# LIE ALGEBRAS: LECTURE 13.

## 1. CASIMIR OPERATORS

Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$ and let $\{y_1, y_2, \ldots, y_n\}$ be the dual basis of $\mathfrak{g}$ with respect to the Killing form *i.e.* $\kappa(x_i, y_i) = \delta_{ij}$ $(1 \leq i, j \leq n)$ (such a basis exists because the Killing form is nondegenerate). If $(V, \rho)$ is a representation of $\mathfrak{g}$, we set

$$C_V = \sum_{i=1}^{n} \rho(x_i)\rho(y_i),$$

the *Casimir operator* of $V$. It is not hard show that $C_V$ is independent of the choice of basis. Moreover, if $\sigma \colon V \to W$ is a $\mathfrak{g}$-homomorphism, it is clear that $C_W \circ \sigma = \sigma \circ C_V$, hence we will normally suppress the subscript in the notation and speak of "the Casimir operator" rather than the Casimir of a particular representation.

*Remark* 1.1. It is straight-forward to check that the Casimir is independent of the choice of basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$. Indeed if $\{z_1, \ldots, z_n\}$ is a different basis, then we may write $z_i = \sum_{j=1}^{n} c_{ij} x_j$ for some scalars $c_{ij}$. Then if $\{t_i\}_{1 \leq i \leq n}$ is the dual basis for $\{z_i\}_{1 \leq i \leq n}$ and we write $t_i = \sum_{j=1}^{n} d_{ij} y_j$ then setting $C = (c_{ij})$ and $D = (d_{ij})$ the condition that $\{t_i\}_{i=1}^{n}$ is the basis dual $\{z_i\}_{i=1}^{n}$ is preciesly that $D^t = C^{-1}$. and this immediately implies that $\sum_{i=1}^{n} \rho(z_i)\rho(t_i) = \sum_{i=1}^{n} \rho(x_i)\rho(y_i)$.

**Lemma 1.2.** *Suppose that $\mathfrak{g}$ is a semisimple Lie algebra and $V$ is a representation of $\mathfrak{g}$. The element $C \in \mathfrak{gl}(V)$ is a $\mathfrak{g}$-endomorphism of $(V, \rho)$.*

*Proof.* For convenience, in this proof we will suppress map $\rho \colon \mathfrak{g} \to \mathfrak{gl}(V)$ and for $x \in \mathfrak{g}$ write $x$ for both the element of the Lie algebra and $\rho(x) \in \mathfrak{gl}(V)$. Let $z \in \mathfrak{g}$ and consider $[C, z]$. Then we have:

$$\begin{aligned}
[C, z] &= \sum_{i=1}^{n} x_i y_i z - z x_i y_i \\
&= \sum_{i=1}^{n} x_i([y_i, z] + z y_i) - ([z, x_i] + x_i z) y_i \\
&= \sum_{i=1}^{n} x_i [y_i, z] - [z, x_i] y_i
\end{aligned}$$

Now if $[y_i, z] = \sum_{j=1}^{n} a_{ij} y_j$, using the fact that the bases $\{x_i\}_{i=1}^{n}$ and $\{y_j\}_{j=1}^{n}$ are dual with respect to the Killing form $\kappa$, we see that $a_{ij} = \kappa([y_i, z], x_j)$ and hence since $\kappa$ is invariant this is equal to $\kappa(y_i, [z, x_j])$, which in turn is equal to $b_{ji}$ where $[z, x_j] = \sum_{i=1}^{n} b_{ji} x_i$. It follows immediately that $[C, z] = 0$ as required.

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A crucial point for us will be that the Casimir operator distinguishes the trivial representation. More precisely, if $(V, \rho)$ is a representation of $\mathfrak{g}$ which is not trivial (*i.e.* $\rho(\mathfrak{g}) \neq \{0\}$), then we will show that $\mathrm{tr}(C_V) \in \mathbb{Q}_{>0}$. To show this we will need an auxiliary notion:

**Definition 1.3.** Suppose that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ is a Cartan decomposition of a semisimple Lie algebra. Say $h \in \mathfrak{h}_{\mathbb{Q}} = \mathbb{Q}\text{-span}\{h_\alpha : \alpha \in \Phi\}$ is *regular* if $\alpha(h) \neq 0$ for all $\alpha \in \Phi$. Given such an $h$, set $\Phi^+ = \{\alpha \in \Phi : \alpha(h) > 0\}$ and $\Phi^- = \{\alpha \in \Phi : \alpha(h) < 0\}$. Since $h$ is regular, we have $\Phi = \Phi^+ \sqcup \Phi^-$, a disjoint union.

If $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ then we define its *height* (with respect to $h$) to be $\mathrm{ht}(\lambda) = \lambda(h)$.

Recall from Problem sheet 5 the following properties of representations.

**Lemma 1.4.** *Let $\{e, h, f\}$ be the standard basis of $\mathfrak{sl}_2$.*

i) *If $V$ is any $\mathfrak{sl}_2$-representation and $v \in V$ has $e(v) = 0$ and $h(v) = m.v$ then $v$ generates an irreducible representation of dimension $m + 1$.*
ii) *Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, and let $(V, \rho)$ be a representation of $\mathfrak{g}$. Then if $h_\alpha = 2t_\alpha/\kappa(t_\alpha, t_\alpha)$, the eigenvalues of $\rho(h_\alpha)$ on $V$ are integers.*

*Proof.* The first part follow by explicit calculation using the relations of $\{e, h, f\}$. For the second part, by using the $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}$ given by $\alpha$ the question reduces to the case $\mathfrak{g} = \mathfrak{sl}_2$, where it follows by induction using the explicit description of irreducible representations: if $\dim(V) = n + 1$, the element $h$ acts semisimply with $V = \bigoplus_{k=0}^{n} V_{n-2k}$ where each $V_i$ is a 1-dimensional $h$-eigenspace with eigenvalue $i$, and $e(V_i) \subset V_{i+2}$, $f(V_i) \subseteq V_{i-2}$ ☐

Note that in particular, all the weights which occur in a finite dimensional representation of $\mathfrak{g}$ lie[1] in $\mathfrak{h}^*_{\mathbb{Q}} = \mathrm{span}_{\mathbb{Q}}(\Phi)$. It follows they have a well defined height $\mathrm{ht}(\lambda)$. For the rest of this section we will fix a regular $h \in \mathfrak{h}_{\mathbb{Q}}$ so that we have a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$ and a height function $\mathrm{ht}\colon \mathfrak{h}^*_{\mathbb{Q}} \to \mathbb{Q}$.

**Lemma 1.5.** *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ and suppose that $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. If $V$ is a representation of $\mathfrak{g}$ then $\mathrm{tr}_V(C_V) \in \mathbb{Q}_{\geq 0}$, and equals zero if and only if $V$ is the trivial representation, i.e. $V = V^{\mathfrak{g}}$.*

*Proof.* We first check the lemma in the case where $V$ is an irreducible representation. Choose a set of positive roots $\Phi^+$ in $\Phi$. Then pick $e_\alpha \in \mathfrak{g}_\alpha$ non-zero, so that $\kappa(e_\alpha, e_{-\alpha}) = 1$, and pick $\{h_i : 1 \leq i \leq r\}$, $\{h'_i : 1 \leq i \leq r\}$ dual bases of $\mathfrak{h}$. Then it is clear that the Casimir operator $C$ can be written

$$\begin{aligned} C &= \sum_{\alpha\in\Phi^+} (e_{-\alpha}e_\alpha + e_\alpha e_{-\alpha}) + \sum_{i=1}^{r} h_i h'_i \\ &= \sum_{\alpha\in\Phi^+} (2e_{-\alpha}e_\alpha + [e_\alpha, e_{-\alpha}])) + \sum_{i=1}^{r} h_i h'_i \\ &= \sum_{\alpha\in\Phi^+} (2e_{-\alpha}e_\alpha + t_\alpha) + \sum_{i=1}^{r} h_i h'_i \end{aligned}$$

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[1] Because, for example, $\mathfrak{h}^*_{\mathbb{Q}} = \{\lambda \in \mathfrak{h}^* : (\lambda, \alpha) \in \mathbb{Q}, \forall \alpha \in \Phi\}$.

Now since $V$ is irreducible, $C$ acts by a scalar $c$ on $V$,so that $\mathrm{tr}_V(C) = \dim(V).c$ and to compute $c$ we may compute on any vector in $V$. Let $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ be the weight space decomposition of $V$ for the action of $\mathfrak{h}$. Since the weights of $V$ are a finite set, we may pick a weight $\lambda$ of maximal height. Then since $\mathfrak{g}_\alpha(V_\lambda) \subseteq V_{\lambda+\alpha}$, and $\mathrm{ht}(\lambda + \alpha) > \mathrm{ht}(\lambda)$ for all $\alpha \in \Phi^+$, it follows that $\lambda + \alpha$ is not a weight for all $\alpha \in \Phi^+$. But then if $v \in V_\lambda \backslash \{0\}$, we must have $e_\alpha(v) = 0$ for all $\alpha \in \Phi^+$. Moreover, we may pick $v \in V_\lambda$ to be a (nonzero) weight vector, that is, a vector $v \in V_\lambda$ such that $\rho(h)(v) = \lambda(h).v$ for all $h \in \mathfrak{h}$. But then computing $C(v)$ in terms of the above expression for $C$ we find that

$$C(v) = \big( \sum_{\alpha \in \Phi^+} \lambda(t_\alpha) + \sum_{i=1}^r \lambda(h_i)\lambda(h'_i)\big).v$$

But now since by definition $\kappa(h_i, h'_j) = \delta_{i,j}$ we have $v = \sum_{i=1}^r \kappa(v, h'_i)h_i$ for all $v \in \mathfrak{h}$ so that

$$\begin{aligned}\sum_{i=1}^r \lambda(h_i)\lambda(h'_i) &= \sum_{i=1}^r \kappa(t_\lambda, h_i)\lambda(h'_i)\\ &= \kappa(t_\lambda, \sum_{i=1}^r \lambda(h'_i)h_i)\\ &= \kappa(t_\lambda, t_\lambda)\\ &= (\lambda, \lambda)\end{aligned}$$

and hence $c = \big(\sum_{\alpha\in\Phi^+} \lambda(t_\alpha)\big) + (\lambda, \lambda)$.

Recall that for each $\alpha \in \Phi$ we have a copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$ denoted $\mathfrak{sl}_\alpha$ which is given by the span of $\{e_\alpha, e_{-\alpha}, t_\alpha\}$. Since $e_\alpha(v) = 0$, it follows from part i) of Lemma 1.4 that $v$ generates an irreducible representation of $\mathfrak{sl}_\alpha$, of dimension $\lambda(h_\alpha)+1$, hence $\lambda(h_\alpha)$ is a nonnegative integer, and hence certainly $\lambda(t_\alpha) \in \mathbb{Q}_{\geq 0}$.

Since $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$, by positive definiteness we have $(\lambda, \lambda) \geq 0$ with equality if and only if $\lambda = 0$ thus it follows that $c > 0$ unless $\lambda = 0$. But if $\lambda = 0$ then $\lambda(h_\alpha) = 0$ and so again from part i) of Lemma 1.4 it is clear that $\mathsf{k}.v$ is the 1-dimensional representation of $\mathfrak{sl}_\alpha$, so that $\rho(e_\alpha)(v) = \rho(e_{-\alpha})(v) = 0$. But then it is clear that $\mathsf{k}.v$ is a subrepresentation of $V$, and so by irreducibility $V = \mathsf{k}.v$ and $V = V^{\mathfrak{g}}$ as claimed.

Now suppose that $V$ is arbitrary. Clearly we may filter it by subrepresentations $0 = V_0 \subset V_1 \subset \ldots \subset V_k = V$ where each successive quotient $V_i/V_{i-1}$ is irreducible. But then

$$\mathrm{tr}_V(C) = \sum_{i=1}^k \mathrm{tr}_{V_i/V_{i-1}}(C)$$

is strictly positive unless all the quotients $V_i/V_{i-1}$ are trivial. But in this case ($V_0 \subset V_1 \subset \ldots \subset V_k$ is a complete flag in $V$ and hence $\rho(\mathfrak{g})$ is nilpotent, and hence $0$, so that $V = V^{\mathfrak{g}}$ as required. □

**Proposition 1.6.** *Let $V$ be a finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$. The invariants of $V^{\mathfrak{g}}$ form form a direct summand in $V$ (i.e. $V^{\mathfrak{g}}$ has a complement). Moreover, the quotient representation $V/V^{\mathfrak{g}}$ contains no composition factor isomorphic to the trivial representation.*

*Proof.* Let $C$ be the Casimir operator and let $V = \bigoplus_{\gamma \in \mathsf{k}} V_\gamma$ be the decomposition of $V$ into generalised eigenspaces for $C$. Since $C$ is a $\mathfrak{g}$-endomorphism of $V$, each generalised eigenspace is a subrepresentation of $V$.

Now since $C$ clearly has trace zero on $V_0$, by Lemma 1.5 we see that $V_0 \subseteq V^{\mathfrak{g}}$. On the other hand, if $\gamma \neq 0$, then $C_V$ acts on any irreducible arising as a composition factor of $V_\gamma$ by a scalar, which must therefore clearly be $\gamma$, and hence again by Lemma 1.5, the irreducibles which occur are all non-trivial, and so certainly $V^{\mathfrak{g}} \cap V_\gamma = \{0\}$. It follows that

$$V = V^{\mathfrak{g}} \oplus \big( \bigoplus_{\gamma \neq 0} V_\gamma \big)$$

and $V^{\mathfrak{g}}$ has a complement as claimed. Since the quotient $V/V^{\mathfrak{g}}$ is isomorphic to $\bigoplus_{\gamma \neq 0} V_\gamma$, the final claim follows because as noted above the $V_\gamma$ do not contain any copy of the trivial representation. ☐

**Theorem 1.7.** *Let $q \colon V \to W$ be a surjective $\mathfrak{g}$-homomorphism. Then $q$ has a right inverse.*

*Proof.* Consider the natural inclusion

$$q_* \colon \mathrm{Hom}(W, V) \to \mathrm{Hom}(W, W),$$

given by $s \mapsto q \circ s$. It is easy to see that $q_*$ is a surjective map of representations of $\mathfrak{g}$. Now clearly $\mathrm{Hom}(W, W)$ contains a copy of the trivial representation of $\mathfrak{g}$, given by the scalar multiples of the identity. It follows from the previous proposition that $\mathrm{Hom}(W, V)$ must have a copy of the trivial representation mapping to this subrepresentation. But this means there is a $\mathfrak{g}$-invariant linear map $s \colon W \to V$ such that $q_*(s) = q \circ s = \mathrm{id}_W$, i.e. the map $q$ has a splitting.

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