LIE ALGEBRAS: LECTURE 3.

1. NILPOTENT LIE ALGEBRAS

We now begin to study particular classes of Lie algebra. The first class we study are nilpotent Lie algebras, which are somewhat analogous to nilpotent groups. We need a few more definitions.

Definition 1.1. If V, W are subspaces of a Lie algebra $\mathfrak{g}$, then write $[V, W]$ for the linear span of the elements $\{[v, w] : v \in V, w \in W\}$. Notice that if I, J are ideals in $\mathfrak{g}$ then so is $[I, J]$. Indeed to check this, note that if $i \in I, j \in J, x \in \mathfrak{g}$ we have:

$$[x,[i,j]] = -[i,[j,x]] - [j,[x,i] = [i,[x,j]] + [[x,i],j] \in [I,J]$$

using the Jacobi identity in the first equality and skew-symmetry in the second.

Definition 1.2. For $\mathfrak{g}$ a Lie algebra, let $C^0(\mathfrak{g}) = \mathfrak{g}$, and $C^i(\mathfrak{g}) = [\mathfrak{g}, C^{i-1}(\mathfrak{g})]$ for $i \geq 1$. This sequence of ideals of $\mathfrak{g}$ is called the *lower central series* of $\mathfrak{g}$, and we say $\mathfrak{g}$ is *nilpotent* if $C^N(\mathfrak{g}) = 0$ for some $N > 0$. If N is the smallest integer such that $C^N(\mathfrak{g}) = 0$ then we say that $\mathfrak{g}$ is an N*-step nilpotent* Lie algebra.

For example, a Lie algebra is 1-step nilpotent if and only if it is abelian. The definition can be rephrased as follows: there is an $N > 0$ such that for any N elements $x_1, x_2, \ldots, x_N$ of $\mathfrak{g}$ the iterated Lie bracket

$$[x_1,[x_2,[\ldots,[x_{N-1},x_N]]\ldots] = \text{ad}x_1(\text{ad}x_2(\ldots\text{ad}x_{N-1}(x_N))\ldots) = 0.$$

In particular, all the elements $\text{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ for $x \in \mathfrak{g}$ are nilpotent.

Remark 1.3. The ideal $C^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$ is known as the *derived subalgebra*[1] of $\mathfrak{g}$ and is also denoted[2] $D(\mathfrak{g})$ and sometimes $\mathfrak{g}'$.

Lemma 1.4. *Let $\mathfrak{g}$ be a Lie algebra. Then*

1. *If $\mathfrak{g}$ is nilpotent, so is any subalgebra or quotient of $\mathfrak{g}$.*
2. *If $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is nilpotent.*

Proof. The first part is immediate from the definition. Indeed if $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ then clearly we have $C^i(\mathfrak{h}) \subseteq C^i(\mathfrak{g})$, so that if $C^N(\mathfrak{g}) = 0$ we also have $C^N(\mathfrak{h}) = 0$. Similarly, an easy induction say shows that if $\mathfrak{h}$ is an ideal, then $C^i(\mathfrak{g}/\mathfrak{h}) = (C^i(\mathfrak{g}) + \mathfrak{h})/\mathfrak{h}$, and so again if $C^N(\mathfrak{g}) = 0$ we must also have $C^N(\mathfrak{g}/\mathfrak{h}) = 0$.

For the second claim, taking $\mathfrak{h} = \mathfrak{z}(\mathfrak{g})$, an ideal of $\mathfrak{g}$, we see from the first part that if $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is nilpotent so that there is some N with $(C^N(\mathfrak{g}) + \mathfrak{z}(\mathfrak{g}))/\mathfrak{z}(\mathfrak{g}) = C^N(\mathfrak{g}/\mathfrak{z}(\mathfrak{g})) = 0$, and so $C^N(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$. But then it is clear that $C^{N+1}(\mathfrak{g}) = 0$, and so $\mathfrak{g}$ is nilpotent as required. $\square$

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[1] Oddly, not as the *derived ideal* even though it is an ideal.

[2] Partly just to cause confusion, but also because it comes up a lot, playing slightly different roles, which leads to the different notation. We'll see it again shortly in a slightly different guise.

Remark 1.5. Notice that if I is an arbitrary ideal in $\mathfrak{g}$, and I and $\mathfrak{g}/I$ are nilpotent it does *not* follow that $\mathfrak{g}$ is nilpotent. Indeed recall that if $\mathfrak{g}$ is the non-abelian 2-dimensional Lie algebra, then $\mathfrak{g}$ can be given a basis x, y with $[x, y] = y$. Hence $\mathsf{k}.y$ is a 1-dimensional ideal in $\mathfrak{g}$ (which is thus abelian and so nilpotent) and the quotient is again 1-dimensional and so nilpotent. However, ad(x) has y as an eigenvector with eigenvalue 1, so $\mathfrak{g}$ cannot be nilpotent, and indeed $C^i(\mathfrak{g}) = \mathsf{k}.y$ for all $i \geq 1$.

Note that the subalgebra $\mathfrak{t} \subset \mathfrak{gl}_n$ of diagonal matrices is nilpotent, since it is abelian, so a nilpotent linear[3] Lie algebra need not consist of nilpotent endomorphisms. Nevertheless we will show there is a close connection between the two notions.

Lemma 1.6. *If $x \in \mathfrak{gl}(V)$ be a nilpotent endomorphism then ad$(x) \in \mathfrak{gl}(\mathfrak{gl}(V))$ is also nilpotent.*

Proof. The map $\lambda_x \colon \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ given by $y \mapsto xy$ is clearly nilpotent if x is nilpotent, and similarly for the map $\rho_x \colon \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ given by $y \mapsto yx$. Moreover, λ_x and ρ_x clearly commute with each other, so since ad$(x) = \lambda_x - \rho_x$, it is also nilpotent. Indeed for $m \geq 0$ we have

$$(\lambda_x - \rho_x)^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \lambda_x^{m-i} \rho_x^i$$

and so if $x^n = 0$, so that $\lambda_x^n = \rho_x^n = 0$, then if $m \geq 2n$, every term on the right-hand side must be zero, and so ad$(x)^m = 0$ as required. □

For the next proposition we need the notion of the normaliser of a sub algebra.

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{a}$ be a subalgebra. The subspace

$$N_{\mathfrak{g}}(\mathfrak{a}) = \{x \in \mathfrak{g} : [x, a] \in \mathfrak{a}, \forall a \in \mathfrak{a}\}$$

is a subalgebra of $\mathfrak{g}$ (check this using the Jacobi identity) which is called the *normaliser* of $\mathfrak{a}$. It is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{a}$ is an ideal.

Proposition 1.8. *Let $\mathfrak{n}$ be a Lie algebra, and (V, ρ) a representation of $\mathfrak{n}$ such that for every $x \in \mathfrak{n}$, the linear map $\rho(x)$ is nilpotent. Then the subspace*

$$V^{\mathfrak{n}} = \{v \in V : \rho(x)(v) = 0, \forall x \in \mathfrak{n}\}.$$

is non-zero.

Proof. We use induction on $d = \dim(\mathfrak{n})$, the case $d = 1$ being clear. Clearly the statement of the proposition is unchanged if we replace $\mathfrak{n}$ by its image in $\mathfrak{gl}(V)$, so we may assume that $\mathfrak{n}$ is a subalgebra of $\mathfrak{gl}(V)$. Now consider $\mathfrak{a} \subsetneq \mathfrak{n}$ a proper subalgebra[4]. Since the elements $a \in \mathfrak{a}$ are nilpotent endomorphisms, the previous lemma shows that ad$(a) \in \mathfrak{gl}(\mathfrak{n})$ is also nilpotent, hence ad(a) also acts nilpotently on the quotient[5] $\mathfrak{n}/\mathfrak{a}$. Since $\dim(\mathfrak{a}) < \dim(\mathfrak{n})$, by induction (applied with $V = \mathfrak{n}/\mathfrak{a}$

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[3] We say a Lie algebra is linear if it is realized as a subalgebra of $\mathfrak{gl}(V)$ for some finite-dimensional vector space V.

[4] In lecture I insisted that $\mathfrak{a}$ was non-zero, but actually we don't need this.

[5] The notation here can be a bit confusing: the Lie algebra $\mathfrak{n}$ is an $\mathfrak{a}$-representation by the restriction of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{a}$. Since $\mathfrak{a}$ is a sub-$\mathfrak{a}$-representation of $\mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{a}$ is an $\mathfrak{a}$-representation. Note however that it is not itself a Lie algebra unless $\mathfrak{a}$ is an ideal of $\mathfrak{g}$.

and $\mathfrak{a}$) we can find $x \notin \mathfrak{a}$ such that $\text{ad}(a)(x) = [a, x] \in \mathfrak{a}$ for all $a \in \mathfrak{a}$. It follows that for any subalgebra $\mathfrak{a}$ its normaliser

$$N_{\mathfrak{n}}(\mathfrak{a}) = \{x \in \mathfrak{n} : [x, a] \in \mathfrak{a}, \forall a \in \mathfrak{a}\}$$

strictly contains $\mathfrak{a}$. Thus if $\mathfrak{a}$ is a proper subalgebra of $\mathfrak{n}$ of maximal dimension, we must have $N_{\mathfrak{n}}(\mathfrak{a}) = \mathfrak{n}$, or in other words, $\mathfrak{a}$ must actually be an ideal of $\mathfrak{n}$.

Now if $\mathfrak{n}/\mathfrak{a}$ is not one-dimensional, the preimage of a one-dimensional subalgebra in $\mathfrak{n}/\mathfrak{a}$ would be a proper subalgebra of $\mathfrak{n}$ strictly containing $\mathfrak{a}$, which again contradicts the maximality of $\mathfrak{a}$. Thus $\mathfrak{n}/\mathfrak{a}$ is one-dimensional, and we may find $z \in \mathfrak{n}$ so that $\mathsf{k}.z \oplus \mathfrak{a} = \mathfrak{n}$.

By induction, we know that $V^{\mathfrak{a}} = \{v \in V : a(v) = 0, \forall a \in \mathfrak{a}\}$ is a nonzero subspace of V. We claim that z preserves $V^{\mathfrak{a}}$. Indeed

$$a(z(v)) = [a, z](v) + z(a(v)) = 0, \quad \forall a \in \mathfrak{a}, v \in V^{\mathfrak{a}},$$

since $[a, z] \in \mathfrak{a}$. But the restriction of z to $V^{\mathfrak{a}}$ is nilpotent, so the subspace $U = \{v \in V^{\mathfrak{a}} : z(v) = 0\}$ is nonzero. Since $U = V^{\mathfrak{n}}$ we are done. □

Remark 1.9. The notation $V^{\mathfrak{n}}$ used above will be used later: if $\mathfrak{g}$ is any Lie algebra and (V, ρ) is a representation of $\mathfrak{g}$, then we write

$$V^{\mathfrak{g}} = \{v \in V : \rho(x)(v) = 0, \forall x \in \mathfrak{g}\}.$$

It is called the (possibly zero in general) subrepresentation of *invariants* in V.

Definition 1.10. If $\mathfrak{g}$ is a Lie algebra and $x \in \mathfrak{g}$, we say that an element x is ad-nilpotent if $\text{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$ is a nilpotent endomorphism. For convenience[6] we say that $\mathfrak{g}$ is ad-nilpotent if all of its elements are.

Theorem 1.11. *(Engel's theorem) A Lie algebra $\mathfrak{g}$ is nilpotent if and only if every $x \in \mathfrak{g}$ is ad(x)-nilpotent.*

Proof. In the terminology of the above definition, the theorem states that a Lie algebra is nilpotent if and only if it is ad-nilpotent. As we already noted, it is immediate from the definition that a nilpotent Lie algebra is ad-nilpotent, so our task is to show the converse. For this, note that it is clear that if $\mathfrak{g}$ is ad-nilpotent, then any quotient of it is also ad-nilpotent. Hence using induction on dimension along with Lemma 1.4 (2), it will be enough to show that $\mathfrak{z}(\mathfrak{g})$ is a non-zero. Now

$$\begin{aligned}\mathfrak{z}(\mathfrak{g}) &= \{z \in \mathfrak{g} : [z, x] = 0, \forall x \in \mathfrak{g}\} \\ &= \{z \in \mathfrak{g} : -\text{ad}(x)(z) = 0, \forall x \in \mathfrak{g}\} \\ &= \mathfrak{g}^{\text{ad}(\mathfrak{g})},\end{aligned}$$

hence applying Proposition 1.8 to $\text{ad}(\mathfrak{g})$, the image of $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$ it follows immediately that $\mathfrak{z}(\mathfrak{g}) \neq 0$ and we are done.

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[6] Though because of the following theorem this terminology won't be used much.