LIE ALGEBRAS: LECTURE 4.

1. NILPOTENT LIE ALGEBRAS (CONTINUED).

Example 1.1. Let V be a vector space, and

$$\mathcal{F} = (0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = V)$$

a sequence of subspaces with $\dim(F_i) = i$ (such a sequence is known as a *flag* or *complete flag* in V). Let $\mathfrak{n} = \mathfrak{n}_{\mathcal{F}}$ be the subalgebra of $\mathfrak{gl}(V)$ consisting of linear maps $X \in \mathfrak{gl}(V)$ such that $X(F_i) \subseteq F_{i-1}$ for all $i \geq 1$. We claim the Lie algebra $\mathfrak{n}$ is nilpotent. To see this we show something more precise. Indeed for each positive integer k, consider the subspace

$$\mathfrak{n}^k = \{x \in \mathfrak{gl}(V) : x(F_i) \subset F_{i-k}\}$$

(where we let $0 = F_l$ for all $l \leq 0$). Then clearly $\mathfrak{n}^k \subset \mathfrak{n}$, and $\mathfrak{n}^k = 0$ for any $k \geq n$. We claim that $C^k(\mathfrak{n}) \subseteq \mathfrak{n}^{k+1}$, which will therefore prove $\mathfrak{n}$ is nilpotent. The claim is immediate for $k = 0$, so suppose we know by induction that $C^k(\mathfrak{n}) \subseteq \mathfrak{n}^{k+1}$. Then if $x \in \mathfrak{n}$ and $y \in \mathfrak{n}^{k+1}$, we have $xy(F_i) \subset x(F_{i-k-1}) \subset F_{i-k-2}$, and similarly $yx(F_i) \subset F_{i-k-2}$, thus certainly $[x, y] \in \mathfrak{n}^{k+2}$ and so $C^{k+1}(\mathfrak{n}) \subseteq \mathfrak{n}^{k+2}$ as required.

In fact you can check that $C^k(\mathfrak{n}) = \mathfrak{n}^{k+1}$, so that $\mathfrak{n}$ is $(n-1)$-step nilpotent *i.e.* $C^{n-2}(\mathfrak{n}) \neq 0$, and $C^{n-1}(\mathfrak{n}) = 0$ (note that if $\dim(V) = 1$ then $\mathfrak{n} = 0$). If we pick a basis $\{e_1, e_2, \ldots, e_n\}$ of V such that $F_i = \text{span}(e_1, e_2, \ldots, e_i)$ then the matrix A representing an element $x \in \mathfrak{n}$ with respect to this basis is strictly upper triangular, that is, $a_{ij} = 0$ for all $i \geq j$. It follows that $\dim(\mathfrak{n}) = \binom{n}{2}$, so when $n = 2$ we just get the 1-dimensional Lie algebra, thus the first nontrivial case is when $n = 3$ and in that case we get a 3-dimensional 2-step nilpotent Lie algebra.

A consequence of the proposition we used to prove Engel's theorem is the following result, which is worth noting separately.

Corollary 1.2. *Let $\mathfrak{g}$ be a Lie algebra and (V, ρ) a representation of $\mathfrak{g}$ such that $\rho(x)$ is a nilpotent endomorphism for all $x \in \mathfrak{g}$. Then there is a complete flag $\mathcal{F} = (0 = F_0 \subset F_1 \subset \ldots \subset F_n = V)$ such that $\rho(\mathfrak{g}) \subseteq \mathfrak{n}_{\mathcal{F}}$.*

Proof. Let us say that $\mathfrak{g}$ respects a flag $\mathcal{F}$ if $\rho(\mathfrak{g}) \subseteq \mathfrak{n}_{\mathcal{F}}$. Use induction on $\dim(V)$. By the Proposition from Lecture 3, we see that the space $V^{\mathfrak{g}} \neq 0$. Thus by induction we may find a flag $\mathcal{F}'$ in $V/V^{\mathfrak{n}}$ which $\mathfrak{n}$ respects. Taking its preimage and extending arbitrarily (by picking any complete flag in $V^{\mathfrak{g}}$) we get a complete flag in V, we obtain a flag which $\mathfrak{g}$ clearly respects as required. □

2. SOLVABLE LIE ALGEBRAS

We now consider another class of Lie algebras which is slightly larger than the class of nilpotent algebras. For a Lie algebra $\mathfrak{g}$, let $D^0\mathfrak{g} = \mathfrak{g}$, and $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$. $D^i\mathfrak{g}$ is the i-th derived ideal of $\mathfrak{g}$. Note that $C^1(\mathfrak{g}) = D^1\mathfrak{g}$ is $D\mathfrak{g}$ the derived subalgebra of $\mathfrak{g}$.

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Definition 2.1. A Lie algebra $\mathfrak{g}$ is said to be *solvable* if $D^N\mathfrak{g} = 0$ for some $N > 0$.

Since it is clear from the definition that $D^i\mathfrak{g} \subset C^i(\mathfrak{g})$, any nilpotent Lie algebra is solvable, but as one can see by considering the non-abelian 2-dimensional Lie algebra, there are solvable Lie algebras which are not nilpotent.

Example 2.2. Let V be a vector space and $\mathcal{F} = (0 = F_0 < F_1 < \ldots < F_n = V)$ a complete flag in V. Let

$$\mathfrak{b}_{\mathcal{F}} = \{x \in \mathfrak{gl}(V) : x(F_i) \subseteq F_i\},$$

that is, $\mathfrak{b}_{\mathcal{F}}$ is the subspace of endomorphisms which preserve the complete flag $\mathcal{F}$. We claim that $\mathfrak{b}_{\mathcal{F}}$ is solvable. Since any nilpotent Lie algebra is solvable, and clearly $\mathfrak{g}$ is solvable if and only if $D^1\mathfrak{g}$ is, the solvability of $\mathfrak{g}$ will follow if we can show that $D^1\mathfrak{g} = \mathfrak{n}_{\mathcal{F}}$. To see this, suppose that $x, y \in \mathfrak{b}_{\mathcal{F}}$ and consider $[x, y]$. We need to show that $[x, y](F_i) \subset F_{i-1}$ for each i, $1 \leq i \leq n$. Since $x, y \in \mathfrak{b}_{\mathcal{F}}$, certainly we have $[x, y](F_i) \subseteq F_i$ for all $i, 1 \leq i \leq n$, thus it is enough to show that the map $\overline{[x, y]}$ induced by $[x, y]$ on F_i/F_{i-1} is zero. But this map is the commutator of the maps induced by x and y in $\mathrm{End}(F_i/F_{i-1})$, which since F_i/F_{i-1} is one-dimensional, is abelian, so that all commutators are zero.

If we pick a basis $\{e_1, e_2, \ldots, e_n\}$ of V such that $F_i = \mathrm{span}(e_1, \ldots, e_i)$, then $\mathfrak{gl}(V)$ gets identified with $\mathfrak{gl}_n$ and $\mathfrak{b}_{\mathcal{F}}$ corresponds to the subalgebra $\mathfrak{b}_n$ of upper triangular matrices. It is straight-forward to show by considering the subalgebra $\mathfrak{t}_n$ of diagonal matrices that $\mathfrak{b}_n$ is not nilpotent.

We will see shortly that, in characteristic zero, any solvable linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a subalgebra of $\mathfrak{b}_{\mathcal{F}}$ for some complete flag $\mathcal{F}$. We next note some basic properties of solvable Lie algebras.

Lemma 2.3. *Let $\mathfrak{g}$ be a Lie algebra, $\phi\colon \mathfrak{g} \to \mathfrak{h}$ a homomorphism of Lie algebras.*

1. *If ϕ is surjective, then $\phi(D^k\mathfrak{g}) = D^k(\phi(\mathfrak{g}))$. In particular $\phi(\mathfrak{g}))$ is solvable if $\mathfrak{g}$ is, thus any quotient of a solvable Lie algebra is solvable.*
2. *If $\mathfrak{g}$ is solvable then so are all subalgebras of $\mathfrak{g}$.*
3. *If im(ϕ) and ker(ϕ) are solvable then so is $\mathfrak{g}$. Thus if I is an ideal and I and $\mathfrak{g}/I$ are solvable, so is $\mathfrak{g}$.*

Proof. The first two statements are immediate from the definitions. For the third, note that if $\mathrm{im}(\phi)$ is solvable, then for some N we have $D^N\mathrm{im}(\phi) = \{0\}$, so that by part (1) we have $D^N(\mathfrak{g}) \subset \ker(\phi)$, hence if $D^M\ker(\phi) = \{0\}$ we must have $D^{N+M}\mathfrak{g} = \{0\}$ as required. □

Note as we saw before the last Lemma that the third part is *false* for nilpotent Lie algebras.

2.1. **Lie's theorem.** In this section we will assume that our field k has characteristic zero.

Lemma 2.4. *(Lie's Lemma) Let $\mathfrak{g}$ be a Lie algebra and let $I \subset \mathfrak{g}$ be an ideal, and V a finite dimensional representation. Suppose $v \in V$ is a vector such that $x(v) = \lambda(x).v$ for all $x \in I$, where $\lambda\colon I \to \mathsf{k}$. Then λ vanishes on $[\mathfrak{g}, I] \subset I$.*

Proof. Let $x \in \mathfrak{g}$. For each $m \in \mathbb{N}$ let $W_m = \mathrm{span}(v, x(v), \ldots, x^m(v)\}$. Thus the W_m form a nested sequence of subspaces of V. We claim that $hx^m(v) \in \lambda(h)x^m v +$

W_{m-1} for all $h \in I$ and $m \geq 0$. Using induction on m, the claim being immediate for $m = 0$, note that

$$\begin{aligned} hx^m(v) &= [h,x]x^{m-1}(v) + xhx^{m-1}(v) \\ &\in (\lambda([h,x])x^{m-1}v + W_{m-2}) + x(\lambda(h)x^{m-1}(v) + W_{m-2}) \\ &\in \lambda(h)x^m(v) + W_{m-1}, \end{aligned}$$

where in the second equality we use induction on m for both $h, [h,x] \in I$.

Now since V is finite dimensional, there is a maximal n such that the vectors $\{v, x(v), \ldots, x^n(v)\}$ are linearly independent, and so $W_m = W_n$ for all $m \geq n$. It then follows from the claim that W_n is preserved by x and every $h \in I$. Moreover, the claim shows that for any $h \in I$ the matrix of $[x,h]$ with respect to the basis $\{v, x(v) \ldots, x^n(v)\}$ of W_n is upper triangular with diagonal entries all equal to $\lambda([x,h])$. It follows that $\mathrm{tr}([x,h]) = (n+1)\lambda([x,h])$. Since the trace of a commutator is zero[1], it follows that $(n+1)\lambda([x,h]) = 0$, and so since $\mathrm{char}(\mathsf{k}) = 0$ we conclude that $\lambda([x,h]) = 0$. □

[1]It is important here that $\rho([x,h])$ is the commutator of $\rho(x)$ and $\rho(h)$ *both of which preserve* W_n – by the claim in the case of $\rho(h)$, and by our choice of n in the case of $\rho(x)$ – in order to conclude the trace is zero.