LIE ALGEBRAS

1. Representations of nilpotent Lie algebras.

This note gives an alternative proof of the structure of representations of nilpotent Lie algebras. We work over an algebraically closed field k.

The following Lemma was a question on one of the problem sets.

Lemma 1.1. *Let $\mathfrak{g}$ be a Lie algebra and (V, ρ) a representation of $\mathfrak{g}$. Then for all $x, y \in \mathfrak{g}$ and all $\lambda, \mu \in \mathsf{k}$ we have*

$$(\rho(x) - \lambda - \mu)^n \rho(y) = \sum_{i=0}^{n} \binom{n}{i} \rho((ad(x) - \mu)^i(y))(\rho(x) - \lambda)^{n-i}. \tag{1.1}$$

Theorem 1.2. *Let $\mathfrak{h}$ be a nilpotent Lie algebra and (V, ρ) a finite dimensional representation of $\mathfrak{h}$. For each $\lambda \in (\mathfrak{g}/D\mathfrak{g})^*$, set*

$$V_\lambda = \{v \in V : \textit{for all } x \in \mathfrak{h}, \exists n > 0 \textit{ such that } (\rho(x) - \lambda(x))^n(v) = 0\}.$$

Then each V_λ is a subrepresentation and

$$V = \bigoplus_\lambda V_\lambda$$

that is, V is the direct sum of subrepresentations indexed by the one-dimensional representations occuring in V.

Proof. We use induction on $\dim(V)$. If $\dim(V) = 1$ the result is trivial. Next note that if $V = U \oplus W$ is a direct sum of proper subrepresentations, then applying induction we know the result holds for U and W respectively, and clearly $V_\lambda = U_\lambda \oplus W_\lambda$, so that we may conclude the result holds for V.

Now suppose that $x \in \mathfrak{h}$ and let $V = \bigoplus V_\lambda(x)$ denote the generalised eigenspace decomposition of V for $\rho(x)$. We claim each subspace $V_\lambda(x)$ is a subrepresentation of $\mathfrak{h}$. Indeed since $\mathfrak{h}$ is a nilpotent Lie algebra, the map $\mathrm{ad}(x)$ is nilpotent, hence if $k > \dim(\mathfrak{h})$ then $\mathrm{ad}(x)^k(y) = 0$ for all $y \in \mathfrak{h}$. It follows that if $v \in V_\lambda(x)$, so that $(\rho(x) - \lambda)^k(v) = 0$ for all $k \geq N$ say, then if $k \geq N + \dim(\mathfrak{h})$, we see from equation (1.1) (with $\mu = 0$) that $(\rho(x) - \lambda)^k(\rho(y)(v)) = 0$, since we must have either $i \geq \dim(\mathfrak{h})$ or $k - i \geq N$ if $k \geq N + \dim(\mathfrak{h})$, and so $\rho(y)(v) \in V_{\lambda(x)}(x)$ as required.

It follows that if there is some $x \in \mathfrak{h}$ which has more than one eigenvalue, then the decomposition $V = \bigoplus V_{\lambda(x)}(x)$ shows that V is a direct sum of proper subrepresentations, and we are done by induction. Thus we are reduced to the case where for all $x \in \mathfrak{h}$ the linear map $\rho(x)$ has a single eigenvalue $\lambda(x)$ say. But then $x \mapsto \lambda(x)$ must be an element of $(\mathfrak{h}/D\mathfrak{h})^*$ by Lie's theorem: Pick a one-dimensional subrepresentation L of V, then $\rho(x)$ acts by a scalar $\mu(x)$ on L, where $\mu \in (\mathfrak{h}/D\mathfrak{h})^*$ and since $\lambda(x)$ is the only eigenvalue of $\rho(x)$ by assumption we must have $\lambda(x) = \mu(x)$. Now clearly $V = V_\lambda$ and we are done.

□

Remark 1.3. The (nonzero) subrepresentations V_λ are known as the *weight spaces* of V, and $\lambda \in (\mathfrak{h}/D\mathfrak{h})^*$ for which $V_\lambda \neq 0$ are known as the *weights* of V.

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