LIE ALGEBRAS: LECTURE 16.

1. Cartan matrices and Dynkin diagrams

Definition 1.1. Let (V, Φ) be a root system. The *Cartan matrix* associated to (V, Φ) is the matrix

$$C = (\langle \alpha_i, \alpha_j \rangle)_{i,j=1}^{l}.$$

where $\{\alpha_1, \alpha_2, \dots, \alpha_\ell\} = \Delta$ is a base of (V, Φ). Since the elements of W are isometries, and W acts transitively on bases of Φ, the Cartan matrix is independent of the choice of base (though clearly determined only up to reordering the base Δ).

Definition 1.2. The entries c_{ij} of the Cartan matrix are all integer with diagonal entries equal to 2, and off-diagonal entries $c_{ij} \in \{0, -1, -2, -3\}$ (where $i \neq j$) such that if $c_{ij} < -1$ then $c_{ji} = -1$ so that the pair $\{c_{ij}, c_{ji}\}$ is determined by the product $c_{ij}.c_{ji}$ and the relative lengths of the two roots (*e.g.* see the table in the Lemma about angles between roots). As a result, the matrix can be recorded as a kind of graph: the vertex set of the graph is labelled by the base $\{\alpha_1, \dots, \alpha_l\}$, and one puts $\langle \alpha_i, \alpha_j \rangle . \langle \alpha_j, \alpha_i \rangle$ edges between α_i and α_j, directing the edges so that they go from the larger root to the smaller root. Thus for example if $\langle \alpha_i, \alpha_j \rangle = -2$ and $\langle \alpha_j, \alpha_i \rangle = -1$ so that $||\alpha_j||^2 > ||\alpha_i||^2$, that is, α_j is longer than α_i, we record this in the graph as:

$$\overset{\alpha_i}{\bullet} \Longleftarrow \overset{\alpha_j}{\bullet}$$

The resulting graph is called the *Dynkin diagram*.

For the next theorem we need to formulate what it means to have an isomorphism of root systems. This is given in the natural way: if (V, Φ) and (V', Φ') are root systems, a linear map $\phi \colon V \to V'$ is an isomorphism of root systems if

(1) The map ϕ is an isomorphism of vector spaces.
(2) $\phi(\Phi) = \Phi'$, and $\langle \alpha, \beta \rangle = \langle \phi(\alpha), \phi(\beta) \rangle$ for all $\alpha, \beta \in \Phi$.

Note that ϕ need not be an isometry (*e.g.* we could scale V by a nonzero constant $c \in \mathbb{Q}$ to obtain $(V, c\Phi)$ a distinct, but isomorphic root system to (V, Φ).

Theorem 1.3. *Let (V, Φ) be a root system. Then (V, Φ) is determined up to isomorphism by the Cartan matrix, or Dynkin diagram associated to it.*

Proof. Given root systems (V, Φ) and (V', Φ') with the same Cartan matrix, we may certainly pick a base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ of (V, Φ) and a base $\Delta' = \{\beta_1, \dots, \beta_\ell\}$ of (V', Φ') such that $\langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle$ for all i, j, $(1 \leq i, j \leq \ell)$. We claim the map $\phi \colon \Delta \to \Delta'$ given by $\phi(\alpha_i) = \beta_i$ extends to an isomorphism of root systems. Clearly, since Δ and Δ' are bases of V and V' respectively, ϕ extends uniquely to an isomorphism of vector spaces $\phi \colon V \to V'$, so we must show that $\phi(\Phi) = \Phi'$, and $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle$ for each $\alpha, \beta \in \Phi$.

Let $s_i = s_{\alpha_i} \in \mathrm{O}(V)$ and $s'_i = s_{\beta_i} \in \mathrm{O}(V')$ be the reflections in the Weyl groups W and W' respectively. Then from the formula for the action of s_i it is clear that $\phi(s_i(\alpha_j)) = s'_i(\beta_j) = s'_i(\phi(\alpha_i))$, so since Δ is a basis it follows $\phi(s_i(v)) = s'_i(\phi(v))$ for all $v \in V$. But then since the s_is and s'_is generate W and W' respectively, ϕ induces an isomorphism $W \to W'$, given by $w \mapsto w' = \phi \circ w \circ \phi^{-1}$. But then given any $\alpha \in \Phi$ we know there is a $w \in W$ such that $\alpha = w(\alpha_j)$ for some j, $(1 \leq j \leq \ell)$. Thus we have $\phi(\alpha) = \phi(w(\alpha_j)) = w'(\phi(\alpha_j)) = w'(\beta_j) \in \Phi'$, so that $\phi(\Phi) \subseteq \Phi'$. Clearly the same argument applied to ϕ^{-1} shows that $\phi^{-1}(\Phi') \subseteq \Phi$ so that $\phi(\Phi) = \Phi'$.

Finally, note that it is clear from the linearity of ϕ and of $\langle \alpha, \gamma \rangle$ in the second variable, that $\langle \alpha, \gamma \rangle = \langle \phi(\alpha), \phi(\gamma) \rangle$ for all $\alpha \in \Delta$, $\gamma \in \Phi$. In the same fashion as above however, if $\alpha \in \Phi$ is arbitrary, then we may find $w \in W$ such that $\alpha = w(\alpha_j) \in \Delta$, and thus $\phi(\alpha) = w'(\beta_j)$, whence we have

$$\begin{aligned} \langle \phi(\alpha), \phi(\gamma) \rangle &= \langle w'(\beta_j), \phi(\gamma) \rangle = \langle \beta_j, (w')^{-1}\phi(\gamma) \rangle \\ &= \langle \alpha_j, w^{-1}(\gamma) \rangle = \langle w(\alpha_j), \gamma \rangle = \langle \alpha, \gamma \rangle. \end{aligned}$$

as required. ☐

Date: December, 2012.

Thus to classify root systems up to isomorphism it is enough to classify Cartan matrices (or Dynkin diagrams).

Definition 1.4. We say that a root system (V, Φ) is *reducible* if there is a partition of the roots into two non-empty subsets $\Phi_1 \sqcup \Phi_2$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1, \beta \in \Phi_2$. Then if we set $V_1 = \text{span}(\Phi_1)$ and $V_2 = \text{span}(\Phi_2)$, clearly $V = V_1 \oplus V_2$ and we say (V, Φ) is the sum of the root systems (V_1, Φ_1) and (V_2, Φ_2). This allows one to reduce the classification of root systems to the classification of *irreducible* root systems, *i,e.* root systems which are not reducible. It is straight-forward to check that a root system is irreducible if and only if its associated Dynkin diagram is connected.

Definition 1.5. (*Not examinable.*) The notion of a root system makes sense over the real, as well as rational, numbers. Let (V, Φ) be a real root system, and let $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ be a base of Φ. If $v_i = \alpha_i/||\alpha_i||$ $(1 \leq i \leq l)$ are the unit vectors in V corresponding to Δ, then they satisfy the conditions:

(1) $(v_i, v_i) = 1$ for all i and $(v_i, v_j) \leq 0$ if $i \neq j$,
(2) If $i \neq j$ then $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$. (This is the reason we need to extend scalars to the real numbers – if you want you could just extend scalars to $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, but it makes no difference to the classification problem).

Such a set of vectors is called an *admissible set*.

It is straightforward to see that classifying $\mathbb{Q}$-vector spaces with a basis which forms an admissible set is equivalent to classifying Cartan matrices, and using elementary techniques it is possible to show that that the following are the only possibilities (we list the Dynkin diagram, a description of the roots, and a choice of a base):

- Type A_ℓ $(\ell \geq 1)$:

$$\bullet \longrightarrow \bullet \cdots \bullet \longrightarrow \bullet$$

$$V = \{v = \sum_{i=1}^{\ell} c_i e_i \in \mathbb{Q}^8 : \sum c_i = 0\}, \Phi = \{\varepsilon_i - e_j : 1 \leq i \neq j \leq \ell\}$$
$$\Delta = \{\varepsilon_{i+1} - \varepsilon_i : 1 \leq i \leq \ell - 1\}$$

- Type B_ℓ $(\ell \geq 2)$:

$$\bullet \longrightarrow \bullet \cdots \bullet \Longrightarrow \bullet$$

$$V = \mathbb{Q}^\ell, \Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i, j \leq \ell, i \neq j\} \cup \{\varepsilon_i : 1 \leq i \leq \ell\},$$
$$\Delta = \{\varepsilon_1, \varepsilon_{i+1} - \varepsilon_i : 1 \leq i \leq \ell - 1\}$$

- Type C_ℓ $(\ell \geq 3)$:

$$\bullet \longrightarrow \bullet \cdots \bullet \Longleftarrow \bullet$$

$$V = \mathbb{Q}^\ell, \Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i, j \leq \ell, i \neq j\} \cup \{2\varepsilon_i : 1 \leq i \leq \ell\},$$
$$\Delta = \{2\varepsilon_1, \varepsilon_{i+1} - \varepsilon_i : 1 \leq i \leq \ell - 1\}$$

- Type D_ℓ $(\ell \geq 4)$:

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad \bullet \\ \bullet \longrightarrow \bullet \cdots \bullet \longrightarrow \bullet \!<\! \\ \qquad\qquad\qquad\qquad\quad \bullet \end{array}$$

$$V = \mathbb{Q}^\ell, \Phi = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i, j \leq \ell, i \neq j\},$$
$$\Delta = \{\varepsilon_1 + \varepsilon_2, \varepsilon_{i+1} - \varepsilon_i : 1 \leq i \leq \ell - 1\}$$

- Type G_2.

$$\bullet \Lleftarrow \bullet$$

Let $e = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \in \mathbb{Q}^3$, then:

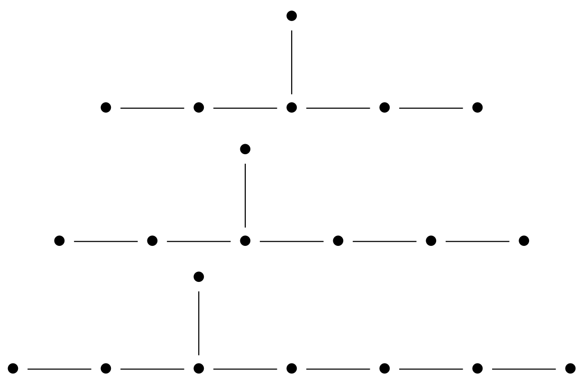
$$V = \{v \in \mathbb{Q}^3 : (v, e) = 0\}, \Phi = \{\varepsilon_i - \varepsilon_j : i \neq j\} \cup \{\pm(3\varepsilon_i - e) : 1 \leq i \leq 3\}$$
$$\Delta = \{\varepsilon_1 - \varepsilon_2, e - 3\varepsilon_1\}$$

- Type F_4:

$$\bullet \longrightarrow \bullet \Longrightarrow \bullet \longrightarrow \bullet$$

$$V = \mathbb{Q}^4,$$
$$\Phi = \{\pm\varepsilon_i : 1 \leq i \leq 4\} \cup \{\pm\varepsilon_i \pm e_j : i \neq j\} \cup \{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$$
$$\Delta = \{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}.$$

- Type E_n ($n = 6, 7, 8$).



These can all be constructed inside E_8 by taking the span of the appropriate subset of a base, so we just give the root system for E_8.

$$V = \mathbb{Q}^8, \Phi = \{\pm\varepsilon_i \pm \varepsilon_j : i \neq j\} \cup \{\frac{1}{2}\sum_{i=1}^{8}(-1)^{a_i}\varepsilon_i : \sum_{i=1}^{8} a_i \in 2\mathbb{Z}\},$$

$$\Delta = \{\varepsilon_1 + \varepsilon_2, \varepsilon_{i+1} - \varepsilon_i, \frac{1}{2}(\varepsilon_1 + \varepsilon_8 - (\varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_7)) : 1 \leq i \leq 6\}.$$

Note that the Weyl groups of type B_ℓ and C_ℓ are equal. The reason for the restriction on ℓ in the types B,C,D is to avoid repetition, *e.g.* B_2 and C_2 are the same up to relabelling the vertices.

Remark 1.6. I certainly don't expect you to remember the root systems of the exceptional types, but you should be familiar with the ones for type A, B, C and D. The ones of rank two (*i.e.* A_2, B_2 and G_2) are also worth knowing (because for example you can draw them!)

2. The Classification of Semisimple Lie algebras

Only the statements of the theorems in this section are examinable, but it is important to know these statements!

Remarkably, the classification of semisimple Lie algebras is identical to the classification of root systems: each semisimple Lie algebra decomposes into a direct sum of simple Lie algebras, and it is not hard to show that the root system of a simple Lie algebra is irreducible. Thus to any simple Lie algebra we may attach an irreducible root system.

A first problem with this as a classification strategy is that we don't know our association of a root system to a semisimple Lie algebra is canonical. The difficulty is that, because our procedure for attaching a root system to a semisimple Lie algebra involves a choice of Cartan subalgebra, we don't currently know it is a bijective correspondence – possibly the same Lie algebra has two different Cartan subalgebras which lead to different root systems. The theorem which ensures this is not the case is the following, where the first part is the more substantial result (though both require some work):

Theorem 2.1. *Let $\mathfrak{g}$ be a Lie algebra over any algebraically closed field k.*

1. *Let $\mathfrak{h}, \mathfrak{h}'$ be Cartan subalgebras of $\mathfrak{g}$. There is an automorphism $\phi\colon \mathfrak{g} \to \mathfrak{g}$ such that $\phi(\mathfrak{h}) = \mathfrak{h}'$.*
2. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be semisimple Lie algebras with Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2$ respectively, and suppose now k is of characteristic zero. Then if the root systems attached to $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are isomorphic, there is an isomorphism $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ taking $\mathfrak{h}_1$ to $\mathfrak{h}_2$.*

Once you know that the assignment of a Dynkin diagram captures a simple Lie algebra up to isomorphism, we still need to show all the root systems we construct arise as the root system of a simple Lie algebra. That is exactly the content of the next theorem.

Theorem 2.2. *There exists a simple Lie algebra corresponding to each irreducible root system.*

There are a number of approaches to this existence theorem. A concrete strategy goes as follows: one can show that the first four infinite families A, B, C, D correspond to the classical Lie algebras, $\mathfrak{sl}_{\ell+1}$, $\mathfrak{so}_{2\ell+1}$, $\mathfrak{sp}_{2\ell}$, $\mathfrak{so}_{2\ell}$, whose root systems can be computed directly (indeed you did a number of these calculations in the problem sets). This of course also requires checking that these Lie algebras are simple (or at least semisimple) but this is also straight-forward with the theory we have developed. It then only remains to construct the five "exceptional" simple Lie algebras. This can be done in a variety of ways – given a root system where all the roots are of the same length there is an explicit construction of the associated Lie algebra by forming a basis from the Cartan decomposition (and a choice of base of the root system) and explicitly constructing the Lie bracket by giving the structure constants with respect to this basis (which, remarkably, can be chosen for the basis vectors corresponding to the root subspaces to lie in $\{0, \pm 1\}$). This gives in particular a construction of the Lie algebras of type E_6, E_7, E_8 (and also A_ℓ and D_ℓ though we already had a construction of these). The remaining Lie algebras can be found by a technique called "folding" which studies automorphisms of simple Lie algebras, and realises the Lie algebras G_2 and F_4 as fixed-points of an automorphism of D_4 and E_6 respectively.

There is also an alternative, more *a posteriori* approach to the uniqueness result which avoids showing Cartan subalgebras are all conjugate for a general Lie algebra: one can check that for a classical Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n$ as above, the Cartan subalgebras are all conjugate by an element of $\mathrm{Aut}(\mathfrak{g})$ (in fact you can show the automorphism is induced by conjugating with a matrix in $\mathrm{GL}_n(\mathsf{k})$) using the fact that a Cartan subalgebra of a semisimple Lie algebra is abelian and consists of semisimple elements. This then shows the assignment of a root system to a classical Lie algebra is unique, so it only remains to check the exceptional Lie algebras. But these all have different dimensions, and the dimension of the Lie algebra is captured by the root system, so we are done.[1]

We conclude by mentioning another, quite different, approach to the existence result, using the *Serre's presentation*: just as one can describe a group by generators and relations, one can also describe Lie algebras in a similar fashion. If $\mathfrak{g}$ is a semisimple Lie algebra and $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a base of the corresponding root system with Cartan matrix $C = (a_{ij})$ then picking bases for the $\mathfrak{sl}_{\alpha_i}$-subalgebras corresponding to them, it is not too hard to show that $\mathfrak{g}$ is generated by the set $\{e_\alpha, f_\alpha, h_\alpha : \alpha \in \Delta\}$.

The Serre presentation gives an explicit realisation, given an arbitrary root system, of the relations which one needs to impose on a set of generators for a Lie algebra labelled $\{e_\alpha, f_\alpha, h_\alpha : \alpha \in \Phi\}$ as above obtain a semisimple Lie algebra whose associated root system is the one we started with. This approach has the advantage of giving a uniform approach, though it takes some time to develop the required machinery.

[1] This is completely rigorous, but feels like cheating (to me).