LIE ALGEBRAS: LECTURE 5.

1. SOLVABLE LIE ALGEBRAS (CONTINUED)

In this section we suppose that our field k is algebraically closed of characteristic zero.

Theorem 1.1. *(Lie's theorem) Let* $\mathfrak{g}$ *be a solvable Lie algebra and* V *is a* $\mathfrak{g}$*-representation. Then there is a homomorphism* $\lambda\colon \mathfrak{g} \to \mathfrak{gl}_1(\mathsf{k})$ *and a nonzero vector* $v \in V$ *such that* $x(v) = \lambda(x).v$ *for all* $x \in \mathfrak{g}$.

Proof. We use induction on $\dim(\mathfrak{g})$. If $\dim(\mathfrak{g}) = 1$, then $\mathfrak{g} = \mathsf{k}.x$ for any nonzero $x \in \mathfrak{g}$, and since k is algebraically closed, x has an eigenvector in V and we are done. For $\dim(\mathfrak{g}) > 1$, consider the derived subalgebra $D^1\mathfrak{g}$. Since $\mathfrak{g}$ is solvable, $D^1\mathfrak{g}$ is a proper ideal of $\mathfrak{g}$. The quotient $\mathfrak{g}/D^1\mathfrak{g}$ is abelian, and taking the preimage of any codimension one subspace of it gives a codimension 1 ideal I of $\mathfrak{g}$. By induction we may pick a homomorphism $\lambda\colon I \to \mathsf{k}$ such that the subspace $U = \{w \in V : x(w) = \lambda(h).w, \forall h \in I\}$ is nonzero. Now if $x \in \mathfrak{g}$, then

$$\begin{aligned} h(x(w)) &= [h,x](w) + xh(w) \\ &= \lambda([h,x])(w) + \lambda(h).x(w) \\ &= \lambda(h).x(w). \end{aligned}$$

where in the second equality we used Lie's Lemma. Thus $\mathfrak{g}$ preserves U. Now since I is codimension one in $\mathfrak{g}$, we may write $\mathfrak{g} = \mathsf{k}x \oplus I$ for some $x \in \mathfrak{g}$. Taking an eigenvector $v \in U$ of x completes the proof. □

Just as for nilpotent Lie algebras, we have the following corollary.

Corollary 1.2. *Let* $\mathfrak{b} \subset \mathfrak{gl}(V)$ *be a solvable subalgebra of* $\mathfrak{gl}(V)$*. Then there is a complete flag* $\mathcal{F}$ *of* V *such that* $\mathfrak{b} \subseteq \mathfrak{b}_{\mathcal{F}}$.

Proof. By induction on $\dim(V)$, where Lie's theorem provides the induction step. □

Note that this theorem shows that if $\mathfrak{g}$ is a solvable Lie algebra, then any irreducible representation of $\mathfrak{g}$ is one-dimensional. Since the converse is clearly true, we can rephrase Lie's theorem as the statement that any irreducible representationa of a solvable Lie algebra is one-dimensional. As such it is natural to note the following simple Lemma.

Lemma 1.3. *Let* $\mathfrak{g}$ *be a Lie algebra. The one-dimensional representations of* $\mathfrak{g}$ *are parametrized by the vector space* $(\mathfrak{g}/D\mathfrak{g})^*$ *as follows: if* (V,ρ) *is a a representation of* $\mathfrak{g}$ *and* $\dim(V) = 1$*, then* $\rho\colon \mathfrak{g} \to End(V) \cong \mathsf{k}$*, and* $\rho(D\mathfrak{g}) = 0$*, so that* ρ *induces a linear functional* $\alpha\colon \mathfrak{g}/D\mathfrak{g} \to \mathsf{k}$.

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Proof. If $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$, then since $\dim(V) = 1$, the associative algebra $\mathrm{End}(V)$ is commutative, so that $\mathfrak{gl}(V)$ is abelian. Thus

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x) = 0, \quad (\forall x, y \in \mathfrak{g}),$$

hence $\rho(D\mathfrak{g}) = 0$, and ρ induces $\alpha \in (\mathfrak{g}/D\mathfrak{g})^*$ as claimed. Conversely, given $\alpha\colon (\mathfrak{g}/D\mathfrak{g})^*$, the map $\rho\colon \mathfrak{g} \to \mathfrak{gl}_1(\mathsf{k})$ given by $x \mapsto \alpha(x + D\mathfrak{g})$ is a Lie algebra homomorphism, because

$$\rho([x,y]) = 0 = \alpha(x + D\mathfrak{g})\alpha(y + D\mathfrak{g}) - \alpha(y + D\mathfrak{g})\alpha(x + D\mathfrak{g}).$$

□

An isomorphism classes of one-dimensional representations of $\mathfrak{g}$ are thus given by the space of linear functionals $(\mathfrak{g}/D\mathfrak{g})^*$. We will refer to an element $\alpha \in (\mathfrak{g}/D\mathfrak{g})^*$ as a *weight* of $\mathfrak{g}$. They should be thought of as the generalization of the notion of an eigenvalue of a linear map. Lie's theorem shows that if $\mathfrak{g}$ is a solvable Lie algebra (hence in particular if $\mathfrak{g}$ is nilpotent), then any irreducible representation is one-dimensional, so that $(\mathfrak{g}/D\mathfrak{g})^*$ parametrizes the irreducible representations of $\mathfrak{g}$. For $\alpha \in (\mathfrak{g}/D\mathfrak{g})^*$ et us write k_α for the representation of $\mathfrak{g}$ on k given by α.

2. REPRESENTATIONS OF NILPOTENT LIE ALGEBRAS

In this section we assume that k is an algebraically closed field.

A representation (ρ, V) of the one-dimensional Lie algebra $\mathfrak{gl}_1(\mathsf{k})$ on a k-vector space V is given, once we chose a basis vector e of $\mathfrak{gl}_1(\mathsf{k})$, by a single linear map $\phi\colon V \to V$ via the correspondence $\phi = \rho(e)$. Thus the classification of representation of $\mathfrak{gl}_1(\mathsf{k})$ is equivalent to the classification of linear endomorphisms[1] This classification is of course given by the Jordan normal form (at least over an algebraically closed field). In this section we will see that a (slightly weaker) version of this classification holds for representations of any nilpotent Lie algebra.

2.1. The Jordan normal form.

We begin by reviewing, in a slightly more invariant form than you may have seen already, the Jordan canonical form. Let $x\colon V \to V$ be a linear map. Let V_λ be the *generalized eigenspaces* for x with eigenvalue λ:

$$V_\lambda = \{v \in V : \exists N > 0, (x - \lambda)^N(v) = 0\}.$$

(thus V_λ is zero unless λ is an eigenvalue of x).

Lemma 2.1. *Let $x\colon V \to V$ be a linear map. There is a canonical direct sum decomposition*

$$V = \bigoplus_{\lambda \in \mathsf{k}} V_\lambda,$$

of V into the generalized eigenspaces of x. Moreover, for each λ, the projection to $a_\lambda\colon V \to V_\lambda$ (with kernel the remaining generalized eigenspace of x) can be written as a polynomial in x.

Proof. Let $m_x \in \mathsf{k}[t]$ be the minimal polynomial of x. Then if $\phi\colon \mathsf{k}[t] \to \mathrm{End}(V)$ given by $t \mapsto x$ denotes the natural map, we have $\mathsf{k}[t]/(m_x) \cong \mathrm{im}(\phi) \subseteq \mathrm{End}(V)$.

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[1] Essentially the same is true for representations of the abelian group $\mathbb{Z}$.

If $m_x = \prod_{i=1}^{k}(t-\lambda_i)^{n_i}$ where the λ_i are the distinct eigenvalues of x, then the Chinese Remainder Theorem shows that

$$\mathsf{k}[t]/(m_x) \cong \bigoplus_{i=1}^{k} \mathsf{k}[t]/(t-\lambda_i)^{n_i},$$

It follows that we may write $1 \in \mathsf{k}[t]/(m_x)$ as $1 = e_1 + \ldots + e_k$ according to the above decomposition. Now clearly $e_i e_j = 0$ if $i \neq j$ and $e_i^2 = e_i$, so that if $U_i = \mathrm{im}(e_i)$, then we have $V = \bigoplus_{1 \leq i \leq k} U_i$. Moreover, each e_i can be written as polynomials in x by picking any representative in $\mathsf{k}[t]$ of e_i (thought of as an element of $\mathsf{k}[t]/(m_x)$). Note in particular this means that each U_i is invariant under $\mathrm{im}(\phi)$.

It thus remains to check that $U_i = V_{\lambda_i}$. Since $(t-\lambda_i)^{n_i} e_i = 0 \in \mathsf{k}[t]/(m_x)$, it is clear that $U_i \subseteq V_{\lambda_i}$. To see the reverse inclusion, suppose that $v \in V_{\lambda_i}$ so that, say, $(x-\lambda_i)^n(v) = 0$. Write $v = v_1 + \ldots + v_k$, where $v_j \in U_j$. Now since $(x-\lambda_i)^n(v) = 0$, and each U_j is stable under $\mathrm{im}(\phi)$ it follows that $(x-\lambda_i)^n(v_j) = 0$ for each j, $(1 \leq j \leq k)$. If $1 \leq l \leq k$ and $l \neq k$, then since $(t-\lambda_i)^n$ and $(t-\lambda_l)^{n_l}$ are coprime, we may find $a, b \in \mathsf{k}[t]$ such that $a.(t-\lambda_i)^n + b.(t-\lambda_l)^{n_l} = 1$.

Setting $t = x$ in this equation, and applying the result to the vector v_l we find:

$$\begin{aligned} v_l = 1.v_l &= \big(a.(t-\lambda_i)^n + b.(t-\lambda_l)^{n_l}\big)(v_l) \\ &= (t-\lambda_i)^n(v_l) + (t-\lambda_l)^{n_l}(v_l) \qquad (2.1) \\ &= 0, \end{aligned}$$

where the first term is zero by the above, and the second term is zero since $v_l \in U_l$. It follows that $v = v_i \in U_i$ and so $V_{\lambda_i} \subseteq U_i$ as required.

□

Next it is convenient to introduce some notation.

Definition 2.2. A sequence of maps of $\mathfrak{g}$-representations

$$U \xrightarrow{\alpha} V \xrightarrow{\beta} W$$

is said to be *exact at* V if $\mathrm{im}(\alpha) = \ker(\beta)$. A sequence of maps

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0$$

is called a *short exact sequence* if it is exact at each of U, V and W, so that α is injective and β is surjective and $\mathrm{im}(\alpha) = \ker(\beta)$. If V is the middle term of such a short exact sequence, it contains a subrepresentation isomorphic to U, such that the corresponding quotient representation is isomorphic to W, and hence, roughly speaking, V is built by gluing together U and W in some fashion.

The crucial step in our generalisation of a Jordan decomposition is to show that if $\mathfrak{g}$ is nilpotent, and U and V are non-isomorphic one-dimensional representations, then they cannot be "glued together" in any way other than by taking their direct sum.

Lemma 2.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $\alpha, \beta \in (\mathfrak{g}/D\mathfrak{g})^*$ be distinct. Any exact sequence of $\mathfrak{g}$-representations*

$$0 \longrightarrow \mathsf{k}_\alpha \longrightarrow V \longrightarrow \mathsf{k}_\beta \longrightarrow 0$$

splits, that is, $V \cong \mathsf{k}_\alpha \oplus \mathsf{k}_\beta$.

Proof. Suppose (V,ρ) is a representation as in the statement of the Lemma. Clearly, if $x \in \mathfrak{g}$, then $\rho(x)$ has eigenvalues $\alpha(x), \beta(x)$. Since $\alpha \neq \beta$ we may find $x_0 \in \mathfrak{g}$ such that $\alpha(x) \neq \beta(x)$, and hence $\rho(x_0)$ has distinct eigenvalues, and so is diagonalizable. But then it is easy to see that $\mathsf{ad}(\rho(x_0)) \in \mathfrak{gl}(\mathfrak{gl}(V))$ is diagonalizable with three distinct eigenvalues $\{0, \pm(\alpha(x)-\beta(x))\}$ and moreover the zero eigenspace is exactly the abelian subalgebra of matrices which preserve the eigenlines of $\rho(x_0)$. Since $\mathfrak{g}$ is nilpotent, the image $\rho(\mathfrak{g})$ lies in the 0-generalized eigenspace of $\rho(x_0)$, and hence the eigenlines of $\rho(x_0)$ are subrepresentations, isomorphic to k_α and k_β as required. $\square$