LIE ALGEBRAS: LECTURE 14.

1. Root Systems

In this section we study the geometry which we are led to by the configuration of roots associated to a Cartan decomposition of a semisimple Lie algebra. These configurations will turn out to have a very special, highly symmetric, form which allows them to be completely classified.

We will work with an inner product space[1], that is a vector space equipped with a positive definite symmetric bilinear form $(-,-)$. Such a form makes sense over any field which has a notion of positive elements, and so in particular over $\mathbb{R}$ and $\mathbb{Q}$. Since the roots Φ associated to a Cartan decomposition of a semisimple Lie algebra naturally live in the $\mathbb{Q}$-inner product space $\mathfrak{h}^*_{\mathbb{Q}}$, we will assume our field is $\mathbb{Q}$ unless otherwise stated. We let $\mathrm{O}(V)$ denote the group of orthogonal linear transformations of V, that is the linear transformations which preserve the inner product, so that $g \in \mathrm{O}(V)$ precisely when $v, w \in V$ then $(v, w) = (g(v), g(w))$ for all $v, w \in V$.

Definition 1.1. A *reflection* is a nontrivial element of $\mathrm{O}(V)$ which fixes a subspace of codimension 1 (*i.e.* dimension $\dim(V)-1$). If $s \in \mathrm{O}(V)$ is a reflection and $W < V$ is the $+1$-eigenspace, then $L = W^{\perp}$ is a line preserved by s, hence the restriction $s_{|L}$ of s to L is an element of $O(L) = \{\pm 1\}$, which since s is nontrivial must be -1. In particular s has order 2. If v is any nonzero element of L then it is easy to check that s is given by

$$s(u) = u - \frac{2(u,v)}{(v,v)}v.$$

Given $v \neq 0$ we will write s_v for the reflection given by the above formula, and refer to it as the "reflection in the hyperplane perpendicular to v".

We now give the definition which captures the geometry of the root of a semisimple Lie algebra.

Definition 1.2. Let V be a $\mathbb{Q}$-vector space equipped with an inner product $(-,-)$. A finite subset $\Phi \subset V\backslash\{0\}$ is called a *root system* if it satisfies the following properties.

(1) Φ spans V;
(2) If $\alpha \in \Phi$ then $c\alpha \in \Phi$ if and only if $c = \pm 1$;
(3) If $\alpha \in \Phi$ then $s_\alpha \colon V \to V$ preserves Φ;
(4) If $\alpha, \beta \in \Phi$ then $2(\beta,\alpha)/(\alpha,\alpha) \in \mathbb{Z}$.

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[1]That is, one with a notion of distance and angle. Apart from working over $\mathbb{Q}$ rather than $\mathbb{R}$, this is pretty much the vector geometry of Geometry I.

If $(-,-)$ denotes the inner product on $\mathfrak{h}^*_{\mathbb{Q}}$, then we have shown that the roots in $\mathfrak{h}^*$ form a root system in the above sense: one simply has to translate the information about the elements $t_\alpha \in \mathfrak{h}$ obtained in our analysis of the Cartan decomposition: the crucial fact is that, for the roots arising from a semisimple Lie algebra we have

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \kappa(t_\beta, \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}) = \beta(h_\alpha) \in \mathbb{Z}.$$

We established that $\beta(h_\alpha) \in \mathbb{Z}$ using root strings, and moreover the properties of root strings also showed that $s_\alpha(\beta) = \beta - \beta(h_\alpha).\alpha \in \Phi$, hence properties $3)$ and $4)$ hold. Since $1)$ and $2)$ were also explicitly checked, we see that the roots Φ in $\mathfrak{h}^*_{\mathbb{Q}}$ do indeed form a root system.

Remarkably, the finite set of vectors given by a root system has both a rich enough structure that it captures the isomorphism type of a semisimple Lie algebra, but is also explicit enough that we can completely classify them, and hence classify semisimple Lie algebras.

Definition 1.3. Let (V,Φ) be a root system. Then the *Weyl group* of the root system is the group $W = \langle s_\alpha : \alpha \in \Phi \rangle$. Since its generators preserve the finite set Φ and these vectors span V, it follows that it is a finite subgroup of $\mathrm{O}(V)$.

Example 1.4. Let $\mathfrak{g} = \mathfrak{sl}_n$, Then let $\mathfrak{d}_n$ denote the diagonal matrices in $\mathfrak{gl}_n$ and $\mathfrak{h}$ the (traceless) diagonal matrices in $\mathfrak{sl}_n$. As you saw in the problem sets, $\mathfrak{h}$ forms a Cartan subalgebra in $\mathfrak{sl}_n$. Let $\{\varepsilon_i : 1 \leq i \leq n\}$ be the basis of $\mathfrak{d}_n^*$ dual to the basis $\{E_{ii} : 1 \leq i \leq n\}$ of $\mathfrak{d}_n$ in $\mathfrak{gl}_n$. Then $\mathfrak{h}^*_{\mathbb{Q}}$ is the quotient space

$$\mathfrak{h}^*_{\mathbb{Q}} = \{\sum_{i=1}^n c_i\varepsilon_i : c_i \in \mathbb{Q}\}/\{\mathbb{Q}.(\varepsilon_1 + \ldots + \varepsilon_n)\},$$

the roots in $\mathfrak{h}^*_{\mathbb{Q}}$ are the (images of the) vectors $\{\varepsilon_i - \varepsilon_j : 1 \leq i,j \leq n, i \neq j\}$. The Weyl group W in this case is the group generated by the reflections s_α which, for $\alpha = \varepsilon_i - \varepsilon_j$ interchange the basis vectors ε_i and ε_j, so it is easy to see that W is just the symmetric group on n letters.

2. Bases for root systems

A key step in the classification process is to find a good basis for V: we assume that Φ spans V, so certainly we may find a subset of Φ which is a basis, but it turns out we can find a class of particularly well adapted bases.

Definition 2.1. Let (V,Φ) be a root system, and let Δ be a subset of Φ. We say that Δ is a *base* for Φ if

(1) Δ is a basis for V.
(2) Every $\beta \in \Delta$ can be written as $\sum_{\alpha\in\Delta} c_\alpha\alpha$ where $c_\alpha \in \mathbb{Z}$ and the non-zero c_αs all have the same sign.

Given a base of Φ we may declare a root positive or negative according to the sign of the nonzero coefficients which occur when we write it in terms of the base. We write Φ^+ for the set of positive roots and Φ^- for the set of negative roots.

The first crucial point in showing that a root system possess a base is that the angle between roots are very constrained. For convenience we will write

$$\langle\alpha,\beta\rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}, \tag{2.1}$$

which we call a *Cartan integer*.

Lemma 2.2. *Let* (V, Φ) *be a root system and let* $\alpha, \beta \in \Phi$ *be such that* $\alpha \neq \pm\beta$. *Then* $\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle \in \{0, 1, 2, 3\}$. *It follows that the angle between two such roots* α, β *lies in the set*

$$\{\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\}.$$

Moreover, the ratios of root lengths which are not perpendicular must be $1, 2, 1/2, 3$ *or* $1/3$.

Proof. By assumption, we know that both $\langle\alpha, \beta\rangle$ and $\langle\beta, \alpha\rangle$ are integers. On the other hand, by the cosine formula (*i.e.* by Cauchy-Schwarz) we see that if θ denotes the angle between α and β, then:

$$\langle\alpha, \beta\rangle\langle\beta, \alpha\rangle = 4\cos(\theta)^2 \leq 4. \tag{2.2}$$

Since $\cos(\theta)^2$ determines the angle between the two vectors (or rather the one which is at most π) and $\langle\beta, \alpha\rangle/\langle\alpha, \beta\rangle = ||\alpha||^2/||\beta||^2$ (where we write $||v||^2 = (v, v)$), the rest of the Lemma follows by a case-by-case check as we see from the following table:

$\langle\alpha, \beta\rangle$	$\langle\beta, \alpha\rangle$	θ	$\|\|\alpha\|\|^2/\|\|\beta\|\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

□