GEOMETRIZATION OF THE QUANTUM FROBENIUS.

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ABSTRACT. Lusztig has constructed a Frobenius morphism for quantum groups at an ℓ-th root of unity, which gives an integral lift of the Frobenius map on universal enveloping algebras in positive characteristic. In the nondivisible case we give a construction of this map at the level of perverse sheaves.

1. INTRODUCTION

Let $\mathbf{U}_v(\mathfrak{g})$ be a quantum group attached to a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. Lusztig [L1], [L2] discovered that when the parameter v is specialized to ε an ℓ-th root of unity, there is a homomorphism Fr from the resulting specialised algebra $\mathbf{U}_\varepsilon(\mathfrak{g})$ to the integral form of the enveloping algebra $\mathcal{U}_\mathbb{Z}(\mathfrak{g})$. This construction gives an integral lift of the Frobenius morphism for the corresponding affine algebraic group in positive characteristic. More precisely, if $\ell = \mathrm{char}(\mathsf{k})$, then after base changing to k one obtains the transpose of the map F on the hyperalgebra of G.

In [McG] we gave a new construction of this map using the Hall algebra construction [R], [L8], of the positive part of a quantum group. This gave a conceptual explanation for the existence of the quantum Frobenius, and also raised the natural question as to whether a similar construction exists at the level of sheaves on the moduli space of quiver representations, or in other words, whether the quantum Frobenius can be "categorified" (or perhaps "geometrized"). The purpose of this paper is to give a positive answer to this question and demonstrate a geometric construction of this map. In this paper we will restrict attention to the positive part $\mathbf{U}^+$ of the quantum group. For a discussion of the issues involved in extending to the full (or rather modified) form of the quantum group, see [L6], [McG].

2. QUIVERS AND QUANTUM GROUPS

2.1. **Cartan data and the algebra f.**

In this section we review Lusztig's approach to the positive part of a quantum group. A detailed account of these algebras is contained in [L6, Part I]. Let $\mathbb{Q}(v)$ be the field of rational functions in an indeterminate v with $\mathbb{Q}$-coefficients, and let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, a subring of $\mathbb{Q}(v)$. Suppose that $C = (a_{ij})_{i,j\in I}$ is a symmetrizable generalized Cartan matrix, and let $(d_i)_{i\in I} \in \mathbb{N}^I$ be a vector of positive integers such that

$$d_i a_{ij} = d_j a_{ji}, \quad \forall i, j \in I. \tag{2.1}$$

We may define a symmetric bilinear pairing $x, y \mapsto x \cdot y$ on $\mathbb{Z}[I]$ by setting

$$i \cdot j = d_i a_{ij}, \quad i, j \in I.$$

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and extending linearly. The pair $(\mathbb{Z}[I], \cdot)$ is then a Cartan datum in the sense of Lusztig [L6], with the integers $d_i = \frac{1}{2}(i \cdot i)$. For each $i \in I$ let $v_i = v^{d_i}$ and set

$$[n]_i = (v_i^n - v_i^{-n})/(v_i - v_i^{-1}) \in \mathcal{A}.$$

We then define

$$[n]_i! = [n]_i[n-1]_i \dots [1]_i; \qquad \begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i![n-k]_i!}.$$

It is easy to check that these quantum binomial coefficients all lie in $\mathcal{A}$.

Given a Cartan datum, we define $\mathbf{f}$ to be the $\mathbb{Q}(v)$-algebra generated by symbols $\{\theta_i : i \in I\}$ subject to relations:

$$\sum_{r+s=1-a_{ij}} \theta_i^{(r)}\theta_j\theta_i^{(s)} = 0,$$

where $\theta_i^{(n)} = \theta_i^n/[n]_i!$ is a quantum divided power. We also need an integral form of $\mathbf{f}$: let $\mathbf{f}_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $\mathbf{f}$ generated by $\{\theta_i^{(n)} : n \geq 0, i \in I\}$. Lusztig [L6] has shown that $\mathbf{f}_{\mathcal{A}}$ is a free $\mathcal{A}$-module, as a consequence of the existence of the canonical basis. Both $\mathbf{f}$ and $\mathbf{f}_{\mathcal{A}}$ are clearly $\mathbb{Z}[I]$-graded. If R is any ring carrying the structure of an $\mathcal{A}$-algebra (*i.e.* a ring R and an invertible element $\epsilon \in R$) we write $\mathbf{f}_R$ for the R-algebra $R \otimes_{\mathcal{A}} \mathbf{f}$. When $R = \mathbb{Z}[\zeta]$ is the ring of 2ℓ-cyclotomic integers (where ζ is a primitive 2ℓ-th root of unity) we write $\mathbf{f}_\ell = \mathbf{f}_{\mathbb{Z}[\zeta]}$ where $\phi \colon \mathcal{A} \to \mathbb{Z}[\zeta]$ is given by $v \mapsto \zeta$, and we write $\mathbf{f}_{\mathbb{Z}}$ for the specialisation $v \mapsto 1 \in \mathbb{Z}$.

When ℓ is coprime to each of $\{\frac{1}{2}(i \cdot i)\}$, Lusztig's quantum Frobenius morphism is an algebra homomorphism:

$$Fr_v \colon \mathbf{f}_\ell \to \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} \mathbf{f}_{\mathbb{Z}}.$$

For most of the paper we will assume that our Cartan datum is symmetric. In section §6 we outline how to extend our results to an arbitrary datum, provided the integer ℓ is coprime to each of $\frac{1}{2}(i \cdot i)$ (we shall refer to this as the indivisible case).

Remark 2.1. In [L6] Lusztig generalised this map to the case where ℓ is no longer required to be coprime to the $\frac{1}{2}(i \cdot i)$, but only to satisfy a few mild conditions (see [L6, 35.1.2]). Our construction at the moment requires ℓ to be indivisible, but does not require these conditions to hold, thus the constructions in some sense complement each other.

2.2. **Quivers and the canonical basis.** A quiver is a directed graph $Q = (I, H)$, where I denotes the vertex set and H the set of directed edges. For $h \in H$ we write $s(h), t(h) \in I$ for the initial and terminal vertex of the edge respectively (which we will assume are distinct, *i.e.* Q has "no loops"). We also assume that there is an involution $h \mapsto \bar{h}$ such that $s(\bar{h}) = t(h)$. An orientation Ω is a subset of H such that $\bar{\Omega} \cap \Omega = \emptyset$ and $\bar{\Omega} \cup \Omega = H$.

A representation of an oriented quiver Q is a pair (V, x) where V is an I-graded vector space $V = \bigoplus_{i \in I} V_i$ and x is an element of the space $E_{V,\Omega} = \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{s(h)}, V_{t(h)})$. When there is no possibility for confusion we will often write E_V instead of $E_{V,\Omega}$. We let $|V| = (\dim(V_i))_{i \in I} \in \mathbb{N}^I$ denote the graded dimension of V. The moduli space of representations of Q with graded dimension $|V|$ is therefore the quotient stack E_V/G_V where $G_V = \prod_{i \in I} \mathrm{GL}(V_i)$. Lusztig constructs the canonical basis by building a categorification of $\mathbf{f}$.

Definition 2.2. We say that a subset J of I is discrete if there is no element of H joining the vertices in J. For $\nu \in \mathbb{N}^I$ we define the support of ν to be the set $\{i \in I : \nu_i > 0\}$, and we will say that ν is discrete if its support is. Let $\mathcal{X}$ be the set of sequences $\underline{\nu} = (\nu^1, \nu^2, \ldots, \nu^m)$ where $\nu^s \in \mathbb{N}^I$ is discrete. Given such a ν let

$$\mathcal{F}_{\underline{\nu}} = \{(V = F^0 \supset F^1 \supset \ldots \supset F^m = 0) : |F^{l-1}/F^l| = \nu^l\}$$

denote the variety of flags in V of type $\underline{\nu}$. Let $\tilde{\mathcal{F}}_{\underline{\nu}}$ denote the variety consisting of pairs $(x, F^\bullet)$ where $F^\bullet$ is adapted to x in the sense that $x(F^l) \subseteq F^l$ for each l, $(0 \leq l \leq m)$. This variety is a bundle over $\mathcal{F}_{\underline{\nu}}$ and hence is smooth. The natural map $\pi_{\underline{\nu}} \colon \tilde{\mathcal{F}}_{\underline{\nu}} \to E_V$ is projective, and thus by the decomposition theorem the push-forward $\tilde{L}_{\underline{\nu}} = (\pi_{\underline{\nu}})_!(\mathbb{C})$ of the constant sheaf is a semisimple complex, that is, it is isomorphic to the direct sum of its (shifted) perverse cohomologies, and these perverse constituents are all semisimple. Let $\mathcal{P}_V$ denote the full subcategory of the category of perverse sheaves on E_V whose objects are direct sums of the simple perverse sheaves[1] that arise as constituents of the complexes $\tilde{L}_{\underline{\nu}}$ as $\underline{\nu}$ runs over the (finite) set of $\underline{\nu}$ such that $\sum_{s=1}^m \nu^s = |V|$. Note that $\mathcal{P}_V$ has finitely many simple objects. Let $\mathcal{Q}_V$ be the full subcategory of $D^b(E_V)$ whose objects are semisimple complexes whose simple constituents are all isomorphic to objects in $\mathcal{P}_V$. Note that since the maps $\pi_{(\underline{i},\underline{c})}$ are G_V-equivariant, and the constant sheaf of $\tilde{\mathcal{F}}_{(\underline{i},\underline{c})}$ is canonically G_V-equivariant, the objects in $\mathcal{P}_V$ have a G_V-equivariant structure. This equivariance means, for example, that if V and W are I-graded vector spaces of the same graded dimension, then the sets $\mathcal{P}_V$ and $\mathcal{P}_W$ are canonically identified via any graded isomorphism $V \cong W$.

For $\nu \in \mathbb{N}^I$, let $\mathcal{K}_\nu$ be the free $\mathcal{A}$-module with basis indexing the simple objects in $\mathcal{P}_V$ for some V with $|V| = \nu$. There is an obvious map from $\mathcal{Q}_V$ to $\mathcal{K}_\nu$ which assigns to P an object of $\mathcal{Q}_V$ the element $\sum_{i=1}^k v^{r_i} b_i$ ($b_i \in \mathcal{P}_V$, $r_i \in \mathbb{Z}$) where $P \cong \bigoplus_{i=1}^k b_i[r_i]$ where $b_i \in \mathcal{P}_V$. Let $\mathcal{K}$ denote the direct sum of these $\mathcal{A}$-modules:

$$\mathcal{K} = \bigoplus_{\nu \in \mathbb{N}^I} \mathcal{K}_\nu.$$

Lusztig defined functors Ind and Res which induce a multiplication and (twisted) comultiplication on $\mathcal{K}$. We first describe the multiplication functor. For $\nu, \nu_1, \nu_2 \in \mathbb{N}^I$ with $\nu = \nu^1 + \nu^2$, take V of dimension ν, and a graded subspace W of dimension ν^2, so that $T = V/W$ has graded dimension ν^1. Let P denote the subgroup of G_V preserving W, and let U denote the subgroup of P consisting of those elements of P which induces the identity on W and V/W. Let K denote the subspace of E_V consisting of those $x \in E_V$ for which W is x-adapted in the sense that (W, x) is a subrepresentation of (V, x). Now consider the diagram:

$$\begin{tikzcd} \bar{E} & E' \arrow[l, "p_1"'] \arrow[r, "p_2"] & E'' \arrow[r, "p_3"] & E_V. \end{tikzcd}$$

where $\bar{E} = E_T \times E_W$, $E' = G_V \times_U K$, $E'' = G_V \times_P K$ and the maps are the obvious ones. The map p_1 is smooth with connected fibres, the map p_2 is a principal $P/U \cong G_W \times G_T$-bundle and the map p_3 is proper.

Let $\mathcal{Q}_{T,W}$ be category given by taking the above definition of $\mathcal{Q}_V$, but now for the disjoint union of two copies of our oriented graph, with graded vector

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[1] One could also consider the abelian category whose composition factors are constituents lie in $\tilde{L}_{\underline{\nu}}$, but we do not need to consider this.

space $T \bigoplus W$. Thus the simple objects in $\mathcal{P}_{T,W}$ are isomorphic to $A_1 \boxtimes A_2$ where $A_1 \in \mathcal{P}_T$ and $A_2 \in \mathcal{P}_W$ are simple. Suppose that A is an object in $\mathcal{Q}_{T,W}$. Then since p_1 is smooth, $p_1^*(A)$ is again a semisimple complex, which is clearly P/U-equivariant. Therefore we may descend it via p_2 to a complex $\tilde{A}$ on E'', and then push $\tilde{A}$ forward to obtain, by the decomposition theorem, a semisimple complex on E_V. We denote this complex by $\mathrm{ind}_{T,W}^V(A)$. This functor is almost compatible with Verdier duality. To obtain this compatiblility on the nose, one must shift by the difference between the fibre dimensions of p_1 and p_2. Let

$$m(\nu^1, \nu^2) = \sum_{h \in H} \nu^1_{s(h)} \nu^2_{t(h)} + \sum_{i \in I} \nu^1_i \nu^2_i,$$

and set

$$\mathrm{Ind}_{T,W}^V(A) = \mathrm{ind}_{T,W}^V(A)[m(\nu_1, \nu_2)].$$

Lusztig shows [L6, §9.2] that this functor restricts to a functor between the categories $\mathcal{Q}_{T,W}$ and Q_V. Moreover, this functor has a natural lift to the equivariant derived category $D^b_{G_V}(E_V)$, and there it has a natural adjoint $\mathrm{Res}_{T,W}^V$ which we now describe. Consider the diagram

$$\begin{tikzcd} \bar{E} & K \arrow[l, "\kappa"'] \arrow[r, "\iota"] & E \end{tikzcd}$$

where κ and ι are the obvious maps. Given a complex A in $\mathcal{Q}_V$ we set $\mathrm{Res}_{T,W}^V(A) = \kappa_! \iota^*(A)[m'(\nu^1, \nu^2)]$, where

$$m'(\nu^1, \nu^2) = \sum_{h \in H} \nu^1_{s(h)} \nu^2_{t(h)} - \sum_{i \in I} \nu^1_i \nu^2_i,$$

The adjunction then shows that

$$\mathcal{R}\mathrm{Hom}_{D^b_{G_V}}(\mathrm{Ind}_{T,W}^V(A), B) \cong \mathcal{R}\mathrm{Hom}_{D^b_{G_V}}(A, \mathrm{Res}_{T,W}^V(B)).$$

Lusztig shows [L6, §9.2] that the functor $\mathrm{Res}_{T,W}^V$ (at the level of the ordinary, rather than equivariant, derived categories) again induces a functor between the categories $\mathcal{Q}_V$ and $\mathcal{Q}_{T,W}$ (we will review the proof of this in §4).

Remark 2.3. Notice that since the group G_V is connected a perverse sheaf on E_V can possess only one equivariant structure, hence the category $\mathcal{Q}_V$ has a natural lift to a full subcategory of the equivariant derived category, and thus we may consider $\mathcal{Q}_V$ inside either $D^b(E_V)$ or $D^b_{G_V}(E_V)$.

3. QUIVERS WITH AUTOMORPHISMS AND SHEAVES

3.1. In this section we review the construction of the canonical basis for a non-simply-laced group. In this situation one must use a quiver equipped with an automorphism

Definition 3.1. Let $Q = (I, H)$ be a quiver, that is, a graph with vertex set I and a set of directed edges H. An automorphism of the quiver is a bijection $a\colon I \to I$ which induces a bijection on H. We say an orientation of a quiver with automorphism a is a-admissible if $s(a(h)) = a(s(h))$ and $t(a(h)) = a(t(h))$.

Let (Q, a) be a quiver with automorphism. We will assume that $a^n = \mathrm{id}$ and that the orbits of vertices are discrete (that is, there is no edge between two vertices in the same a-orbit). For such (Q, a) it is straightforward to show that there always exists a compatible orientation.

Lusztig modifies his definition of $\mathcal{P}_V$ and $\mathcal{Q}_V$ in order to take into account the automorphism. For this he uses the general notion of a *periodic functor*.

Definition 3.2. Let $\mathcal{C}$ be a linear category, that is, a category where $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is a vector space for each pair of objects A and B, such that composition is bilinear and finite direct sums exist. Suppose that n is a positive integer. A periodic functor is a functor $a^*\colon \mathcal{C} \to \mathcal{C}$ such that $(a^*)^n$ is the identity functor. To a pair $(\mathcal{C}, a^*)$ one may attach a new category $\tilde{\mathcal{C}}$. The objects of $\tilde{\mathcal{C}}$ are pairs (A, ϕ) where A is an object of $\mathcal{C}$ and $\phi\colon a^*(A) \to A$ is an isomorphism in $\mathcal{C}$ such that the composition:

$$(a^*)^n(A) \xrightarrow{(a^*)^{n-1}\phi} (a^*)^{n-1}(A) \xrightarrow{(a^*)^{n-2}\phi} (a^*)^{n-2} \dots \longrightarrow a^*(A) \xrightarrow{\phi} A$$

is the identity map. If (A, ϕ) and (A', ϕ) two such pairs, there is a natural map $u\colon \mathrm{Hom}(A, A') \to \mathrm{Hom}(A, A')$ given by $u(f) = \phi' a^*(f)\phi^{-1}$. It follows from the definition that this map u satisfies $u^n = 1$, thus we may set

$$\mathrm{Hom}_{\tilde{\mathcal{C}}}((A, \phi), (A, \phi')) = \{f \in \mathrm{Hom}_{\mathcal{C}}(A, A') : u(f) = f\}.$$

Notice that if (A, ϕ) is an object of $\tilde{\mathcal{C}}$ the so is $(A, \zeta\phi)$ where ζ is any n-th root of unity in our field.

3.2. Given an object (B, ϕ) in $\tilde{\mathcal{C}}$, we say that it is *traceless* if there is an object D in $\tilde{\mathcal{C}}$ such that $(a^*)^k(D) \cong D$ for some $k \geq 2$ dividing n and

$$B \cong D \oplus a^*(D) \oplus \ldots \oplus (a^*)^{k-1}(D),$$

where under this isomorphism, the map ϕ goes to the map which cycles the summands $(a^*)^j(D)$. Set $\mathcal{O}$ to be the subring of $\mathbb{C}$ consisting of the ring of integers in the n-th cyclotomic field. Then we may attach to $\tilde{\mathcal{C}}$ an $\mathcal{O}$-module $\mathcal{K}(\tilde{C})$ as follows. As an $\mathcal{O}$-module $\mathcal{K}(\tilde{C})$ is generated by symbols $[B, \phi]$ for each isomorphism class of objects (B, ϕ) in $\tilde{C}$, subject to the relations:

1. $[B, \phi] + [B', \phi'] = [B \oplus B', \phi \oplus \phi']$
2. $[B, \phi] = 0$ if (B, ϕ) is a traceless object.
3. $[B, \phi] = \zeta[B, \phi']$ if $\phi = \zeta\phi'$.

Clearly if $\mathcal{C}$ and $\mathcal{D}$ are linear categories equipped with periodic functors $a^*_{\mathcal{C}}$ and $a^*_{\mathcal{D}}$ respectively, and $F\colon \mathcal{C} \to \mathcal{D}$ is a functor equipped with an isomorphism of functors $F \circ a^*_{\mathcal{C}} = a^*_{\mathcal{D}} \circ F$, then F induces an $\mathcal{O}$-linear map between the modules $\mathcal{K}(\tilde{\mathcal{C}})$ and $\mathcal{K}(\tilde{\mathcal{D}})$.

3.3. Now suppose that (Q, a) is a quiver with automorphism, and that an admissible orientation has been chosen. Then we may consider a category of representations of Q equipped with an automorphism corresponding to a. More precisely, we consider now I-graded vector spaces V equipped with a linear map $a\colon V \to V$ such that $a(V_i) = V_{a(i)}$ where if $a^k(i) = i$, then $a^k_{|V_i} = \mathrm{id}$. Note that for such a graded vector space V, we must have $\dim(V_i) = \dim(V_{a(i)})$ for each $i \in I$, thus $|V|$ is an element of the a-invariants $\mathbb{N}I^a$ of $\mathbb{N}I$. As before, one may define the objects $\mathcal{P}_V$ and the category $\mathcal{Q}_V$. If $((\underline{i}, \underline{c})) \in I^m \times \mathbb{N}^m$ is such that $\sum_{k=1}^m a_k i_k = |V|$, we may consider the resolution $\pi_{(\underline{i},\underline{c})}\colon \tilde{\mathcal{F}}_{(\underline{i},\underline{c})} \to E_V$ as before. Clearly the action of a on V induces a map $\tilde{a}\colon \tilde{\mathcal{F}}_{(\underline{i},\underline{c})} \to \tilde{\mathcal{F}}_{(\underline{i}',\underline{c}')}$ compatible with the projections, where $i'_k = i_{a(k)}$ and $c'_k = c_{a(k)}$. It is thus clear that a^* induces an isomorphism between $L_{(\underline{i},\underline{c})}$ and $L_{(\underline{i}',\underline{c}')}$ so that a^* preserves the categories $\mathcal{P}_V$ and $\mathcal{Q}_V$, and hence they are equipped with a periodic structure. Thus we have $\mathcal{O}$-modules and $\mathcal{K}(\tilde{\mathcal{Q}}_V)$. Moreover, the discussion in [L6, 12.1.3] shows that, if we write $[B[n], \phi[n]] = v^n[B, \phi]$ where (B, ϕ) is an object in $\tilde{\mathcal{P}}_V$, then $\mathcal{K}(\tilde{\mathcal{Q}}_V)$ becomes an $\mathcal{O}[v, v^{-1}]$-module which is free with basis given by objects $[B, \phi]$ (up to n-th roots of unity) where B is a simple object of $\mathcal{P}_V$ such that $\phi\colon a^*(B) \to B$ is an isomorphism.

The functors $\mathrm{Ind}^V_{T,W}$ and $\mathrm{Res}^V_{T,W}$ are compatible with a^* and so induce functors on the periodic categories $\tilde{\mathcal{Q}}_{T,W}$ and $\tilde{\mathcal{Q}}_V$ and hence $\mathcal{O}$-linear maps on the groups $\mathcal{K}(\tilde{\mathcal{Q}}_{T,W})$, $\mathcal{K}(\tilde{\mathcal{Q}}_V)$, so that ${}_{\mathcal{O}}\mathbf{k} = \bigoplus_V \mathcal{K}(\tilde{\mathcal{Q}}_V)$ becomes a twisted $\mathcal{O}[v, v^{-1}]$-Hopf algebra.

4. CONSTRUCTION OF THE QUANTUM FROBENIUS FUNCTOR

4.1. Fix a symmetric Cartan datum $(I, \cdot)$ and let $Q = (I, H)$ be the (undirected) quiver attached to it in the natural way. Pick Ω an orientation of Q, and let $\mathcal{V}$ be the category of I-graded vector spaces.

We need a slight generalisation of the restriction functor for sheaves on quiver moduli (see §2.2). Let ℓ be a positive integer and let $V \in \mathcal{V}$, together with a flag

$$\mathcal{W} = (V = W_0 \supseteq W_1 \supseteq W_2 \ldots \supseteq W_\ell),$$

of I-graded subspaces. If $T = \bigoplus_{i=1}^{\ell} W_{i-1}/W_i$, then T is an $I \times \mathbb{Z}/\ell\mathbb{Z}$-graded vector space, and we may define functors I^V_T and R^V_T (known as induction and restriction) between the categories $\mathcal{P}_V$ and $\mathcal{P}_T$ as slight generalizations of the functors defined in §2.2.

To defined R^V_T let $K = \{x \in E_V : x(W_j) \subseteq W_j\}$, and let $\iota\colon K \to E_V$ be the natural inclusion. Then there is a natural map $\kappa\colon K \to E_T$ given by $x \mapsto (x_i)_{i=1}^{\ell}$ where x_i is the map induced by x on W_{i-1}/W_i.

Define R^V_T by

$$A \mapsto \kappa_!(\iota^*(A))$$

(where the functor $\kappa_!$ is understood in its derived sense). As such R^V_T appears only to be a functor from $\mathcal{D}(E_V)$ to $\mathcal{D}(E_T)$, but Lusztig's work shows that it in fact induces a map on the categories $\mathcal{Q}_V$. We will however give here a proof which is somewhat different to the one in [L6], in that it works in a slightly more general context, though it involves the same tools.

4.2. In this section we recall the definition of the hyperbolic localization functor of [Br] and note its compatibilities with some basic functors. Suppose that X is a normal variety over an algebraically closed field (we only need to work over $\mathbb{C}$, though [Br] works also in the etale setting), and that $T = \mathbb{G}_m$ acts algebraically on X. Let $F = X^T$ be the variety of fixed points of X and suppose it has connected components $F_1, F_2, \ldots F_r$. Then we set

$$X_k^+ = \{x \in X : \lim_{z \to 0} z \cdot x \in F_k\}, \quad X_k^- = \{x \in X : \lim_{z \to \infty} z \cdot x \in F_k\}.$$

Let X^+ (respectively X^-) be the disjoint union of the X_k^+ (respectively X_k^-) and let $f^\pm\colon F \to X^\pm$ and $g^\pm\colon X^\pm \to X$ be the obvious maps induced by the inclusions. Define the hyperbolic localization functors $(\bullet)^{!*}, (\bullet)^{*!}\colon \mathcal{D}(X) \to \mathcal{D}(F)$ by

$$S^{!*} = (f^+)^!(g^+)^*(S); \quad S^{*!} = (f^-)^*(g^-)^!(S).$$

These functors are not in general isomorphic. We will say that a complex $S \in \mathcal{D}(X)$ is twisted equivariant for the action of T, in the sense that $a^*(S) \cong L \boxtimes S$

where $a\colon T \times X \to X$ denotes the action map, and L is a local system on T. A complex S is said to be T-constructible if is constructible with respect to a T-invariant stratification of X. Then we have the following theorem.

Theorem 4.1. ([Br]) *There is a natural morphism $\iota_S\colon S^{*!} \to S^{!*}$ which is an isomorphism if S is weakly equivariant, or (when we work over $\mathbb{C}$) if S is T-constructible. As a consequence of this, if S is a pure complex, then so is $S^{!*}$.*

Note that in particular, if S is a simple perverse sheaf which is the middle extension of an local system which underlies a (polarizable) pure variation of Hodge structure, then S is pure, and so its hyperbolic localization is also, and hence it is a semisimple complex, that is, it is isomorphic to the direct sum of its shifted perverse cohomology sheaves, and these perverse sheaves are semisimple. We now check that hyperbolic localization is compatible with some standard functors.

Lemma 4.2. *Suppose that X and Y are normal varieties with a T-action, and $\pi\colon X \to Y$ is a proper T-equivariant map. Then there is a natural isomorphism:*

$$\pi_!(A^{!*}) \cong (\pi_!(A))^{!*}.$$

Proof. Consider first the case where $F_X = X^T$ is connected. Let $F_Y = Y^T$ and let X^+ and Y^+ be the corresponding attracting sets. Then we have the diagram:

$$\begin{tikzcd}
X \arrow[d] & X^+ \arrow[l, "f_X"'] \arrow[d] & F_X \arrow[l, "g_X"'] \arrow[d] \\
Y & Y^+ \arrow[l, "f_Y"] & F_Y \arrow[l, "g_Y"]
\end{tikzcd}$$

where the vertical arrows are all induced by the map π. Now we also have maps $h_X\colon X^+ \to F_X$ and $h_Y\colon Y^+ \to F_Y$ where $h_X(x) = \lim_{z\to 0} z\cdot x$ and h_Y is defined similarly. It is know (see [Br]) that if A is weakly T-equivariant, then $A^{!*} = g_X^! f_X^*(A) \cong (h_X)_! f_X^*(A)$. It follows from this and functoriality of proper pushforward that we have a natural isomorphism

$$\pi_! A^{!*} \cong (h_Y)_! \pi_! f_X^*(A). \tag{4.1}$$

Now note that since π is proper, the left-hand square is Cartesian, *i.e.* if the limit $\lim_{z\to 0} z\cdot \pi(x)$ exists, then so also must the limit $\lim_{z\to 0} z\cdot x$. Thus using proper base-change we obtain a canonical isomorphism

$$\pi_! f_X^*(A) \cong f_Y^* \pi_!(A), \tag{4.2}$$

and combining the isomorphisms given in (4.1) and (4.2) we get the required isomorphism. Finally, to deal with the general case, note that hyperbolic localization commutes with pullbacks to open subsets and pushforwards from closed subsets, so that we may reduce to the case of a single fixed point component in X. (Note that such partitioning arguments may result in the restriction of π no longer being proper, however we only used this to ensure that the left-hand square in our diagram is Cartesian, and this property will indeed be preserved.)

□

Remark 4.3. Note that if, in the context of the previous Lemma, we assume that π is a smooth map, then the hyperbolic localization functor commutes with the operation of pulling back along π, since $\pi^! \cong \pi^*[2d]$ where d is the fibre dimension of π.

4.3. We now check, as mentioned above, that the restriction functor R^V_T is compatible with the categories $\mathcal{Q}_V$ and $\mathcal{Q}_T$.

Theorem 4.4. *The functor R^V_T sends the category $\mathcal{Q}_V$ to the category $\mathcal{Q}_T$.*

Proof. Let us pick a $\mathbb{C}^\times$ action on E_V such that $E_V^{\mathbb{C}^\times}$ is isomorphic to E_T via the restriction of the map κ (such an action is easily found by choosing an arbitrary splitting of the flag $\mathcal{W}$ and assigning the distinct subspaces of the splitting distinct weights). Moreover, by choosing the weights of the action appropriately, we may arrange that the attracting set for this $\mathbb{C}^\times$ action is the set $K = \{x \in E_V : x(W_j) \subseteq W_j\}$. Then if A is weakly equivariant for the $\mathbb{C}^\times$ action, so is $\iota^*(A)$ and for such sheaves we have $g^!\iota^*(A) \cong \kappa_!\iota^*(A)$ where $g\colon E_V^{\mathbb{C}^\times} \to K$ denotes the inclusion map.

Now if A is an object in $\mathcal{P}_V$, then it is G_V-equivariant, and so in particular $\mathbb{C}^\times$-equivariant (not just weakly equivariant) so that we have $R^V_T(A) \cong A^{!*}$. But then it follows from Theorem 4.1 that $R^V_T(A)$ is a semisimple complex, and so to check that it lies in $\mathcal{Q}_T$ we just need to show the simple constituents are objects of $\mathcal{P}_T$. For this we just need to consider the complexes of the form $L_{\underline{\nu}} = (\pi_{\underline{\nu}})_!(\mathbb{C})$ where $\pi_{\underline{\nu}}\colon \tilde{\mathcal{F}}_{\underline{\nu}} \to E_V$ is the proper map constructed in Definition 2.2. But now since the $\mathbb{C}^\times$ action clearly extends to $\tilde{\mathcal{F}}_{\underline{\nu}}$ we may apply Lemma 4.2 to see that $(L_{\underline{\nu}})^{!*}$ is isomorphic to $(\pi_{\underline{\nu}})_!(\mathbb{C}^{!*})$.

Now the hyperbolic localization of the constant sheaf on a smooth projective variety X is given by:

$$\mathbb{C}_X^{!*} = \bigoplus_{k=1}^{r} \mathbb{C}_{F_k}[-n_k],$$

where F_k are the components of the fixed point locus, and $n_k = \dim_{\mathbb{R}} X_k^+ - \dim_{\mathbb{R}} F_k$, as X_k^+ is an affine bundle of dimension $n_k/2$ over F_k. By compactifying our space $\tilde{\mathcal{F}}_{\underline{\nu}}$ compatibly with the $\mathbb{C}^\times$-action, it is easy to see that the same result holds for $\tilde{\mathcal{F}}_{\underline{\nu}}$. Now the components of the fixed point loci can be identified with products $\tilde{\mathcal{F}}_{\underline{\nu}^1} \times \dots \tilde{\mathcal{F}}_{\underline{\nu}^\ell}$ where $\tilde{\mathcal{F}}_{\underline{\nu}^s}$ is a resolution of E_{T^s}, where we write $T^s = W_{s-1}/W_s$, so that $\sum_t \nu_t^s = |T^s|$, hence we see that

$$L_{\underline{\nu}}^{!*} \cong \bigoplus L_{\underline{\nu}^1} \boxtimes \dots \boxtimes L_{\underline{\nu}^\ell}[d_{\underline{\nu}^1,\dots,\underline{\nu}^\ell}]$$

where each $L_{\underline{\nu}^s}$ is a complex in $\mathcal{Q}_{T^s}$ and $d_{\underline{\nu}^1,\dots,\underline{\nu}^\ell} \in \mathbb{Z}$, and the sum is over all possible ℓ-tuples $((\underline{\nu}^1,\dots,\underline{\nu}^\ell)$. Thus, in particular, the category $\mathcal{Q}_V$ is sent to the category $\mathcal{Q}_T$ under R^V_T as claimed.

□

Remark 4.5. Note that the identification of R^V_T with a hyperbolic localization shows that R^V_T sends the category of equivariant pure objects on E_V to that on E_T, so that there is a "restriction functor" defined on the category of pure equivariant complexes on the quiver moduli, which is a larger category than Lusztig's category $\mathcal{Q}_V$ (a kind of "pure Hall category", whereas Lusztig's corresponds to a categorification of the composition algebra). The above theorem then shows that R^V_T restricts to a functor between Lusztig's categories. Note also that Lusztig has given a generalization of his technique of the "long exact sequence of a partition" in [L7, §1].

4.4. We now come to the construction of the geometric version of the Frobenius map. We start with a new category of vector spaces $\mathcal{V}^c$. Objects of $\mathcal{V}^c$ are $I \times \mathbb{Z}/\ell\mathbb{Z}$ graded vector spaces equipped with a map $a\colon V \to V$ which maps $V_{i,k}$ to $V_{i,k+1}$ $(i \in I, k \in \mathbb{Z}/\ell Z)$, so that a^ℓ is the identity. The category $\mathcal{V}^c$ supports representations of the quiver with automorphism $Q_\ell = (I \times \mathbb{Z}/\ell\mathbb{Z}, H_\ell, a)$, where we let H_ℓ be a set of edges on the vertex set $I \times \mathbb{Z}/\ell\mathbb{Z}$ given by joining (i,r) to (j,s) if $r = s$ and $i-j$ is an edge in H, and the automorphism a is given by $a(i,r) = (i, r+1)$. Clearly an orientation Ω of (I,H) induces an a-admissible orientation of $(I \times \mathbb{Z}/\ell\mathbb{Z}, H_\ell)$. By forgetting the $\mathbb{Z}/\ell\mathbb{Z}$ part of the grading there is a natural functor from $\mathcal{V}^c$ to $\mathcal{V}$. Note that the Cartan datum attached to the quiver with automorphism $(I \times \mathbb{Z}/\ell\mathbb{Z}, H_\ell, a)$ is ℓ^2 times the Cartan datum attached to the quiver (I,H).

The construction of Lusztig [L6, Part II] reviewed in §3.3 shows that the direct sum of the groups ${}_{\mathcal{O}}\mathbf{k}_\ell = \bigoplus_T \mathcal{K}(\tilde{\mathcal{Q}}_T)$ where T runs over the isoclasses of objects in $\mathcal{V}^c$, is an $\mathcal{O}[v, v^{-1}]$-algebra, and in fact Lusztig shows that it contains an $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$-lattice which is isomorphic to the $\mathcal{A}$-form ${}_{\mathcal{A}}\mathbf{f}_\ell$ of the positive part of the quantum group attached to the Cartan datum of given by $(I \times \mathbb{Z}/\ell\mathbb{Z})$. Note that this datum is just the datum given by I scaled by ℓ^2. On the other hand, given I, we may build the algebra $\mathbf{k} = \bigoplus_V \mathcal{K}(\mathcal{Q}_V)$ where V now runs over isoclasses of objects in the category $\mathcal{V}$, and give it the structure of twisted Hopf algebra ${}_{\mathcal{A}}\mathbf{f}$ (here since there is no automorphism, we do not need to consider any root of unity). We would like to construct a functor from $\mathcal{Q}_V$ to $\tilde{\mathcal{Q}}_T$ where V is the image of T under the forgetful functor from $\mathcal{V}^c$ to $\mathcal{V}$.

Definition 4.6. Let V and T be as above. Since V is in the image of the forgetful functor from $\mathcal{V}^c$ to $\mathcal{C}$, if we let $\dim(V) = \mathbf{a}$ where $\mathbf{a} = (a_i)_{i \in I}$, then $a_i = \ell b_i$ for some nonnegative integers b_i. Pick a $\mathbb{C}^\times$-action on T which gives each graded subspace $T_{i,r}$ weight λ_r $(i \in I, r \in \mathbb{Z}/\ell\mathbb{Z})$, for some integers $\lambda_1 < \lambda_2 < \ldots < \lambda_r$. Clearly this gives a $\mathbb{C}^\times$ action on V, and moreover we see that $E_V^{\mathbb{C}^\times} = E_T$.

Given a complex A in $\mathcal{Q}_V$ we can consider its hyperbolic localization $A^{!*}$ with respect to this $\mathbb{C}^\times$-action. By Theorem 4.4 we know that $A^{!*}$ is an object of $\mathcal{Q}_T$ (since we may identify the functor $A \mapsto A^{!*}$ with the functor R_T^V for a suitable filtration of V). However we need an object of the corresponding periodic category $\tilde{\mathcal{Q}}_T$. For this we again need to use equivariance. The $\mathbb{C}^\times$ action on E_V gives a one-parameter subgroup which we denote by $\lambda\colon \mathbb{C}^\times \to G_V$. Note that G_T is the centralizer of λ in G_V. Then clearly the functor of hyperbolic localization induces a functor from G_V-equivariant complexes to G_T-equivariant complexes.

Lemma 4.7. *Let V, T be as above, and let A be an object in $\mathcal{Q}_T$. Then there is a natural choice of isomorphism $\phi\colon a^*(A^{!*}) \to A^{!*}$.*

Proof. Since the element $a\colon T \to T$ (thought of as an endomorphism of V) clearly lies in G_V, the G_V-equivariant structure of A provides us with a canonical isomorphism $\tilde{\phi}\colon a^*(A) \to A$, hence by functoriality we obtain a morphism $\phi\colon (a^*(A))^{!*} \to A^{!*}$ whose ℓ-th power is the identity, since $a^\ell = \mathrm{id}_V \in G_V$.

Thus we will have the required isomorphism if we can show there is a canonical isomorphism $(a^*(A))^{!*} \cong a^*(A^{!*})$. For this we use our above discussion of the relation between hyperbolic localization and Lusztig's restriction functor. The latter is adjoint to the corresponding induction functor (in the equivariant category) as is shown in [L6, Lemma 9.2.9]. But this functor clearly depends only on the filtration attached to the $\mathbb{C}^\times$-action. Now since we assume that the grading on each

V_i given by the $\mathbb{C}^\times$ has graded summands of equal dimension, all such filtrations are conjugate under the action of G_V, hence the all such induction functors are isomorphic. Adjointness then yields the isomorphism we need. $\square$

The previous lemma thus shows that we obtain a functor from $\mathcal{Q}_V$ to $\tilde{Q}_T$ given by

$$A \mapsto (A^{!*}, \phi).$$

We denote this functor by **Fr**, and let Fr_{cat} denote the induced map from $\mathbf{k}$ to ${}_{\mathcal{O}}\mathbf{k}$ (note that we do not assert that the image of this map lies in the lattice yielding ${}_{\mathcal{A}}\mathbf{f}_\ell$).

Remark 4.8. One should note that the functor **Fr** uses the equivariant structure, so that it is a functor from the equivariant version of Lusztig's category $\mathcal{Q}_V$. A simple object in $\mathcal{Q}_V$ has an equivariant structure which is unique up to a scalar, since the group G_V is connected reductive. Moreover, each simple object in $\mathcal{P}_V$ is also Verdier self-dual, so that if we require the isomorphism to be compatible with duality, then it is unique up to sign.

We now wish to show that, once we pass to the coefficient ring $\mathbb{Z}[\zeta]$ of ℓ-th cyclotomic integers, this map becomes an algebra homomorphism (that is, it is compatible with Lusztig's induction functor.) For this we follow the strategy of [McG] in that we work with the "monomial" complexes $L_{\underline{\nu}}$ defined in §2.2. We will write $\mathcal{X}_\ell$ for the set of sequences $\underline{\mu} = (\mu^1, \mu^2, \dots, \mu^m)$ where each $\mu^s \in \mathbb{N}[I \times \mathbb{Z}/\ell\mathbb{Z}]$ has discrete support, and $\mathcal{X}$ for the corresponding set of sequences for the quiver Q. There is a natural map from $\chi\colon \mathcal{X}_\ell \to \mathcal{X}$ induced by the map $(i,r) \mapsto i$ from $I \times \mathbb{Z}/\ell\mathbb{Z}$ to I. Note that there is a natural action of a on $\mathcal{X}_\ell$ induced from its action on $I \times \mathbb{Z}/\ell\mathbb{Z}$, and χ restricts to a bijection between $\mathcal{X}_\ell^a$ and

$$\ell\mathcal{X} = \{\underline{\nu} \in \mathcal{X} : \exists \underline{v} \in \mathcal{X}, \nu^s = \ell v^s, 1 \leq s \leq m\}.$$

If A is a $\mathbb{Z}[v, v^{-1}]$ algebra, and $\mathcal{O}$ as above denotes the ring of ℓ-th cyclotomic integers, then let $\mathcal{O} \otimes A$ denote the algebra obtained by base change via the homomorphism $\mathbb{Z}[v, v^{-1}] \to \mathcal{O}$ given by $v \mapsto \zeta$. Recall that if $\underline{\nu} \in \mathcal{X}$ then we defined $\tilde{L}_{\underline{\nu}} = (\pi_{\underline{\nu}})_!(\mathbb{C})$. This complex is not Verdier self-dual, however if we set

$$f(\underline{\nu}) = \sum_{h \in H, l' < l} \nu^{l'}_{s(h)} \nu^{l}_{t(h)} + \sum_{i; l < l'} \nu^{l'}_i \nu^l_i, \tag{4.3}$$

then the complex $L_{\underline{\nu}} = \tilde{L}_{\underline{\nu}}[f(\underline{\nu})]$ is self-dual (see [L6, §9.1.2]). Similarly, in the category $\mathcal{Q}_T$ we have the complexes $L_{\underline{\mu}}$. Now if $\underline{\mu} \in \mathcal{X}_\ell^a$, the a-invariant elements of $\mathcal{X}_\ell$ then there is an obvious isomorphism $a^*(\mathbb{C}_{\tilde{\mathcal{F}}_{\underline{\mu}}}) \cong \mathbb{C}_{\tilde{\mathcal{F}}_{\underline{\mu}}}$ from the pullback of the constant sheaf on $\tilde{\mathcal{F}}_{\underline{\mu}}$ to the constant sheaf, and this induces an isomorphism $\phi_0\colon a^*(L_{\underline{\mu}}) \to L_{\underline{\mu}}$. Thus we can associate to each $\underline{\mu} \in \mathcal{X}_\ell^a$ an object $(L_{\underline{\mu}}, \phi_0) \in \tilde{\mathcal{Q}}_T$. Note that the same argument also shows that the semisimple complexes $L_{\underline{\nu}}$ also have a canonical equivariant structure, and so may be viewed as objects in the equivariant version of $\mathcal{Q}_V$.

Theorem 4.9. *Let $\underline{\nu}$ and $\underline{\nu}'$ be elements of $\mathcal{X}$, and consider the corresponding complexes $L_{\underline{\nu}}$ and $L_{\underline{\nu}'}$. Then the map $Fr\colon \mathcal{O} \otimes \mathbf{k} \to_{\mathcal{O}} \mathbf{k}$ induced by Fr_{cat} from satisfies*

$$Fr([L_{\underline{\nu}}].[L_{\underline{\nu}'}]) = Fr([L_{\underline{\nu}}]).Fr([L_{\underline{\nu}}]).$$

where $[L_{\underline{\nu}}]$ denotes the element of $\mathbf{k}$ given by $L_{\underline{\nu}}$.

Proof. Recall the construction of the induction and restriction functors from §2.2. It is immediate from the definitions (see [L6, §9.2.7]) that $\mathrm{Ind}(L_{\underline{\nu}} \boxtimes L_{\underline{\nu}'}) = L_{\underline{\nu},\underline{\nu}'}$ where we write $\underline{\nu}, \underline{\nu}'$ for the concatenation of the elements of $\mathcal{X}$, and moreover the analogous result holds in $\tilde{\mathcal{Q}}_T$ for objects $(L_{\underline{\mu}}, \phi_0), (L_{\underline{\mu}'}, \phi_0)$ as above ($\underline{\mu}, \underline{\mu}' \in \mathcal{X}^a$). Thus it in fact suffices to show that $Fr(L_{\underline{\nu}}) = 0$ unless there is a $\underline{\mu} \in \mathcal{X}_\ell$ such that $\chi(\underline{\mu}) = \underline{\nu}$, in which case we have

$$Fr([L_{\underline{\nu}}]) = [L_{\underline{\mu}}, \phi_0]. \tag{4.4}$$

First consider the case where $\underline{\mu}$ does not exist, *i.e.* $\underline{\nu}$ not in the image of χ. In this case, we see from the proof of Theorem 4.4 shows that

$$\mathbf{Fr}(L_{\underline{\nu}}) = \Big(\bigoplus_{\underline{\mu} \in \mathcal{X}_\ell} L_{\underline{\mu}}[d_{\underline{\mu}}], \phi \Big)$$

where $\underline{\mu}$ runs over the elements of $\mathcal{X}_\ell$ such that $\chi(\underline{\mu}) = \underline{\nu}$, and where the integers $d_{\underline{\mu}}$ record shifts. The action of a^* clearly permutes the summands by interchanging the factors in the tensor product. When $\underline{\mu}$ does not exist, there is no summand fixed by this permutation, so that all objects on the right-hand side are traceless, and we get 0 in the traceless $\mathcal{K}$-group $\mathcal{K}(\tilde{\mathcal{Q}}_T)$ as claimed.

On the other hand, if $\chi(\underline{\mu}) = \underline{\nu}$, for some (unique) $\underline{\mu} \in \mathcal{X}_\ell^a$, then clearly the only term in the above direct sum which is preserved by a^* is the term $(L_{\underline{\mu}}[d_{\underline{\mu}}], \phi)$. Thus to finish the proof it is enough to show that the isomorphism ϕ corresponds to ϕ_0, and that the shift $d_{\underline{\mu}}$ is congruent to zero modulo ℓ (this is where we use the specialization from $\mathcal{O}[v, v^{-1}]$ to $\mathcal{O}$). To check the identification of the isomorphisms ϕ and ϕ_0 one only needs to note that the G_V-equivariant structure on $L_{\underline{\nu}}$ and the periodic isomorphism for $L_{\underline{\mu}}$ come from that on the constant sheaf on $\tilde{\mathcal{F}}_{\underline{\nu}}$ and $\tilde{\mathcal{F}}_{\underline{\mu}}$ respectively, and thus are clearly compatible.

Now for $\underline{\mu}^0, \underline{\mu}^2, \ldots, \underline{\mu}^{\ell-1} \in \mathcal{X}$ set:

$$M(\underline{\mu}^0, \underline{\mu}^2, \ldots, \underline{\mu}^{\ell-1}) = \sum_{\substack{h \in H \\ p' \leq p; r' < r}} \mu^{r'}_{s(h)}(p') \mu^{r}_{t(h)}(p) + \sum_{\substack{i \in I \\ p' < p; r' < r}} \mu^{r'}_{i}(p') \mu^{r}_{i}(p).$$

where we write $\underline{\mu}^r = (\mu^r(1), \ldots, \mu^r(m))$ with $\mu^r(p) \in \mathcal{X}$ discrete. By iterating the argument in [L5, Lemma 4.4] we find that for $\underline{\mu} \in \mathcal{X}_\ell$ the shifts $d_{\underline{\mu}}$ are given by

$$d_{\underline{\mu}} = f_\ell(\underline{\mu}) - f(\underline{\nu}) - 2M(\underline{\nu}^0, \ldots, \underline{\nu}^{\ell-1})$$

where we identify $\underline{\mu} \in \mathcal{X}_\ell$ with an ℓ-tuple of elements $(\underline{\nu}^0, \ldots, \underline{\nu}^{\ell-1})$ of $\mathcal{X}$ in the obvious way, and f_ℓ is the function for the quiver $(I \times \mathbb{Z}/\ell\mathbb{Z}, H_\ell)$ given by the formula corresponding to Equation (4.3).

In the case where $\underline{\mu}$ is preserved by a, the $\underline{\mu}^r$ are all equal, to say $\underline{\tau} \in \mathcal{X}$, then $\nu^i = \ell\tau^i$, and so that

$$\ell f_\ell(\underline{\mu}) = f(\underline{\nu}) = \ell^2 f(\underline{\tau})$$

(since each edge and vertex in Q has ℓ "counterparts" in Q_ℓ). Thus we find that

$$\begin{aligned} d_{\underline{\mu}} &= f_\ell(\underline{\mu}) - f(\underline{\nu}) - 2\ell(\ell-1)M(\underline{\tau}) \\ &= \ell(\ell-1)(f(\underline{\tau}) - 2M(\underline{\tau})) \end{aligned}$$

where $M(\tau) = \sum_{h \in H; p' \leq p} \tau_{s(h)}(p)\tau_{t(h)}(p') + \sum_{i \in I; p' < p} \tau_i(p')\tau_i(p)$. Since this is divisible by ℓ we see that after changing coefficients $Fr([L_{\underline{\nu}}]) = [L_{\underline{\mu}}, \phi_0]$ as claimed.

□

5. ACTION OF Fr ON THE CANONICAL BASIS

5.1. Lusztig's construction of the canonical basis is an immediate consequence of his categorification of $\mathbf{U}^+$, thus given the construction of the previous section, it is natural to consider the action of the quantum Frobenius on the canonical basis. Naively, one might hope that the map Fr is compatible with this basis. More precisely, one might hope that Fr sends the specialization of a canonical basis element to the specialization of another canonical basis element, and that the kernel of Fr is spanned by canonical basis elements. Given that the functor **Fr** exists before specialization, this could be extended to a corresponding categorical statement for Lusztig's categories $\mathcal{Q}_V$. Unfortunately this conjecture turns out to be false. Indeed this hope fails already at the level of algebras as was first observed in unpublished work of Baumann[2].

5.2. We begin by relating our construction of the quantum Frobenius with a recent construction of Lusztig giving a "Frobenius morphism" on a monoid he constructs [L9] via the piece-wise linear parametrization of the canonical basis. For simplicity we restrict attention to the finite-type simply-laced case, thus our quiver $Q = (I, H)$ is of ADE type. In this case, the G_V action on an I-graded vector space has finitely many orbits, and the category $\mathcal{P}_V$ consists of the semisimple perverse sheaves on E_V which are G_V-equivariant. Let us briefly recall Lusztig's description [L4] of the orbits of G_V on E_V (a reinterpretation of the results of Gabriel and Bernstein-Gelfand-Ponomarev). Let W denote the Weyl group of the root datum associated to Q, with simple reflections $(s_i)_{i\in I}$ and let $(s_{i_1}, s_{i_2}, \dots, s_{i_N})$ be a reduced expression for the longest word in W which is adapted to the orientation Ω of our quiver (see [L4, §4] for the definitions). The partial products of the reduced expression give an ordering of the positive roots, which by Gabriel's theorem are in bijection with the indecomposable representations via their graded dimension (where we identify the root lattice with $\mathbb{Z}^I$ via the basis of simple roots). Thus the orbits of the G_V-action are given by N-tuples of nonnegative integers $\mathbf{c} = (c_j)_{j=1}^N$ such that $\sum_{j=1}^N c_j\alpha_j = |V|$ where α_j denotes the j-th root in the ordering given by our longest element.

Since the elements of the canonical basis in this situation are given by the simple perverse which arise as the middle extension of a constant sheaf on a G_V-orbit, we obtain a natural bijection between the canonical basis $\mathbf{B}$ and $\mathbb{N}^N$. Lusztig shows in [L4] that this bijection can also be seen using certain purely algebraic "PBW"-bases, and thus that switching orientation corresponds to certain piecewise linear maps on the set $\mathbb{N}^N$ (note that a reduced expression of the longest element is adapted to at most one orientation of Q).

Now in [L9, §2] Lusztig constructs for each positive integer $\ell \geq 1$ an embedding of B into itself which is given in any parametrization $\mathbf{B} \to \mathbb{N}^N$ by $(c_i) \mapsto (\ell c_i)$. We now wish to show how this map arises naturally in the context of our construction of the quantum Frobenius morphism (thus giving some motivation for Lusztig's choice of name). Let $\mathbf{i} = (i_1, i_2, \dots, i_N)$ be a reduced expression for the longest element of the Weyl group of Q which is adapted to our orientation Ω. Then the elements of $\mathbb{N}^N$ parametrize isomorphism classes of representations of Q where c_k denotes the multiplicity of the indecomposable representation with dimension

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[2] The author would like to thank B. Leclerc [Le2] for informing him of this.

vector α_k the k-th root in the ordering associated to $\mathbf{i}$. We will write $\mathcal{O}_{\mathbf{c}}$ for the orbit of G_V on E_V corresponding this isomorphism class, and $L_{\mathbf{c}}$ for the corresponding G_V-equivariant simple perverse sheaf given by the middle extension of the constant sheaf on $\mathcal{O}_{\mathbf{c}}$.

Similarly, if we consider the quiver $Q_\ell = (I \times \mathbb{Z}/\ell\mathbb{Z}, H_\ell)$ as in §4.4, and take T an $I \times \mathbb{Z}/\ell\mathbb{Z}$-graded vector space, then E_T again has finitely many orbits for the action of G_T (indexed now by ℓ-tuples of elements of $\mathbb{N}^N$), and the automorphism a permutes these orbits. As usual we will assume that V is the image of T under the forgetful functor from $I\times\mathbb{Z}/\ell\mathbb{Z}$-graded vector spaces to I-graded vector spaces, and we pick a $\mathbb{C}^\times$ action on T (and hence V) so that $E_T = E_V^{\mathbb{C}^\times}$.

Definition 5.1. Let $\ell \geq 1$ be an integer, and suppose that V is an I-graded vector space, and $L_{\mathbf{c}} \in \mathcal{P}_V$ is a simple object. We will say that $L_{\mathbf{c}}$ is ℓ-good if an a-equivariant simple constituent of $\mathbf{Fr}(L_{\mathbf{c}})$ has support intersecting $\mathcal{O}$.

Lemma 5.2. *Let T be an I_ℓ-graded vector space, and V the associated I-graded vector space obtained by forgetting the $\mathbb{Z}/\ell\mathbb{Z}$-grading. Then if $\mathcal{O}$ is a G_V-orbit in E_V, then the connected components of the intersection $E_T \cap \mathcal{O}$ are exactly the G_T-orbits.*

Proof. By the above discussion, $\mathcal{O} = \mathcal{O}_{\mathbf{c}}$ for an N-tuple $\mathbf{c} \in \mathbb{N}^N$. Now the orbits of the intersection $\mathcal{O}_{\mathbf{c}} \cap E_T$ correspond to decompositions $\mathbf{c} = \mathbf{d}_0 + \mathbf{d}_1 + \ldots + \mathbf{d}_{\ell-1}$ where for each $\mathbf{d}^k$ we have $\sum_{j=1}^N d_j^k \alpha_j = \ell^{-1}|V|$. The connected components of $\mathcal{O} \cap E_T$ are a union of orbits, and since there are only finitely many G_T-orbits in E_T, each component is a union of finitely many orbits, hence if some component is not a single orbit, there must be two G_T-orbits in $\mathcal{O} \cap E_T$ which are comparable in the closure ordering.

Now note that if $\preceq$ denotes the partial order on orbits given by closure in E_V, and $\preceq_\ell$ the corresponding partial order on orbits in E_T, then $(\mathbf{d}_0, \ldots, \mathbf{d}_{\ell-1}) \preceq_\ell (\mathbf{e}_0, \ldots, \mathbf{e}_\ell)$ if and only if $\mathbf{d}_k \preceq \mathbf{e}_k$ for each $k \in \mathbb{Z}/\ell\mathbb{Z}$. Moreover, if V_1, V_2 are I-graded vector spaces, and $\mathcal{O}_i, \mathcal{O}'_i$ denote orbits in E_{V_i} $(i = 1,2)$, such that $\mathcal{O}_i \preceq \mathcal{O}'_i$ for $i = 1,2$, then $\mathcal{O}_1 \oplus \mathcal{O}_2 \preceq \mathcal{O}'_1 \oplus \mathcal{O}'_2$ where $\mathcal{O}_1 \oplus \mathcal{O}_2$ denotes the orbit of representations of Q in $E_{V_1 \oplus V_2}$ which are isomorphic to $(V_1, x_1) \oplus (V_2, x_2)$ where $(V_1, x_1) \in \mathcal{O}_1$ and $(V_2, x_2) \in \mathcal{O}_2$, with strict containment holding if we have strict containment for either $\mathcal{O}_1$ or $\mathcal{O}_2$.

It is then clear that the G_T orbits lying in a single G_V-orbit $\mathcal{O}$ must be incomparable in $\preceq$, and hence they give the connected components of $\mathcal{O} \cap E_T$ as claimed. □

Proposition 5.3. *Let A be a simple object in $\mathcal{P}_V$ which is ℓ-good. Then $A = L_{\mathbf{c}}$ is ℓ-good precisely when $\mathbf{c} \in \ell\mathbb{N}^N$. Moreover, if $L_{\mathbf{c}}$ is ℓ-good, then there is a unique a^*-equivariant simple perverse sheaf in $\mathcal{P}_T$ whose support intersects $\mathcal{O}_{\mathbf{c}}$.*

Proof. First let us show that the condition that $L_{\mathbf{c}}$ is ℓ-good is equivalent to the conditions that $\mathcal{O}_{\mathbf{c}}^{\mathbb{C}^\times}$ and $\mathcal{O}_{\mathbf{c}}^{a}$ are both nonempty. Suppose that $\mathbf{Fr}(L_{\mathbf{c}})$ contains a simple constituent A whose support intersects $\mathcal{O}_{\mathbf{c}}$ and for which $a^*(A) \cong A$. Then since A is simple, it is isomorphic to a complex of the form $B_1 \boxtimes B_2 \boxtimes \ldots \boxtimes B_\ell$ where each B_i is a simple perverse sheaf on E_{T^i} (where $T = \bigoplus_{i \in \mathbb{Z}/\ell\mathbb{Z}} T^i$) which is G_{T^i}-equivariant. Now the action of a on A is clearly given by permuting the factors B_i, thus if $A \cong a^*(A)$ we must have $B_i \cong B_j$ for all i,j (via a power of a, or indeed any isomorphism identifying T^i with T^j, since the sheaves B_i are G_{T^i}-equivariant). Then $B_0 = L_{\mathbf{d}}$ for some $\mathbf{d} \in \mathbb{N}^n$ and $|\sum_{j=1}^N d_j \alpha_j = \ell^{-1}|V|$, and the support of A is $\overline{\mathcal{O}}_{\mathbf{d}}^{\times \ell}$. Now $\mathcal{O}_c$ is Zariski open in the support of $L_{\mathbf{c}}$, and hence the intersection $\mathrm{supp}(A) \cap \mathcal{O}_{\mathbf{c}}$ must be Zariski open in $\mathrm{supp}(A)$. Since $\mathcal{O}_{\mathbf{d}}^{\times \ell}$ is Zariski open in the support of A by definition and $\mathrm{supp}(A)$ is irreducible, it follows that if as we are supposing $\mathcal{O}_{\mathbf{c}} \cap \mathrm{supp}(A) \neq \emptyset$ then $\mathcal{O}_{\mathbf{d}}^{\times \ell} \cap \mathcal{O}_{\mathbf{c}}$ is nonempty and hence since $G_T \subset G_V$ in fact $\mathcal{O}_{\mathbf{d}}^{\times \ell} \subset \mathcal{O}_{\mathbf{c}}$ so that $\ell \mathbf{d} = \mathbf{c}$, hence clearly if A exists it must be unique. Moreover, it is now also clear that this intersection will contain an $\mathbb{C}^\times$-fixed, and a-fixed point as required.

On the other hand, suppose that $\mathcal{O}_{\mathbf{c}}$ contains both a-fixed and $\mathbb{G}_m$-fixed points. It follows from [L4] that the orbit closures $\overline{\mathcal{O}}_{\mathbf{c}}$ have smooth resolutions which are proper birational maps, restricting to an isomorphism over the open orbit. Since hyperbolic localization commutes with proper push-forward and pullback via open embeddings, we then see that over $\mathcal{O}_{\mathbf{c}}$ the hyperbolic localization is a direct sum of shifted constant sheaves with disjoint support corresponding to the connected components of $\mathcal{O}_{\mathbf{c}}^{\mathbb{C}^\times}$ (*c.f.* the proof of Theorem 4.4). Thus since $\mathcal{O}_{\mathbf{c}}$ is open in the support of $L_{\mathbf{c}}$ and simple perverse sheaves pull back via open embeddings to simple perverse sheaves (or zero), and by Lemma 5.2 we know that the intersection $\mathcal{O} \cap E_T$ is a disconnected union of finitely many orbits, we see that a component of $\mathcal{O}_{\mathbf{c}}^{\mathbb{C}^\times}$ containing an a-fixed point will yield the required simple constituent.

□

This gives us a natural bijection between the natural $\mathcal{A}$-basis of $\mathcal{K}(\tilde{\mathcal{Q}}_T)$ and the ℓ-good elements of the $\mathcal{A}$-basis of $\mathcal{K}(\mathcal{Q}_V)$, as the following theorem shows.

Theorem 5.4. *The Frobenius functor* **Fr** *induces a bijection between the ℓ-good elements of the natural basis of $\mathcal{K}(\mathcal{Q}_V)$ and the natural basis of $\mathcal{K}(\tilde{\mathcal{Q}}_T)$.*

Proof. Now the group $\mathcal{K}(\tilde{\mathcal{Q}}_T)$ has a $\mathcal{A}$-basis given by the classes $[A, \phi]$ where A is a simple perverse sheaves in $\mathcal{P}_T$ and $\phi\colon a^*(A) \to A$ is an isomorphism. Such a sheaf will be of the form $L_{\mathbf{d}} \boxtimes a^*(L_{\mathbf{d}}) \boxtimes \ldots \boxtimes (a^*)^{\ell-1}(L_{\mathbf{d}})$ on $E_T = E_{T^0} \otimes E_{T^1} \otimes \ldots E_{T^{\ell-1}}$ where a restricts to an isomorphism between T^i and T^{i+1} for each i $(0 \leq i \leq \ell-2)$, and $\mathbf{d} \in \mathbb{N}^N$ has $|\sum_{j=1}^N d_j \alpha_j| = |T^0|$. If $L_{\mathbf{c}}$ is a simple perverse sheaf in $\mathcal{P}_V$ which is ℓ-good, then Proposition 5.3 shows that $\mathbf{Fr}(L_{\mathbf{c}})$. □

Remark 5.5. Of course, as we have seen, this bijection is given very simply as the combinatorial embedding of $\mathbb{N}^N$ into itself given by $\mathbf{d} \mapsto \ell \mathbf{d}$. Our point here is that this combinatorics has a sheaf-theoretic meaning. Note moreover that although our definition of ℓ-good elements involves support conditions, it follows from the combinatorial description given in Proposition 5.3 and Lusztig's piecewise linear bijections which describe the change in parametrization given by changing the reduced expression for the longest word, that in fact if A is ℓ-good for an orientation Ω, and $\mathcal{F}$ is a Fourier transform which switches to another orientation Ω', then $\mathcal{F}(A)$ is again ℓ-good.

5.3. As mentioned at the beginning of this section, although we have found a relation between Lusztig's combinatorial Frobenius endomorphism and the quantum Frobenius, it is not the case that the quantum Frobenius is compatible with the canonical basis in a strong sense. We want to explain how this happens and its relation to the reducibility of characteristic cycles of the simple perverse sheaves in $\mathcal{P}_V$. This also allows us to explain a connection to the product of dual canonical basis elements first noticed by Leclerc [Le1].

5.4. We begin by recalling some basic facts about characteristic cycles. Given a perverse sheaf A on a variety X one may attach to it a Lagrangian cycle $CC(A)$ in T^*X know as its characteristic cycle (indeed this can be done more generally for a constructible complex, but if the complex is perverse the cycle has positive coefficients). A discussion of this construction is given in [KS, Chapter IX]. The characteristic cycles of the simple perverse sheaves in the categories $\mathcal{P}_V$ are all supported in a particular Lagrangian variety Λ_V which is given as follows. Pick a function $\varepsilon\colon H \to \mathbb{C}^\times$ such that $\varepsilon(h)+\varepsilon(\bar{h}) = 0$. The cotangent bundle T^*E_V of E_V may be identified with

$$\begin{aligned} X_V &= \bigoplus_{h\in H} \mathrm{Hom}(V_{s(h)}, V_{t(h)}) \\ &= \bigoplus_{h\in \Omega} \mathrm{Hom}(V_{s(h)}, V_{t(h)} \oplus \mathrm{Hom}(V_{t(h)}, V_{s(h)})), \end{aligned}$$

the moduli space of representations of the "doubled" quiver, by using the trace pairing and the function ε to obtain a symplectic form on X_V given by

$$(x,y) \mapsto \sum_{h\in H, s(h)=i} \varepsilon(h)\mathrm{tr}(x_{\bar{h}}y_h)$$

The natural action of G_V is then a symplectic action, and the moment map $\mu\colon X_V \to \mathfrak{g}_V = \mathrm{Lie}(G_V)$ for the action is given by $\mu((x_h)) = (\sum_{s(h)=i}\varepsilon(h)x_{\bar{h}}x_h)_{i\in I}$. We set $X_V^0 = \mu^{-1}(0)$, the preimage of zero under the moment map. Lusztig [L5] shows that

$$\Lambda_V = \{(x_h) \in E_{V,H} : (x_h) \text{ is nilpotent and } x \in X_V^0.\}$$

is a Lagrangian variety in X_V. (In the finite type case, the nilpotence condition is automatic, but in general it must be imposed.) Moreover, if $|V| = \nu$ then it is shown in [KSa] that Λ_V has $\dim_{\mathbb{Q}(v)}(\mathbf{f}_\nu)$ components, and the crystal $\mathbf{B}(\infty)$ of the positive part of the quantum group associated to the symmetric Cartan datum given by Q may be realized on the union of the components of the Λ_Vs as V runs over all possible graded dimensions.

We would like construct a characteristic cycle which is defined on the group $\mathcal{K}(\tilde{\mathcal{Q}}_V)$. Now if a is an automorphism of the quiver $Q = (I,H)$, and V is an I-graded vector space equipped with a corresponding automorphism a, then the induced map (which we will again denote by a) on X_V preserves Λ_V since the defining equations and the nilpotence condition are evidently preserved. Now if $(A,\phi) \in \tilde{\mathcal{Q}}_V$ then as $a^*(A) \cong A$ it follows by functoriality of the characteristic cycle that $CC(A)$ must be a-invariant.

Definition 5.6. Let B_V be an indexing set for the components of Λ_V, and denote the component corresponding to $b \in B_V$ by Λ_V^b. Thus if A is an object in $\mathcal{Q}_V$, we may write $CC(A)$ as a sum $\sum_{b\in B_V} a_b[\Lambda_V^b]$ where $a_b \in \mathbb{Z}$. Let B_V^a denote the subset of components of Λ_V such that $a^{-1}(\Lambda_V^b) = \Lambda_V^b$, and let $CC^a(A)$ be the Lagrangian cycle consisting of the a-fixed part of $CC(A)$:

$$CC^a(A) = \sum_{b\in B_V^a} a_b[\Lambda_V^b].$$

It follows readily from the results of [KSa] (see [Xu, §2] for details) that the a-invariant components of Λ_V index the irreducible objects in $\tilde{\mathcal{P}}_V$ (*i.e.* the simple objects S in $\mathcal{P}_V$ for which $a^*(S) \cong S$). Since, by §3.3 these form a basis for the group $\mathcal{K}(\tilde{\mathcal{Q}}_V)$ we may define a map CC^{per} on $\mathcal{K}(\tilde{Q}_V)$ taking values in $\mathcal{O}$-linear combinations of the cycles $[\Lambda_V^b]$ ($b \in B_V^a$) by setting $CC^{\mathrm{per}}(S[n], \phi[n])$ to be the a-invariant part $CC^a(S[n])$ of $CC(S[n])$ for (S, ϕ) a simple object in $\tilde{\mathcal{P}}_V$, ($n \in \mathbb{Z}$) and extending $\mathcal{O}$-linearly. Note that if (A, ϕ) is an object in $\tilde{\mathcal{Q}}_V$ and $[A, \phi]$ denotes the corresponding element of $\mathcal{K}(\tilde{\mathcal{Q}}_V)$ then it is not clear that $CC^{\mathrm{per}}([A, \phi]) = CC^a(A)$, since it is not clear if $CC^a(A) = 0$ for a traceless object in $\tilde{\mathcal{Q}}_V$, thus our definition is somewhat *ad hoc*.

5.5. We now recall the (local) index theorem for constructible sheaves on an analytic space X. Since we only need a local result, we may suppose that X is a stratified analytic subset of affine space $\mathbb{C}^n$. Thus $X = \bigsqcup_{S \in \mathcal{S}} S$, where each S is a smooth locally-closed connected subset of $\mathbb{C}^n$, and the closure of a stratum S is a union of strata. We may also assume that the Whitney (a) and (b) conditions are satisified (or for that matter the μ-condition introducted by Kashiwara and Schapira [KS, Chapter 8]). For $x \in X$ we will write $B_\varepsilon(x)$ for the set

$$\{y \in X : \|y - x\| < \varepsilon\},$$

where $\|.\|$ is the standard Hermitian norm on $\mathbb{C}^n$. (Thus our constructions here use the analytic variety attached to the algebraic varieties we considered earlier).

Definition 5.7. Given a stratification $\mathcal{S}$ of an analytic space X, write Λ_S for the closure of the conormal bundle T_S^*X of S, and set $\Lambda_{\mathcal{S}} = \bigsqcup_{S \in \mathcal{S}} T_S^*X$ (which is a closed set if $\mathcal{S}$ is a Whitney stratification). Let Λ_S^0 denote the points $(x, \xi) \in T_S^*X$ which do not lie in any Λ_T for $T \neq S$, the generic covectors in Λ_S.

Let $z \in S$ be a point of the stratum S. By taking a normal slice N to S at x, that is, choosing a complex analytic submanifold N which intersects each stratum transversely and such that $N \cap S = \{z\}$, we may reduce to the case $S = \{z\}$. Let $\phi \colon N \to \mathbb{C}$ be a holomorphic function vanishing at z such that $d\phi(z) \in \Lambda_S^0$ for any stratum $T \neq S$. If $S = T$ we define $c_{S,S} = 1$ for all S.

Endow X with a Hermitian metric (say by taking the restriction of the standard on on $\mathbb{C}^n$) and pick a small disk $B = B_\varepsilon(z)$ about z, and a generic $\eta \in \mathbb{C}^\times$ such that $|\eta| << \varepsilon$. The complex link L of the stratum S in T is then defined to be the set

$$L = B \cap T \cap N \cap \phi^{-1}(\eta).$$

Stratified Morse theory shows that the homeomorphism type of L is independent of the choices made (in fact it is independent of the metric $\|.\|$ also, so does not depend on the choice of local embedding we make). The local Euler obstruction $c_{S,T}$ is defined to be the Euler characteristic with compact supports of the complex link, that is

$$c_{S,T} = \chi_c(L).$$

As the notation suggests, this number is independent of the choice of z in the stratum S (here we use the assumption that our strata are connected).

The index theorem shows that the characteristic cycle of a perverse sheaf determines its local Euler characteristics. Indeed suppose that A is a perverse sheaf whose characteristic cycle lies in $\bigsqcup_{S\in\mathcal{S}} T^*_S M$. Then since we assume that our stratification satisfies the Whitney conditions, A is locally constant on the strata S, and we may set

$$\chi(A)(x) = \sum_i (-1)^i \dim(\mathcal{H}^i(A)_x).$$

This is an integer-valued function on X which is constant on each stratum S (thus if there are finitely many strata it is a integral linear combination of the characteristic functions of the strata). We will write $\chi_S(A) = \chi(A)(x)$ where x is any point in the stratum S. We also have

$$CC(A) = \sum_{S\in\mathcal{S}} m_S(A)[T^*_S X].$$

Theorem 5.8. *Let A be a perverse sheaf as above. Then we have*

$$m_S(A) = \sum_{T\subset\bar{S}} c_{S,T}\chi_T(A).$$

Thus we see that the constructible function $\chi(A)$ determines the characteristic cycle $CC(A)$, and in fact since the matrix $(c_{S,T})$ is unitriangular, and hence invertible, the converse is also true.

5.6. Now suppose that A be a simple object in $\mathcal{P}_V$ which is ℓ-good. We wish to compute the characteristic cycle of $A^{!*}$ in terms of that of A. Using Theorem 5.8 and Lemma 5.9 we see that it is enough to calculate the local Euler obstructions for the G_T-orbits in terms of those for the G_V-orbits.

5.7. Next we note the following Lemma:

Lemma 5.9. [Br] *Let X be a normal variety with a $\mathbb{C}^\times$ action and let $A \in D^b_{\mathbb{C}}(X)$ be a complex of constructible sheaves. Then if χ denotes the Euler characteristic, and $x \in X$, we have*

$$\chi(A_x) = \chi((A^{!*})_x).$$

Proof. This follows immediately from the fact that Verdier duality perserves local Euler characteristics. ☐

It follows that given a quiver $Q = (I, H)$ and a complex A in the category $\mathcal{Q}_V$ for an I-graded vector space V which is the image of an $(I \times \mathbb{Z}/\ell\mathbb{Z})$-graded vector space T under the forgetful functor, the constructible function $\chi(A^{!*})$ is just $\chi(A)_{|E_T}$, the restriction of the constructible function $\chi(A)$ to $E_T \cong E_V^{\mathbb{C}^\times}$.

5.8. Now to compute the local Euler obstructions at $x \in E_T = E_V^{\mathbb{C}^\times}$, we must pick a normal slice to the orbit, $\mathcal{O}_{\mathbf{c}}$ say ($\mathbf{c} \in \mathbb{N}^N$) in which x lies, and a generic covector in the conormal bundle of this orbit. Lusztig [L4, §10] constructs an explicit slice which is an affine linear subspace of the form $\mathcal{T}_x = x + H_x$ by choosing a one-parameter subgroup λ of G_V so that the tangent space to $\mathcal{O}$ at x has only positive weights for the $\mathbb{C}^\times$-action given by λ. If $x \in E_T$ we may assume that this one-parameter subgroup λ commutes with the $\mathbb{C}^\times$ action defining $E_T \subset E_V$, and then it follows immediately from Lusztig's construction that $x + (H_x \cap E_T)$ is again a normal slice through the orbit of x in E_T.

5.9. We now show that it is possible to find a generic covector ξ which is $\mathbb{C}^\times$-invariant. We proceed by induction. For this we need to recall some constructions from [L5] and [KSa]. For a vertex $i \in I$ and an integer p, let

$$X^0_{V,i,p} = \{x \in X^0_V : \operatorname{codim}_{V_i}(\sum_{h \in H, t(h)=i} \operatorname{im}(x_h \colon V_{s(h)} \to V_i)) = p\}.$$

Clearly the sets $X^0_{V,i,\geq p} = \bigcup_{r \geq p} X^0_{V,i,r}$ are open in X^0_V, and thus the $X^0_{V,i,p}$ are locally closed subsets of X^0_V. Given a component Λ_b of Λ_V, there is a unique $p \in \mathbb{Z}$ such that $\Lambda_b \cap X^0_{V,i,p}$ is dense in Λ_b, and we define an integer-valued function t_i on the components of Λ_V by setting $t_i(\Lambda_b) = p$. If we set $\Lambda_{V,i,p} = \Lambda_V \cap X^0_{V,i,p}$ then it is shown in [L5] that if $V \neq \{0\}$ then

$$\Lambda_V = \bigcup_{i \in I, p > 0} \Lambda_{V,i,p}.$$

It follows from this that (again provided $V \neq \{0\}$) given Λ_b a component of Λ_V, there is some i for which $t_i(\Lambda_b) > 0$.

We now recall the correspondence given in [KSa]. Suppose that $\nu = \dim(V)$ and $\nu = \mu + \tau$. Then let W, T be I-graded vector spaces with $\dim(W) = \mu$ and $\dim(T) = \tau$. Then we have a diagram

$$\begin{tikzcd} X^0_W \times X^0_T & X'_V(\nu,\tau) \arrow[l, "q_1"'] \arrow[r, "q_2"] & X^0_V, \end{tikzcd}$$

Here $X'_\nu(\mu,\tau)$ is the variety of triples (x, ψ, φ) with $x \in X_V$, and $\psi \colon W \to V$, $\varphi \colon V \to T$ morphisms of I-graded vector spaces such that

$$\begin{tikzcd} 0 \arrow[r] & W_i \arrow[r, "\psi_i"] & V_i \arrow[r, "\varphi_i"] & T_i \arrow[r] & 0 \end{tikzcd}$$

forms an exact sequence. This correspondence induces a correspondence between the varieties Λ_V and $\Lambda_W \times \Lambda_T$. In the case where $\tau = ci$ for $c \in \mathbb{Z}$ and $i \in I$, the variety X_T is a point, so that we may identify $X^0_W \times X^0_T$ with X^0_W, and hence we obtain a correspondence between Λ_W and Λ_V. We now restrict attention to this case. It is clear from the above discussion that $q_1^{-1}(X^0_{W,i,p}) = q_2^{-1}(X^0_{V,i,p+c})$ and we denote this subset of $X'_V(\mu, ci)$ by $X'_V(\mu, ci)_{i,p}$. Note that $X'_V(\mu, ci)_{i,0}$ is open in $X'_V(\mu, ci)$. We have the following theorem (see [KSa, §5] and [L5, §12]).

Theorem 5.10. *The map $q_2 \colon X'_V(\mu, ci)_0 \to X^0_{V,i,c}$ is a principal $GL_c \times G_\mu$-bundle, and the map $q_1 \colon X'_V(\mu, ci)_{i,0} \to X_{W,i,0}$ is a locally trivial fibration.*

Clearly, these maps then also restrict to a principal bundle and a fibration over the varieties $\Lambda_{V,i,c}$ and $\Lambda_{W,i,0}$ respectively. Now suppose that we have a component Λ_b of Λ_V, and we wish to find a generic $\mathbb{C}^\times$-invariant covector. We are assuming that b is ℓ-good, so that if $b = L_{\mathbf{c}}$ then $\mathbf{c} = \ell\mathbf{d}$. Now it follows from the combinatorial description of Kashiwara's functions ε_i in Lusztig's piecewise-linear parametrizations given, for example, in [L9, §2], that in this case if $i = i_1$, we have $\varepsilon_i(b) = c_1 = \ell d_1$. Furthermore, it is shown in [KSa] that $t_i(\Lambda_b) = \varepsilon_i(b)$, hence we may take $ci = \ell i_1$ in the above discussion, and we see that correspondence $X'_V(\nu - \ell i_1, \ell i_1)$ restricts to a smooth fibre bundle over a component $\Lambda_{b'}$ of Λ_W and a principal bundle over Λ_b. (Here W is an I-graded subspace of dimension $\nu - \ell i_1$.)

Lemma 5.11. *The component $\Lambda_{b'}$ corresponds to an ℓ-good element of $\mathcal{P}_W$.*

Proof. In [K] Kashiwara shows that, given a positive integer ℓ, there is an embedding S_∞ of $\mathbf{B}(\infty)$, the crystal basis of $\mathbf{f}$, into itself which scales the operators $\tilde{e}_i, \tilde{f}_i$

by ℓ. This map corresponds, in the finite type case, to Lusztig's Frobenius endomorphism [L9], as follows immediately from the results of those papers, so it's image is exactly the ℓ-good elements, since Proposition 5.3 identifies these components with the image of Lusztig's map. Now clearly we have $b' = \tilde{e}_i^{\ell}(b)$, and by [K] this again lies in the image of S_∞.

Alternatively, one can also see this result by using the piecewise linear bijections of Lusztig which allow you to calculate the parameterization of $\tilde{e}_i^{\ell}(b)$ in terms of that of b. ☐

Granting the lemma, we can conclude by induction that Λ_b contains a generic $\mathbb{C}^\times$-fixed covector.

Theorem 5.12. *If b is an ℓ-good element of $\mathbf{B}$, then Λ_b contains a $\mathbb{C}^\times$-fixed covector in Λ_b^0.*

Proof. We use induction, the claim being trivial if $\dim(E_V) = 0$. Let $\nu = \dim(V)$. If $\dim(E_V) > 0$, and b is ℓ-good, the above discussion shows we may find some $i \in I$ such that $b' = \tilde{e}_i^{\ell}(b)$ is also ℓ-good, (for the corresponding $\mathbb{C}^\times$-action on W, where $|W| = |V| - ci$). By induction we know that $\Lambda_{b'}^0$ contains a $\mathbb{C}^\times$-fixed covector, and we have the correspondence given by $\Lambda_V(\nu - ci, ci)$. Now since the maps q_1 and q_2 are smooth fibre bundles, the correspondence restricts to one between the loci $\Lambda_{b'}^0$ and Λ_b^0. But now examining this correspondence we see that if y is a $\mathbb{C}^\times$-fixed point in $\Lambda_{b'}^0$, then the fibre of q_1 certainly contains $\mathbb{C}^\times$-fixed points, and so Λ_b^0 must also contain $\mathbb{C}^\times$-fixed points as required. ☐

5.10. Let $b \in \mathbf{B}$ be ℓ-good with parameter $\mathbf{c}$ associated to a reduced expression of the longest element of the Weyl group compatible with our orientation Ω. Pick a $\mathbb{C}^\times$-fixed point $(x, \xi) \in \Lambda_b^0$. Then (having equipped E_V with an appropriate Hermitian metric so that we may define an open ball B_x centred at x) the local Euler obstruction for the pair of orbits $(\mathcal{O}_\mathbf{c}, \mathcal{O}_\mathbf{d})$ where $\mathcal{O}_\mathbf{c} \subset \overline{\mathcal{O}}_\mathbf{d}$ is given by:

$$c_{\mathbf{c},\mathbf{d}} = \chi_c(B_x \cap \mathcal{T}_x \cap d\xi^{-1}(t) \cap \mathcal{O}_\mathbf{d}).$$

where $|t| << 1$, and $\mathcal{T}_x$ is Lusztig's normal slice to $\mathcal{O}_\mathbf{c}$. Now the unit complex numbers S^1 act (via the $\mathbb{C}^\times$ action defining E_T) preserving $B_x, \mathcal{T}_x, d\xi^{-1}(t)$ and $\mathcal{O}_\mathbf{d}$ (since it is a one-parameter subgroup of G_V). But then by a standard property of Euler characteristics with compact support we may take S^1-fixed points without changing the Euler characteristics. The fixed points coincide with the intersection of these subsets with E_T. Thus $\mathcal{O}_\mathbf{d}^{S^1}$ is a (possibly empty) union of G_T-orbits which from a single orbit under the action of a and $\mathcal{T}_x^{S^1}$ is a normal slice to $\mathcal{O}_\mathbf{c}^{S^1}$, so that this expression is a sum over the local Euler obstructions of the orbit of $x \in E_T$ with the orbits in $\mathcal{O}_\mathbf{d}^{S^1}$. Let us denote elements of $\mathcal{X}_\ell$, which consist of ℓ-tuples of elements of $\mathbb{N}^N$, by $\tilde{\mathbf{e}}$, so that since $\mathbf{c}$ is ℓ-good, the a-fixed G_T-orbit in $\mathcal{O}_\mathbf{c}$ is given by $\tilde{\mathbf{d}} = (\mathbf{e}, \dots, \mathbf{e})$ where $\mathbf{e} = \ell^{-1}\mathbf{c} \in \mathcal{X}$, and $x \in \mathcal{O}_{\tilde{\mathbf{e}}}$, and write $c_{\tilde{\mathbf{c}},\tilde{\mathbf{d}}}$ for the corresponding local Euler obstructions for a pair of G_T-orbits. Then we have established the following result.

Lemma 5.13. *Let $\mathbf{c}$ and $\mathbf{d}$ be as above. Then if $\mathbf{d}$ is such that $\mathcal{O}_\mathbf{c} \subset \overline{\mathcal{O}}_\mathbf{d}$ we have:*

$$c_{\mathbf{c},\mathbf{d}} = \left\{ \begin{array}{cc} 0, & \textit{if } \mathcal{O}_\mathbf{d} \cap E_T = \emptyset, \\ \sum_{\tilde{\mathbf{d}}} c_{\tilde{\mathbf{e}},\tilde{\mathbf{d}}}, & \textit{if } \mathcal{O}_\mathbf{d} \cap E_T = \bigcup_{\tilde{\mathbf{d}}} \mathcal{O}_{\tilde{\mathbf{d}}^0} \end{array} \right.$$

Proof. This follows immediately from the existence of a $\mathbb{C}^\times$-fixed generic covector (x, ξ) – the local Euler obstruction computed with respect to (x, ξ) is equal to the Euler characteristic of the S^1-fixed points, which calculate exactly the sum of the local Euler obstructions for the pairs of strata $\mathcal{O}_{\tilde{\mathbf{e}}}$ and $\mathcal{O}_{\tilde{\mathbf{d}}}$ where $\mathcal{O}_{\mathbf{d}} \cap E_T = \bigcup_{\tilde{\mathbf{d}}} \mathcal{O}_{\tilde{\mathbf{d}}^0}$ as in the statement of the Lemma. □

Let S_ℓ denote the map from the free abelian group on the components of Λ_V to the corresponding free abelian group on the a-fixed components of Λ_T given by sending Λ_b to 0 if b is not ℓ-good, and $\Lambda_{b'}$ if b is ℓ-good and $b' = S_\infty^{-1}(b)$ where S_∞ denotes Kashiwara's embedding of $\mathbf{B}$ into itself for the integer ℓ.

Theorem 5.14. *Let $L_\mathbf{c} \in \mathcal{P}_V$ be the simple perverse sheaf corresponding to an ℓ-good element of $\mathbf{B}$. Then we have*

$$S_\ell(CC(L_\mathbf{c})) = CC(R_T^V(L_\mathbf{c}))^a.$$

Proof. This follows immediately from Lemmas 5.13 and 5.9 and Theorem 5.8. □

6. THE NON-SYMMETRIC CASE

We have assume throughout that our Cartan datum $(I, \cdot)$ is symmetric. The construction of the algebra $\mathbf{f}$ in the non-symmetric case is done in [L6] using quivers with automorphism, and it is natural to try to extend the construction of this paper to this context. In the case where the integer ℓ is coprime to the integers $\frac{1}{2}(i \cdot i)$, it is in fact straightforward (though somewhat notationally messy) to extend the construction given in this paper to this case – if our quiver with automorphism is $Q = (I, H, \alpha)$ we again take ℓ copies of our quiver glued together using a cycling automorphism a. Now the resulting quiver $(I \times \mathbb{Z}/\ell\mathbb{Z}, H_\ell)$ has two commuting automorphisms, a and one which is induced by α. In the coprime case, these generate a cyclic group, and we may replace the pair of automorphisms with the single automorphism $a \circ \alpha$, and then the construction works as before. Thus our construction in fact gives a geometrization of the quantum Frobenius in this "non-divisible" case.

In the case where ℓ has a common factor some integer $\frac{1}{2}(i \cdot i)$ then the commuting automorphisms a and α generate a non-cyclic abelian group, and it is not currently clear to the author how to extend the construction.

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