LIE ALGEBRAS: LECTURE 10.

1. Derivations and the abstract Jordan decomposition

Lemma 1.1. *Let $\mathfrak{a}$ be a semisimple Lie algebra.*

(1) *Suppose that $\mathfrak{a}$ is an ideal of a Lie algebra $\mathfrak{g}$. Then there is a unique ideal I in $\mathfrak{g}$ such that $\mathfrak{g}$ is the direct sum of ideals $\mathfrak{a} \oplus I$.*
(2) *All derivations of $\mathfrak{a}$ are inner, that is, $Der_{\mathsf{k}}(\mathfrak{a}) = Inn_{\mathsf{k}}(\mathfrak{a})$.*

Proof. For the first part, consider the Killing form κ of $\mathfrak{g}$. Let κ be the Killing form of $\mathfrak{g}$ and let $\mathfrak{a}^{\perp} = \{x \in \mathfrak{g} : \kappa(x, a) = 0, \forall a \in \mathfrak{a}\}$, then $\mathfrak{a}^{\perp}$ is an ideal in $\mathfrak{g}$. Now $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ is an ideal of $\mathfrak{g}$ on which the Killing form vanishes, so that by Cartan's Criterion, κ is nondegenerate on $\mathfrak{a}$. By the Lemma in the notes on symmetric bilinear forms this shows that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ as required[1]. For uniqueness, note that if $\mathfrak{g} = \mathfrak{a} \oplus I$, is a direct sum of ideals, then $[\mathfrak{a}, I] = 0$ and so clearly $I \subseteq \mathfrak{a}^{\perp}$.

For the second part, note that the Lie algebra of derivations $D = \mathrm{Der}_{\mathsf{k}}(\mathfrak{a})$ is a subalgebra of $\mathfrak{gl}(\mathfrak{a})$ containing the image I of ad as the subalgebra of "inner derivations" which, since it is isomorphic to $\mathfrak{a}$, is semisimple. We first claim that this subalgebra is an ideal: indeed if $\mathrm{ad}(x)$ is any inner derivation, and $\delta \in D$, then

$$\begin{aligned}[\delta, \mathrm{ad}(x)](y) &= \delta[x, y] - [x, \delta(y)] \\ &= [\delta(x), y] \\ &= \mathrm{ad}(\delta(x))(y).\end{aligned}$$

thus $[\delta, \mathrm{ad}(x)] \in I$, and hence I is an ideal in D. Now since I is semisimple, by the first part we see that $D = I \oplus I^{\perp}$, thus it is enough to prove that $I^{\perp} = \{0\}$. Thus suppose that $\delta \in I^{\perp}$. Then since $[I, I^{\perp}] \subset I \cap I^{\perp} = \{0\}$ we see that

$$[\delta, \mathrm{ad}(x)] = \mathrm{ad}(\delta(x)) = 0, \forall x \in \mathfrak{a},$$

so that, again by the injectivity of ad, we have $\delta = 0$ and so $I^{\perp} = \{0\}$ as required. □

Lemma 1.2. *Let $\mathfrak{a}$ be a Lie algebra and $Der_{\mathsf{k}}(\mathfrak{a}) \subset \mathfrak{gl}(\mathfrak{a})$ the Lie algebra of k-derivations on $\mathfrak{a}$. Let $\delta \in Der_{\mathsf{k}}(\mathfrak{a})$. If $\delta = s + n$ is the Jordan decomposition of δ as an element of $\mathfrak{gl}(\mathfrak{a})$, then $s, n \in Der_{\mathsf{k}}(\mathfrak{a})$.*

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[1] I have updated the notes online to give details about this.

Proof. We may decompose $\mathfrak{a} = \bigoplus_\lambda \mathfrak{a}_\lambda$ where $\mathfrak{a}_\lambda$ is the generalized eigenspace of x with eigenvalue $\lambda \in \mathsf{k}$ say. If $x \in \mathfrak{a}_\lambda$ and $y \in \mathfrak{a}_\mu$, then one can check by induction[2] that

$$(\delta - (\lambda+\mu))^n([x,y]) = \sum_{r=0}^{n} \binom{n}{r} [(\delta-\lambda)^r(x), (\delta-\mu)^{n-r}y]$$

hence $[x,y] \in \mathfrak{a}_{\lambda+\mu}$. It follows immediately that s is a derivation on $\mathfrak{a}$, and since $n = \delta - s$ we see that n is also. □

Theorem 1.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then given any $x \in \mathfrak{g}$ has an abstract Jordan decomposition: that is, there exist unique elements $s, n \in \mathfrak{g}$ such that $x = s+n$ and $[s,n] = 0$, and ad(s) is semisimple, while ad(n) is nilpotent. Moreover if ad(x) preserves a subspace of $\mathfrak{g}$, so do ad(x) and ad(n).*

Proof. As noted above, since $\mathfrak{g}$ is semisimple, $\text{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an embedding, and the conditions on s and n show that if they exist, they must satisfy $\text{ad}(s) = \text{ad}(x)_s$ and $\text{ad}(n) = \text{ad}(x)_n$, where $\text{ad}(x) = \text{ad}(x)_s + \text{ad}(x)_n$ is the Jordan decomposition of $\text{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$. Thus it remains to show that $\text{ad}(x)_s$ and $\text{ad}(x)_n$ lie in the image of ad. But $\text{ad}(x)$ acts as a derivation on $I = \text{im}(\text{ad})$, so by Lemma 1.2 so do $\text{ad}(x)_s$ and $\text{ad}(x)_n$. But then by Lemma 1.1, we see that $\text{ad}(x)_s = \text{ad}(s)$ for some $s \in \mathfrak{g}$ and $\text{ad}(x)_n = \text{ad}(n)$ for some $n \in \mathfrak{g}$. The conditions on $s, n \in \mathfrak{g}$ then follow from the injectivity of ad, and we are done. The moreover part follows from the same properties for the Jordan decomposition of endomorphisms of a vector space. □

2. The Cartan decomposition of a semisimple Lie algebra

In this section we work over an algebraically closed field k of characteristic zero.

Although the Cartan decomposition makes sense in any Lie algebra, we will now restrict attention to semisimple Lie algebras $\mathfrak{g}$, where we can give much more precise information about the structure of the root spaces than in the general case.

Proposition 2.1. *Suppose that $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra.*

1. *Let κ be the Killing form. Then $\kappa(\mathfrak{g}_\lambda, \mathfrak{g}_\mu) = 0$ unless $\lambda + \mu = 0$.*
2. *If $\alpha \in \Phi$ the $-\alpha \in \Phi$.*
3. *The restriction of κ to $\mathfrak{h}$ is nondegenerate.*

Proof. For the first part, from the previous Lemma we know that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$. Thus if $x \in \mathfrak{g}_\lambda, y \in \mathfrak{g}_\mu$, we see that $\text{ad}(x)\text{ad}(y)(\mathfrak{g}_\nu) \subseteq \mathfrak{g}_{\lambda+\mu+\nu}$. But then picking a basis of $\mathfrak{g}$ compatible with the Cartan decomposition it is clear the matrix of

[2]For $n = 1$ the identity is clear. For $n > 1$ we have

$$\begin{aligned}(\delta - (\lambda+\mu))^{n+1}([x,y]) &= (\delta - (\lambda+\mu)(\sum_{r=1}^{n}[(\delta-\lambda)^r(x), (\delta-\mu)^{n-r}(y)] \\ &= \sum_{r=0}^{n}\binom{n}{r}([(\delta-\lambda)^{r+1}(x), (\delta-\mu)^{n-r}x)] + [\delta-\lambda)^r, (\delta-\mu)^{n-r+1}(x)]) \\ &= \sum_{r=0}^{n+1}(\binom{n}{r} + \binom{n}{r-1})[(\delta-\lambda)^r(x), (\delta-\mu)^{n+1-r}(x)] \\ &= \sum_{r=0}^{n+1}\binom{n+1}{r}[(\delta-\lambda)^r(x), (\delta-\mu)^{n+1-r}(x)].\end{aligned}$$

where we interpret $\binom{k}{i}$ as zero if $i \notin \{0, 1, \ldots, k\}$.

$\text{ad}(x)\text{ad}(y)$ will have no non-zero diagonal entry unless $\lambda+\mu=0$, hence $\kappa(x,y)=0$ unless $\lambda+\mu=0$ as required.

For the second part, recall that if α is a root, then $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. If $-\alpha \notin \Phi$ then $\mathfrak{g}_{-\alpha}=0$ and so $\mathfrak{g}_\alpha^\perp = \mathfrak{g}$, which is impossible since κ is nondegenerate.

For the third part note that $\mathfrak{h}^\perp$ contains all the $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$ by part (1). Since κ is nondegenerate, by dimension counting this must be equal to $\mathfrak{h}^\perp$. It follows that $\kappa_{|\mathfrak{h}}$ must be nondegenerate as claimed. □

Lemma 2.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Then $\mathfrak{h}$ is abelian. Moreover, all the elements of $\mathfrak{h}$ are semisimple.*

Proof. We show that $[\mathfrak{h},\mathfrak{h}] = D\mathfrak{h} = 0$. By part (3) of Proposition 2.1 it is enough to show that $D\mathfrak{h}$ lies in the radical of $\kappa_{|\mathfrak{h}}$. Suppose that $x, y \in \mathfrak{h}$. Then we claim that:

$$\kappa(x,y) = \text{tr}(\text{ad}(x)\text{ad}(y)) = \sum_{\lambda \in \Phi} \dim(\mathfrak{g}_\lambda)\lambda(x)\lambda(y). \tag{2.1}$$

Indeed each $\mathfrak{g}_\lambda$ (for $\lambda \in \Phi \cup \{0\}$) is an $\mathfrak{h}$-subrepresentation, so we may compute the trace of $\text{ad}(x)\text{ad}(y)$ on each in turn and then add the resulting expressions. Now $\mathfrak{h}$ is solvable, so we may find a basis for $\mathfrak{g}_\lambda$ with respect to which each $\text{ad}(x)$ has an upper triangular matrix, and since $\text{ad}(x)$ has the unique eigenvalue $\lambda(x)$ on $\mathfrak{g}_\lambda$, the diagonal entries of the matrix must all equal $\lambda(x)$. It follows that the matrix of $\text{ad}(x)\text{ad}(y)$ is upper triangular with diagonal entries $\lambda(x)\lambda(y)$, and so summing over $\lambda \in \Phi \cup \{0\}$ we obtain Equation (2.1). Now λ is a one-dimensional representation of $\mathfrak{h}$, so it vanishes on $D\mathfrak{h}$. It is then immediate from Equation (2.1) that $\kappa(x,y)=0$, and so $x \in \text{rad}(\kappa_{|\mathfrak{h}}) = \{0\}$ as claimed.

Next suppose that $x \in \mathfrak{h}$ has Jordan decomposition $x = s+n$ in $\mathfrak{g}$. Now $\text{ad}(s), \text{ad}(n)$ must preserve $\mathfrak{h}$ since $\text{ad}(x)$ does, and thus since $\mathfrak{h}$ is self-normalising, we see that $s, n \in \mathfrak{h}$. Thus it is enough to show that $\mathfrak{h}$ contains no nilpotent elements. But if $n \in \mathfrak{h}$ is nilpotent, then $\text{ad}(n)$ is nilpotent on $\mathfrak{g}$, and hence on each $\mathfrak{g}_\lambda$. But then $0 = \text{tr}(\text{ad}(n)_{|\mathfrak{g}_\lambda}) = \dim(\mathfrak{g}_\lambda)\lambda(n)$, so that $\lambda(n)=0$. But then we see from Equation (2.1) that n is in the radical of $\kappa_{|\mathfrak{h}}$, so $n=0$ as required. □

Remark 2.3. Since the restriction of κ to $\mathfrak{h}$ is non-degenerate, it yields an isomorphism from $\mathfrak{h}^*$ to $\mathfrak{h}$, indeed if $\lambda \in \mathfrak{h}^*$ then there is a unique $t_\lambda \in \mathfrak{h}$ such that $\kappa(t_\lambda, y) = \lambda(y)$ for all $y \in \mathfrak{h}$, and the assignment $\lambda \to t_\lambda$ is clearly linear. (See the notes on bilinear forms for more details.)

Remark 2.4. Some textbooks study semisimple Lie algebras $\mathfrak{g}$ via *maximal toral subalgebras*. These are subalgebras of $\mathfrak{g}$ consisting entirely of semisimple elements, maximal with this property. It follows readily from the above Lemma and the Cartan decomposition that Cartan subalgebras of semisimple Lie algebras are maximal toral subalgebras, though we will not use this fact.

Proposition 2.5. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra, and $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ the associated Cartan decomposition.*

(1) *If $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and $h \in \mathfrak{h}$ then*
$$\kappa(h,[x,y]) = \alpha(h)\kappa(x,y).$$
(2) *The roots $\alpha \in \Phi$ span $\mathfrak{h}^*$.*
(3) *The subspace $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ is one-dimensional and $\alpha(\mathfrak{h}_\alpha) \neq 0$.*

(4) *If $\alpha \in \Phi$, we may find $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_\alpha$ and $h_\alpha \in \mathfrak{h}_\alpha$ so that the map $e \mapsto e_\alpha$, $f \mapsto f_\alpha$ and $h \mapsto h_\alpha$ gives an embedding $\mathfrak{sl}_2 \to \mathfrak{g}_\alpha \oplus \mathfrak{h}_\alpha \oplus \mathfrak{g}_{-\alpha}$.*

Proof. We only proved $i)$ in the lecture. For (1) we have

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \kappa(\alpha(h)x, y) = \alpha(h)\kappa(x, y),$$

as required.

Note that in the statement of the last part, the element $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$. □