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# A Review on Travelling Wave Solutions of One-Dimensional Reaction-Diffusion Equations with Non-Linear Diffusion Term

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**Abstract.** In this paper we review the existence of different types of travelling wave solutions $u(x,t) = \phi(x - ct)$ of degenerate non-linear reaction-diffusion equations of the form $u_t = [D(u)u_x]_x + g(u)$ for different density-dependent diffusion coefficients $D$ and kinetic part $g$. These include the non-linear degenerate generalized Fisher-KPP and the Nagumo equations. Also, we consider an equation whose diffusion coefficient changes sign as the diffusive substance increases. This describes a diffusive-aggregative process. In this case the travelling wave solutions are explored and the ill-posedness of two boundary-value problems associated with the above equation is stated.

## 1. Introduction

Since the classical works by FISHER (1937) and KOLMOGOROV *et al.* (1937), who introduced the model $u_t = Du_{xx} + g(u)$ to describe the propagation of an advantageous gene within an one-dimensional habitat, a great deal of work has been carried out to extend their model to take into account other biological, chemical or physical factors. One of these extensions considers non-linear diffusion terms, which can be seen as a non-linear Fick's diffusion law. The non-linearity can arise in terms of a space, time or density dependent diffusion coefficient. If, in the latter case, the diffusion coefficient and its derivative vanish at certain values of the diffusive substance, then the corresponding reaction-diffusion equation *degenerates* into an ODE equation at these values.

This degeneracy has two main effects on the qualitative features of its solutions. Firstly, they do not propagate through space with infinite speed, as for the case when the equation has positive constant diffusion coefficient $D$. In fact, there is a *finite speed of propagation*. In the case of constant diffusion coefficient, for suitable initial conditions, the corresponding boundary value problem has smooth solutions on its domain. This is not the situation for the degenerate case; normally, we do not expect smooth solutions (some of them can be of *sharp type*), and it is necessary, instead, to introduce a suitable *weak solution* concept.

The classical approach to the investigation of the existence of travelling wave solutions (t.w.s.), $u(x,t) = \phi(x - ct)$, of linear and nonlinear reaction-diffusion systems was introduced by KOLMOGOROV *et al.* (1937). It consists of re-stating the original initial and boundary value problem (which is in an infinite dimensional space), in terms of searching for a set of parameters (in which the speed $c$ is included) for which a finite system of ODEs has trajectories connecting pairs of equilibrium points (heteroclinic and/or homoclinic). The boundary conditions for the t.w.s. are re-stated in terms of the asymptotic behaviour of the heteroclinic trajectories as time $t$ tends to $-\infty$ and to $+\infty$. This ODE system is obtained by re-stating the problem in the appropriate travelling wave variables.

A wide range of methods have been developed to search for travelling wave solutions using Kolmogorov's method in many models in biology, ecology, physiology, chemistry, etc. A first class of methods involves a direct, *ad hoc* examination of the dynamics of the specific ODE system in each application, see, for example, FIFE (1979), SMOLLER (1983), BRITTON (1986), MURRAY (1989), GRINDROD (1991), SWINNEY and KRINSKY (1992), SÁNCHEZ-GARDUÑO and MAINI (1994a, 1995a).

Another approach is to use the Conley Index. This is a more topological approach and has also been used to investigate the existence of heteroclinic connections and global bifurcations for ODE systems (SMOLLER, 1983; KAPPOS *et al.*, 1991; see KAPPOS 1995 for an accessible presentation of the Conley index method). Shooting arguments have also been used (see DUNBAR, 1984; SÁNCHEZ-GARDUÑO *et al.*, 1995) to prove the existence of heteroclinic trajectories corresponding to t.w.s. of certain non-linear reaction-diffusion equations.

In cases where it is known that heteroclinic trajectories exist, one can analyse them using numerical or analytical tools. In this direction different numerical methods have been developed to approximate the appropriate trajectories for the corresponding ODE system (see DOEDEL and FRIEDMAN, 1989; BEYN, 1990), as well as solving the PDE model. For certain special cases, perturbation methods can be used to derive analytic approximations to the t.w.s. (SÁNCHEZ-GARDUÑO and MAINI, 1994b).

In this paper we review results on the existence of different types of t.w.s. for several types of one-dimensional density-dependent reaction-diffusion equations. The paper is structured as follows: In Section 2 we present an overview of the t.w.s. dynamics for the degenerate Fisher-KPP equation. Section 3 contains a similar review for the generalized non-linear diffusion Nagumo equation. In Section 4 we consider a density-dependent reaction-diffusion equation in which the diffusive term changes sign. The negative diffusive term is associated with an aggregative process. The aggregative travelling wave dynamics is explored and the *ill-posedness* of a couple of initial and boundary value problems is stated.

In this paper we will omit the technical proof of the results, and give, instead, the relevant references.

## 2. Non-Linear Travelling Waves in the Degenerate Fisher-KKP Equations

Density-dependent dispersal has been observed in many biological populations, for example, squirrels, small rodents, ants, etc. (see MYERS and KREBS, 1974; CARL, 1979; SHIGUESADA *et al.*, 1979). For some of these species, individuals may prefer to migrate from crowded areas to sparsely populated areas, despite the possibility of adverse physical or ecological conditions. In SÁNCHEZ-GARDUÑO and MAINI (1994a), the different types of one-dimensional non-linear reaction-diffusion equations which arise as descriptions of the space-time dynamics of a single species are reviewed. It is shown there that, under certain conditions, these models can be reduced to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u), \quad \forall \ (x,t) \in \boldsymbol{R} \times \boldsymbol{R}^{+}, \tag{1}$$

where the growth rate $g$ may be density-dependent. We require the following conditions on $D$ and $g$, both defined on the interval $[0,1]$:

1. $g(0) = g(1) = 0$, $g(u) > 0 \ \forall \ u \in (0,1)$,
2. $g \in C^2_{[0,1]}$ with $g'(0) > 0$, $g'(1) < 0$ and $g''(0) \neq 0$,
3. $D(0) = 0$, $D(u) > 0 \ \forall \ u \in (0,1]$,
4. $D \in C^2_{[0,1]}$ with $D'(u) > 0 \ \forall \ u \in [0,1]$.

In ecological terms Eq. (1), with the above conditions, models the space-time dynamics of a population in which individuals disperse to avoid crowded areas in a habitat with limited resources. Under the above conditions the following theorem holds:

**Theorem 1.** (see SÁNCHEZ-GARDUÑO and MAINI, 1994a) If the functions $D$ and $g$ satisfy the above conditions, then there exists a unique value, $c^* > 0$, of $c$, such that Eq. (1) has:

1. No t.w.s. for $0 < c < c^*$,
2. a travelling wave solution of sharp type satisfying: $\phi(-\infty) = 1$, $\phi(\xi) = 0 \ \forall \ \xi \geq \xi^*$; $\phi'(\xi^{*-}) = -c^*/D'(0)$, $\phi'(\xi^{*+}) = 0$,
3. a monotone decreasing travelling wave of front type satisfying $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$ for each $c > c^*$.

**Outline of the proof.** If we substitute $u(x,t) = \phi(x - ct) \equiv \phi(\xi)$ into (1) we obtain a second order ODE which, by setting $v = \phi'$, can be written as a singular (at $\phi = 0$) ODE system. The singularity can be removed by introducing a new parameter, $\tau$, in such a way that $d\tau/d\xi = 1/D(\phi(\xi))$. Thus, the proof is based on analysing the local and global phase portrait of the non-singular ODE system

$$\left.\begin{aligned} \dot{\phi} &= D(\phi)v \\ \dot{v} &= -cv - D'(\phi)v^2 - g(\phi) \end{aligned}\right\}, \tag{2}$$

as the travelling wave speed $c$ varies. The key thing in the local analysis is that for all positive $c$ the system (2) has: 1. two hyperbolic saddle points ($P_1 = (1,0)$ and $P_c = (0,-c/D'(0))$) and 2. one non-hyperbolic saddle-node point ($P_0 = (0,0)$). The local phase portrait around $P_0$ can be determined by using the second order local approximation of (2) and applying the Centre Manifold Theorem. For the global phase portrait one needs to consider trajectories for extreme (very small and sufficiently large) values of $c$, the

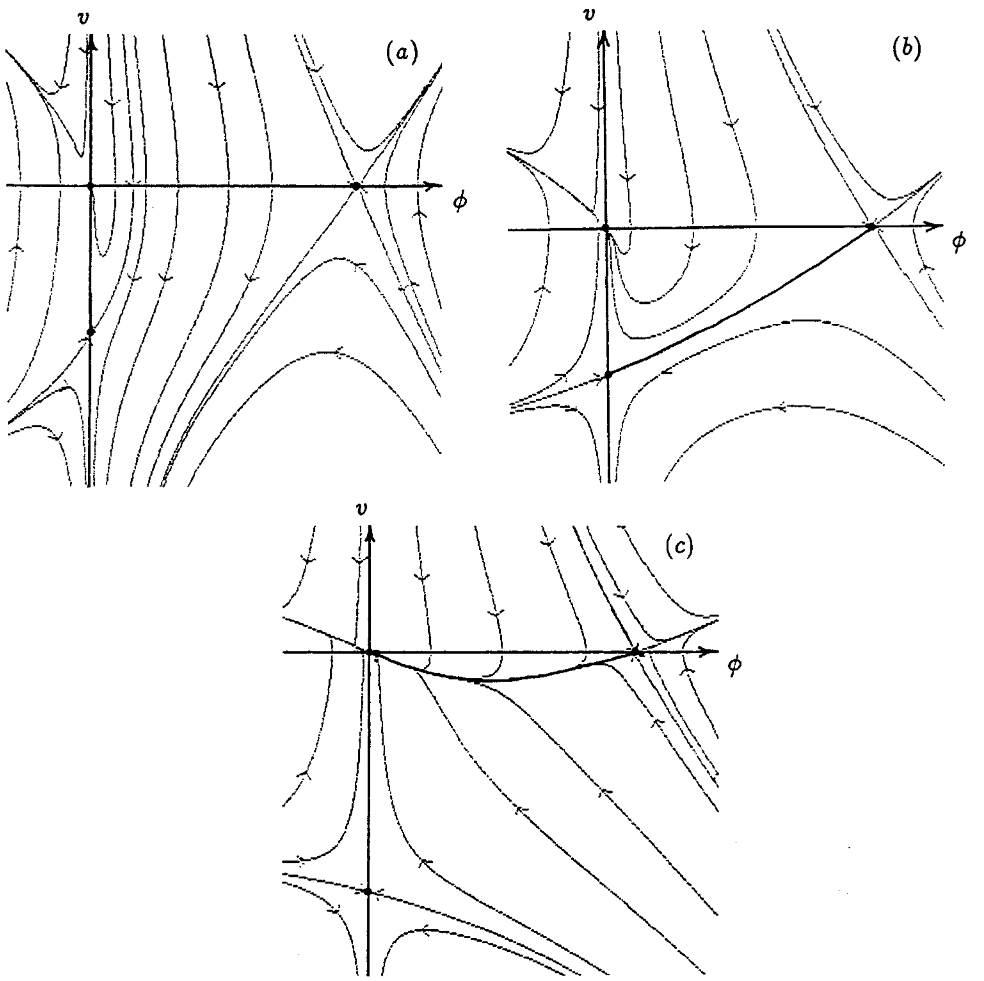


Fig. 1. Phase portrait of system (2) with $D(u) = [1 - \exp(-\phi)]$ and $g(\phi) = \phi(1 - \phi)$ for different values of $c$: (a) $c = 0.4$; there are no heteroclinic connections, therefore there are no t.w.s. for the full PDE. (b) $c = 0.645$; this is a good approximation to the critical values $c^*$ for which there exists the heteroclinic saddle-saddle trajectory, which gives us the sharp type solution of the non-linear reaction-diffusion equation. (c) $c = 1.5$; here we have a saddle-saddle-node connection, which corresponds to a travelling wave solution of front type for Eq. (1).

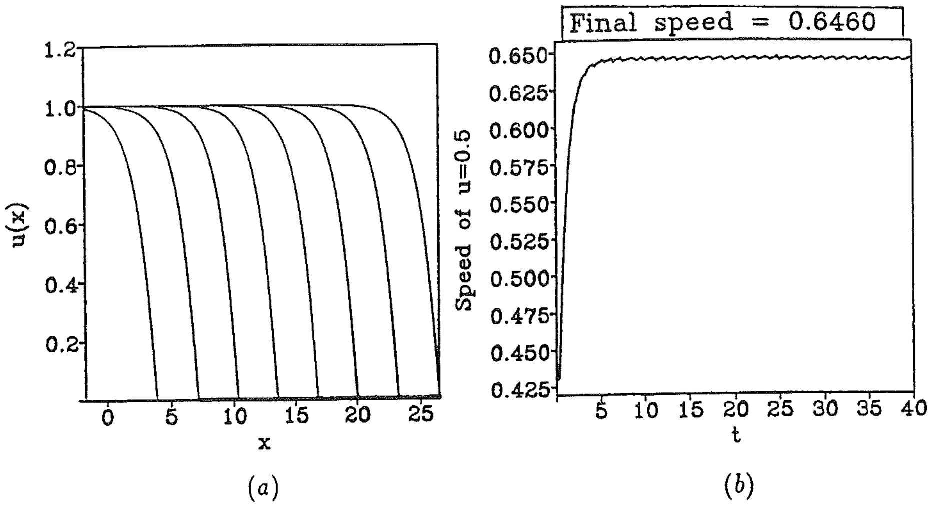


Fig. 2. Numerical simulation for Eq. (1) with $D$ and $g$ as in Fig. 1. (a) Approximations to the sharp type solution of (1) at regular time intervals. (b) Speed of the approximate sharp solution as a function of time.

vertical null-clines of (2) and the monotonicity property of the paths of the trajectories of (2) with respect to $c$. Then, by using the continuity of the solutions of (2) with respect to $c$, one can conclude the existence of a unique value, $c^* > 0$, of $c$, such that system (2): 1. has no heteroclinic trajectories for $0 < c < c^*$, 2. has a heteroclinic trajectory connecting the points $P_1$ and $P_{c^*}$ and 3. has a heteroclinic trajectory connecting the equilibria $P_1$ and $P_0$ for each $c > c^*$. In light of the equivalence between the heteroclinic trajectories of (2) and the t.w.s. of (1), the proof of Theorem 1 follows. □

For alternative proof using a shooting argument, see SÁNCHEZ-GARDUÑO *et al.* (1995). We can interpret the above t.w.s. as a wave of invasion of the individuals of the population into the habitat. Figure 1 shows the phase portrait of a particular case of system (2) as $c$ varies. In Fig. 2 different approximations to the sharp type solution of the corresponding density dependent reaction-diffusion equation are shown.

## 3. Travelling Wave Solutions in Degenerate Nagumo Equations

In this section we consider Eq. (1) but instead of conditions 1–4 in the previous section, for some number $\alpha \in (0,1)$, we impose the following conditions on the functions $g$ and $D$, which are defined on the interval $[0,1]$:

1. $g(0) = g(\alpha) = g(1) = 0$, $g(u) < 0\ \forall\ u \in (0, \alpha)$, $g(u) > 0\ \forall\ (\alpha,1)$,
2. $g \in C^2_{[0,1]}$, $g'(0) < 0$, $g'(\alpha) > 0$, $g'(1) < 0$ and $g''(0) > 0$,
3. $D(0) = 0$, $D(u) > 0\ \forall\ u \in (0,1]$,

4. $D \in C^2_{[0,1]}$, $D'(u) > 0 \; \forall \; u \in [0,1]$ and $D''(0) > 0$.

Our interest is to look for t.w.s. $u(x,t) = \phi(x - ct) \equiv \phi(\xi)$ for Eq. (1) where $D$ and $g$ satisfy the above conditions. We also require $u(x,0) = u_0(x)$ with $0 \le u_0(x) \le 1$ and $0 \le u(x,t) \le 1 \; \forall \; (x,t) \in \boldsymbol{R} \times \boldsymbol{R}^+$.

One can verify that, for $D$ and $g$ as above, the functions $u_0(x,t) \equiv 0$, $u_1(x,t) \equiv \alpha$ and $u_2(x,t) \equiv 1$, are homogeneous and stationary solutions of Eq. (1).

With the above features of $g$ and $D$, the corresponding Eq. (1) can describe the following situations:

1. **Ecological:** the individuals of the population migrate from crowded areas into sparse ones by observing an *Allee effect*. By this we mean the situation in which the net rate of growth of a biological population is negative if its density falls below a certain threshold level ($\alpha$).

2. **Genetical:** suppose a population has two alleles $A$ and $a$ with probabilities of occurrence given by: $P(A) = u$, $P(a) = 1 - u$. Then the probability of the three genotypes $AA$, $Aa$ and $aa$ are: $u^2$, $2u(1 - u)$ and $(1 - u)^2$, respectively. Under the assumptions: i) the population has no structure, ii) Hardy-Wienberg equilibrium and iii) growth in overlapping generations, one can prove that, in an one-dimensional space, $u$ satisfies the equation $u_t = u_{xx} + g(u)$ where $g$ has the qualitative features listed above.

3. **Physiological:** if we interpret $u$ as the membrane potential in a nerve axon, Eq. (1) can be seen as a generalization of Nagumo's equation arising in nerve conduction models. Here the stationary and homogeneous solution $u_0(x,t) \equiv 0$ is the resting state, $u_1(x,t) \equiv \alpha$ is the threshold that a stimulus must exceed to excite the nerve and $u_2(x,t) \equiv 1$ is the excited state.

To state the main result of this section let us define the function $\mathfrak{D}: [0,1] \to \boldsymbol{R}$ as follows:

$$\mathfrak{D}(\phi) = \int_0^\phi D(s)g(s)ds.$$

The following theorem gives us the whole t.w.s. dynamics associated with Eq. (1):

**Theorem 2.** (see SÁNCHEZ-GARDUÑO and MAINI, 1995b) If the functions $D$ and $g$ satisfy the conditions stated in this section, then there exists a critical value, $c^* > 0$, of the speed $c$, such that Eq. (1):

1. has: (a) an isolated pulse based at $P_0$ if $c = 0$ and $\mathfrak{D}(1) > 0$; (b) an isolated pulse based at $P_1$ if $c = 0$ and $\mathfrak{D}(1) < 0$; (c) two stationary monotonic fronts: one connecting the states 0 and 1 and the other connecting 1 to 0, if $c = 0$ and $\mathfrak{D}(1) = 0$,

2. has an oscillatory front from 0 to $\alpha$ and another from 1 to $\alpha$ for each $c$ such that $0 < c < c^* < \sqrt{4D(\alpha)g'(\alpha)}$,

3. has a unique travelling wave solution of sharp type from 1 to 0 for the critical value, $c^*$, of the speed $c$. For this value of $c$ there exists an oscillatory travelling wave from 0 to $\alpha$,

4. does not possesses t.w.s. connecting the homogeneous and stationary steady

states $u(x,t) \equiv 1$ and $u(x,t) \equiv 0$, for $\mathfrak{D}(1) \leq 0$ and $c > 0$,

5. has two oscillatory travelling fronts for $c^* < c < \sqrt{4D(\alpha)g'(\alpha)}$: one from 0 to $\alpha$ and another from 1 to $\alpha$,

6. has a monotonic decreasing front from 1 to $\alpha$ for each $c$ such that $c \geq \sqrt{4D(\alpha)g'(\alpha)}$. For the same values of $c$ it has a monotonic increasing front from 0 to $\alpha$.

**Outline of the proof.** Except for some additional technical difficulties, the proof follows the same lines of the proof of Theorem 1. Firstly, we note that, for $c > 0$, the corresponding

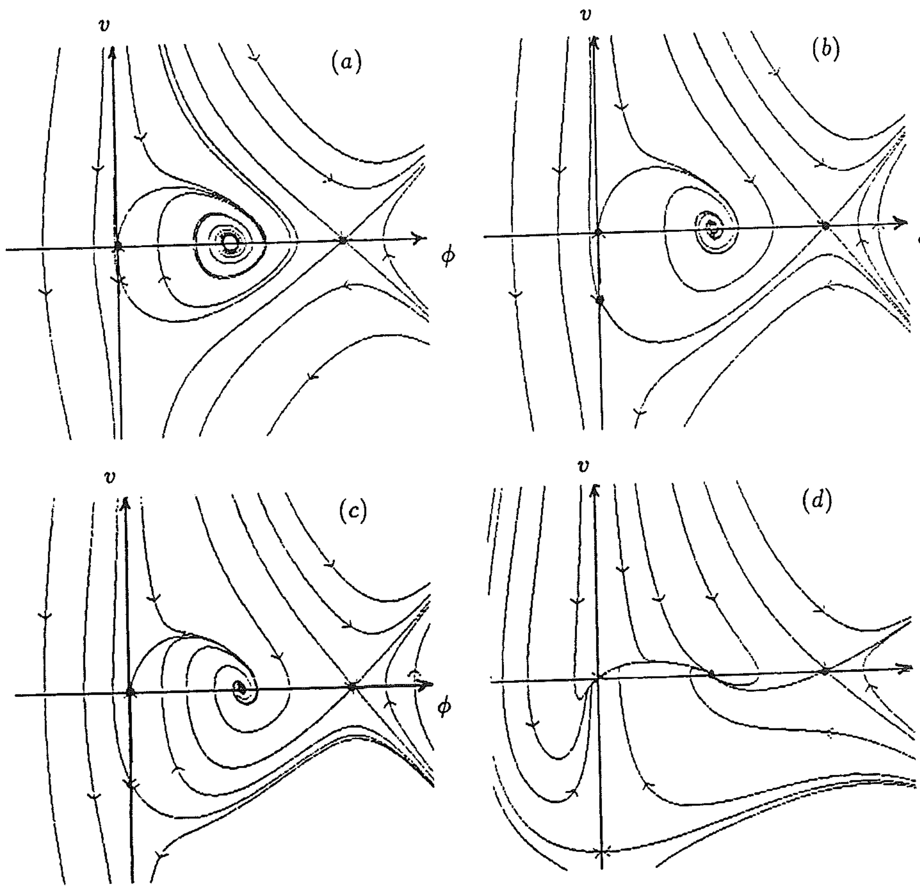


Fig. 3. Phase portrait of system (2) with $D(\phi) = (2\phi + \phi^2)$ and $g(\phi) = \phi(1-\phi)(\phi - 0.5)$ for different values of $c$: (a) $c = 0.1$: There exists only the $P_0$ to $P_\alpha$ connection which corresponds to an oscillatory front. (b) $c = 0.201$: This is an approximation to the critical value of $c$ for which there exists a saddle-saddle heteroclinic trajectory, associated with this connection Eq. (1) has a sharp type solution. Note that for this value of $c$ the heteroclinic connection from $P_0$ to $P_\alpha$ corresponds to an oscillatory travelling wave solution for Eq. (1).

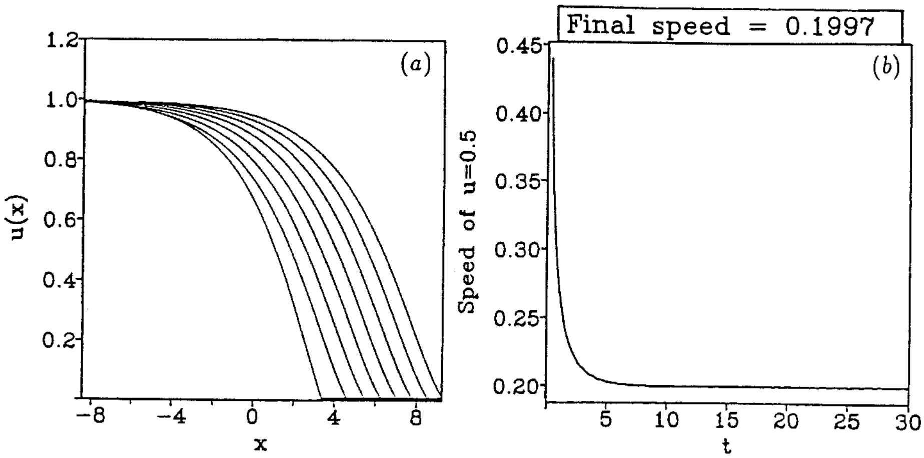


Fig. 4. Numerical solution of (1) for $D$ and $g$ as in Fig. 3. (a) Approximations to the sharp type solution of (1) at regular time intervals. (b) Diagram of the calculated speed (as a function of time) of one point on the graph of the sharp type solution.

ODE system (2) has four equilibria: $P_1 = (0,1)$ and $P_c = (0,-c/D'(0))$ are hyperbolic saddles, $P_\alpha(\alpha,0)$ is a hyperbolic: locally stable node if $c^2 \geq 4D(\alpha)g'(\alpha)$, locally stable focus if $c^2 < 4D(\alpha)g'(\alpha)$. $P_0 = (0,0)$ is a non-hyperbolic saddle-node point. Because of the number of equilibrium points of (2), as $c$ varies it has a greater richness of heteroclinic and homoclinic trajectories. Full details of the proof can be found in SÁNCHEZ-GARDUÑO and MAINI (1995b). □

Figure 3 shows the phase portrait of system (2) for different values of $c$ where $D$ and $g$ have the geometrical properties mentioned in this section. Figure 4 illustrates different numerical approximations to the sharp type solution for the corresponding degenerate reaction-diffusion Nagumo equation.

## 4. Aggregative Travelling Waves and Ill-Posed Problems in a Negative Non-Linear Diffusion Equation

The mutual attraction of individuals of populations is a well documented phenomenon. We distinguish between two types of attraction: *indirect* and *direct*. In the former, for individuals to meet each other requires a secondary agent which, for example, produces an attracting substance; in the latter, because of social behaviour, the individuals of the population attract other conspecifics. This gregarious behaviour of the individuals of ecological populations is an important factor for survival (to defend themselves against predators) and also for reproduction.

There are several approaches to model the aggregation phenomenon. For instance, the well known *chemotactic-reaction-diffusion* systems describe a type of indirect attraction, typically the aggregative process of the amoeba *Dictyostelium discoideum*. For direct aggregation there exist a wider range of models. These include: 1. *Fourth order diffusion equations* derived by using a Ginzburg-Landau approach (COHEN and MURRAY, 1981), 2. *Integralpartial differential equations* which consider non-local effects over space (see MIMURA and YAMAGUTI, 1982; NAGAI and MIMURA, 1983; ALT, 1985a, b) and 3. *Negative diffusion equations* (see ALT, 1985b; ARONSON, 1985; TURCHIN, 1989; SÁNCHEZ-GARDUÑO and MAINI, in prep.).

In this section we deal with the third approach. For derivation purposes we consider an ecological interpretation. Thus we consider a one-dimensional habitat, using a biased random walk approach and the following assumptions:

1. the size of the population is a constant $N$, i.e., there are no births and no deaths,
2. when there are no other individuals of the same species (conspecific) at adjacent positions the animal moves randomly,
3. if there is a conspecific on an adjacent position, the individual moves there with conditional probability (conditioned on the existence of the individual) with probability $k$, or ignores its neighbour with probability $(1 - k)$,
4. at low population density we can ignore the probability of having more than one conspecific in the immediate vicinity of each moving individual.

In the papers by TURCHIN (1989) and SÁNCHEZ-GARDUÑO and MAINI (in prep.) the authors derive an one-dimensional negative-density dependent diffusion equation. In the former paper there is an application of a strictly diffusive equation whose density-dependent diffusion coefficient changes sign, to describe the aggregation process of *Aphis varians*, a herbivore which lives in the stem of some *leguminosae* plants. The equation is:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{\mu}{2} - 2k_0 u + \frac{2k_0}{\omega}u^2\right)\frac{\partial u}{\partial x}\right], \tag{3}$$

where $\mu$, $k_0$ and $\omega$ are positive constants. For $k_0 > \mu/\omega$ the density-dependent diffusion coefficient in Eq. (3) has two positive real roots $u_1$ and $u_2$. Thus Eq. (3): 1. degenerates at $u_1$ and $u_2$ and 2. given that the non-linear diffusive coefficient is negative on the interval $(u_1,u_2)$, Eq. (3) is not of parabolic type there. In interpretative terms, we have that for values of $u$ within $(u_1,u_2)$, Eq. (3) describes an aggregative process.

To state the first result, we consider a finite dimensional space with length $L > 0$.

**Lemma 1.** For $k_0 > \mu/\omega$, the solutions of the problem

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{\mu}{2} - 2k_0 u + 2\frac{k_0}{\omega}u^2\right)\frac{\partial u}{\partial x}\right], \quad \forall\ (x,t) \in (0,L)\times \mathbf{R}^+, \tag{4}$$

with $u(x,0) = f(x)\ \forall\ x \in [0,L]$ and $u(0,t) = u(L,t) = u_0 > 0$, $\forall\ t > 0$, are not continuous with respect to small perturbations in the initial conditions.

**Outline of the proof.** The key idea is to write down the solution of the above problem as a space-time dependent perturbation to the corresponding solution for the stationary problem, i.e., by setting $u(x,t) = u_i + v(x,t)$ where $u_i$ is a stationary solution of the previous problem. One can verify that $v$ satisfies a linear heat equation whose solution can be written as $[C_1 \sin qx + C_2 \cos qx] \exp \sigma t$. This leads to the dispersion relation $\sigma = -\psi'(u_i)q^2$, where $\psi$ is the non-linear coefficient arising in Eq. (4). From this dispersion relation it follows that for $\psi'(u_i) < 0$ the perturbation $v$ grows without bound as time $t$ goes to infinity. □

Depending on the geometrical features of the density-dependent diffusion coefficient in Eq. (4) the aggregative phenomenon can be classified as *weak* or *strong*.

Now we turn towards exploring aggregative t.w.s. $u(x,t) = \phi(x - ct) \equiv \phi(\xi)$ for an equation which includes a density-dependent growth rate $g$. This is:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right] + g(u), \quad \forall\ (x,t) \in \mathbf{R} \times \mathbf{R}^+, \tag{5}$$

where $D$ and $g$ are defined on $[0,1]$ and satisfy:

1. $g(0) = g(1) = 0$, $g(u) > 0\ \forall\ u \in (0, 1)$,
2. $g \in C^2_{[0,1]}$, $g'(0) > 0$, $g'(1) < 0$,
3. $D(0) = 0$, $D(u) < 0\ \forall\ u \in (0,1]$,
4. $D \in C^2_{[0,1]}$, $D'(u) < 0\ \forall\ u \in (0,1]$, and $D''(u) < 0\ \forall\ u \in [0,1]$. We distinguish two cases: a) $D'(0) = 0$ and b) $D'(0) < 0$.

Also we require the following conditions: $u(x,0) = u_0(x)$ where $0 \le u_0(x) \le 1\ \forall\ x \in \mathbf{R}$, $0 \le u(x,t) \le 1\ \forall\ (x,t) \in \mathbf{R} \times \mathbf{R}^+$, $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$.

We will refer to Eq. (5) as the reaction-aggregation equation. Adopting the definition of well-posed problems for partial differential equations from AMES (1992), the following theorem can be proved (SÁNCHEZ-GARDUÑO and MAINI, in prep.):

**Theorem 3.** For each $c$ such that $c^2 < 4g'(1)D(1)$ problem (5) is Ill-posed (has no solution). Meanwhile for each $c$ such that $c^2 \ge 4g'(1)D(1)$ it is well-posed. Moreover, for each $c$ as above, problem (5) has a t.w.s. of front type.

**Outline of the proof.** As in previous proofs, we substitute $u(x,t) = \phi(x - ct) \equiv \phi(\xi)$ into (5) to obtain a second order ODE for $\phi$ which, by putting $\phi' = v$ can be written as a singular ODE system. Although the singularity can be removed, it is important to realize that the reparametrization as it was stated in the proof of Theorem 1, does not work. Thus, instead of introducing $\tau$ such that $d\tau/d\xi = 1/D(\phi(\xi))$, we choose $\tau$ such that $d\tau/d\xi = -1/D(\phi(\xi))$. With this selection of $\tau$, the orientation of the trajectories of the singular system and those of the non-singular one is the same and they are equivalent on the first and fourth quadrants. Thus the ODE system becomes:

$$\left.\begin{aligned}\dot{\phi} &= -D(\phi)v \\ \dot{v} &= cv + D'(\phi)v^2 + g(\phi)\end{aligned}\right\},\qquad (6)$$

and, together with the conditions $\phi(-\infty) = 1$, $v(-\infty) = 0$ and $\phi(+\infty) = 0$, $v(+\infty) = 0$ with $0 \le \phi(\tau) \le 1\ \forall\ \tau \in (-\infty,+\infty)$, contitutes the restatement of the problem.

We now analyse the local and global phase portrait of the above system. Depending on the local features of $D$ at $\phi = 0$ we have different dynamics. Thus we consider two separate cases:

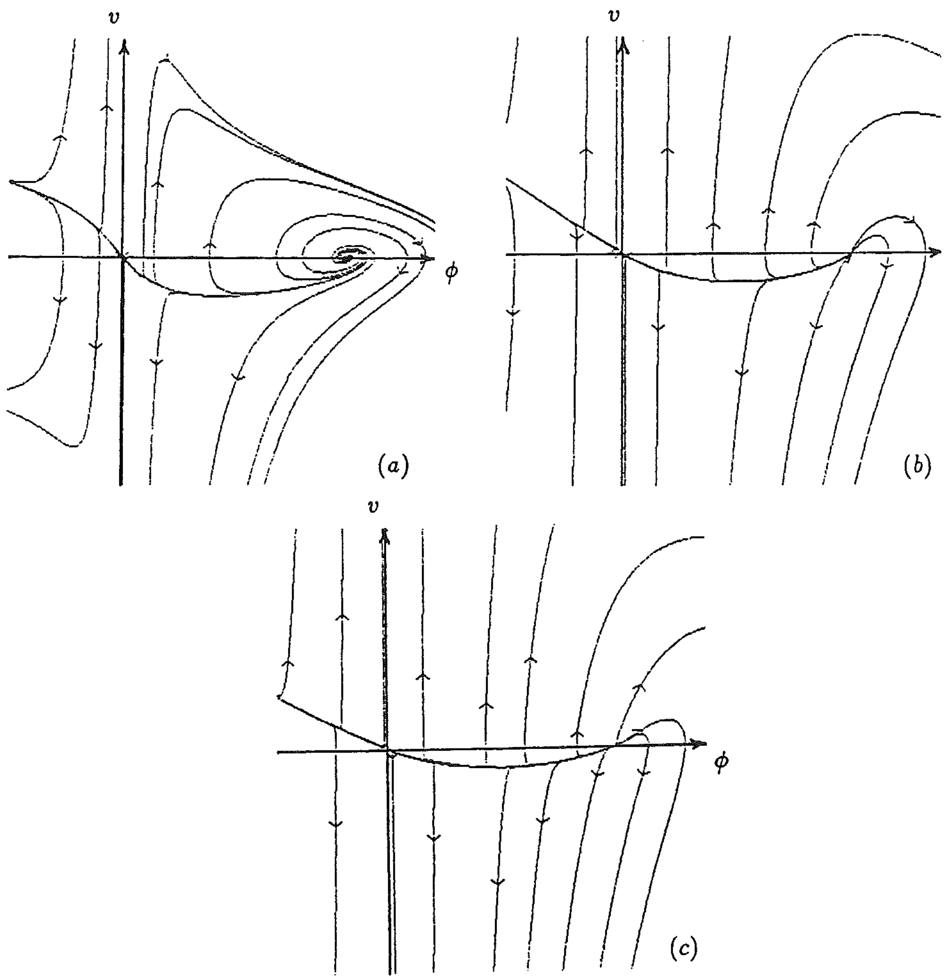


Fig. 5. Phase portrait of system (2) with $D$ and $g$ as in Case 1 ($D(\phi) = \phi^2$ and $g(\phi) = \phi(1-\phi)$) for different values of $c$: (a) $c = 0.5$, (b) $c = 2.0$ and (c) $c = 3.0$. Note that the phase portraits in (a) and (b) are not consistent with the conditions on $\phi$. See text for details.

**Case 1. *g* satisfies 1–2 and *D* satisfies 3–4a).** Here (6) has two equilibria: $P_0 = (0,0)$ and $P_1 = (1,0)$. The point $P_0$ is a non-hyperbolic saddle-like point and $P_1$ is a hyperbolic point which, depending on the values of $c$, has different qualitative behaviour:

1. For $c^2 > 4g'(1)D(1)$, $P_1$ is an unstable node,
2. For $c^2 < 4g'(1)D(1)$, $P_1$ is an unstable focus.

**Case 2. *g* satisfies 1–2 and *D* satisfies 3–4b).** Here, in addition to $P_0$ and $P_1$, system (6) has a third equilibrium: $P_c = (0, -c/D'(0))$, which is a hyperbolic saddle point. $P_0$ is a saddle-node point, while the behaviour of the trajectories of (6) around $P_1$ is the same as in Case 1.

Since any oscillation around $P_1$ implies the violation of one of the conditions ($0 \leq \phi \leq 1$) of our problem, we avoid those values of $c$ for which such oscillations occur. The remaining part of the proof consists of constructing the proper positive invariant set for system (6). The details are presented in SÁNCHEZ-GARDUÑO and MAINI (in prep.).

To illustrate the ideas we consider a couple of examples corresponding to Cases 1 and

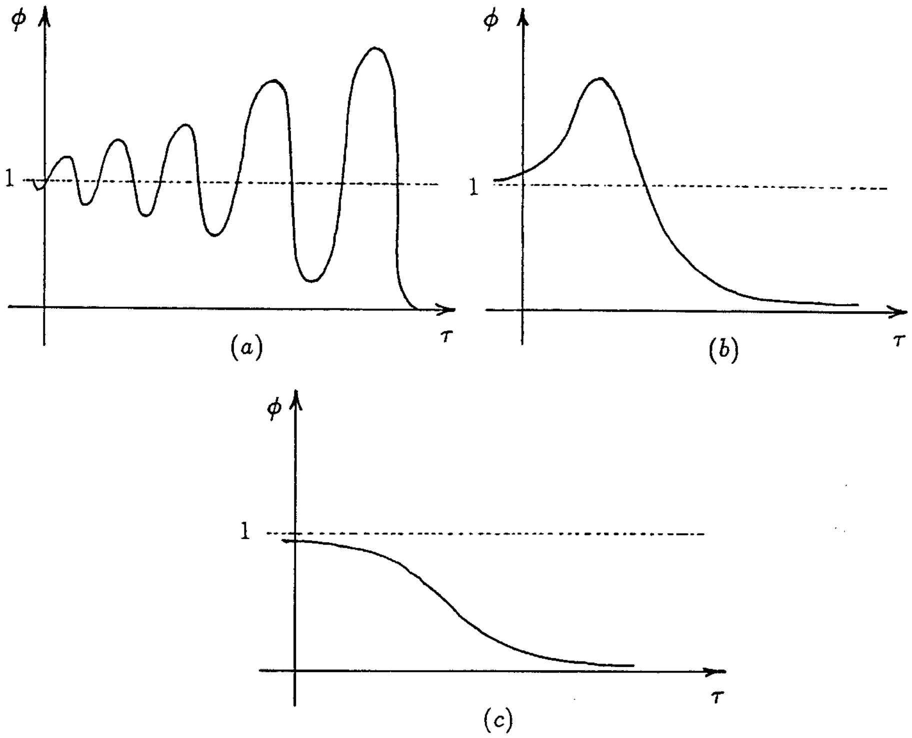


Fig. 6. Different potential t.w.s. for Eq. (5) with $D$ and $g$ as in Fig. 5. (a) Oscillatory front, (b) damped front and (c) monotonic front. Only (c) is consistent with the require conditions on $\phi$.

2, respectively.

Figure 5 shows the phase portrait of system (6) where $D$ and $g$ are as in Case 1 for different values of $c$.

Figure 6 shows the t.w.s. of system (6) corresponding to each heteroclinic connection from $P_1$ to $P_0$. Not all of them are consistent with the conditions imposed on $\phi$.

In Fig. 7 we illustrate the phase portrait of system (6) with $D$ and $g$ as in Case 2. Again,

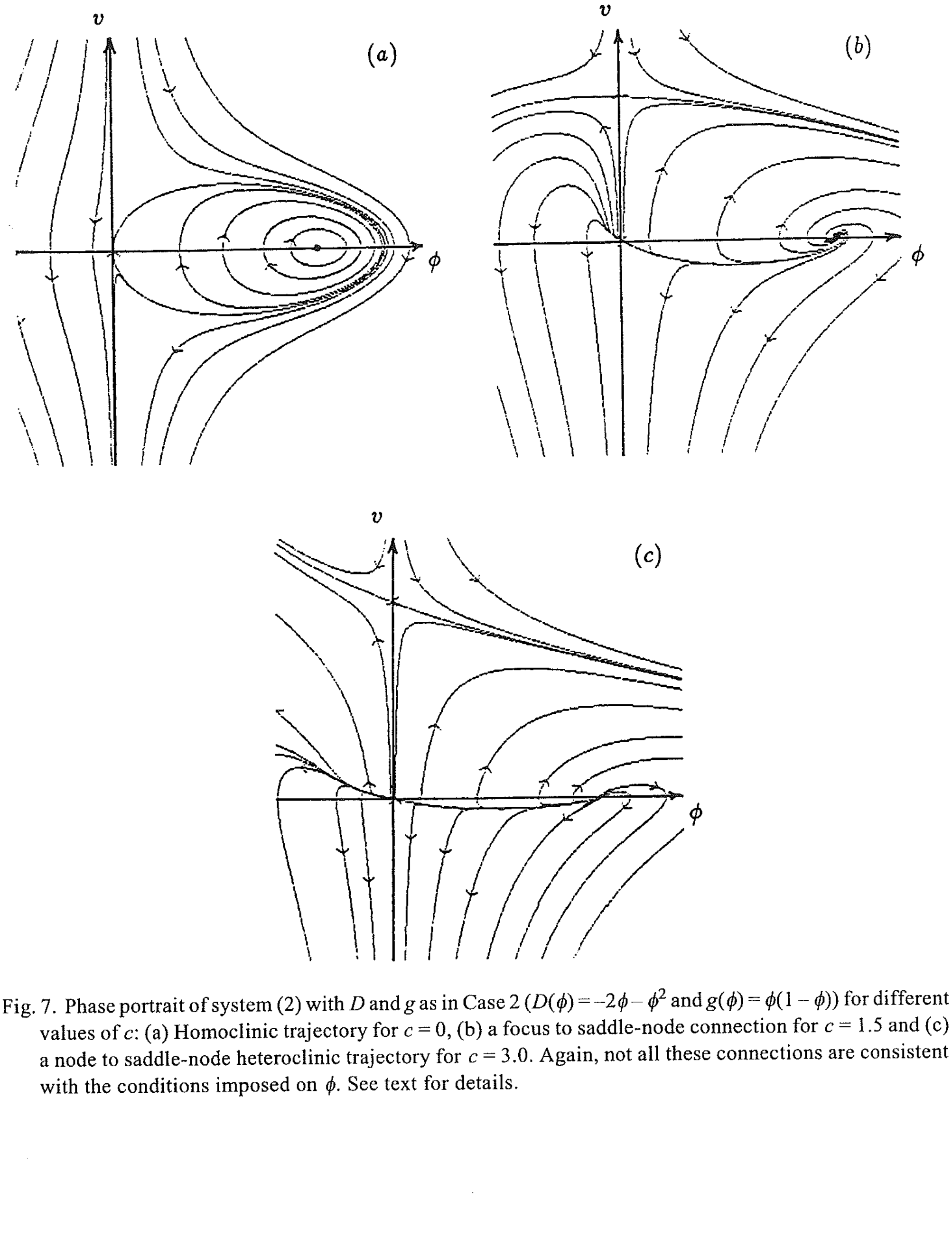


Fig. 7. Phase portrait of system (2) with $D$ and $g$ as in Case 2 ($D(\phi) = -2\phi - \phi^2$ and $g(\phi) = \phi(1-\phi)$) for different values of $c$: (a) Homoclinic trajectory for $c = 0$, (b) a focus to saddle-node connection for $c = 1.5$ and (c) a node to saddle-node heteroclinic trajectory for $c = 3.0$. Again, not all these connections are consistent with the conditions imposed on $\phi$. See text for details.

not all the connections from $P_1$ to $P_0$ correspond to t.w.s. of the problem for the full PDE as was stated in this section. This agrees with our last result.

## 5. Some Open Problems in Non-Linear T.W.S.

Here, we list some open problems in this field. Of course this list is not exhaustive.

1. **Convergence to t.w.s.** In addition to proving the existence of t.w.s. for non-linear reaction-diffusion equations it is important to determine the set of initial conditions for which the corresponding solutions converge to the t.w.s. This is important, for instance, for those equations analysed in this paper.

2. **Density-dependent diffusive systems.** When we consider the interactions between two species in such a way that each one disperses avoiding crowded areas, this means that the corresponding diffusion coefficient depends on both population densities. Moreover, they should vanish at certain points within their domain implying that the equations are degenerate there. Therefore the problem becomes one of investigating the existence of t.w.s. for degenerate reaction-diffusion systems. The direct analysis of the dynamics of systems of dimension greater than two is hard, so alternative methods may be useful, for example using the Conley index.

3. **Regularization problems.** As we have seen, some boundary and initial conditions problems associated with degenerate non-linear diffusion equations (here we include those with negative diffusion) have no strong or classical solutions. Nevertheless, if we extend the set of solutions in such a way that include a proper weak solution concept (with discontinuous derivatives), the above mentioned problems have solutions in this new space. In this technique, there exist other tools for anlaysing ill-posed problems arising in partial differential equations (see AMES, 1992).

4. **Approximation of: 1. the speed and 2. the profile of t.w.s.** Once we have proved the existence of a certain type of t.w.s. (of sharp type, for instance) it is important, from a practical point of view, to have an approximation to the travelling wave solution itself and to the speed for which that t.w.s. exists.

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