LIE ALGEBRAS: LECTURE 15.

1. Bases for root systems

We now show that any root system must possess a base. In fact we show a little more. If $v \in V$, we say that v is regular[1] if $(v, \alpha) \neq 0$ for all $\alpha \in \Phi$. Since V is not a finite union of proper subspaces, regular vectors certainly exist. Given such a v, write $\Phi^+(v) = \{\alpha \in \Phi : (v, \alpha) > 0\}$, and $\Phi^-(v) = -\Phi^+(v)$. Clearly $\Phi = \Phi^+(v) \bigsqcup \Phi^-(v)$. We say a root $\alpha \in \Phi^+(v)$ is *decomposable* if we can find roots $\alpha_i \in \Phi^+(v)$ and nonnegative integers n_i, $(1 \leq i \leq s,\ s \geq 2)$, such that $\alpha = \sum_{i=1}^s n_i \alpha_i$. Let $\Delta(v)$ be the set of indecomposable roots, *i.e.* the roots which are not decomposable, in $\Phi^+(v)$.

Proposition 1.1. *Let (V, Φ) be a root system, and let $v \in V$ be a regular vector. Then $\Delta(v)$ is a base for (V, Φ) and every base is of this form.*

Proof. Let $\Delta(v)$ be the indecomposable roots in $\Phi^+(v)$ as above. We first show that every root in $\Phi^+(v)$ is a nonnegative integer combination of indecomposable roots. Indeed let $m = \min\{(v, \alpha) : \alpha \in \Phi^+\}$. If $\gamma \in \Phi^+(v)$, and we have $\gamma = \sum_{i=1}^s n_i \alpha_i$ where $n_i \in \mathbb{Z}_{>0}$ and $\alpha_i \in \Phi^+(v)$, then clearly $(v, \gamma) \geq \sum_{i=1}^s n_i.m$, hence for any such expression the sum $\sum_{i=1}^s n_i$ is bounded above by $(v, \alpha)/m$. If we write $\gamma = \sum_{i=1}^s n_i \alpha_i$ where $\sum_{i=1}^s n_i$ is maximal possible, then we claim each γ_i is indecomposable. Indeed otherwise there is some i such that γ_i is decomposable, so that we may write $\gamma_i = \sum_{j=1}^t m_j \beta_j$ for some $t \geq 2$, positive integers m_j, and $\beta_j \in \Phi^+(v)$, $(1 \leq j \leq t)$. But then $\gamma = \sum_{k \neq i} n_i \alpha_i + \sum_{j=1}^t m_j n_i \beta_j$ and $\sum_{k \neq i} n_k + (\sum_{j=1}^t m_j) n_i \geq \sum_{k \neq i} n_k + 2n_i > \sum_{i=1}^s n_i$ contradicting our maximality assumption. Since Φ also spans V, this shows that $\Delta(f)$ spans V, hence since $\Phi = \Phi^+(v) \sqcup -\Phi^+(v)$ in order to show that $\Delta(v)$ is a base it only remains to show that $\Delta(f)$ is linearly independent.

We check this in a number of steps.

Step 1: The angle between any two vectors $\alpha, \beta \in \Delta(v)$ is obtuse, that is $(\alpha, \beta) \leq 0$: To see this suppose that $(\alpha, \beta) > 0$. Then one can check[2] using the table from the proof of the Lemma from last time on the angles between roots that if we take $(\alpha, \alpha) \geq (\beta, \beta)$ then $\langle \beta, \alpha \rangle = -1$. But then $s_\alpha(\beta) = \beta - \alpha \in \Phi$, so that either $\beta - \alpha$ or $\alpha - \beta$ lie in Φ^+, and hence either α or β will be decomposable.

Step 2: Linear independence: Suppose that we have $\sum_{\alpha \in \Delta(f)} c_\alpha \alpha = 0$ for $c_\alpha \in \mathbb{Q}$, $(\alpha \in \Delta)$. Gathering all the positive and nonpositive coefficients let

$$z = \sum_{c_\alpha > 0} c_\alpha \alpha = \sum_{c_\beta \leq 0} (-c_\beta) \beta$$

Then we have:

$$(z, z) = \sum_{\alpha, \beta : c_\alpha > 0, c_\beta \leq 0} c_\alpha.(-c_\beta)(\alpha, \beta) \leq 0,$$

so by positive definiteness we must have $z = 0$. But since $(v, z) = \sum_{c_\alpha > 0} c_\alpha (v, \alpha) \geq 0$ with equality if and only if the sum is empty we conclude that for all $\beta \in \Delta$ we have $c_\beta \leq 0$, but then we have $0 = \sum_{c_\beta \leq 0} (-c_\beta) \beta$, and pairing with v again we deduce that for all β we have $c_\beta = 0$, and so Δ is a linearly independent set.

Step 3: Finally we need to show that any base Δ is of the form $\Delta(v)$ for some regular $v \in V$. For $\alpha \in \Delta$, let $P_\alpha = \{v \in V : (v, \alpha) > 0\}$. This is a half-space in V. It can be checked that $\bigcap_{\alpha \in \Delta} P_\alpha$ is non-empty (indeed this holds for any linearly independent set in V, not just a base[3]) so we may pick some $v \in V$ in this intersection. Then since any root is a nonpositive or nonnegative combination of the roots in Δ, it follows the vector v is regular, and moreover it is clear that $\Phi^+ \subseteq \Phi^+(v)$ the set of positive roots associated with Δ, and since $\Phi^- = -\Phi^+$, we see similarly that $\Phi^- \subseteq \Phi^-(v)$, and since Φ is the disjoint

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[1] In the lecture I used $f \in V^*$. Since any such f is of the form $(v, -)$ for some $v \in V$ this makes no difference, and since later I switched to thinking of f in that form, I'm writing everything in terms of v here for simplicity.

[2] This either follows just inspecting the table, or by noting that by inequality proved there for α, β with $\beta \neq \pm\alpha$ we have $\langle \alpha, \beta \rangle . \langle \beta, \alpha \rangle < 4$ and so since $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ are integers at least one must be ± 1.

[3] Just extend your linearly independent set to a basis, and take the dual basis with respect to the inner product– the intersection of half-spaces then become the set of vectors whose coordinates with respect to the appropriate dual basis vectors are positive.

union of both $\Phi^+ \sqcup \Phi^-$ and $\Phi^+(v) \sqcup \Phi^-(v)$ it follows that $\Phi^\pm(v) = \Phi^\pm$. Finally, since $\Delta(v)$ are the indecomposable roots in $\Phi^+ = \Phi^+(v)$, we must certainly have $\Delta(v) \subseteq \Delta$, and since both are bases of V it follows that $\Delta = \Delta(v)$ as required. □

Our proof shows that for a given root system one can always find a base, but it will not be unique: different choices of regular v can lead to different positive systems $\Phi^+(v)$ and hence different bases. This ambiguity is however controlled by the action of the Weyl group, as we will now see. From now on fix a base Δ for our root system, and thus have the corresponding positive and negative roots Φ^+, Φ^-. Let W_0 be the subgroup of W generated by the reflections s_α for $\alpha \in \Delta$. The elements of Δ are called *simple roots*, and the corresponding reflections s_α are called *simple reflections*.

Lemma 1.2. *If $\alpha \in \Delta$, then the reflection s_α preserves the set $\Phi^+ \backslash \{\alpha\}$. Moreover, if $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$, then $(\rho, \gamma) = (1/2)||\gamma||^2 > 0$ for all $\gamma \in \Delta$ and hence $\Phi^+ = \Phi^+(\rho)$*

Proof. Let $\gamma \in \Phi^+ \backslash \{\alpha\}$. Since Δ is a base, we may write $\gamma = \sum_{\beta \in \Delta} c_\beta \beta$ where $c_\beta \in \mathbb{Z}_{\geq 0}$ for all $\beta \in \Delta$, and so

$$s_\alpha(\gamma) = (c_\alpha - \langle \alpha, \gamma \rangle)\alpha + \sum_{\beta \in \Delta, \beta \neq \alpha} c_\beta \beta \in \Phi.$$

Now since $\gamma \neq \pm\alpha$ (since it is positive) there must be some $\beta_0 \in \Delta \backslash \{\alpha\}$ with $c_{\beta_0} > 0$. But then by the definition of a base we must have $s_\alpha(\gamma) \in \Phi^+$, and clearly $s_\alpha(\gamma) \neq \alpha$ since this would imply $\gamma = s_\alpha(\alpha) = -\alpha \in \Phi^-$. For the final part, note that if $\gamma \in \Delta$ we must have

$$\begin{aligned} s_\gamma(\rho) &= s_\gamma(\frac{1}{2}\gamma) + s_\gamma(\frac{1}{2}\sum_{\alpha \in \Phi^+ \backslash \{\gamma\}} \alpha) \\ &= -\frac{1}{2}\gamma + \frac{1}{2}\sum_{\alpha \in \Phi^+ \backslash \{\gamma\}} \alpha \\ &= \rho - \gamma \end{aligned}$$

now since by definition $s_\gamma(\rho) = \rho - \frac{2(\rho,\gamma)}{(\gamma,\gamma)}\gamma$ the result follows immediately. The fact that $\Phi^+ = \Phi^+(\rho)$ (in the notation of Proposition 1.1 then follows immediately from the proof of that proposition. □

Definition 1.3. Given a root system equipped with a base, we may define a *height function* $\text{ht}: \Phi \to \mathbb{Z}$: if $\beta \in \Phi$ then we may write $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$, and we set $\text{ht}(\beta) = \sum_{\gamma \in \Delta} c_\gamma$. Note that from the definition of a base, $\text{ht}(\beta) = 1$ if and only if $\beta \in \Delta$.

Note that this height function is a special case of the height function used in the proof of Weyl's theorem – if $\{\varpi_i : 1 \leq i \leq r\}$ denotes the dual basis to the base Δ with respect to the inner product, then $\text{ht}(\beta) = (\delta, \beta)$ where $\delta = \sum_{i=1}^r \varpi_i$.

Proposition 1.4. *Suppose that $\beta \in \Phi$. Then there is a $w \in W_0$ and an $\alpha \in \Delta$ such that $w(\beta) = \alpha$.*

Proof. First suppose that $\beta \in \Phi^+$. We prove the statement by induction on the height of β. The statement being clear if β is of height 1, we may assume $h(\beta) > 0$. We claim there is some $\gamma \in \Delta$ such that $(\beta, \gamma) > 0$. If not then writing $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$, we see that

$$(\beta, \beta) = \sum_{\gamma \in \Delta} c_\gamma (\beta, \gamma) \leq 0,$$

so by positive definiteness we would have $\beta = 0$.

Now taking γ with $(\beta, \gamma) > 0$, we see that $s_\gamma(\beta) \in \Phi^+$ using the previous Lemma (since β is certainly not equal to α as $h(\beta) > 1$, and moreover $h(s_\gamma(\beta)) = h(\beta) - \langle \gamma, \beta \rangle < h(\beta)$. Thus by induction we are done. Finally we need to consider the case $\beta \in \Phi^-$. But then we have $-\beta \in \Phi^+$, and so there is a $w \in W_0$ such that $w(-\beta) = \gamma$ for some $\gamma \in \Delta$. But then clearly $(s_\gamma w)(\beta) = \gamma$ and we are done. □

Corollary 1.5. *The Weyl group W is generated by the reflections $\{s_\gamma : \gamma \in \Delta\}$, that is $W = W_0$.*

Proof. If $\beta \in \Phi$ then we have just shown in the previous proposition that there is a $w \in W_0$ such that $w(\beta) = \gamma$ for some $\gamma \in \Delta$. But the clearly $s_\beta = w^{-1} s_\gamma w \in W_0$, and so since W is generated by the s_βs we have $W = W_0$ as required. □

Theorem 1.6. *Suppose that Δ and Δ_1 are bases of (V, Φ). Then there is a $w \in W$ such that $w(\Delta) = \Delta_1$.*

Proof. By Proposition 1.1 we know any base is determined by a regular element $v \in V$: given v, the base it yields is the set of indecomposable roots in $\Phi^+(v)$. Thus it is enough to show that there is a w in W such that $w(\Phi_1^+) = \Phi^+$, where Φ_1^+ and Φ^+ are the positive roots corresponding to Δ_1 and Δ respectively.

Suppose v is a regular vector such that $\Phi^+(v) = \Phi_1^+$. We claim that there is a $w \in W$ such that $w(v)$ satisfies $(w(v), \alpha) > 0$ for all $\alpha \in \Delta$. This immediately implies that $(w(v), \alpha) > 0$ for all $\alpha \in \Phi^+$, so that $\Phi^+ = \Phi^+(w(v))$, and thus $w(\Phi_1^+) = \Phi$, since $(w(v), \alpha) > 0$ if and only if $(v, w^{-1}(\alpha)) > 0$.

To prove the claim, first consider $\rho = \frac{1}{2}\sum_{\alpha\in\Phi} \alpha$. By the Lemma 1.2 we know that $s_\alpha(\rho) = \rho - \alpha$ for all $\alpha \in \Delta$. Now choose $w \in W$ such that $(\rho, w(v))$ is as large as possible. Then if $\alpha \in \Delta$, we must have

$$(w(v), \rho) \geq (s_\alpha.w(v), \rho) = (w(v), s_\alpha(\rho)) = (w(v), \rho - \alpha) = (w(v), \rho) - (w(v), \alpha).$$

Thus it follows that $(w(v), \alpha) \geq 0$ for all $\alpha \in \Delta$, and since $(v, \alpha) \neq 0$ for all $\alpha \in \Phi$ (as f was assumed to be generic) the claim follows.

□

Remark 1.7. In fact W acts simply transitively on the bases of (V, Φ), that is if $w(\Delta) = \Delta$ then $w = 1$. The proof (which we will not give) consists of examining the minimal length expression for w in terms of these generators $\{s_\alpha : \alpha \in \Delta\}$.