LIE ALGEBRAS: LECTURE 8.

1. Cartan criteria for solvable and semisimple Lie algebras

In this section k *is an algebraically closed field of characteristic zero.*

Proposition 1.1. *Suppose that $\mathfrak{g}$ is a Lie algebra, and suppose that $D\mathfrak{g} = \mathfrak{g}$. Then there is an $x \in \mathfrak{g}$ for which $\kappa(x,x) \neq 0$.*

Proof. Suppose that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and that $\mathfrak{g} = \bigoplus_{\lambda \in \Phi \cup \{0\}} \mathfrak{g}_\lambda$. Then by our assumptions we have

$$\mathfrak{g} = D\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = [\bigoplus_{\lambda \in \Phi \cup \{0\}} \mathfrak{g}_\lambda, \bigoplus_{\mu \in \Phi \cup \{0\}} \mathfrak{g}_\mu] = \sum_{\lambda,\mu} [\mathfrak{g}_\lambda, \mathfrak{g}_\mu].$$

Now since we know that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$, and moreover $\mathfrak{h} = \mathfrak{g}_0$, it follows that we must have

$$\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] + \sum_\lambda [\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}],$$

where the sum is over those roots λ such that $-\lambda$ is also a root. Now since $D\mathfrak{g} = \mathfrak{g}$ we see that $\mathfrak{g}$ is not solvable, and hence $\mathfrak{h} \neq \mathfrak{g}$. Thus there is at least one root $\beta \in \Phi$. Since β by definition vanishes on $D\mathfrak{h}$, there must be an $\alpha \in \Phi$ for which $\beta([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \neq \{0\}$. Picking $x \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\beta(x) \neq 0$, we find that:

$$\kappa(x,x) = \mathrm{tr}(\mathrm{ad}(x), \mathrm{ad}(x)) = \sum_{\lambda \in \Phi} \dim(\mathfrak{g}_\lambda)(\lambda(x))^2,$$

since the action of $\mathrm{ad}(x)$ on $\mathfrak{g}_\lambda$ has unique eigenvalue $\lambda(x)$. Now by Lemma **??** we know there are rational numbers $r_{\lambda,\alpha} \in \mathbb{Q}$ such that $\lambda(x) = r_{\lambda,\alpha}\alpha(x)$ for each $\lambda \in \Phi$, and hence

$$\kappa(x,x) = \big(\sum_{\lambda \in \Phi} \dim(\mathfrak{g}_\lambda).r_{\lambda,\alpha}^2\big)\alpha(x)^2,$$

Now $\beta(x) = r_{\beta,\alpha}\alpha(x)$, and $\beta(x) \neq 0$, hence $\alpha(x) \neq 0$ and $r_{\beta,\alpha} \neq 0$. It follows that $\kappa(x,x) \neq 0$ also. □

The Killing form can be used to detect when a Lie algebra is solvable.

Theorem 1.2 (*Cartan's criterion for solvability*)**.** *A Lie algebra $\mathfrak{g}$ is solvable if and only if the Killing form restricted to $D\mathfrak{g}$ is identically zero.*

Proof. Consider the derived series $D^k\mathfrak{g}$, $(k \geq 1)$. If there is some k with $D^k\mathfrak{g} = D^{k+1}\mathfrak{g} = D(D^k\mathfrak{g})$, then by Lemma **??** and Proposition 1.1 there is an $x \in D^k\mathfrak{g}$ with

$$\kappa^{\mathfrak{g}}(x,x) = \kappa^{D^k\mathfrak{g}}(x,x) \neq 0,$$

and hence κ is not identically zero. Thus we conclude $D^{k+1}\mathfrak{g}$ is a proper subspace of $D^k\mathfrak{g}$ for each k, and hence since $\mathfrak{g}$ is finite dimensional, it must be solvable as required.

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For the converse, if $\mathfrak{g}$ is solvable, then by Lie's theorem we can find a filtration $0 \subset F_1 \subset F_2 \subset \ldots \subset F_n = \mathfrak{g}$ of ad-stable subspaces with $\dim(F_i) = i$. Now if $x, y \in \mathfrak{g}$ the maps induced by ad(x)ad(y) and ad(y)ad(x) on F_i/F_{i-1} are equal, since $\mathfrak{gl}_1$ is commutative. Thus it follows that if $z = [x, y] \in D\mathfrak{g}$, then ad$(z) = [\text{ad}(x), \text{ad}(y)]$ maps F_i into F_{i-1} and hence for all $z \in D\mathfrak{g}$ we have ad$(z)(F_i) \subset F_{i-1}$. But now if $z_1, z_2 \in D\mathfrak{g}$, then ad$(z_1)ad(z_2)(F_i) \subseteq F_{i-2}$, and hence ad$(z_1)ad(z_2)$ is nilpotent, and so $\kappa(z_1, z_2) = \text{tr}(\text{ad}(z_1)\text{ad}(z_2)) = 0$ as required.

□

2. Semisimple Lie algebras, extensions and semi-direct products.

Suppose that $\mathfrak{g}$ is a Lie algebra, and $\mathfrak{a}$ and $\mathfrak{b}$ are solvable Lie ideals of $\mathfrak{g}$. It is easy to see that $\mathfrak{a} + \mathfrak{b}$ is again solvable (for example, because $0 \subseteq \mathfrak{a} \subseteq \mathfrak{a} + \mathfrak{b}$, and $\mathfrak{a}$ and $(\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ are both solvable). It follows that if $\mathfrak{g}$ is finite dimensional, then it has a largest solvable ideal $\mathfrak{r}$ (in the strong sense: every solvable ideal of $\mathfrak{g}$ is a subalgebra of $\mathfrak{r}$).

Definition 2.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The largest solvable ideal $\mathfrak{r}$ of $\mathfrak{g}$ is known as the *radical* of $\mathfrak{g}$, and will be denoted rad$(\mathfrak{g})$. We say that $\mathfrak{g}$ is *semisimple* if rad$(\mathfrak{g}) = 0$, that is, if $\mathfrak{g}$ contains no non-zero solvable ideals.

Lemma 2.2. *The Lie algebra $\mathfrak{g}$/rad$(\mathfrak{g})$ is semisimple, that is, it has zero radical.*

Proof. Suppose that $\mathfrak{s}$ is a solvable ideal in $\mathfrak{g}/\text{rad}(\mathfrak{g})$. Then if $\mathfrak{s}'$ denotes the preimage of $\mathfrak{s}$ in $\mathfrak{g}$, we see that $\mathfrak{s}'$ is an ideal of $\mathfrak{g}$, and moreover it is solvable since rad$(\mathfrak{g})$ and $\mathfrak{s} = \mathfrak{s}'/\text{rad}(\mathfrak{g})$ as both solvable. But then by definition we have $\mathfrak{s}' \subseteq \text{rad}(\mathfrak{g})$ so that $\mathfrak{s}' = \text{rad}(\mathfrak{g})$ and $\mathfrak{s} = 0$ as required. □

Thus we have shown that any Lie algebra $\mathfrak{g}$ contains a canonical solvable ideal rad$(\mathfrak{g})$ such that $\mathfrak{g}/\text{rad}(\mathfrak{g})$ is a semisimple Lie algebra. We thus have an exact sequence[1]

$$0 \longrightarrow \text{rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\text{rad}(\mathfrak{g}) \longrightarrow 0,$$

so that, in some sense at least, every finite dimensional Lie algebra is "built up" out of a semisimple Lie algebra and a solvable one. Slightly more precisely, if

$$0 \longrightarrow \mathfrak{g}_1 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}_2 \longrightarrow 0$$

is an exact sequence of Lie algebras, we say that $\mathfrak{g}$ is an *extension* of $\mathfrak{g}_2$ by $\mathfrak{g}_1$. Thus Lemma 2.2 can be rephrased as saying that any Lie algebra is an extension of the semisimple Lie algebra $\mathfrak{g}/\text{rad}(\mathfrak{g})$ by the solvable Lie algebra rad$(\mathfrak{g})$.

Definition 2.3. Suppose that $\mathfrak{g}$, $\mathfrak{h}$ are Lie algebras, and we have a homomorphism $\phi\colon \mathfrak{g} \to \text{Der}_{\mathsf{k}}(\mathfrak{h})$, the Lie algebra of derivations[2] on $\mathfrak{h}$. Then it is straight-forward to check that we can form a new Lie algebra $\mathfrak{h} \rtimes \mathfrak{g}$, the *semi-direct product* of $\mathfrak{g}$ and $\mathfrak{h}$ by ϕ which as a vector space is just $\mathfrak{g} \oplus \mathfrak{h}$, and where the Lie bracket is given by:

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2] + \phi(y_1)(x_2) - \phi(y_2)(x_1), [y_1, y_2]),$$

[1] Of Lie algebras rather than representations, but exactly the same definitions apply in this case also: the image of the incoming morphism is the kernel of the outgoing morphism. In particular the zeros at each end record the facts that the map from rad$(\mathfrak{g})$ to $\mathfrak{g}$ is injective (since it is an inclusion) and the map from $\mathfrak{g}$ to $\mathfrak{g}/\text{rad}(\mathfrak{g})$ is surjective (since it is a quotient map).

[2] Recall that the derivations of a Lie algebra are the linear maps $\alpha\colon \mathfrak{h} \to \mathfrak{h}$ such that $\alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)]$.

where $x_1, x_2 \in \mathfrak{h}, y_1, y_2 \in \mathfrak{g}$. This is the Lie algebra analogue of the semidirect product of groups, where you build a group $H \rtimes G$ via a map from G to the automorphisms (rather than derivations) of H. The Lie algebra $\mathfrak{h}$, viewed as the subspace $\{(x, 0) : x \in \mathfrak{h}\}$ of $\mathfrak{h} \rtimes \mathfrak{g}$, is clearly an ideal of $\mathfrak{h} \rtimes \mathfrak{g}$ and it is easy to check that we then have an extension

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{h} \rtimes \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Thus semi-direct products give us examples of extensions of Lie algebras.

In fact, semidirect products correspond to a relatively simple kind of extension.

Definition 2.4. An extension of Lie algebras

$$0 \xrightarrow{i} \mathfrak{g}_1 \longrightarrow \mathfrak{g} \xrightarrow{q} \mathfrak{g}_2 \longrightarrow 0$$

is said to be *split* if there is a map $s\colon \mathfrak{g}_2 \to \mathfrak{g}$ such that $q \circ s = \mathrm{id}_{\mathfrak{g}_2}$. If such a map exists, then its image $s(\mathfrak{g}_2)$ is a subalgebra of $\mathfrak{g}$ which is isomorphic to $\mathfrak{g}_2$ and is complementary to $\mathfrak{g}_1$, in the sense that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector spaces. In fact we have $\mathfrak{g} \cong \mathfrak{g}_1 \rtimes s(\mathfrak{g}_2)$ where $s(\mathfrak{g}_2)$ acts by derivations on $\mathfrak{g}_1$ via the adjoint action (since $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$).

Note that there may be many ways to split an exact sequence of Lie algebras (see the problem sheet).

In characteristic zero, the extension of Lie algebras given by the rad($\mathfrak{g}$) and $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is always split:

Theorem 2.5. *(Levi's theorem) Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k of characteristic zero, and let $\mathfrak{r}$ be its radical. Then there exists a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g} \cong \mathfrak{r} \ltimes \mathfrak{s}$.*

Note that in particular $\mathfrak{s}$ is isomorphic to $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$. We will not prove this theorem in this course.

Remark 2.6. Note that an extension does not *a priori* have to split! A simple example of this in the world of abelian groups is the following:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{q} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where the map from $\mathbb{Z}$ to itself is given by multiplication by 2. (This is an exact sequence since clearly q is surjective and multiplication by 2 is injective). The map q does not split, as $\mathbb{Z}$ does not contain any subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

The Killing form also gives us a way of detecting when a Lie algebra is semisimple. Recall that a bilinear form $B\colon V \times V \to \mathsf{k}$ is said to be nondegenerate if $\{v \in V : \forall w \in V, B(v, w) = 0\} = \{0\}$. To prove it we first note the following simple result.

We have the following simple characterisation of semisimple Lie algebras.

Lemma 2.7. *A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if it does not contain any non-zero abelian ideals.*

Proof. Clearly if $\mathfrak{g}$ contains an abelian ideal if contains a solvable ideal, so that $\mathrm{rad}(\mathfrak{g}) \neq 0$. Conversely, if $\mathfrak{s}$ is a non-zero solvable ideal in $\mathfrak{g}$, then the last term in the derived series of $\mathfrak{s}$ will be an abelian ideal of $\mathfrak{g}$ (*check this*!). □

Theorem 2.8. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate.*

Proof. Let $\mathfrak{g}^\perp = \{x \in \mathfrak{g} : \kappa(x, y) = 0, \forall y \in \mathfrak{g}\}$. Then by Lemma **??** $\mathfrak{g}^\perp$ is an ideal in $\mathfrak{g}$, and clearly the restriction of κ to $\mathfrak{g}^\perp$ is zero, so by Cartan's Criterion, and Lemma **??** the ideal $\mathfrak{g}^\perp$ is solvable. It follows that if $\mathfrak{g}$ is semisimple we must have $\mathfrak{g}^\perp = \{0\}$ and hence κ is non-degenerate.

Conversely, suppose that κ is non-degenerate. To show that $\mathfrak{g}$ is semisimple it is enough to show that any abelian ideal of $\mathfrak{g}$ is trivial, thus suppose that $\mathfrak{a}$ is an abelian ideal. Then if $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$ is arbitrary, the composition $\text{ad}(x)\text{ad}(y)\text{ad}(x)$ must be zero, since $\text{ad}(y)\text{ad}(x)(z) \in \mathfrak{a}$ for any $z \in \mathfrak{g}$, as $\mathfrak{a}$ is an ideal, and since $\mathfrak{a}$ is abelian $\text{ad}(a)(b) = 0$ for all $a, b \in \mathfrak{a}$. But then clearly $(\text{ad}(y)\text{ad}(x))^2 = 0$, so that $\text{ad}(y)\text{ad}(x)$ is nilpotent and $\kappa(a, x) = 0$ for all $x \in \mathfrak{g}$. But then $\mathfrak{a} \subseteq \mathfrak{g}^\perp = \{0\}$ and $\mathfrak{a} = \{0\}$ as required. ∎