LIE ALGEBRAS: LECTURE 6.

1. Representations of Nilpotent Lie algebras: continued

The next theorem defines the analogue of generalized weight spaces for a linear map.

Theorem 1.1. *Let $\mathfrak{h}$ be a nilpotent Lie algebra and (V, ρ) a finite dimensional representation of $\mathfrak{h}$. For each $\lambda \in (\mathfrak{g}/D\mathfrak{g})^*$, set*

$$V_\lambda = \{v \in V : \textit{ for all } x \in \mathfrak{h}, \exists n > 0 \textit{ such that } (\rho(x) - \lambda(x))^n(v) = 0\}.$$

Then

$$V = \bigoplus_\lambda V_\lambda$$

that is, V is the direct sum of subrepresentations indexed by the one-dimensional representations occuring in V.

Proof. We first show that the V_λ form a direct sum. Indeed suppose we have a relation $v_1 + v_2 + \ldots + v_k = 0$ where $v_i \in V_{\lambda_i}$ say, for pairwise distinct $\lambda_1, \lambda_2, \ldots, \lambda_k \in (\mathfrak{g}/D\mathfrak{g})^*$. Since $(\mathfrak{g}/D\mathfrak{g})$ is not a finite union of codimension 1 subspaces[1], we may find an $x \in \mathfrak{g}$ such that $\lambda_i(x) \neq \lambda_j(x)$ for $i \neq j$. But then since V_{λ_i} is clearly contained in the generalized $\lambda_i(x)$-eigenspace of $\rho(x)$, and generalized eigenspaces form a direct sum, we see that each v_i must be zero. Note this also shows that there can be only finitely many nonzero V_λs.

We now prove the theorem by induction on $\dim(V)$, the result being clear if $\dim(V) = 1$. By Lie's theorem, we may pick a $\mathfrak{g}$-invariant line $L < V$. Consider the representation $U = V/L$ and let $q\colon V \to U$ denote the quotient map. By induction we have $U = \bigoplus_\lambda U_\lambda$, (where the U_λs are defined in the same way as the V_λs) a direct sum of $\mathfrak{g}$-subrepresentations. Now since L is 1-dimensional, we have $L \cong \mathsf{k}_\alpha$ for some $\alpha \in (\mathfrak{g}/D\mathfrak{g})^*$ so that $L \subseteq V_\alpha$. We claim that

i) $q^{-1}(U_\alpha) = V_\alpha$,
ii) $q^{-1}(U_\lambda) = L \oplus V_\lambda$ if $\alpha \neq \lambda$.

Clearly the theorem follows once we have established the claim, and moreover the case when $\lambda = \alpha$ is clear. Thus we may suppose that $\lambda \neq \alpha$ and let $W = q^{-1}(U_\lambda)$, a subrepresentation of V. We prove $W = L \oplus W_\lambda$ (Since W_λ clearly contains V_λ this suffices to prove part (ii) of the claim).

By Lie's theorem, the representation W/L contains a 1-dimensional subrepresentation, whose preimage in W is a subrepresentation U which lies in an exact sequence as in the statement of Lemma **??**, and hence it splits as a direct sum $U = L \oplus L'$, where $L' \cong \mathsf{k}_\lambda$, so that $L' \subset V_\lambda$. But then W/L' has dimension less than $\dim(W)$, so that by induction it splits as a direct sum $L \oplus (W/L')_\lambda$, and hence using claim (i) for W it follows that $W = L \oplus W_\lambda$ as required..

□

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[1]For this one needs to assume that the field k is infinite.

2. CARTAN SUBALGEBRAS

In this section we work over an algebraically closed field k.

Let $\mathfrak{g}$ be a Lie algebra. Recall that if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ then the normalizer $N_{\mathfrak{g}}(\mathfrak{h})$ of $\mathfrak{g}$ is

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} : [x, h] \in \mathfrak{h}, \forall h \in \mathfrak{h}\}.$$

It follows immediately from the Jacobi identity that $N_{\mathfrak{g}}(\mathfrak{h})$ is a subalgebra, and clearly $N_{\mathfrak{g}}(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ in which $\mathfrak{h}$ is an ideal.

Definition 2.1. We say that a subalgebra $\mathfrak{h}$ is a *Cartan subalgebra* if it is nilpotent and self-normalizing, that is, $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

It is not clear from this definition if a Lie algebra necessarily has a Cartan subalgebra. To show this we need a few more definitions.

Definition 2.2. If $x \in \mathfrak{g}$, let $\mathfrak{g}_{0,x}$ be the generalized 0-eigenspace of $\text{ad}(x)$, that is

$$\mathfrak{g}_{0,x} = \{y \in \mathfrak{g} : \exists N > 0 \text{ such that } \text{ad}(x)^N(y) = 0\}$$

We say that $x \in \mathfrak{g}$ is *regular* if $\mathfrak{g}_{0,x}$ is of minimal dimension. Note that it follows from the formula:

$$\text{ad}(x)^n[y, z] = \sum_{k=0}^{n} \binom{n}{k} [\text{ad}(x)^k(y), \text{ad}(x)^{n-k}(z)] \tag{2.1}$$

that $\mathfrak{g}_{0,x}$ is a subalgebra of $\mathfrak{g}$.

Lemma 2.3. *Let $x \in \mathfrak{g}$ be a regular element. Then $\mathfrak{g}_{0,x}$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. We first show that $\mathfrak{h}$ is a self-normalizing in $\mathfrak{g}$. Indeed if $z \in N_{\mathfrak{g}}(\mathfrak{h})$ then $[x, z] \in \mathfrak{h}$ (since certainly $x \in \mathfrak{h}$), so that for some n we have $\text{ad}(x)^n([x, z]) = 0$, and hence $\text{ad}(x)^{n+1}(z) = 0$ and $z \in \mathfrak{h}$ as required[2].

To show that $\mathfrak{h}$ is nilpotent, we use Engel's theorem, that is we will show that for each $y \in \mathfrak{h}$ the map $\text{ad}(y)$ is nilpotent as an endomorphism of $\mathfrak{h}$. To show $\text{ad}(y)$ is nilpotent on $\mathfrak{h}$, we consider the characteristic polynomial of $\text{ad}(y)$ on $\mathfrak{g}$ and $\mathfrak{h}$. Since $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, the characteristic polynomial $\chi^y(t) \in \mathsf{k}[t]$ of $\text{ad}(y)$ on $\mathfrak{g}$ is the product of the characteristic polynomials of $\text{ad}(y)$ on $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$, which we will write as $\chi_1^y(t)$ and $\chi_2^y(t)$ respectively.

We may write $\chi^y(t) = \sum_{k=0}^{n} c_k(y)t^k$ (where $n = \dim(\mathfrak{g})$). Pick $\{y_1, y_2, \ldots, y_r\}$ a basis of $\mathfrak{h}$, so that we may write $y = \sum_{i=1}^{r} a_i y_i$. Then we may view the coefficients c_k as polynomials in the coordinates $\{a_i : 1 \leq i \leq r\}$. Similarly, we have $\chi_1^y = \sum_{k=0}^{r} d_k(y)t^k$and $\chi_2^y = \sum_{i=0}^{n-r} e_i(y)t^i$ where the d_i, e_j are polynomials in the $\{a_i : 1 \leq i \leq r\}$. Now we know that if $y = x$ then $\text{ad}(x)$ is invertible on $\mathfrak{g}/\mathfrak{h}$, since it has no 0-eigenspace there, so that χ_2^x has $e_0(x) \neq 0$, and thus the polynomial e_0 is nonzero.

Now for each $y \in \mathfrak{h}$ the number $\min\{i : c_i(y) \neq 0\}$ is clearly $\dim(\mathfrak{g}_{0,y})$, so by the assumption that x is regular, this minimum must be at least $r = \dim(\mathfrak{h})$. However, suppose we write $\chi_1^y(t) = t^s \sum_{k=0}^{r-s} d_{k+s}t^k$, where $d_s \neq 0$ as a polynomial in the $\{a_i : 1 \leq i \leq r\}$. Then we would have

$$\chi^y(t) = t^s(d_s + d_{s+1}t + \ldots)(e_0 + e_1.t + \ldots) = t^s d_s e_0 + \ldots,$$

[2]Note that this didn't use the fact that x is regular, hence $\mathfrak{g}_{0,x}$ gives a self-normalizing subalgebra of $\mathfrak{g}$ for any $x \in \mathfrak{g}$.

hence if we pick $z \in \mathfrak{g}_{0,x}$ such that $d_s(z)e_0(z)$ is nonzero, then $\mathfrak{g}_{0,z}$ would have dimension $s < r$ which contradicts the choice of x. Thus we see that $\chi_1^y = t^r$ for all $y \in \mathfrak{g}_{0,x}$, and so every $\mathrm{ad}(y)$ is nilpotent on $\mathfrak{g}_{0,x}$, so that $\mathfrak{g}_{0,x}$ is a Cartan subalgebra as required.

$\square$