss the numerical methods used to solve the different models and present some additional numerical results.

For the purpose of clarity, we recall that our paper focussed on studying weak travelling wave solutions (TWS) for the following partial different equation (PDE) model:

$$\begin{cases} \dfrac{\partial N}{\partial t} = \dfrac{\partial}{\partial x}\left[(1-M)\dfrac{\partial N}{\partial x}\right] + (1-N)\,N, \\ \dfrac{\partial M}{\partial t} = -\kappa M N. \end{cases} \tag{S0.1}$$

Introducing the travelling wave coordinate $\xi = x - ct$, where $c > 0$, and the ansatz $N(x,t) = \mathcal{N}(\xi)$ and $M(x,t) = \mathcal{M}(\xi)$, the TWS we seek must satisfy the following ordinary differential equation (ODE) system:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left((1-\mathcal{M})\dfrac{\mathrm{d}\mathcal{N}}{\mathrm{d}\xi}\right) + c\dfrac{\mathrm{d}\mathcal{N}}{\mathrm{d}\xi} + (1-\mathcal{N})\mathcal{N} = 0; & \text{(S0.2a)} \\ c\dfrac{\mathrm{d}\mathcal{M}}{\mathrm{d}\xi} - \kappa\mathcal{M}\mathcal{N} = 0, & \text{(S0.2b)} \end{cases}$$

and either of the following two sets of asymptotic conditions:

$$\lim_{\xi\to-\infty}(\mathcal{N}(\xi),\mathcal{M}(\xi)) = (1,0), \ \lim_{\xi\to+\infty}(\mathcal{N}(\xi),\mathcal{M}(\xi)) = (0,1), \tag{S0.3}$$

$$\lim_{\xi\to-\infty}(\mathcal{N}(\xi),\mathcal{M}(\xi)) = (1,0), \ \lim_{\xi\to+\infty}(\mathcal{N}(\xi),\mathcal{M}(\xi)) = (0,\bar{\mathcal{M}}) \text{ with } \bar{\mathcal{M}} \in [0,1). \tag{S0.4}$$

To simplify the analysis, we removed the singularity in system (S0.2a)-(S0.2b) by introducing a new independent variable y. Denoting derivatives with respect to y using primes and further introducing a dependent variable $p = n'$, we studied solutions $(n_{\alpha,c}, p_{\alpha,c}, m_{\alpha,c})$ of the following system:

$$\begin{cases} n' = p, & \text{(S0.5a)} \\ p' = -cp - (1-n)n(1-m), & \text{(S0.5b)} \\ m' = \dfrac{\kappa}{c}m(1-m)n. & \text{(S0.5c)} \end{cases}$$

subject to the following asymptotic conditions as $y \to -\infty$, for $\alpha \geq 0$:

$$\begin{aligned} n(y) &= 1 - e^{\lambda_2 y} + \mathcal{O}(e^{(\lambda_2+\mu)y}), \\ p(y) &= -\lambda_2 e^{\lambda_2 y} + \mathcal{O}(e^{(\lambda_2+\mu)y}), \\ m(y) &= \alpha e^{\lambda_3 y} + \mathcal{O}(e^{(\lambda_3+\mu)y}), \end{aligned} \tag{S0.6}$$

where $\lambda_2 = (-c + \sqrt{c^2+4})/2$, $\lambda_3 = \kappa/c$ and $\mu = \min(\lambda_2, \lambda_3) > 0$.

S1 Supporting results for Conjecture 1

In this section, we detail calculations that support Conjecture 1. Let $\bar{m} \in (0,1)$. We consider the following boundary value problem:

$$\begin{cases} P' = -c - \frac{(1-n)n(1-M(n))}{P}, \\ M' = \frac{\kappa}{c}\frac{M(1-M)n}{P}, \\ P(0) = 0, M(0) = \bar{m}, \end{cases} \tag{S1.1}$$

subject to the additional conditions

$$P(n) < 0 \text{ and } M(n) \in (0, \bar{m}) \ \forall n \in (0,1), \quad P(1) = M(1) = 0. \tag{S1.2}$$

The result underpinning Conjecture 1 is the following. If the reaction term $g(n) = (1-n)n(1-M(n))$ is of Fisher-KPP type and $g''(n) < 0 \ \forall n \in [0,1]$, then there exists a unique solution to (S1.1) that satisfies (S1.2) for any $c \geq 2\sqrt{g'(0)}$.

Suppose (P, M) is the unique solution of the Cauchy problem (S1.1), which exists by the Cauchy-Lipschitz Theorem. We will check that g is of Fisher-KPP type, calculate $g'(0)$ and study the sign of g'', making a hypothesis about the necessary conditions for $g''(n) < 0 \ \forall n \in [0,1]$ to hold. To do this, we first need to prove three preliminary results.

Result 1: $P'(0)$ is real, finite and negative. By letting $n \to 0$ in the differential equation $(S1.1)_1$ for $P(n)$ and using l'Hôpital's rule to compute $\lim_{n\to 0} \dfrac{n}{P(n)}$, we obtain

$$P'(0) = -c - (1-\bar{m})\frac{1}{P'(0)} \quad \Longrightarrow \quad P'_{\pm}(0) = \frac{-c \pm \sqrt{c^2 - 4(1-\bar{m})}}{2}. \tag{S1.3}$$

Since $\bar{m} \in (0,1)$, (S1.3) implies that, if $c \geq 2\sqrt{1-\bar{m}}$, then

$$P'_{\pm}(0) \in \mathbb{R}, \quad -\infty < P'_{\pm}(0) < 0. \tag{S1.4}$$

Result 2: Boundedness of M. Solving the differential equation $(S1.1)_2$ for M yields

$$M(n) = \frac{1}{1 + A\exp\left(\dfrac{\kappa}{c}\displaystyle\int_0^n \frac{q}{-P(q)}\,\mathrm{d}q\right)},$$

where A is a constant of integration. If $c \geq 2\sqrt{1-\bar{m}}$, then (S1.4) holds, which implies that

$$\lim_{q\to 0}\frac{q}{-P(q)} = \frac{1}{-P'(0)} < +\infty.$$

and, imposing the condition $M(0)=\bar{m}$, we find $A=\dfrac{1-\bar{m}}{\bar{m}}$. Hence,

$$M(n)=\frac{\overline{m}}{\overline{m}+(1-\overline{m})\exp\left(\dfrac{\kappa}{c}\displaystyle\int_0^n \frac{q}{-P(q)}\,\mathrm{d}q\right)} \tag{S1.5}$$

and, therefore,

$$0\le M(n)\le 1 \quad \forall n\in[0,1]. \tag{S1.6}$$

Result 3: Non-positivity of P. If $c\ge 2\sqrt{1-\bar{m}}$, then (S1.4) implies that $P'(0)<0$ and, therefore, $P(n)<0$ in a right-neighbourhood of $n=0$. Suppose, for a contradiction, that there exists $n_1\in(0,1]$ such that $P(n_1)>0$. Then, we can find $n_0\in(0,n_1)$ such that $P(n_0)=0$ and $P(n)\ge 0$ for $n\in(n_0,n_1)$. Multiplying both sides of $(\mathrm{S1.1})_1$ by P and integrating between n_0 and n_1, we obtain

$$\frac{1}{2}\int_{n_0}^{n_1}(P^2(n))'\,\mathrm{d}n=-c\int_{n_0}^{n_1}P(n)\,\mathrm{d}n-\int_{n_0}^{n_1}n\,(1-n)\,(1-M(n))\,\mathrm{d}n. \tag{S1.7}$$

Using the fact that $P(n)\ge 0$ for $n\in(n_0,n_1)$ and (S1.6), we have

$$-c\int_{n_0}^{n_1}P(n)\,\mathrm{d}n\le 0, \text{ and } \quad -\int_{n_0}^{n_1}n\,(1-n)\,(1-M(n))\,\mathrm{d}n\le 0.$$

Therefore, (S1.7) implies that

$$\frac{1}{2}P^2(n_1)=\frac{1}{2}\int_{n_0}^{n_1}(P^2(n))'\,\mathrm{d}n\le 0, \text{ i.e. } P(n_1)=0.$$

This contradicts the fact that $P(n_1)>0$, and, thus, we must have

$$P(n)\le 0 \quad \forall n\in[0,1]. \tag{S1.8}$$

Further, given the expression (S1.5) for M, (S1.4) and (S1.8) imply that $M(n)$ is strictly decreasing in a neighbourhood of $n=0$ and non-increasing in $n\in(0,1]$, i.e.

$$0\le M(n)<\bar{m}<1 \;\forall n\in(0,1]. \tag{S1.9}$$

Now, the reaction term g is of Fisher-KPP type if $g\in C([0,1])$, $g(0)=g(1)=0$, and $g(n)>0\;\forall n\in(0,1)$. Since M is a component of a classical solution to the Cauchy problem (S1.1), M is a $C^1([0,1])$ function and it follows that $g\in C([0,1])$. By direct calculation, we have

$$g(0)=(1-0)0(1-\bar{m})=0, \quad g(1)=(1-1)1(1-M(1))=0,$$

and (S1.9) implies that $g(n)=(1-n)n(1-M(n))>0\;\forall n\in(0,1)$. Therefore, we have shown that g is of Fisher-KPP type.

Moreover, we have $g'(n)=(1-2\,n)(1-M(n))-n\,(1-n)\,M'(n)$, and, thus

$$\begin{aligned} g'(0)&=\lim_{n\to 0}\left[(1-2\,n)(1-M(n))-n\,(1-n)\,\frac{\kappa}{c}\,\frac{M(n)\,(1-M(n))\,n}{P(n)}\right],\\ &=1-\bar{m}-\frac{\kappa}{c}\bar{m}(1-\bar{m})\lim_{n\to 0}\frac{2n}{P'(n)}, \end{aligned} \tag{S1.10}$$

by l'Hôpital's rule. If $c\ge 2\sqrt{1-\bar{m}}$, then $P'(0)\in(-\infty,0)$ is well-defined by (S1.4) and, using (S1.10), we can conclude that

$$g'(0)=1-\bar{m}. \tag{S1.11}$$

This value of $g'(0)$ yields a minimal wave speed $c = 2\sqrt{g'(0)} = 2\sqrt{1-\bar{m}}$, which is consistent with our recurrent assumption that $c \geq 2\sqrt{1-\bar{m}}$.

Finally, we study the sign of g'' as a function of c and κ and make a hypothesis about the necessary conditions these two parameters must satisfy for the condition $g''(n) < 0 \;\; \forall n \in [0,1]$ to hold.

$$\begin{aligned}
g'' &= -2(1-M) - 2\,(1-2n)\,M' - n\,(1-n)\,M'' \\
&= -2(1-M) - 2\,(1-2n)\,(1-M)\,M\,\frac{\kappa}{c}\,\frac{n}{P} + \\
&\quad -n\,(1-n)\,(1-M)\,M\,\frac{\kappa}{c}\left[\frac{\kappa}{c}\,\left(\frac{n}{P}\right)^2(1-2M) + \frac{1}{P} + \frac{n}{P^2}\left(c + \frac{n\,(1-n)\,(1-M)}{P}\right)\right] \\
&= -2(1-M) + \\
&\quad -2(1-M)\,M\,\frac{\kappa}{c}\,\left(\frac{n}{P}\right)^2\left\{\left[1-2n+\frac{(1-n)}{2}\right]\frac{P}{n} + \frac{\kappa}{c}\,\frac{n\,(1-n)}{2}\,(1-2M) - \frac{(1-n)}{2}\,P'\right\}
\end{aligned}$$

that is,

$$g''(n) = -2(1-M(n))\,(1-H(n))$$

where

$$H(n) := -M(n)\,\frac{\kappa}{c}\,\left(\frac{n}{P(n)}\right)^2\left\{\left[1-2n+\frac{(1-n)}{2}\right]\frac{P(n)}{n} + \frac{\kappa}{c}\,\frac{n(1-n)}{2}\,(1-2M(n)) - \frac{(1-n)}{2}\,P'(n)\right\}.$$

Given $M(0) = \bar{m}$ and (S1.9), $0 \leq M(n) < 1$ for all $n \in [0,1]$, and, thus, $g''(n) < 0 \; \forall n \in [0,1]$ if and only if $H(n) < 1 \; \forall n \in [0,1]$. Using l'Hôpital's rule to compute $\lim\limits_{n\to 0}\dfrac{n}{P(n)}$ and $\lim\limits_{n\to 0}\dfrac{P(n)}{n}$ we find that

$$H(0) = -\bar{m}\,\frac{\kappa}{c}\,\left(\frac{1}{P'(0)}\right)^2\left\{\frac{3}{2}\,P'(0) - \frac{1}{2}\,P'(0)\right\} = -\bar{m}\,\frac{\kappa}{c}\,\frac{1}{P'(0)}.$$

Using the fact that

$$P'_{\pm}(0) = \frac{-c \pm \sqrt{c^2 - 4(1-\bar{m})}}{2},$$

we obtain

$$H(0) < 1 \qquad \Longleftrightarrow \qquad \kappa\,\frac{\bar{m}}{c}\,\frac{2}{c - \sqrt{c^2-4(1-\bar{m})}} < 1,$$

since $-[P'_+(0)]^{-1} \geq -[P'_-(0)]^{-1}$. Solving this inequality, we find that, if $c \geq 2\sqrt{1-\bar{m}}$ and $0 < \kappa < \frac{1-\bar{m}}{\bar{m}}$, then $H(0) < 1$. We now make the hypothesis that, if $c \geq 2\sqrt{1-\bar{m}}$ and $0 < \kappa < \frac{1-\bar{m}}{\bar{m}}$, then

$$H'(n) \leq 0 \quad \forall n \in [0,1].$$

This would allow us to conclude that

$$H(n) < 1 \;\; \forall n \in [0,1], \quad \text{i.e.} \quad g''(n) < 0 \;\; \forall n \in [0,1]. \tag{S1.12}$$

To summarise, if $0 < \kappa \leq \frac{1-\bar{m}}{\bar{m}}$, then, for any $c \geq 2\sqrt{1-\bar{m}}$, g is of Fisher-KPP type, $g''(0) < 0$ and we conjecture that we also have $g''(n) < 0 \; \forall n \in (0,1]$. This implies that there exists a unique solution to (S1.1) that satisfies (S1.2) for any $c \geq 2\sqrt{g'(0)} = 2\sqrt{1-\bar{m}}$. Equivalently, there exists a unique solution to (S0.5a)-(S0.5c) that satisfies $\lim\limits_{y\to-\infty}(n,p,m) = (1,0,0)$ and $\lim\limits_{y\to+\infty}(n,p,m) = (0,0,\bar{m})$ for any $c \geq 2\sqrt{1-\bar{m}}$.

S2 Numerical simulations

Numerical methods.

We solve numerically the PDE model (S0.1) on the 1-D spatial domain $\mathcal{X} := [0, L]$ with $L > 0$, subject to the initial conditions (S2.6), using the method of lines. We discretise the spatial domain $\mathcal{X}$ using a uniform grid comprising P points. This spatial discretisation results in a system of $2P$ time-dependent ODEs. Using the explicit central difference scheme introduced in [2] to approximate the nonlinear diffusion terms, these ODEs for N and M take the following form for $r \in [\![2, P-1]\!]$ and $r \in [\![1, P]\!]$, respectively:

$$\begin{aligned}\frac{\mathrm{d}N_r}{\mathrm{d}t} &= N_r(1-N_r) + \frac{1}{2(\delta_x)^2}\Bigg(\left(D(M)\Big|_{r-1} + D(M)\Big|_{r}\right) N_{r-1} \\ &\quad - \left(D(M)\Big|_{r-1} + 2D(M)\Big|_{r} + D(M)\Big|_{r+1}\right) N_r - \left(D(M)\Big|_{r} + D(M)\Big|_{r+1}\right) N_{r+1}\Bigg),\end{aligned} \tag{S2.1}$$

$$\frac{\mathrm{d}M_r}{\mathrm{d}t} = -\kappa M_r N_r, \tag{S2.2}$$

where $|_r$ denotes evaluation at the r^{th} spatial grid point and $\delta_x = L/P$ is the spatial grid step (i.e. $x_r = r\delta_x$). To close the system, we impose no flux boundary conditions for N by setting:

$$N_1(t) = N_2(t), \quad N_P(t) = N_{P-1}(t) \quad \forall t \geq 0, \tag{S2.3}$$

which implies, in particular, that:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{\mathrm{d}N_2}{\mathrm{d}t}, \quad \frac{\mathrm{d}N_P}{\mathrm{d}t} = \frac{\mathrm{d}N_{P-1}}{\mathrm{d}t}. \tag{S2.4}$$

We solve the system given by (S2.1)-(S2.2) and (S2.4) for $r \in [\![1, P]\!]$ using ODE15s, a variable step, variable order MATLAB built-in solver for stiff ODEs that is based on the numerical differentiation formulas (NDF1-NDF5). In line with the initial conditions (S2.6), for each $r \in [\![1, P]\!]$, we impose the following initial conditions:

$$\begin{cases} N_r(0) = 1,\ M_r(0) = 0, & \text{if } 0 \leq x_r < \sigma - \omega, \\ N_r(0) = \exp\left(1 - \frac{1}{1-\left(\frac{x_r - \sigma + \omega}{\omega}\right)^2}\right),\ M_r(0) = \bar{M}\left(1 - N_r(0)\right), & \text{if } \sigma - \omega \leq x_r < \sigma, \\ N_r(0) = 0,\ M_r(0) = \bar{M}, & \text{if } \sigma \leq x_r \leq L, \end{cases} \tag{S2.5}$$

where $\bar{M} \in [0, 1]$. Unless otherwise stated, we set $L = 200$, $P = 2000$, $\sigma = 2$ and $\omega = 1$ for the simulations and run them for $t \in (0, 100]$.

To numerically solve the ODE models (S0.2a)-(S0.2b) and (S0.5a)-(S0.5c), subject to their respective initial conditions, we use the MATLAB built-in solvers ODE15s and ODE45, respectively.

Travelling wave profiles for TWS of the ODE model in the desingularised variables.

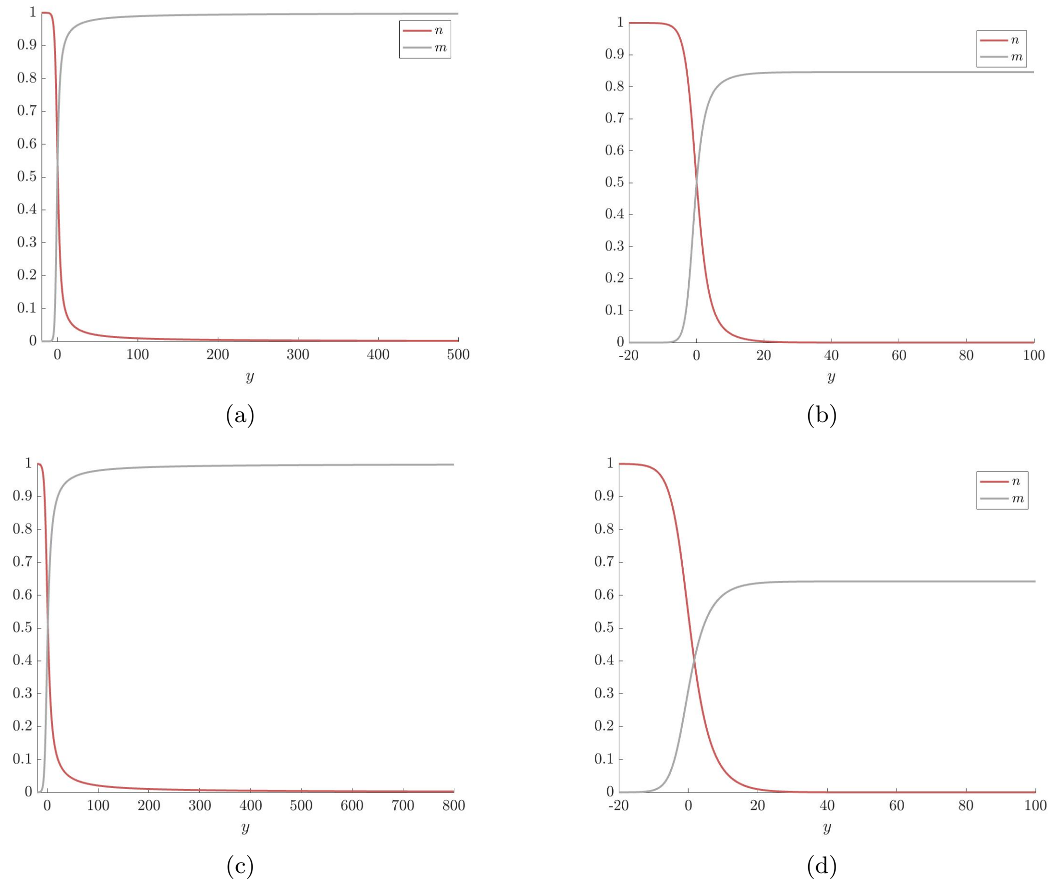


Figure S2.1: In (a) and (b), we plot the n and m components of the solution of the desingularised system (S0.5a)-(S0.5c) subject to the asymptotic conditions (S0.6) with $\kappa = 1$, $c = 1$ and $\alpha = 3.72$ (a) or $\alpha = 3$ (b). In (c) and (d), we plot the n and m components of the solution of the desingularised system (S0.5a)-(S0.5c) subject to the asymptotic conditions (S0.6) with $\kappa = 1$, $c = 2$ and $\alpha = 1.161$ (c) or $\alpha = 1$ (d). The travelling wave profiles in plots (a) and (c) correspond to TWS that satisfy the asymptotic condition $\lim_{y \to +\infty} (n(y), p(y), m(y)) = (0, 0, 1)$ and those in plots (b) and (d) correspond to TWS that satisfy the asymptotic condition $\lim_{y \to +\infty} (n(y), p(y), m(y)) = (0, 0, \bar{m})$, with $\bar{m} \in [0, 1)$. We observe that, in the former case, $n(y)$ and $m(y)$ converge slowly to 0 and 1, respectively, as $y \to +\infty$, whereas, in the latter case, $n(y)$ and $m(y)$ converge fast to 0 and $\bar{m}$, respectively, as $y \to +\infty$.

Travelling waves of the PDE model.

We recall that we solve (S0.1) on the 1-D spatial domain $\mathcal{X} := [0, L]$, where $L > 0$. Similarly to [2], we assume that the tumour has already spread to a position $x = \sigma < L$ in the tissue and we impose initial conditions that satisfy, for $\bar{M} \in [0, 1]$,

$$\begin{cases} N(x,0) = 1,\ M(x,0) = 0, & \text{if } 0 \le x < \sigma - \omega, \\ N(x,0) = \exp\left(1 - \frac{1}{1-\left(\frac{x-\sigma+\omega}{\omega}\right)^2}\right),\ M(x,0) = \bar{M}\left(1 - N(x,0)\right), & \text{if } \sigma - \omega \le x < \sigma, \\ N(x,0) = 0,\ M(x,0) = \bar{M}, & \text{if } \sigma \le x \le L. \end{cases} \tag{S2.6}$$

Here, $0 < \omega < \sigma$ represents how sharp the initial boundary between the tumour and healthy tissue is.

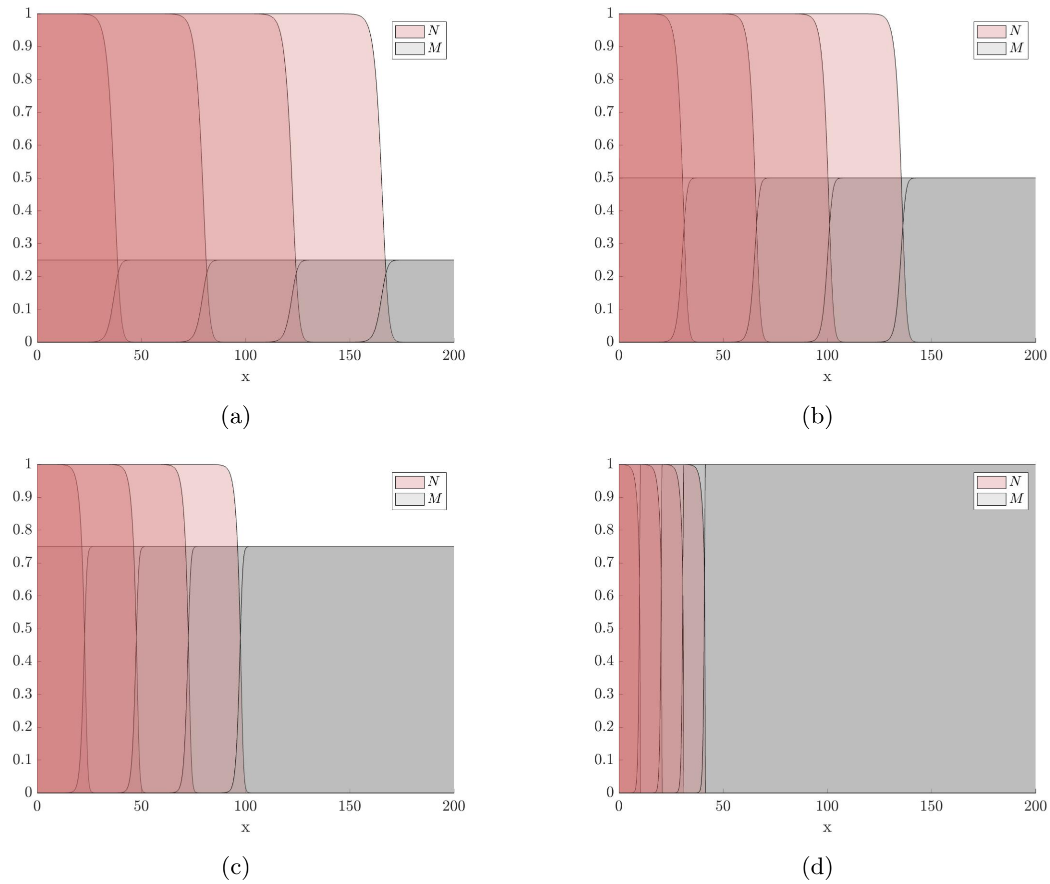


Figure S2.2: We solve system (S0.1) on the 1-D spatial domain, $x \in \mathcal{X} = [0, 200]$, and impose the initial conditions (S2.6) with $\sigma = 2$, $\omega = 1$ and $\bar{M} = 0.25$ (a), $\bar{M} = 0.5$ (b), $\bar{M} = 0.75$ (c) and $\bar{M} = 1$ (d). We plot the respective solutions for $t \in \{25, 50, 75, 100\}$ and observe the emergence of a constant profile, constant speed TWS in all cases.

The influence of initial conditions on the travelling waves of the PDE model.

In order to assess whether the choice of initial conditions with compact support for N influences our numerical results, we solve (S0.1) on the 1-D spatial domain $\mathcal{X} := [0, L]$, where $L > 0$, and impose different sets of initial conditions with compact support for N. In particular, given the density of ECM far ahead of the wave front, $\bar{M}$, we vary the initial distribution of tumour cells and ECM. We illustrate these initial conditions in Figure S2.3 for $\bar{M} = 0.25$. In Figures S2.4 and S2.5, we observe that, for $\bar{M} \in \{0.25, 1\}$, the initial conditions do not change the wave profile or the wave speed of the TWS of (S0.1) that connect $(1, 0)$ and $(0, \bar{M})$. This suggests that there is no significant influence of initial conditions with compact support for N on our numerical results.

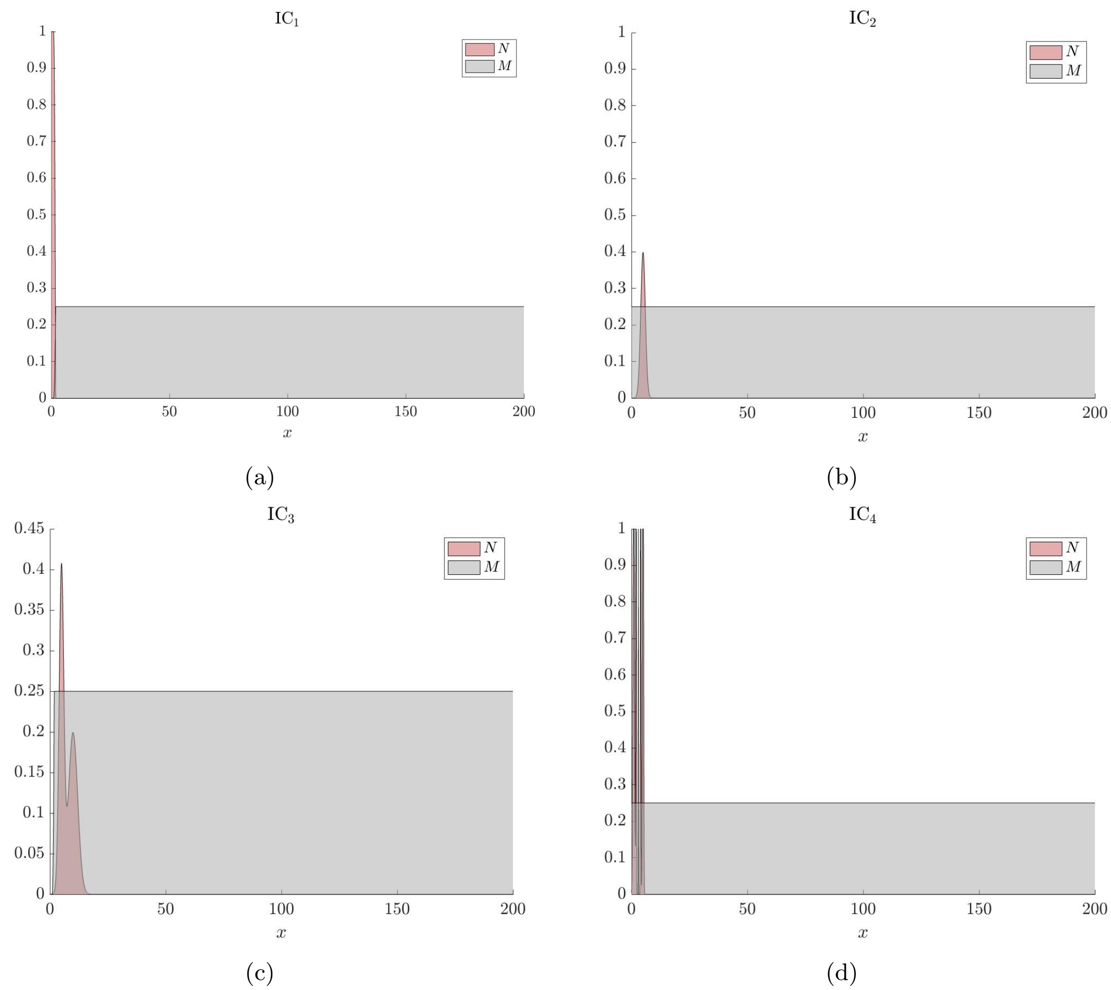
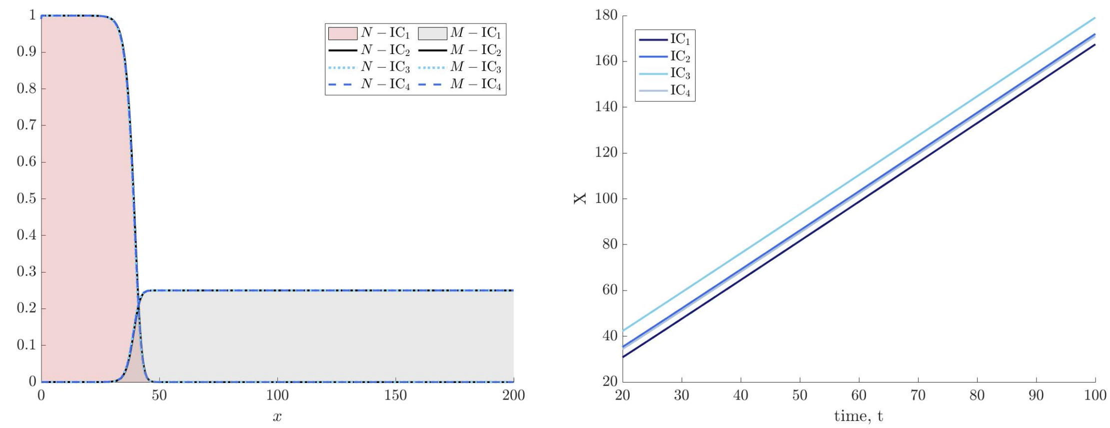
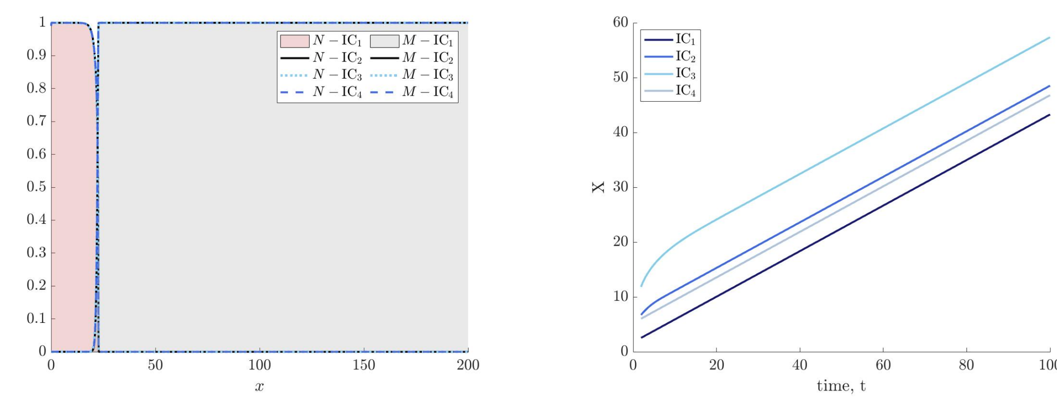


Figure S2.3: Plots (a)-(d) contain four different pairs of initial conditions ($\mathrm{IC}_1 - \mathrm{IC}_4$) for N and M such that the initial distribution of N has compact support and the density of M far ahead of the wave front is $\bar{M} = 0.25$.



(a) (b)

Figure S2.4: We numerically solve system (S0.1) with $\kappa = 1$ on the 1-D spatial domain, $x \in \mathcal{X} = [0, 200]$, and impose the initial conditions $\mathrm{IC}_1 - \mathrm{IC}_4$ from Figure S2.3. Plot (a) represents the travelling wave profiles for each initial condition at times $t = 25$, $t = 24.4$, $t = 18.1$ and $t = 22.9$, respectively, and plot (b) represents $X(t)$ such that $N(X(t), t) = 0.5$ for $t \in [20, 100]$ for each initial condition. We observe that the travelling wave profiles and wave speeds agree regardless of the initial conditions.



(a) (b)

Figure S2.5: We numerically solve system (S0.1) with $\kappa = 1$ on the 1-D spatial domain, $x \in \mathcal{X} = [0, 200]$, and impose the initial conditions $\mathrm{IC}_1 - \mathrm{IC}_4$ from Figure S2.3 adapted such that $\bar{M} = 1$. Plot (a) represents the travelling wave profiles for each initial condition at times $t = 50$, $t = 37.4$, $t = 16.3$ and $t = 41.5$, respectively, and plot (b) represents $X(t)$ such that $N(X(t), t) = 0.5$ for $t \in (1.9, 100]$ for each initial condition. We observe that the travelling wave profiles and wave speeds agree regardless of the initial conditions.

The influence of numerical diffusion on the numerical results.

To investigate the presence of numerical diffusion, we solve (S0.1) on the 1-D spatial domain $\mathcal{X}$ using the method previously described and impose the initial conditions (S2.6), with $\bar{M} \in \{0.25, 0.5, 0.75, 1\}$. We only vary the number of points in our spatial discretisation and, in particular, we set $P \in \{1000, 2000, 3000, 4000, 5000\}$ (i.e. $\delta_x \in \{0.2, 0.1, 0.066, 0.05, 0.04\}$). Given the results in Figure S2.6, using the discretisation step size $\delta_x = 0.1$ ensures that the numerical results presented in the paper and supplementary material are weakly influenced by numerical diffusion.

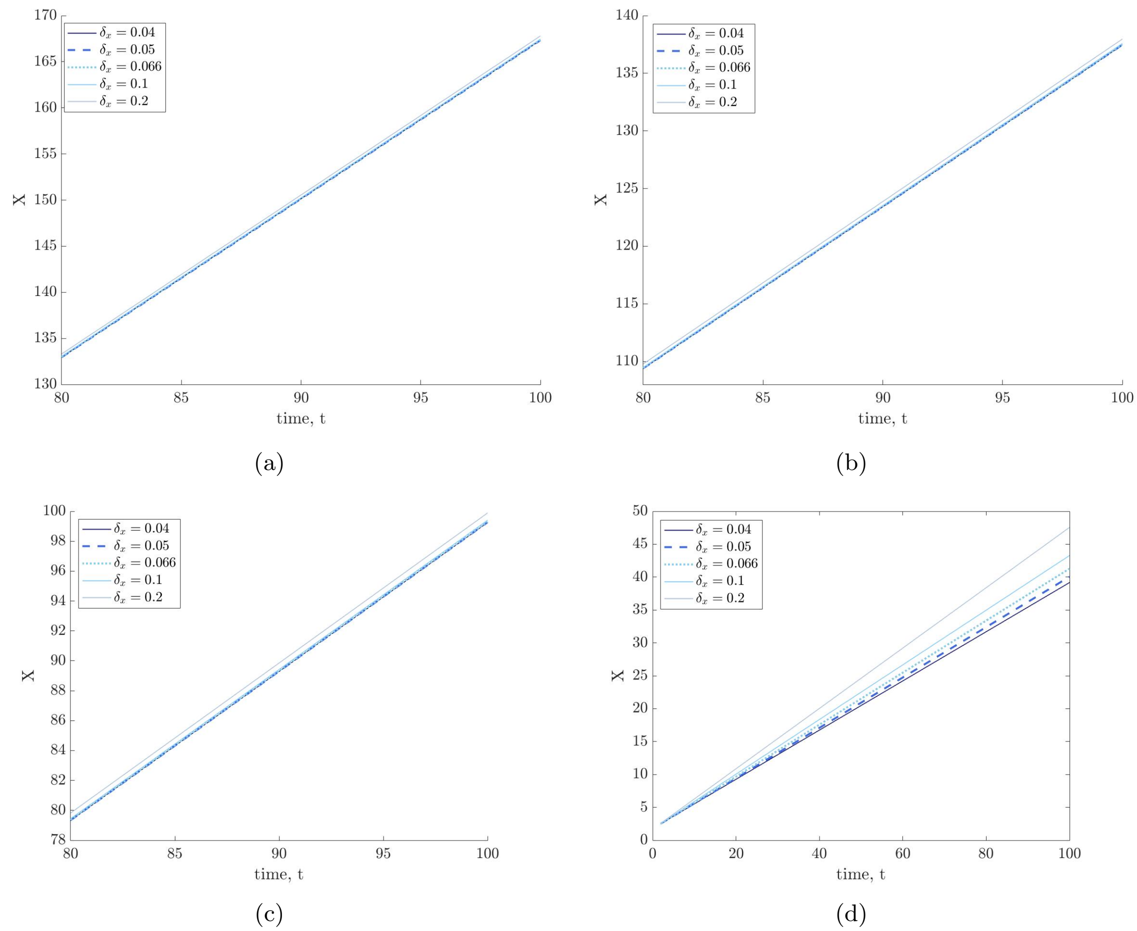


Figure S2.6: We numerically solve system (S0.1) with $\kappa = 1$ on the 1-D spatial domain, $x \in \mathcal{X} = [0, 200]$, and impose the initial conditions (S2.6) with $\sigma = 2$, $\omega = 1$ and $\bar{M} = 0.25$ (a), $\bar{M} = 0.5$ (b), $\bar{M} = 0.75$ (c) and $\bar{M} = 1$ (d). Each plot represents $X(t)$ such that $N(X(t), t) = 0.5$ for $t \in [80, 100]$ (a)-(c) and $t \in (1.9, 100]$ (d) when the discretisation step size is $\delta_x \in \{0.2, 0.1, 0.066, 0.05, 0.04\}$. We can see that the influence of the discretisation step size on the wave speed selected by the PDE model increases with $\bar{M}$ and becomes significant as $\bar{M}$ approaches 1.

References

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